EXISTENCE AND REGULARITY OF POSITIVE SOLUTIONS TO ELLIPTIC EQUATIONS OF SCHRÖDINGER TYPE

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Abstract. We prove the existence of positive solutions with optimal local regularity to the homogeneous equation of Schrödinger type,

$$-\mathrm{div}(\mathcal{A}\nabla u) - \sigma u = 0 \quad \text{in } \Omega,$$

under only a form boundedness assumption on $\sigma \in D'(\Omega)$ and ellipticity assumption on $\mathcal{A} \in L^\infty(\Omega)^{n\times n}$, for an arbitrary open set $\Omega \subseteq \mathbf{R}^n$.

We demonstrate that there is a two way correspondence between the form boundedness and the existence of positive solutions to this equation, as well as weak solutions to the equation with quadratic nonlinearity in the gradient,

$$-\mathrm{div}(\mathcal{A}\nabla v) = (\mathcal{A}\nabla v)\cdot\nabla v + \sigma \quad \text{in } \Omega.$$

As a consequence, we obtain necessary and sufficient conditions for both the form-boundedness (with a sharp upper form bound) and the positivity of the quadratic form of the Schrödinger type operator $\mathcal{H} = -\mathrm{div}(\mathcal{A}\nabla\cdot) - \sigma$ with arbitrary distributional potential $\sigma \in D'(\Omega)$, and give examples clarifying the relationship between these two properties.

1. Introduction

1.1. The goal of this paper is to present an existence and regularity theory for *positive* solutions to the equation of Schrödinger type:

$$-\mathrm{div}(\mathcal{A}\nabla u) - \sigma u = 0 \quad \text{in } \Omega, \tag{1.1}$$

on an arbitrary open set $\Omega \subseteq \mathbf{R}^n$, $n \geq 1$, under the standard ellipticity assumptions on $\mathcal{A} \in L^\infty(\Omega)^{n\times n}$, and the sole condition of form boundedness on the real-valued distributional potential $\sigma \in D'(\Omega)$:

$$|\langle \sigma, h^2\rangle| \leq C \int_\Omega (\mathcal{A}\nabla h)\cdot\nabla h\, dx, \qquad \text{for all } h \in C_0^\infty(\Omega). \tag{1.2}$$

Simultaneously, a corresponding theory will be developed for (possibly sign changing) weak solutions to the equation with quadratic growth in the gradient:

$$-\mathrm{div}(\mathcal{A}\nabla v) = (\mathcal{A}\nabla v)\cdot\nabla v + \sigma \quad \text{in } \Omega. \tag{1.3}$$

In displays (1.1)–(1.3), $\mathcal{A} : \Omega \to \mathbf{R}^{n\times n}$ is a real $n \times n$ (possibly non-symmetric) matrix-valued function on Ω, so that there exist $m, M > 0$ such that, for a.e. $x \in \Omega$:

$$m|\xi|^2 \leq \mathcal{A}(x)\xi\cdot\xi, \qquad \text{and} \qquad |\mathcal{A}(x)\xi| \leq M|\xi|, \qquad \text{for all } \xi \in \mathbf{R}^n\backslash\{0\}. \tag{1.4}$$

It has been a long standing problem to extend the existing theory to general classes of σ, including highly oscillating, singular or distributional potentials, where

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the separation of the positive and negative parts of σ is impossible due to the interaction between them. In our framework of distributional σ, positive solutions to the Schrödinger equation are not locally bounded, and consequently standard PDE tools based on Harnack's inequality, and the classical iterative techniques of Moser [Mos60] and Trudinger [Tru73], or their extension due to Brezis and Kato [BK79], are no longer available.

A primary result of the present paper is the following principle: *For any form bounded potential σ with the upper form bound strictly less than* 1*, one can find a positive solution of (1.1) which lies in the local Sobolev space* $L^{1,2}_{\rm loc}(\Omega)$. This regularity for positive solutions of (1.1) is in fact optimal in the generality of the potentials considered here, as demonstrated by examples discussed below. Furthermore, there is a two way correspondence between the existence of positive solutions $u \in L^{1,2}_{\rm loc}(\Omega)$ of (1.1) satisfying an additional *logarithmic Caccioppoli-type condition*, and the form boundedness of the potential σ; see Theorem 1.1.

As a consequence, necessary and sufficient conditions will be established for both the form-boundedness and the positivity of the quadratic form of the Schrödinger type operator $\mathcal{H} = -\text{div}(\mathcal{A}\nabla\cdot) - \sigma$ with arbitrary distributional potential $\sigma \in D'(\Omega)$. The form boundedness property (1.2) is known to be equivalent to the boundedness of the operator $\mathcal{H} : L^{1,\,2}_0(\Omega) \to L^{-1,2}(\Omega)$ from the homogeneous Sobolev space $L^{1,\,2}_0(\Omega)$ into its dual. It is therefore a natural class of potentials in which to study the Schrödinger equation. In a wide class of domains Ω, it has been characterized by the second and third authors [MV02a, MV06].

Our results for the equations (1.1) and (1.3) in turn provide an alternative proof (with a sharp upper form bound) of the characterization of (1.2) established in [MV02a, MV06], where harmonic analysis and potential theory methods were employed. In addition, we obtain a characterization of potentials $\sigma \in \mathcal{D}'(\Omega)$ satisfying the corresponding *semi-boundedness* property, so that the operator $\mathcal{H}$ is non-negative:

$$\langle \sigma, h^2 \rangle \le \int_\Omega (\mathcal{A}\nabla h)\cdot \nabla h \, dx, \qquad \text{for all } h \in C_0^\infty(\Omega). \tag{1.5}$$

Both equations (1.1) and (1.3), as well as the quadratic form properties of $\mathcal{H}$ (1.2) and (1.5) are of fundamental importance to partial differential equations, spectral theory, and mathematical physics. Consequently, these questions have attracted the attention of many authors, starting from the foundational work of Bôcher, Hartman, Hille, and Wintner on the Sturm-Liouville theory (see e.g., [Hi48], [Har82], Chapter 11), followed by seminal contributions of Agmon [Ag83], Aizenman and Simon [AS82], Ancona [An86], Brezis and Kato [BK79], Chung and Zhao [CZ95], Maz'ya [Maz85], Murata [Mur86] et al. in the multi-dimensional case. A recent survey of this rich area has been given by Pinchover [Pin07]. We also refer to [Maz69, CFKS89, NP92, BNV94, BM97, RS98, RSS94, SW99, Fit00, MS00, Sha00, Mur02, DN02, DD03] and references therein for equation (1.1) and form-boundedness properties (1.2), (1.5), and [AHBV09, Ev90, FM98, FM00] for equation (1.3).

Given the wealth of the previous literature, it is important to stress what is novel about our approach. In all the papers listed above, various assumptions on the potential σ ensure the validity of Harnack's inequality for positive solutions of the Schrödinger equation or some form of compactness properties of $\mathcal{H}$. Moreover, σ is usually decomposed into the sum of its positive and negative parts: $\sigma = \sigma_+ - \sigma_-$, which are treated separately, with more stringent assumptions on σ_+ than σ_-. In many of these results σ_+ is assumed to belong to the Kato class of potentials, while σ_- to the local Kato class (see [AS82], [CZ95]). The corresponding positive solutions are continuous, and the existence of a positive solution is equivalent to the positivity of the quadratic form of $\mathcal{H}$. In the mathematical physics literature the latter is known as the Allegretto-Piepenbrink theorem (see e.g. [CFKS89], Sec. 2). All these tools are not available for general potentials σ.

The primary technical hurdles of our approach in comparison with the existing literature arise from the following essential characteristics of σ satisfying (1.2):

(1) σ in general does not lie *globally* in a dual Sobolev space, i.e. $\sigma \notin L^{-1,s}(\Omega)$ for any $s > 0$;
(2) there are no local compactness conditions on σ.

From the first item above, it is clear that one cannot follow standard methods to achieve global estimates which would yield the existence of solutions of (1.1). Indeed, there are simple examples of σ so that a solution u of (1.1) does not lie in $L^1(\Omega)$. On the other hand, as a result of the second item, finding the correct quantity to work with in order to prove local estimates becomes a subtle issue.

We will see that the two inequalities contained in (1.2) are responsible for two distinct aspects of the existence of solutions to (1.1) and (1.3). Let us therefore consider the following upper and lower bounds of the quadratic form $\langle \sigma, h^2 \rangle$:

$$\langle \sigma, h^2 \rangle \le \lambda \int_\Omega (\mathcal{A}\nabla h) \cdot \nabla h \, dx, \qquad \text{for all } h \in C_0^\infty(\Omega), \tag{1.6}$$

and

$$\langle \sigma, h^2 \rangle \ge -\Lambda \int_\Omega (\mathcal{A}\nabla h) \cdot \nabla h \, dx, \qquad \text{for all } h \in C_0^\infty(\Omega). \tag{1.7}$$

In what follows, a positive function u on Ω is defined to be a function $u \in L^{1,2}_{\text{loc}}(\Omega)$ such that $u > 0$ quasi-everywhere in Ω. Let us now state our first theorem:

Theorem 1.1. *Let $\Omega \subseteq \mathbf{R}^n$, $n \ge 1$, be an open set. Let $\sigma \in D'(\Omega)$, and $\mathcal{A} : \Omega \to \mathbf{R}^{n\times n}$ be a matrix function satisfying the ellipticity conditions (1.4). Then the following statements hold:*

(i) *Suppose σ obeys (1.6) with an upper form bound $\lambda < 1$, and (1.7) with a lower form bound $\Lambda > 0$. Then $\sigma \in L^{-1,2}_{loc}(\Omega)$, and there exists a positive solution $u \in L^{1,2}_{loc}(\Omega)$ of the equation*

$$-\text{div}(\mathcal{A}\nabla u) = \sigma \, u \quad \text{in } D'(\Omega), \tag{1.8}$$

so that the following logarithmic Caccioppoli inequality holds:

$$\int_\Omega \frac{|\nabla u|^2}{u^2} \phi^2 dx \le C_0 \int_\Omega |\nabla \phi|^2 dx, \qquad \text{for all } \phi \in C_0^\infty(\Omega), \tag{1.9}$$

with $C_0 = C_0(n, m, M, \Lambda) > 0$.

(ii) *Suppose $\sigma \in L^{-1,2}_{loc}(\Omega)$, and there exists a solution $u \in L^{1,2}_{loc}(\Omega)$ satisfying (1.9) for a constant $C_0 > 0$. Then there exists a solution $v \in L^{1,2}_{loc}(\Omega)$ of the equation*

$$-\text{div}(\mathcal{A}\nabla v) = \mathcal{A}(\nabla v) \cdot \nabla v + \sigma \quad \text{in } D'(\Omega), \tag{1.10}$$

such that:

$$\int_\Omega |\nabla v|^2 \, \phi^2 dx \le C_0 \int_\Omega |\nabla \phi|^2 dx, \qquad \text{for all } \phi \in C_0^\infty(\Omega). \tag{1.11}$$

(iii) *Suppose (1.10) has a solution* $v \in L^{1,2}_{loc}(\Omega)$ *satisfying (1.11) for a positive constant* $C_0 > 0$. *Then* σ *satisfies the lower form bound (1.7) for a positive constant* $\Lambda = \Lambda(C_0, m, M) > 0$, *and the upper form bound (1.6) with:*

(a) $\lambda = 1$ *if* $\mathcal{A}$ *is symmetric.*

(b) $\lambda = \left(\frac{M}{m}\right)^2$ *if* $\mathcal{A}$ *is non-symmetric.*

In the case where σ is a *positive measure*, the relationship between positive *superharmonic* supersolutions of (1.8) and the validity of (1.6) has been discussed in a general framework using probabilistic methods by Fitzsimmons in [Fit00]. This work was in turn building on the important paper of Ancona [An86]. In our framework, as we are considering oscillating potentials, one cannot rely on the theory of superharmonic functions, and we need to prove sharper estimates in order to obtain the stronger $L^{1,2}_{\rm loc}$ regularity. Without this it is not obvious how to even make sense of solutions to the equation (1.1).

The sharpness of Theorem 1.1 is exhibited by well known examples. One such example is included in Sec. 7 for the convenience of the reader. Here it is also shown that in general there exist positive solutions u of (1.1) which lie in the space $L^{1,1}_{\rm loc}(\Omega)$, but not $L^{1,2}_{\rm loc}(\Omega)$, and that statement (i) of Theorem 1.1 fails in general if $\lambda = 1$. One should also note that in statement (iii), in the case of a non-symmetric matrix $\mathcal{A}$, the constant λ must in general depend on M and m, see Sec. 7.

1.2. In Theorem 1.1, it was seen that the lower form bound (1.7) on the potential σ is necessary in order to obtain solutions satisfying the regularity conditions (1.9) and (1.11). These conditions are of importance in our application of Theorem 1.1 in characterizing the inequality (1.2), and are also of classical interest in partial differential equations.

However, if one is solely interested in the existence of solutions to (1.1) and (1.3), then condition (1.7) can be relaxed to a local condition stated in terms of dual Sobolev spaces: when $n \geq 3$, suppose

$$\sigma = \operatorname{div}(\vec{G}) \quad \text{in } \Omega, \quad \text{where} \quad \vec{G} \in \mathcal{L}^{2,n-2}_{\rm loc}(\Omega)^n. \tag{1.12}$$

Here $\mathcal{L}^{2,n-2}_{\rm loc}(\Omega)$ is the local Morrey space defined in Sec. 2; see (2.2) and (2.11) below. This condition is significantly weaker than (1.2). In dimensions $n = 1, 2$ we only require $\sigma \in L^{-1,2}_{\rm loc}(\Omega)$.

The following theorem should be compared to statements (i) and (ii) of Theorem 1.1 above:

Theorem 1.2. *Let* $\Omega \subseteq \mathbf{R}^n$ *be an open set, and suppose that* $\mathcal{A} : \Omega \to \mathbf{R}^{n\times n}$ *is a matrix function satisfying the ellipticity conditions (1.4). Let* $\sigma \in \mathcal{D}'(\Omega)$ *satisfy (1.6) with a constant* $0 < \lambda < 1$, *and in addition suppose:*

(a) $n = 1, 2$, *and* $\sigma \in L^{-1,2}_{loc}(\Omega)$;

(b) $n \geq 3$, *and* σ *satisfies the local condition (1.12).*

Then there exists a positive solution $u \in L^{1,2}_{loc}(\Omega)$ *of (1.1), and a solution* $v \in L^{1,2}_{loc}(\Omega)$ *of (1.3).*

A crude sufficient condition for (1.12) is $\sigma \in L^{n/2}_{\rm loc}(\Omega) + L^{-1,n}_{\rm loc}(\Omega)$, but much more general σ are admissible for (1.12). We emphasize that condition (1.12) is considerably weaker than the usual local Kato class condition. It is not necessary for the existence of a positive solution $u \in L^{1,2}_{\rm loc}(\Omega)$ of (1.1). However, it is the sharp condition to obtain solutions u of (1.1) so that $\log(u) \in BMO_{\rm loc}(\Omega)$.

To prove Theorems 1.1 and 1.2, we make crucial use of certain Caccioppoli-type inequalities. As was mentioned above, the classical iterative techniques used in [Mos60, Tru73, BK79, AS82, CFG86, MZ97] are not available.

Instead, we interpolate between a Caccioppoli inequality and an estimate on the mean oscillation of the logarithm to obtain uniform doubling properties on an approximating sequence. See Proposition 3.8, which constitutes a key part of the argument. From this doubling property, one can deduce local uniform gradient estimates. This technique yields the optimal regularity for solutions of (1.1) in the generality of potentials satisfying (1.2) or (1.12).

Along the way, we obtain a characterization of when a nonnegative weight function satisfying a weak reverse Hölder inequality is doubling (see Sec. 2.2 for definitions). Our main hard analysis tool here is Proposition 2.3 which may be of independent interest.

1.3. Let us now turn to discussing applications of Theorem 1.1. As a first application, we deduce an alternative approach to the results of the second and third authors in [MV02a], regarding the characterization of the inequality (1.2). It avoids the heavy harmonic analysis and potential theory machinery that was used in [MV02a], and is considerably more elementary. This program is carried out in Sec. 4. In particular, if $\Omega = \mathbf{R}^n$ and with $\mathcal{A}$ the identity matrix, we will show that the form boundedness condition (1.2) is equivalent to the following representation of σ:

$$\sigma = \operatorname{div}(\vec{\Gamma}), \quad \text{with} \int_{\mathbf{R}^n} h^2 |\vec{\Gamma}|^2 dx \le C_1 \int_{\mathbf{R}^n} |\nabla h|^2 dx, \qquad \text{for all } h \in C_0^\infty(\Omega). \tag{1.13}$$

Moreover, (1.13) with $C_1 = \frac{1}{4}$ implies (1.2) with $C = 1$. Conversely, (1.2) with $C < 1$ (or more precisely, (1.6) with $\lambda < 1$ and (1.7) with $\Lambda > 0$) implies (1.13).

In Section 5, we consider distributions $\sigma \in \mathcal{D}'(\Omega)$ satisfying the semi-boundedness property (1.5). In the case of the Laplacian, it means

$$\langle \sigma, h^2 \rangle \le \int_\Omega |\nabla h|^2 \, dx, \qquad \text{for all } h \in C_0^\infty(\Omega). \tag{1.14}$$

Our main result in this regard is the following. (See Theorem 5.1 below for a similar criterion concerning the general operator $\operatorname{div}\mathcal{A}(\nabla \cdot)$ in place of the Laplacian.)

Theorem 1.3. *A real-valued distribution $\sigma \in \mathcal{D}'(\Omega)$ satisfies (1.14) if and only if there exists $\vec{\Gamma} \in L^2_{loc}(\Omega)^n$, so that:*

$$\sigma \le \operatorname{div}(\vec{\Gamma}) - |\vec{\Gamma}|^2 \qquad \text{in } \mathcal{D}'(\Omega). \tag{1.15}$$

The inequality in (1.15) can not in general be strengthened to an equality. Such conditions have their roots in classical Sturm-Liouville theory in one dimension, see e.g. [Har82] (Chapter 11, Theorem 7.2).

It had been conjectured that a condition characterizing (1.14) was the following:

$$\sigma \le \operatorname{div}(\vec{\Phi}), \quad \text{where} \int_\Omega |h|^2 |\vec{\Phi}|^2 dx \le C \int_\Omega |\nabla h|^2 dx, \qquad \text{for all } h \in C_0^\infty(\Omega), \tag{1.16}$$

for some $\vec{\Phi} \in L^2_{\mathrm{loc}}(\Omega)^n$ and $C > 0$. In other words, this means that 'half' the condition (1.13) found in [MV02a] should characterize semi-boundedness. However, it is proved below that, for any $C > 0$, condition (1.16) is *not necessary* for (1.14) to hold, although it is obviously sufficient when $C = \frac{1}{4}$.

Proposition 1.4. *Let $\Omega = \mathbf{R}^n$, $n \geq 1$. Let σ be the radial potential defined by:*

$$\sigma = \cos r + \frac{n-1}{r}\sin r - \sin^2 r,$$

where $r = |x|$. Then σ satisfies (1.14), but cannot be represented in the form (1.16).

This is the content of Proposition 7.1 in Sec. 7, where additional examples are exhibited to help clarify our results.

Theorem 1.3 above concerned the case when $\lambda = 1$ in (1.6), so that the Schrödinger operator fails to be coercive in the homogeneous Sobolev space. In the case when σ is a positive measure, one can instead consider superharmonic supersolutions of the equation (1.1) in this critical case. This is a sharpening of [Fit00] mentioned above. Indeed:

Proposition 1.5. *Suppose that Ω is an open set, and let σ be a positive Borel measure defined on Ω. Let $\mathcal{A}$ be a symmetric matrix function satisfying (1.4). Then σ satisfies:*

$$\int_\Omega h^2 d\sigma \leq \int_\Omega \mathcal{A}(\nabla h)\cdot\nabla h\, dx, \qquad \text{for all } \; h \in C_0^\infty(\Omega), \tag{1.17}$$

if and only if there exists a positive superharmonic function u so that:

$$-\text{div}\mathcal{A}(\nabla u) \geq \sigma u \quad \text{ in } \Omega.$$

In Proposition 1.5, the notion of superharmonicity is the one associated to the operator $\mathcal{A}$, see e.g. [HKM06]. This is proved in Section 3.8 below.

In Section 6, we consider a recent result of Frazier, Nazarov and the third author [FNV10] on positive solutions with prescribed boundary values. Let us recall one of their main theorems. Suppose Ω is a bounded NTA domain, and that σ is a nonnegative measure in Ω. Then, under precise necessary and sufficient conditions on σ up to the boundary, a positive minimal solution u (called the *gauge* in the probabilistic literature, see e.g. [CZ95]) to the equation

$$\begin{cases} -\Delta u = \sigma u \quad \text{in } \Omega, \\ u = 1 \quad \text{on } \partial\Omega, \end{cases} \tag{1.18}$$

is constructed in [FNV10] (see Theorem 6.1 below). The solution is understood in the sense that

$$u(x) = \int_\Omega G(x,y)\, u(y)\, d\sigma(y) + 1,$$

where $G(x,y)$ is Green's function of the Laplacian. In this paper, we will adapt the approach taken in the proof of Theorem 1.1 to show that in fact $u \in L^{1,2}_{\text{loc}}(\Omega)$; see Theorem 6.2 below. This regularity is again optimal under the assumptions of the theorem.

In conclusion, we remark that our approach outlined above is nonlinear in nature, and an extension to general quasilinear operators of p-Laplacian type will be presented in a forthcoming paper [JMV10], where the L^p-analogue of (1.2) will be characterized.

2. Preliminaries

2.1. Notation and function spaces. For an open set $\Omega \subseteq \mathbf{R}^n$, $n \geq 1$, we denote by $C_0^\infty(\Omega)$ the space of smooth functions with compact support in Ω. The energy space $L_0^{1,2}(\Omega)$ is then the completion of $C_0^\infty(\Omega)$ with respect to the Dirichlet norm $\|\nabla h\|_{L^2(\Omega)}$. The majority of estimates in this paper are local; we say that $h \in L^{1,2}_{\text{loc}}(\Omega)$ if $h\phi \in L_0^{1,2}(\Omega)$ whenever $\phi \in C_0^\infty(\Omega)$.

For test function arguments it will be useful to introduce the space $L_c^{1,2}(\Omega)$. We say that $h \in L_c^{1,2}(\Omega)$ if $h \in L_0^{1,2}(\Omega)$ has compact support.

Define $L^{-1,2}(\Omega)$ to be the dual of $L_0^{1,2}(\Omega)$. Then a distribution $\sigma \in L^{-1,2}_{\text{loc}}(\Omega)$ if $\phi\sigma \in L^{-1,2}(\Omega)$ for any $\phi \in C_0^\infty(\Omega)$.

We will write $V \subset\subset U$, for two open sets $U, V \subset \mathbf{R}^n$, if there exists a compact set $K \subset \mathbf{R}^n$ so that $V \subset K \subset U$.

Throughout the paper, we use the usual notation for the integral average:

$$\fint_E \cdots\, dx = \frac{1}{|E|}\int_E \cdots\, dx.$$

For an open set U, we say $u \in BMO(U)$ if there is a positive constant D_U so that:

$$\fint_{B(x,r)} |u(y) - \fint_{B(x,r)} u(z)\, dz|^2 dy \leq D_U, \text{ for any ball } B(x,2r) \subset U. \tag{2.1}$$

In addition, $u \in BMO_{\text{loc}}(\Omega)$ if for each compactly supported open set $U \subset\subset \Omega$, there is a positive constant $D_U > 0$ so that (2.1) holds.

Let us next introduce the *local Morrey space*: we say $f \in \mathcal{L}^{p,q}_{\text{loc}}(\Omega)$ if, for each compactly supported set $U \subset\subset \Omega$, there exists a constant C_U so that:

$$\int_{B(z,s)} |f|^p\, dx \leq C_U s^q, \qquad \text{for all balls } B(z,2s) \subset U. \tag{2.2}$$

We conclude with the definition of a multiplier (see [MSh09]). Let X and Y be two normed function spaces, and let Z be a dense subset of X. We say that g is a multiplier from X to Y, written as $g \in M(X \to Y)$, if $g \cdot f \in Y$ for all $f \in Z$, and there is a positive constant $C > 0$ so that the following inequality holds:

$$\|g \cdot f\|_Y \leq C\, \|f\|_X\,, \qquad \text{for all } f \in Z.$$

In what follows X and Y will be $L_0^{1,2}(\Omega)$ and $L^2(\Omega)$ respectively, and Z will be $C_0^\infty(\Omega)$.

2.2. On weak reverse Hölder inequalities and BMO. In this section we characterize the weak reverse Hölder weights that are doubling. This forms a key tool in our argument. First, let us introduce some notation.

Definition 2.1. Let $U \subset \mathbf{R}^n$ be an open set, and let w be a nonnegative measurable function. Then w is said to be *doubling* in U if there exists a constant $A_U > 0$ so that,

$$\fint_{B(x,2r)} w\, dx \leq A_U \fint_{B(x,r)} w\, dx, \text{ for all balls } B(x,4r) \subset U. \tag{2.3}$$

Let w be a nonnegative measurable function. Then w is said to satisfy a *weak reverse Hölder inequality* in U if there exists constants $q > 1$ and $B_U > 0$ so that:

$$\Big(\fint_{B(x,r)} w^q dx\Big)^{1/q} \leq B_U \fint_{B(x,2r)} w\, dx, \text{ for all balls } B(x,2r) \subset U. \tag{2.4}$$

Remark 2.2. The following simple consequence of the doubling property will prove useful. Let U be an open set, and suppose w is doubling in U. Then, whenever $B(x, 4r) \subset U$ and $z \in B(x, r)$ with $B(z, 4s) \subset U$, it follows:

$$\fint_{B(x,r)} w(y)dy \leq C(A_U, s, r) \fint_{B(z,s)} w(z)dz.$$

This principle will be used in a Harnack chain argument in Proposition 3.8.

Our argument hinges on the following result:

Proposition 2.3. *Let U be an open set, and suppose w satisfies the weak reverse Hölder inequality (2.4) in U. Then w is doubling in U, i.e. (2.3) holds, if and only if $\log(w) \in BMO(U)$ (see (2.1)).*

In particular, if w satisfies (2.4) and

$$\fint_{B(x,s)} |\log w(y) - \fint_{B(x,s)} \log w(z)dz|^2 dy \leq D_U, \text{ for all balls } B(x, 2s) \subset U. \tag{2.5}$$

Then there is a constant $C(A_U, D_U) > 0$, so that for any ball $B(x, 4r) \subset U$:

$$\fint_{B(x,2r)} w\, dx \leq C(A_U, D_U) \fint_{B(x,r)} w\, dx. \tag{2.6}$$

Only the the sufficiency direction is required in what follows; however, since this characterization does not seem to appear explicitly in the literature we prove the full statement. To prove Proposition 2.3, we use the following lemma:

Lemma 2.4. *Let $U \subset \mathbf{R}^n$ be an open set. Suppose that there exist $s > 1$ and $w \geq 0$, along with a constant $C_1 > 0$ so that the following inequality holds:*

$$\Big(\fint_{B(x,r)} w^s\, dx\Big)^{1/s} \leq C_1 \fint_{B(x,2r)} w\, dx, \text{ whenever } B(x, 2r) \subset U.$$

Then, for any $t > 0$, there exists a constant $C_t = C(t, C_1) > 0$ so that:

$$\Big(\fint_{B(x,r)} w^s\, dx\Big)^{1/s} \leq C_t \Big(\fint_{B(x,2r)} w^t\, dx\Big)^{1/t}, \quad \text{whenever } B(x, 2r) \subset U.$$

This lemma had been used in proving estimates for quasilinear equations by G. Mingione [Min07]. A proof can be found in Remark 6.12 of [Giu03]. Let us now turn to proving the proposition.

Proof of Proposition 2.3. Let us first prove the necessity, suppose that w satisfies the weak reverse Hölder inequality (2.4), and in addition that w is doubling in U. Then, for each ball $B(x, 4r) \subset U$:

$$\Big(\fint_{B(x,r)} w^q dx\Big)^{1/q} \leq B_U \fint_{B(x,2r)} w\, dx \leq A_U B_U \fint_{B(x,r)} w\, dx.$$

It follows that w satisfies a reverse Hölder inequality in U, and is therefore a Muckenhoupt A_∞-weight. It follows (see Chapter 5 of [St93]) that $\log(u) \in BMO(U)$.

Let us now turn to the converse statement. Suppose w satisfies (2.5) and (2.4). From (2.5), it is a well known consequence of the John-Nirenberg inequality that there exists a constant $0 < t \leq 1$ so that w^t is an A_2-weight in U, i.e. there exists a positive constant $A > 0$ (depending on D_U in (2.5)) so that for all balls $B(z, 2s) \subset U$:

$$\fint_{B(z,s)} w^t \, dx \leq A\Big(\fint_{B(z,s)} w^{-t} \, dx\Big)^{-1}. \tag{2.7}$$

Indeed, from the John-Nirenberg inequality (see [St93]), there exists a constant $t = t(D_U)$ so that for any ball so that $B(z, 2s) \subset U$:

$$\fint_{B(z,s)} \exp\Big(t\Big|\log(w)(y') - \fint_{B(z,s)} \log(w(y))dy\Big|\Big)dy' \leq C(D_U). \tag{2.8}$$

Inequality (2.8) clearly implies the two inequalities:

$$\fint_{B(z,s)} \exp\Big(\log(w^t(y')) - \fint_{B(z,s)} \log(w^t(y))dy\Big)dy' \leq C(D_U), \quad \text{and:}$$

$$\fint_{B(z,s)} \exp\Big(\log(w^{-t}(y')) + \fint_{B(z,s)} \log(w^t(y))dy\Big)dy' \leq C(D_U).$$

Multiplying these two inequalities together, one obtains (2.7).

It follows from (2.7) and Jensen's inequality that, if $B(z, 4s) \subset U$:

$$\fint_{B(z,2s)} w^t \, dx \leq A2^n (\fint_{B(z,s)} w^{-t} \, dx\Big)^{-1} \leq A2^n \fint_{B(z,s)} w^t \, dx. \tag{2.9}$$

Let $B(z, 8s) \subset U$, then, applying Lemma 2.4 with this choice of t:

$$\fint_{B(z,2s)} w \, dx \leq C_{U,t}\Big(\fint_{B(z,4s)} w^t \, dx\Big)^{1/t} \leq \tilde{C}_{U,t}\Big(\fint_{B(z,s)} w^t \, dx\Big)^{1/t} \leq \tilde{C}_{U,t} \fint_{B(z,s)} w \, dx.$$

The second inequality in the chain follows from the doubling of w^t, and the last inequality follows from Hölder's inequality. By a standard covering argument, the factor of 8 in the enlargement of the ball can be replaced by 4, which yields (2.6). This completes the proposition. □

2.3. Preliminaries for distributional potentials σ. Let Ω be an open set in $\mathbf{R}^n$, with $n \geq 2$. Let $\mathcal{A} : \Omega \to \mathbf{R}^{n\times n}$ satisfying (1.4). For a real-valued distribution σ defined on Ω, we define the multiplication operator by:

$$\langle \sigma h, h\rangle := \langle \sigma, h^2\rangle, \text{ for all } h \in C_0^\infty(\Omega).$$

Suppose now σ satisfies (1.2) for a positive constant $C > 0$, and let us write:

$$||h||_{\mathcal{A}} = \Big(\int (\mathcal{A}\nabla h) \cdot \nabla h dx\Big)^{1/2}, \text{ for } h \in C_0^\infty(\Omega)$$

It is easy to see that (1.2) is equivalent to the inequality:

$$|\langle \sigma g, h\rangle| \leq C||g||_{\mathcal{A}}||h||_{\mathcal{A}} \text{ for all } g, h \in C_0^\infty(\Omega), \tag{2.10}$$

with $C > 0$ the same constant that appears in (1.2). Furthermore, by the boundedness and ellipticity assumptions (1.4) on the operator $\mathcal{A}$, we can extend (2.10) by continuity, so that (2.10) is valid for all $g, h \in L_0^{1,2}(\Omega)$. We denote this extension again by σ. This simple observation will be used several times in the sequel.

We now state some simple lemmas regarding the local character of the distributional potentials we will consider. Let us begin with an alternative way of stating the local condition (1.12):

Lemma 2.5. *Suppose that $n \geq 3$ and $\sigma \in \mathcal{D}'(\Omega)$. If σ satisfies (1.12) then the following condition holds:*

$$\begin{aligned}&\textit{For each open set } U \subset\subset \Omega, \textit{ there exists a constant } C_U > 0,\\ &\textit{such that whenever } B(z,2s) \subset U:\\ &|\langle h, \sigma\rangle| \leq C_U s^{(n-2)/2} ||\nabla h||_{L^2(B(z,s))}, \textit{ for all } h \in C_0^\infty(B(z,s)).\end{aligned} \tag{2.11}$$

The two conditions (2.11) and (1.12) are in fact equivalent, but we omit the proof of the converse statement to Lemma 2.5, as we will not use it, and its proof is slightly lengthy. To prove the lemma, note that if $\sigma = \mathrm{div}(\vec{G})$, with $|\vec{G}|$ satisfying (1.12), then it follows directly from differentiation that (2.11) holds.

We next prove a key local property of distributions σ satisfying (1.2), namely that (1.2) is a stronger condition than the condition (2.11). We define the capacity $\mathrm{cap}(E, \Omega)$ of a compact set E by:

$$\mathrm{cap}(E,\Omega) = \inf\{||\nabla h||^2_{L^2(\Omega)} \,:\, h \in C_0^\infty(\Omega),\, h \geq 1 \text{ on } E\}. \tag{2.12}$$

Lemma 2.6. *Suppose that σ satisfies (1.2), and let $V \subset\subset \Omega$. Then $\sigma \in L^{-1,2}(V)$, and:*

$$||\sigma||_{L^{-1,2}(V)} \leq MC \cdot \mathrm{cap}(V,\Omega)^{1/2}. \tag{2.13}$$

Here M and C are as in (1.4), and (1.2) respectively. In particular, for each ball $B(x,r)$ so that $B(x,2r) \subset \Omega$:

$$||\sigma||_{L^{-1,2}(B(x,r))} \leq C_1(C,M) r^{(n-2)/2}. \tag{2.14}$$

From display (2.14), it follows that (1.2) is stronger than the local condition (2.11).

Proof. Let $h \in C_0^\infty(V)$, and let $g \in C_0^\infty(\Omega)$, so that $g \equiv 1$ on V. Then by (2.10) and (1.2):

$$|\langle \sigma, h\rangle| = |\langle \sigma g, h\rangle| \leq C||g||_{\mathcal{A}}||h||_{\mathcal{A}} \leq MC||\nabla g||_2 ||\nabla h||_2.$$

Therefore $\sigma \in L^{-1,2}_{\mathrm{loc}}(\Omega)$, and minimising over such g yields (2.13) by definition of capacity. The second estimate is a special case of (2.13) and follows from well known estimates for the capacity of a ball, see e.g. [Maz85]. □

It will be convenient to use a mollification of the potential σ. Let us fix a smooth radial approximate identity ϕ, i.e. $\phi \in C_0^\infty(B(0,1))$, so that $\phi \geq 0$ on $B_1(0)$, with $||\phi||_{L^1} = 1$. For $\epsilon > 0$, we denote $\phi_\epsilon = \epsilon^{-n}\phi(x/\epsilon)$. Then, denote the convolution of the distribution by $\sigma_\epsilon = \phi_\epsilon * \sigma$. We will write $d\sigma_\epsilon = \sigma_\epsilon dx$. The next two lemmas show that the mollification does not effect σ in terms of form boundedness.

Lemma 2.7. *Let Ω be an open set, and let $V \subset\subset \Omega$. In addition let $\epsilon \leq d(V, \partial\Omega)/2$. Suppose that $\sigma \in \mathcal{D}'(\Omega)$ so that (5.1) holds. Then, it holds:*

$$\int_V h^2 d\sigma_\epsilon \leq \int_V (\mathcal{A}_\epsilon \nabla h)\cdot \nabla h dx, \textit{ for all } h \in C_0^\infty(V). \tag{2.15}$$

where:

$$\mathcal{A}_\epsilon(x) = (\phi_\epsilon * \mathcal{A})(x),$$

In the notation of Lemma 2.7, the mollified matrix $\mathcal{A}_\epsilon$ satisfies the bounds (1.4) inside of V, with the same constants ellipticity constants m and M. This will be used often in the sequel.

Proof. Let $h \in C_0^\infty(V)$. We first note that by the interchange of mollification and the distribution (see Lemma 6.8 of [LL01]):

$$\langle \sigma, \phi_\epsilon * h^2 \rangle = \int_{B(0,\epsilon)} \phi_\epsilon(t) \langle \sigma, h(\cdot - t)^2 \rangle dt.$$

By elementary geometry, $h(\cdot - t) \in C_0^\infty(\Omega)$ for all $t \in B(0,\epsilon)$, and hence:

$$\begin{aligned} \langle \sigma_\epsilon, h^2 \rangle &\leq \int_{B(0,\epsilon)} \phi_\epsilon(t) \Big(\int_\Omega (\mathcal{A}(x) \nabla h(x-t)) \cdot \nabla h(x-t) dx \Big) dt \\ &= \int_\Omega \mathcal{A}_\epsilon(x) \nabla h(x) \cdot \nabla h(x) dx, \end{aligned} \tag{2.16}$$

which proves the lemma. □

Our second mollification lemma says that if σ satisfies the local condition (2.11), then so does the mollification of σ. Let us introduce the notation $U_\epsilon = \{x \in \mathbf{R}^n : \mathrm{dist}(x, U) < \epsilon\}$.

Lemma 2.8. *Suppose $n \geq 3$ and $\sigma \in \mathcal{D}'(\Omega)$ is a real valued distribution satisfying (2.11). Let $V \subset\subset \Omega$, then if $\epsilon < d(V, \partial\Omega/2)$, the mollified potential σ_ϵ satisfies (2.11) for all open sets $U \subset\subset V$, with constant C_{U_ϵ}.*

Proof. Let $U \subset\subset V$ be a compactly supported open set. Then, for $B(x, 2r) \subset U$ and $h \in C_0^\infty(B(x,r))$, $h(\cdot - x) \in C_0^\infty(\Omega)$ for all $x \in B_\epsilon(0)$. Hence, for all such h:

$$\begin{aligned} \Big| \int_\Omega h d\sigma_\epsilon \Big| &\leq \int_{B_\epsilon(0)} \phi_\epsilon(x) |\langle \sigma, h(\cdot - x) \rangle| dx \\ &\leq C_{U_\epsilon} r^{(n-2)/2} \int_{B_\epsilon(0)} \phi_\epsilon(x) ||\nabla h(\cdot - x)||_{L^2} dx \leq C_{U_\epsilon} r^{(n-2)/2} ||\nabla h||_{L^2}. \end{aligned} \tag{2.17}$$

This completes the proof. □

3. The proofs of Theorems 1.1 and 1.2

In this section we prove our primary existence theorems, as well as prove the connections between the solutions of the equations (1.1) and (1.3) with the validity of (1.2). Throughout this section we will suppose without loss of generality $\Omega \subset \mathbf{R}^n$ is a *connected* open set; note that in the case of an arbitrary open set, our arguments apply to each connected component. This assumption is used in a Harnack chain argument.

The most substantial argument will be the assertion of statement (i) in Theorem 1.1, and its refinement Theorem 1.2. We restate these two results as propositions for convenience. In light of Lemma 2.5, the existence result for equation (1.1) in Theorem 1.2 follows from:

Proposition 3.1. *Let $\mathcal{A} : \Omega \to \mathbf{R}^{n \times n}$ be a possibly non-symetric matrix function satisfying (1.4). Suppose that σ is a real-valued distribution satisfying the local dual Sobolev condition (2.11), and the upper boundedness condition (1.6) for a constant $0 < \lambda < 1$. Then there is a positive solution $u \in L^{1,2}_{loc}(\Omega)$ of (1.1).*

The second proposition concerns the case when σ in addition satisfies (1.7):

Proposition 3.2. *Suppose $\mathcal{A}:\Omega \to \mathbf{R}^{n\times n}$ is a possibly non-symetric matrix function satisfying (1.4). In addition, suppose that σ is a real-valued distribution satisfying (1.7) for a positive constant $\Lambda > 0$, and (1.6) for a constant $0<\lambda<1$. Then there is a positive solution $u \in L^{1,2}_{loc}(\Omega)$ of (1.1). Furthermore, u satisfies:*

$$\int_\Omega \frac{|\nabla u|^2}{u^2}\phi^2\, dx \leq C\int_\Omega |\nabla \phi|^2\, dx \quad \textit{for all } \phi \in C_0^\infty(\Omega). \tag{3.1}$$

It was seen in Lemma 2.6 that if σ satisfies (1.6) and (1.7), then the condition (2.11) holds. Hence the existence part of Proposition 3.2 follows from Proposition 3.1.

3.1. An approximating sequence. To prove Propositions 3.1 and 3.2, we use local properties of σ to find a solutions to a mollified variant of equation (1.1) in a sequence of subdomains of Ω. We will then prove a uniform gradient estimate on this sequence.

Suppose that σ satisfies (1.6) with $0<\lambda<1$. Let Ω_j, for $j\geq 1$ be an exhaustion of Ω by smooth domains, that is, $\Omega_j \subset\subset \Omega_{j+1}$, and $\Omega = \bigcup_j \Omega_j$. In addition, let us fix a ball B so that its concentric enlargement $4B \subset\subset \Omega_1$. Let $\epsilon_0 = 1$, and $\epsilon_j = \min(\epsilon_{j-1}/2, d(\Omega_j, \partial\Omega_{j+1})/2, 2^{-j})$ for $j\geq 1$. With this notation, define:

$$\sigma_j = \phi_{\epsilon_j} * \sigma, \text{ and } \mathcal{A}_j = \phi_{\epsilon_j} * \mathcal{A}.$$

Define u_j to be the solution of:

$$\begin{cases} -\text{div}(\mathcal{A}_j\nabla u_j) = \sigma_j u_j & \text{in } \Omega_j, \\ \displaystyle\int_B u_j^2 dx = 1, \quad u_j \geq 0 & \text{q.e.} \end{cases} \tag{3.2}$$

Furthermore, u_j satisfies the Harnack inequality in Ω_j.

Proof of existence and uniqueness of (3.2). We will see that the existence of (3.2) is a simple consequence of the Lax-Milgram lemma. Define the bilinear form $\mathcal{L}$ on $L^{1,2}_0(\Omega_j)\times L^{1,2}_0(\Omega_j)$ by:

$$\mathcal{L}(w,h) = \int_{\Omega_j} (\mathcal{A}_j \nabla w)\cdot \nabla h dx - \langle \sigma_j w, h\rangle, \text{ with } w,h \in L^{1,2}_0(\Omega_j).$$

Then, by the assumptions on σ, the following properties hold:

$$|\mathcal{L}(w,h)| \leq (M + C(n)||\sigma_j||_{L^\infty(\Omega_j)}|\Omega_j|^{2/n})\,\|\nabla w\|_2\,\|\nabla h\|_2\,, \text{ and:}$$

$$\mathcal{L}(w,w) \geq m(1-\lambda)\,\|\nabla w\|_2\,.$$

The first inequality follows from (1.4) along with the Sobolev inequality. The second inequality is a combination of Lemma 2.7 and (1.4). Hence the hypothesis of the Lax-Milgram lemma are satisfied.

Applying the Lax-Milgram lemma, there exists a unique $w \in L^{1,2}_0(\Omega_j)$ so that:

$$\mathcal{L}(w,h) = \langle \sigma_j, h\rangle.$$

Let $v_j = w+1$. Let us next show that $v_j \geq 0$ q.e. To see this, let $w = v_j$, and note that $w^- = \min(w,0) \in L^{1,2}_0(\Omega_j)$. By testing (3.2) with the valid test function w^-, it follows:

$$\int_{\Omega_j} \mathcal{A}_j(\nabla w^-)\cdot\nabla w^- = \langle \sigma_j w^-, w^-\rangle \leq \lambda \int_{\Omega_j} \mathcal{A}_j(\nabla w^-)\cdot \nabla w^-. \tag{3.3}$$

Since $0 < \lambda < 1$, it follows from (1.4) that $w^- = 0$ q.e. Let us now define:

$$u_j = \Big(\fint_B v_j dx \Big)^{-1} v_j.$$

Then u_j solves (3.2). The validity of Harnack's inequality for u_j follows from classical elliptic regularity theory, see e.g. [Tru73], since σ_j is smooth. □

3.2. Caccioppoli and Morrey estimates for the approximating sequence. We next turn to proving two estimates on the gradient of the approximating sequence. The first estimate is a Caccioppoli inequality:

Lemma 3.3. *Suppose that σ satisfies (1.6) with $0 < \lambda < 1$, and let $\{u_j\}$ be the sequence constructed in (3.2). Let $\psi \in C_0^\infty(\Omega_j)$, then for any $k \geq j$:*

$$\int_{\Omega_j} |\nabla u_k|^2 \, \psi^2 \, dx \leq C(M, m, \lambda) \int_{\Omega_j} u_k^2 \, |\nabla \psi|^2 \, dx. \tag{3.4}$$

Proof. Let us fix k and j as in the statement of the lemma, and let $v = u_k$. With $\psi \in C_0^\infty(\Omega_j)$, $\psi \geq 0$, test the weak formulation of (3.2) with $v\psi^2 \in L_0^{1,2}(\Omega_j)$. Using (1.6), it follows:

$$\begin{aligned}
\int_{\Omega_j} ((\mathcal{A}_k \nabla v) \cdot \nabla v) \psi^2 \, dx &\leq \langle \sigma_k v, \psi^2 v \rangle + 2M \int_{\Omega_j} v \, \psi \, |\nabla v| \, |\nabla \psi| \, dx \\
&= \langle \sigma_k(\psi v), \psi v \rangle + 2M \int_{\Omega_j} v \, \psi \, |\nabla v| \, |\nabla \psi \,| \, dx \\
&\leq \lambda \int_{\Omega_j} \mathcal{A}_k(\nabla(v\psi))(\nabla(v \, \psi)) \, dx + 2M \int_{\Omega_j} v \, \psi \, |\nabla v| \, |\nabla \psi| \, dx.
\end{aligned}$$

Here we have used Lemma 2.7 and $\langle \sigma_k v, \psi^2 v \rangle = \langle \sigma_k(v\psi), v\psi \rangle$. By Cauchy's inequality, it follows that for any $\epsilon > 0$ there exists a constant C_ϵ, depending on ϵ, λ, M and m, so that:

$$(1 - \lambda) \int_{\Omega_j} (\mathcal{A}_k \nabla v) \cdot \nabla v \, \psi^2 \, dx \leq \epsilon \int_{\Omega_j} |\nabla v|^2 \psi^2 \, dx + C_\epsilon \int_{\Omega_j} v^2 |\nabla \psi|^2 \, dx.$$

Choosing $\epsilon < (1 - \lambda)m$ and rearranging, we recover (3.4). □

The second estimate we use relates a uniform bound on the gradient logarithm with uniform properties on the 'negative part' of the distribution σ:

Lemma 3.4. *With $\{u_j\}$ the sequence constructed in (3.2), the following estimate holds for all $\psi \in C_0^\infty(\Omega_j)$:*

$$\int_{\Omega_j} \frac{|\nabla u_j|^2}{u_j^2} \psi^2 \, dx \leq -C(M, m) \int_{\Omega_j} \psi^2 \, d\sigma_j + C(M, m) \int_{\Omega_j} |\nabla \psi|^2 \, dx. \tag{3.5}$$

Proof. Let $h = \psi^2 / u_j$, with $\psi \in C_0^\infty(\Omega_j)$, $\psi \geq 0$. Since u_j satisfies the Harnack inequality in Ω_j, there exists a constant $c > 0$ so that $u_j > c$ on the support of ψ. It follows that h is a valid test function for the weak formulation of (3.2). This yields:

$$-\int_{\Omega_j} (\mathcal{A}_j \nabla u_j) \cdot \nabla \Big(\frac{\psi^2}{u_j} \Big) dx = -\Big\langle \sigma_j, \psi^2 \Big\rangle \tag{3.6}$$

On the other hand:

$$m\int_{\Omega_j}\frac{|\nabla u_j|^2}{u_j^2}\psi^2dx \leq -\int_{\Omega_j}(\mathcal{A}_j\nabla u_j)\cdot\nabla\Big(\frac{\psi^2}{u_j}\Big)dx + 2M\int_{\Omega_j}\frac{|\nabla u_j|}{u_j}|\nabla\psi|\psi dx,$$

and therefore, by Cauchy's inequality:

$$\int_{\Omega_j}\frac{|\nabla u_j|^2}{u_j^2}\psi^2dx \leq -\int_{\Omega_j}(\mathcal{A}_j\nabla u_j)\cdot\nabla\Big(\frac{\psi^2}{u_j}\Big)dx + C(M,m)\int_{\Omega_j}|\nabla\psi|^2dx. \tag{3.7}$$

Combining (3.6) and (3.7), we deduce (3.5). □

From Lemma 3.4, we deduce two estimate, depending on the addition properties of σ. First, we deduce that the so-called logarithmic Cacciopolli inequality holds if σ in addition satisfies (1.7).

Lemma 3.5. *Suppose that the real-valued distribution σ satisfies (1.7) with constant $\Lambda > 0$, and let $\{u_j\}_j$ be as in (3.2). Let $\psi \in C_0^\infty(\Omega_j)$, then for any $k \geq j$:*

$$\int_{\Omega_j}\frac{|\nabla u_k|^2}{u_k^2}\psi^2dx \leq C(M,m,\Lambda)\int_{\Omega_j}|\nabla\psi|^2dx \tag{3.8}$$

Proof. From Lemma 3.4, it clearly suffices to show that, for all $\psi \in C_0^\infty(\Omega_j)$:

$$-\int_{\Omega_j}\psi^2d\sigma_j \leq M\Lambda\int_{\Omega_j}|\nabla\psi|^2dx.$$

But this follows in precisely the same manner as Lemma 2.7. □

An immediate corollary of Lemma 3.5, and the definition of capacity in (2.12), is:

Corollary 3.6. *Under the hypotheses of Lemma 3.5, there exists a positive constant $C = C(\Lambda, m, M)$, so that whenever $F \subset\subset \Omega_j$, it follows:*

$$\int_F\frac{|\nabla u_k|^2}{u_k^2}dx \leq C\text{cap}(F,\Omega_j), \qquad \textit{for all } k \geq j. \tag{3.9}$$

In the case when σ only satisfies (2.11), a local Morrey space estimate holds, which is a weakened version of (3.8):

Lemma 3.7. *Suppose $n \geq 3$, and that σ satisfies (2.11). Consider the sequence $\{u_j\}$ as in (3.2). Then for each ball $B(x,r)$ so that $B(x,2r) \subset \Omega_j$, it follows that for all $k > j$:*

$$\int_{B(x,r)}\frac{|\nabla u_k|^2}{u_k^2}\,dx \leq C(\Omega_j,M,m)r^{n-2}. \tag{3.10}$$

Proof. Fix such a ball $B(x,r)$ as in the statement of the lemma, and let $\psi \in C_0^\infty(B(x,2r))$ such that $\psi \equiv 1$ on $B(x,r)$, $0 \leq \psi \leq 1$ and $|\psi| \leq C/R$. The lemma follows from estimating (3.5) with this choice of ψ. It suffices to prove, for all $k > j$:

$$\Big|\int_{\Omega_k}\psi^2\,d\sigma_k\Big| \leq C(\Omega_j)r^{n-2}. \tag{3.11}$$

Picking $U = \Omega_j$ in the definition of (2.11), we deduce from Lemma 2.8 that:

$$\Big|\int_{\Omega_k}\psi^2d\sigma_k\Big| \leq C_{(\Omega_j)_{\epsilon_k}}r^{n-2}||\nabla(\psi^2)||_{L^2(\Omega_j)} \leq 2C_{(\Omega_j)_{\epsilon_j}}||\nabla\psi||_{L^2(B(x,2r))}.$$

The last inequality follows since $0 \le \psi \le 1$. Here $(\Omega_j)_{\epsilon_j}$ is the ϵ_j-neighbourhood of Ω_j. Note that $(\Omega_j)_{\epsilon_j} \subset \Omega_{j+1}$, and since $\epsilon_k < \epsilon_j$, it follows $(\Omega_j)_{\epsilon_k} \subset (\Omega_j)_{\epsilon_j}$ and hence by definition in (2.11): $C_{(\Omega_j)_{\epsilon_k}} \le C_{(\Omega_j)_{\epsilon_j}}$.

The display (3.11) now follows from the estimate on the gradient of ψ. □

In the case $n = 1$ or 2; note that if $\sigma \in L^{-1,2}_{\text{loc}}(\Omega)$, then for all $k > j$:

$$\int_{B(x,r)} \frac{|\nabla u_k|^2}{u_k^2} dx \le C(\Omega_j, M, m) \text{ whenever } B(x, 2r) \subset \Omega_j. \tag{3.12}$$

In fact this estimate holds for all dimensions, but it is not strong enough to provide us with a uniform bound in higher dimensions. The estimate (3.12) follows from display (3.5) in Lemma 3.4. Indeed, for $k > j$, just pick the test function ψ in (3.5) so that $\psi \equiv 1$ on Ω_j, and $\psi \in C_0^\infty(\Omega_{j+1})$. Then:

$$\int_{\Omega_j} \frac{|\nabla u_k|^2}{u_k^2} dx \le C(\Omega_j, M, m, ||\sigma||_{L^{-1,2}(\Omega_{j+1})}).$$

Here we are using the fact that the mollification does not effect the local dual Sobolev norm within Ω_j, this can be establish precisely as in Lemma 2.8. The estimate (3.12) clearly follows from the previous display.

3.3. A local gradient estimate. The key estimate is the following:

Proposition 3.8. *Suppose σ is a real-valued distribution defined on Ω, satisfying (1.6) with $0 < \lambda < 1$, and in addition suppose that (2.11) holds. Let $\{u_j\}$ be the sequence in (3.2). Then, whenever $\omega \subset\subset \Omega_j$, the following estimate holds:*

$$\int_\omega |\nabla u_k|^2 dx \le C(\omega, \lambda, \Lambda, m, M, B, \Omega_j), \quad \textit{for all } k > j. \tag{3.13}$$

Note that the estimate (3.13) is *independent of* k for $k \ge j$. This is the key to allow us to deduce the existence of positive solutions to (1.1).

Proof. Fix j, and k, as in the statement of the proposition, and let $v = u_k$. It suffices to prove that whenever $B(x, 8r) \subset\subset \Omega_j$, there exists a positive constant $C > 0$, depending on n, λ, Λ, m, M, B, $B(x,r)$ and Ω_j such that:

$$\int_{B(x,r)} |\nabla v|^2 dx \le C. \tag{3.14}$$

The reader should keep in mind that *all constants will be independent of* k. Fix such a ball $B(x, 8r) \subset\subset \Omega_j$. To prove (3.14), we will employ Proposition 2.3 in $U = \Omega_j$ to show that v^2 is doubling in Ω_j, with constants independent of k. To verify the hypothesis of Proposition 2.3, we first show that v^2 satisfies a weak reverse Hölder inequality, i.e. that (2.4) holds in Ω_j. To this end, let us fix $B(z, 2s) \subset\subset \Omega_j$. Suppose first $n \ge 3$. Then by Sobolev's inequality, for any $\psi \in C_0^\infty(\Omega_j)$:

$$\Big(\int_{\Omega_j} v^{\frac{2n}{n-2}} |\psi|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{2n}} \le C\Big(\int_{\Omega_j} |\nabla v|^2 \psi^2 \, dx\Big)^{1/2} + C\Big(\int_{\Omega_j} v^2 \, |\nabla \psi|^2 \, dx\Big)^{1/2}. \tag{3.15}$$

Applying Lemma 3.3 in the first term on the right hand side of (3.15), we deduce:

$$\Big(\int_{\Omega_j} v^{\frac{2n}{n-2}} |\psi|^{\frac{2n}{n-2}} dx\Big)^{\frac{n-2}{n}} \le C \int_{\Omega_j} v^2 |\nabla \psi|^2 dx. \tag{3.16}$$

Specialising (3.16) to the case $\psi \in C_0^\infty(B(z,2s))$, with $\psi \equiv 1$ in $B(z,s)$, and $|\nabla\psi| \leq C/s$, it follows:

$$\Big(\fint_{B(z,s)} (v^2)^{\frac{n}{n-2}} dx\Big)^{\frac{n-2}{n}} \leq C\fint_{B(z,2s)} v^2\, dx. \tag{3.17}$$

The constant in $C > 0$ in (3.17) depends on n, M, m, and λ. Hence, if $n \geq 3$, (2.4) holds in $U = \Omega_j$, with $w = v^2$ and $q = n/(n-2)$.

If $n = 2$, we slightly modify the above argument. The following Sobolev inequality is standard (see e.g. [MZ97], Corollary 1.57): for each $q < \infty$, and for all $f \in C_0^\infty(B(z,2s))$,

$$\Big(\fint_{B(z,2s)} |f(y)|^q\, dy\Big)^{1/q} \leq C(q)\Big(\int_{B(z,2s)} |\nabla f(y)|^2\, dy\Big)^{1/2}. \tag{3.18}$$

Using (3.18) as in (3.15) and following the argument through display (3.17), it follows in the case $n = 2$ that (2.4) holds in $U = \Omega_j$, with $w = v^2$ for any choice $q < \infty$. Note that in the case $n = 1$ even stronger Sobolev inequalities are at our disposal, and so the estimate (2.4) continues to hold; we leave this to the reader.

To apply Proposition 2.3, it remains to show $\log(v) \in BMO(\Omega_j)$. For this, note that it follows from Poincaré's inequality that whenever $B(z,2s) \subset \Omega_j$:

$$\fint_{B(z,s)} |\log v - \fint_{B(z,s)} \log v|^2\, dx \leq Cs^{2-n} \int_{B(z,s)} \frac{|\nabla u_k|^2}{u_k^2}\, dx.$$

First suppose $n \geq 3$. Then from Lemma 3.7,

$$\int_{B(z,s)} \frac{|\nabla u_k|^2}{u_k^2}\, dx \leq C(M, m, \Omega_j) s^{2-n}. \tag{3.19}$$

Hence,

$$\fint_{B(z,s)} |\log v - \fint_{B(z,s)} \log v|^2 dx \leq C(M, m, \Omega_j). \tag{3.20}$$

In the case $n = 1, 2$; we apply the weaker estimate (3.12) in combination with Poincaré's inequality, to conclude (3.20) remains true in these cases. From (3.20), we conclude that (see (2.1)), that $\log v \in BMO(\Omega_j)$, with BMO-norm depending only on n, m, M, Ω_j. In particular, v^2 satisfies both (2.4) and (2.5) in Ω_j. From Proposition 2.3, it follows that v^2 is doubling in Ω_j, with constants depending on $n, m, M, \Omega_j, \lambda$ and Λ, see (2.6).

Since Ω_j is a smooth connected set, one can find a Harnack chain from $B(x, 2r)$ to the fixed ball $B \subset\subset \Omega_1$. In other words, one can three positive constants c_0, c_1 and $N > 0$, depending on the smooth parameterization of Ω_j, along with points $x_0, \dots x_N$ and balls $B(x_i, 4r_i) \subset \Omega_j$ so that:

1. (1) $B(x_0, r_0) = B(x, 2r)$, and $B(x_N, r_N) = B$;
2. (2) $r_i \geq c_0 \min(r_0, r_N)$, and $|B(x_i, r_i) \cap B(x_{i+1}, r_{i+1})| \geq c_1 \min(r_0, r_N)^n$ for all $i = 0 \dots N-1$.

Since v^2 is doubling in Ω_j, it follows from the chain construction above, along with a Harnack chain argument (see Remark 2.2) that:

$$\fint_{B(x,2r)} v^2 dx \leq C(B(x,r), m, M, \Omega_j, B, \lambda, \Lambda)\fint_B v^2 dx.$$

It therefore follows from the normalization on v^2 that:

$$\fint_{B(x,2r)} v^2 dx \leq C(B(x,r), m, M, \Omega_j, B, \lambda, \Lambda). \tag{3.21}$$

To complete the proof, combine the Caccioppoli inequality (Lemma 3.3) with the estimate (3.21) to conclude:

$$\int_{B(x,r)} |\nabla v|^2 dx \leq \frac{C}{r^2} \int_{B(x,2r)} v^2 dx \leq C,$$

for a constant $C > 0$, depending on n, m, M, B, Λ, λ, Ω_j and $B(x,r)$. Hence (3.14) is proved for a constant independent of k. □

3.4. Proof of Propositions 3.1 and 3.2. We begin by proving Proposition 3.1, before moving on to prove Proposition 3.2, and with it statement (i) of Theorem 1.1.

Proof of Proposition 3.1. Let Ω_j be an exhaustion of Ω by smooth domains. We will use Proposition 3.8 repeatedly in each Ω_j to deduce the existence of a solution of (1.1). Fix a ball $B \subset \Omega_1$, with $4B \subset \Omega_1$, and note that the construction of the approximate sequence from (3.2) is valid under the present assumptions on σ, and the gradient estimate (3.13) holds.

First, by (3.13) with $j = 1$, along with weak compactness and Rellich's theorem, we pass to a subsequence $u_j^{(1)}$ of u_j so that $u_j^{(1)} \to u^{(1)}$ weakly in $L^{1,2}_{\text{loc}}(\Omega_1)$, and $u_j^{(1)} \to u^{(1)}$ a.e. in Ω_1. Let $\epsilon_{j,1}$ be the corresponding sequence from (3.2). Since $\sigma \in L^{-1,2}(\Omega_1)$, it follows that whenever $h \in C_0^\infty(\Omega_1)$:

$$\langle \sigma u_j^{(1)}, h \rangle = \langle \sigma, u_j^{(1)} h \rangle \to \langle \sigma, u^{(1)} h \rangle = \langle \sigma u^{(1)}, h \rangle. \tag{3.22}$$

Note also, by combining the uniform bound (3.13), with convergence of the mollification:

$$\begin{aligned} |\langle \sigma_{\epsilon_{j,1}} u_j^{(1)}, h \rangle - \langle \sigma u_j^{(1)}, h \rangle| &\leq ||\sigma_{\epsilon_{j,1}} - \sigma||_{L^{-1,2}(\Omega_1)} ||\nabla(u_j^{(1)} h)||_{L^2} \\ &\leq C ||\sigma_{\epsilon_{j,1}} - \sigma||_{L^{-1,2}(\Omega_1)} \to 0 \text{ as } j \to \infty. \end{aligned}$$

We conclude:

$$\langle \sigma_{j,1} u_j^{(1)}, h \rangle \to \langle \sigma u^{(1)}, h \rangle. \tag{3.23}$$

Similarly, by linearity and boundedness of the operator $\mathcal{A}$, we deduce that:

$$\int_{\Omega_1} \mathcal{A} \nabla u_j^{(1)} \cdot \nabla h \, dx \to \int_{\Omega_1} \mathcal{A} \nabla u^{(1)} \cdot \nabla h \, dx.$$

From the uniform bound (3.13), along with standard convergence properties of the mollification in L^2:

$$\Big| \int_{\Omega_1} (\mathcal{A}_{j,1} - \mathcal{A}) \nabla u_j^{(1)} \cdot \nabla h \, dx \Big| \to 0 \quad \text{as } j \to \infty.$$

It therefore follows that the limit function $u^{(1)} \in L^{1,2}_{\text{loc}}(\Omega_1)$ satisfies:

$$-\text{div}(\mathcal{A} \nabla u^{(1)}) = \sigma u^{(1)} \quad \text{in } \mathcal{D}'(\Omega_1). \tag{3.24}$$

Given $\{u_k^{(j)}\}_k$, let us apply estimate (3.13) in Ω_{j+1} to obtain a subsequence $u_k^{(j+1)}$ of $u_k^{(j)}$ and $u^{(j+1)} \in L^{1,2}_{\text{loc}}(\Omega_{j+1})$ with:

$$u_k^{(j+1)} \to u^{(j+1)} \text{ weakly in } L^{1,2}_{\text{loc}}(\Omega_{j+1}), \quad \text{and} \quad u_k^{(j+1)} \to u^{(j+1)} \text{ a.e. in } \Omega_{j+1}.$$

Note that by Lemma 2.6, it follows that $\sigma \in L^{-1,2}(\Omega_j)$. As in the argument leading to (3.24), we see that $u^{(j+1)}$ satisfies

$$-\mathrm{div}(\mathcal{A}\nabla u^{(j+1)}) = \sigma u^{(j+1)} \quad \text{in } \mathcal{D}'(\Omega_{j+1}). \tag{3.25}$$

By construction, $u^j = u^{j+1}$ in Ω_j. Hence one can define $u \in L^{1,2}_{\mathrm{loc}}(\Omega)$ by: $u = u^j$ in Ω_j. By (3.25) it follows that:

$$-\mathrm{div}(\mathcal{A}\nabla u) = \sigma u \quad \text{in } \mathcal{D}'(\Omega).$$

Next, let us demonstrate that u is not the zero function. To see this note that:

$$\int_B (u_j^\ell)^2 dx = 1, \quad \text{for all } j,\, \ell. \tag{3.26}$$

Since $u_j^\ell \to u$ in $L^q_{\mathrm{loc}}(\Omega_\ell)$, whenever $q < 2n/(n-2)$, we may pass to the limit in (3.26). A standard application of Mazur's lemma shows that the limit solution $u \geq 0$. On the other hand, for any $k > 0$ from Lemma 3.4 and weak compactness there exists $v \in L^{1,2}_{\mathrm{loc}}(\Omega)$ so that $\log(u_j^k) \to v$ a.e., but then $v = \log(u)$ a.e. and it follows that $\log(u_j^k)$ converges weakly to $\log(u)$ in $L^{1,2}_{\mathrm{loc}}(\Omega_k)$ (see e.g. Theorem 1.32 of [HKM06]). Hence $u > 0$ q.e. and u is a positive weak solution of (1.1). □

We now move onto Proposition 3.2.

Proof of Proposition 3.2. Let us keep the notation from the proof of Proposition 3.1. The existence of a positive solution $u \in L^{1,2}_{\mathrm{loc}}(\Omega)$ of (1.1) follows from Proposition 3.1. It was proved above in addition that $\log(u)$ is well defined in $L^{1,2}_{\mathrm{loc}}(\Omega)$, and in each Ω_k, $\log(u)$ is the weak limit of a sequence $\log u_j^k$. From Lemma 3.5, it follows that, for all $\psi \in C_0^\infty(\Omega_k)$:

$$\int_\Omega \frac{|\nabla u_j^k|^2}{(u_j^k)^2} \psi^2\, dx \leq C(M, m, \Lambda) \int_\Omega |\nabla \psi|^2\, dx. \tag{3.27}$$

Since $\nabla \log u_j^k$ converges to $\nabla \log u$ weakly in $L^2_{\mathrm{loc}}(\Omega_k)$, we deduce from weak lower semi-continuity of the L^2 norm that for all $\psi \in C_0^\infty(\Omega_k)$:

$$\int_\Omega \frac{|\nabla u|^2}{u^2} \psi^2\, dx \leq C(M, m, \Lambda) \int_\Omega |\nabla \psi|^2\, dx. \tag{3.28}$$

Since (3.28) holds for each k, the estimate (3.1) holds. □

3.5. A logarithmic change of variable: solutions of (1.3) from solutions of (1.1).

This section is concerned with deducing solutions of (1.3) from solutions of (1.1) by a logarithmic substitution. This substitution is classical, for instance it appears in the study of ODEs in [Hi48], and there are examples that show it can be delicate, see e.g. [FM00]. In [AHBV09], there is a rather comprehensive account of the connection between these two types of equations when σ is a finite measure. We will prove the following lemma, from which statement (ii) in Theorem 1.1 and the remainder of Theorem 1.2 follows.

Lemma 3.9. *Let Ω be an open set, and let $\mathcal{A} : \Omega \to \mathbf{R}^{n\times n}$ satisfy (1.4). Suppose that $\sigma \in L^{-1,2}_{loc}(\Omega)$, and that there exists a positive solution u of (1.1). Then $v = \log(u) \in L^{1,2}_{loc}(\Omega)$ is a solution of (1.3).*

Proof. The first step in Lemma 3.9 is to prove that for each $\phi \in C_0^\infty(\Omega)$:

$$\int_\Omega \frac{|\nabla u|^2}{u^2}\phi^2 dx \le C(\phi, \sigma, n, p). \tag{3.29}$$

Let $\epsilon > 0$. Then for $h \in C_0^\infty(\Omega)$, test the weak formulation of (1.1) with $\psi = h(u+\epsilon)^{-1} \in L_c^{1,2}(\Omega)$. This yields:

$$\int_\Omega \frac{\mathcal{A}(\nabla u)}{u+\epsilon} \cdot \nabla h \, dx = \int_\Omega \frac{\mathcal{A}(\nabla u)\cdot \nabla u}{(u+\epsilon)^2} h dx + \langle \sigma \frac{u}{u+\epsilon}, h\rangle. \tag{3.30}$$

Let us now estimate the third term on the right. Let U be an open set, compactly supported in Ω, and containing supp(h). By assumption $\sigma \in L^{-1,2}(U)$, and hence there exists $\vec{\Gamma} \in (L^2(U))^n$ so that $\sigma = \mathrm{div}(\vec{\Gamma})$ in U. Therefore:

$$\langle \sigma \frac{u}{u+\epsilon}, h\rangle = \int_\Omega \frac{\nabla u \cdot \vec{\Gamma}}{u+\epsilon}\Big(\frac{\epsilon}{u+\epsilon}\Big) h dx + \int_\Omega \frac{u}{u+\epsilon} \nabla h \cdot \vec{\Gamma} dx. \tag{3.31}$$

Since $\epsilon/(u+\epsilon) \le 1$, it follows from Cauchy's inequality, that for any $\delta > 0$:

$$|\langle \sigma \frac{u}{u+\epsilon}, h\rangle| \le \delta \int_\Omega \frac{|\nabla u|^2}{(u+\epsilon)^2} h dx + C_\delta \int_\Omega |\vec{\Gamma}|^2 h dx + \int_\Omega |\nabla h||\vec{\Gamma}| dx. \tag{3.32}$$

Letting $h = \phi^2$ for $\phi \in C_0^\infty(\Omega)$, $\phi \ge 0$ in (3.30) and rearranging, using the ellipticity and boundedness assumptions (1.4), we obtain:

$$\begin{aligned} m \int_\Omega \frac{|\nabla u|^2}{(u+\epsilon)^2}\phi^2 \, dx &\le 2M \int_\Omega \frac{|\nabla u|}{u+\epsilon} |\nabla \phi| \, \phi \, dx \\ &+ \delta \int_\Omega \frac{|\nabla u|^2}{(u+\epsilon)^2}\phi^2 \, dx + C_\delta \int_\Omega |\vec{\Gamma}|^2 \phi^2 \, dx + 2 \int_\Omega |\nabla \phi| \, |\vec{\Gamma}| \, \phi \, dx. \end{aligned} \tag{3.33}$$

Here the bound (3.32) has also been used. Appealing to Cauchy's inequality again in (3.33), we obtain:

$$\int_\Omega \frac{|\nabla u|^2}{(u+\epsilon)^2}\phi^2 \, dx \le C(\phi, \sigma, n, p).$$

Letting $\epsilon \to 0$, (3.29) follows from Fatou's lemma.

Now, let us again look at (3.30). It follows from (3.29) that as $\epsilon \to 0$:

$$\int_\Omega \frac{\mathcal{A}\nabla u}{u+\epsilon} \cdot \nabla h \, dx \to \int_\Omega \frac{\mathcal{A}\nabla u}{u} \cdot \nabla h \, dx, \text{ and}$$

$$\int_\Omega \frac{(\mathcal{A}\nabla u)\cdot \nabla u}{(u+\epsilon)^2} \, h \, dx \to \int_\Omega \frac{(\mathcal{A}\nabla u)\cdot \nabla u}{u^2} \, h \, dx.$$

To handle the last term in (3.30), note that from (3.29) and the dominated convergence theorem:

$$\nabla\Big(\frac{u}{u+\epsilon}\Big) = \Big(\frac{\epsilon}{u+\epsilon}\Big) \cdot \frac{\nabla u}{u+\epsilon} \to 0 \quad \text{in } L^2_{\mathrm{loc}}(\Omega) \quad \text{as } \epsilon \to 0,$$

on the other hand, it is clear that:

$$\frac{u}{u+\epsilon} \to 1 \quad \text{in } L^2_{\mathrm{loc}}(\Omega) \quad \text{as } \epsilon \to 0.$$

Thus it follows:

$$\frac{u}{u+\epsilon} \to 1 \quad \text{in } L^{1,2}_{\mathrm{loc}}(\Omega) \quad \text{as } \epsilon \to 0.$$

But since $\sigma \in L^{-1,2}(V)$ for any $V \subset\subset \Omega$, we conclude:

$$\langle \sigma \frac{u}{u+\epsilon}, h\rangle = \langle \sigma, \frac{u}{u+\epsilon} h\rangle \to \langle \sigma, h\rangle, \text{ as } \epsilon \to 0.$$

It follows that $v = \log(u)$ is a solution of (1.3). □

Proof of Theorem 1.1, statement (ii). This is nothing more than a restatement of Lemma 3.9 above, along with the trivial observation that if u satisfies (1.9), then $v = \log(u)$ satisfies (1.11). □

We may now also complete the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.5 and Proposition 3.1, it follows that under the hypothesis of Theorem 1.2, there exists a position solution $u \in L^{1,2}_{\text{loc}}(\Omega)$ of (1.1). By Lemma 3.9, setting $v = \log(u)$, we see that $v \in L^{1,2}_{\text{loc}}(\Omega)$ is a solution of (1.3). □

We did not use (1.7) in the previous lemma, doing so allows us to conclude that the solution satisfies an additional multiplier condition.

Lemma 3.10. *Under the assumptions of Lemma 3.9, if in addition σ satisfies (1.7) for a positive constant $\Lambda > 0$, then there exists a solution $v \in L^{1,2}_{loc}(\Omega)$ of (1.3) satisfying (1.11).*

Proof. We will keep the notation from the proof in Lemma 3.9. It is left to prove that v satisfies (1.11). To this end, let us again test (1.1) with $\phi^2/(u+\epsilon)$, for a smooth test function $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$. There exist constants $C_1, C_2 > 0$ depending on m and M, so that:

$$\int_\Omega \frac{|\nabla u|^2}{(u+\epsilon)^2}\phi^2 dx \leq C_1 \int_\Omega |\nabla \phi|^2 dx - C_2 \Big\langle \sigma \phi \sqrt{\frac{u}{u+\epsilon}}, \phi \sqrt{\frac{u}{u+\epsilon}} \Big\rangle. \tag{3.34}$$

Indeed, as in display (3.30), one obtains by testing equation (1.1) the following identity:

$$2\int_\Omega \phi \frac{\mathcal{A}(\nabla u)}{u+\epsilon} \cdot \nabla \phi \, dx = \int_\Omega \frac{\mathcal{A}(\nabla u)\cdot \nabla u}{(u+\epsilon)^2}\phi^2 dx + \langle \sigma \frac{u}{u+\epsilon}, \phi^2\rangle \tag{3.35}$$

Hence, from the ellipticity and boundedness assumptions (1.4):

$$m\int_\Omega \frac{|\nabla u|^2}{(u+\epsilon)^2}\phi^2 \, dx \leq M \int_\Omega \frac{|\nabla u|}{u+\epsilon}\phi \cdot |\nabla \phi| \, dx - \langle \sigma \frac{u}{u+\epsilon}, \phi^2\rangle. \tag{3.36}$$

From an elementary application of Cauchy's inequality in (3.36), display (3.34) follows. One can pick, for instance, $C_1 = (M/m)^2$ and $C_2 = 2/m$.

Next, applying (1.7), it follows that the second term on the right hand side in (3.34) is bounded by a constant multiple of:

$$\Lambda \int_\Omega \frac{|\nabla u|^2}{u^2}\Big(\frac{\epsilon}{u+\epsilon}\Big)^2 \phi^2 dx + \Lambda \int_\Omega \frac{u}{u+\epsilon}|\nabla \phi|^2 dx. \tag{3.37}$$

The first term in (3.37) converges to zero as $\epsilon \to 0$, by virtue of (3.29) and the dominated convergence theorem. Again by dominated convergence, the second term in (3.37) converges to:

$$\int_\Omega |\nabla \phi|^2 dx, \quad \text{ as } \epsilon \to 0.$$

Substituting these estimates into (3.34), we deduce that (1.11) holds. □

3.6. Existence of solutions to (1.3) implies the validity of (1.2).

Proof of Theorem 1.1, statement (iii). This will follow immediately from Lemmas 3.11 through 3.13 below. □

Lemma 3.11. *Let Ω be an open set, and suppose that σ is a distribution defined on Ω. Let $\mathcal{A}$ be an $n \times n$ real-valued symmetric matrix defined on Ω satisfying (1.4). Suppose there exists a supersolution $v \in L^{1,2}_{loc}(\Omega)$ of (1.3), then (1.6) holds with $\lambda = 1$.*

Proof. Suppose that there exists a solution v of (1.3), and let $\phi \in C_0^\infty(\Omega)$, then testing (1.3) with ϕ^2, yields:

$$\langle \sigma\phi, \phi\rangle = \langle \sigma, \phi^2\rangle \le 2\int_\Omega |\phi||(\mathcal{A}(\nabla v))\cdot\nabla\phi|dx - \int_\Omega (\mathcal{A}\nabla v)\cdot(\nabla v)\phi^2 dx.$$

Under the present assumptions, $\mathcal{A}$ is a symmetric positive definite matrix. It follows that for $\xi, \eta \in \mathbf{R}^n$,

$$|(\mathcal{A}\xi)\cdot\eta| \le ((\mathcal{A}\xi)\cdot\xi)^{1/2}((\mathcal{A}\eta)\cdot\eta)^{1/2}. \tag{3.38}$$

Thus:

$$\begin{aligned} 2\int_\Omega |\phi||(\mathcal{A}(\nabla v))\cdot\nabla\phi|dx &\le 2\int_\Omega |\phi|((\mathcal{A}\nabla v)\cdot\nabla v)^{1/2}((\mathcal{A}\nabla\phi)\cdot\nabla\phi)^{1/2}dx \\ &\le \int_\Omega \phi^2((\mathcal{A}\nabla v)\cdot\nabla v) + \int_\Omega ((\mathcal{A}\nabla\phi)\cdot\nabla\phi)dx. \end{aligned} \tag{3.39}$$

Therefore (1.6) holds with $\lambda = 1$. □

On the other hand, if symmetry is not assumed, one may still conclude the validity of (1.6) and (1.7) from (1.3). Indeed:

Lemma 3.12. *Let Ω be a connected open set, and suppose that $\mathcal{A} : \Omega \to \mathbf{R}^{n\times n}$, satisfying (1.4). Suppose that $v \in L^{1,2}_{loc}(\Omega)$ is a supersolution of (1.3), then (1.6) holds with:*

$$\lambda = \Big(\frac{M}{m}\Big)^2.$$

Proof. Let us first show that (1.6) holds with the given choice of λ. To this end, let $\phi \in C_0^\infty(\Omega)$ and test the weak formulation (1.1) with the valid test function ϕ^2. Together with the assumptions (1.4), this yields:

$$\begin{aligned} \langle \sigma\phi, \phi\rangle = \langle \sigma, \phi^2\rangle &\le M2\int_\Omega |\nabla v||\phi||\nabla\phi|dx - m\int_\Omega |\nabla v|^2\phi^2 dx \\ &\le \frac{M^2}{m}\int_\Omega |\nabla\phi|^2 dx \le \Big(\frac{M}{m}\Big)^2\int (\mathcal{A}\nabla\phi)\cdot\nabla\phi, \end{aligned} \tag{3.40}$$

where Young's inequality was used in the last line. □

We conclude this section by showing that if the solution v of (1.3) in addition satisfies (1.11), then σ satisfies (1.7) for a positive constant $\Lambda > 0$.

Lemma 3.13. *Under the assumptions of either Lemma 3.11 or Lemma 3.12, if one in addition assumes the solution $v \in L^{1,2}_{loc}(\Omega)$ satisfies (1.11), then σ satisfies (1.7) for a positive constant $\Lambda > 0$.*

Proof. Let $\phi \in C_0^\infty(\Omega)$ and test the weak formulation (1.1) with the valid test function ϕ^2. Then:

$$\begin{aligned}\langle \sigma\phi, \phi\rangle &= 2\int_\Omega ((\mathcal{A}\nabla v)\cdot\nabla\phi)\phi dx - \int_\Omega |\nabla v|^2\phi^2 dx \\ &\geq -(M+1)\int_\Omega |\nabla v|^2\phi^2 - \int_\Omega |\nabla \phi|^2 dx \geq -\frac{\Lambda}{m}\int_\Omega |\nabla\phi|^2 dx,\end{aligned} \tag{3.41}$$

for a suitable choice of constant $\Lambda > 0$. Appealing to (1.4), the result follows. □

3.7. On the equation (1.3). In this section, we make a few comments regarding our results for the equation (1.3) in comparison to existing literature, as the introduction of this paper focussed rather more on the Schrödinger type equation (1.1). First, we will restate a theorem which has been proved for reference:

Theorem 3.14. *Let Ω be an open set. Let $\sigma \in D'(\Omega)$ be a real-valued distribution. Suppose $\mathcal{A} : \Omega \to \mathbf{R}^n$ is a symmetric real-valued matrix function satisfying (1.4). The the following two statements hold:*

(i) Suppose that σ satisfies (1.7) for a positive constant $\Lambda > 0$, and (1.6) for a constant $0 < \lambda < 1$, then there exists a positive solution $v \in L^{1,2}_{loc}(\Omega)$ of (1.3) so that (1.11) holds. In addition, the constructed solution has the exponential integrability property: $e^v \in L^{1,2}_{loc}(\Omega)$.

(ii) Conversely, if there exists a solution $v \in L^{1,2}_{loc}(\Omega)$ of (1.3) so that (1.11) holds, then σ satisfies (1.7) for a positive constant $\Lambda > 0$, and (1.6) with $\lambda = 1$.

In [FM98], solutions of (1.3) are proved in the global energy space $L^{1,\,2}_0(\Omega)$ are proved when Ω is a bounded domain, under the assumption that $\sigma \in L^{n/2}(\Omega)$. They explicitly note that this condition on σ is used to guarantee that (1.2) is valid. Theorem 3.14 therefore compliments their theorem with a more local result in nature, and therefore one which requires less restriction on σ. As was noticed in [FM98, FM00], there exist classes of solutions of (1.3) that are exponentially integrable. One can trace this principle back to the employment of certain nonlinear test functions in proving the existence of solutions to (1.3) (see e.g. [Ev90, FM00]). A refinement of this argument is what is also employed in the current paper, since we deduce Theorem 3.14 from our considerations of the Schrödinger type equation via a logarithmic substitution.

The local exponential integrability in statement (i) is sharp, as can be seen from the example discussed in Section 7. The paper [FM00] concerns quasilinear equations of p-Laplacian type, and we will consider such equations in our forthcoming paper [JMV10].

3.8. On the critical case $\lambda = 1$. In this subsection we briefly discuss the limiting case when $\lambda = 1$. We aim to prove the following proposition:

Proposition 3.15. *Suppose that Ω is an open set, and suppose $\mathcal{A}$ is a symmetric bounded matrix function defined on Ω. Then (1.17) holds if and only if there exists a positive superharmonic function u such that:*

$$-\mathrm{div}(\mathcal{A}\nabla u) \geq \sigma u \quad \text{in } \Omega. \tag{3.42}$$

Proof. Let us assume that Ω is connected. The necessity is well known, and holds even in very general potential theoretic frameworks, see e.g. [Fit00]. For the converse, let $\lambda_j \in (0,1)$ be a sequence such that $\lambda_j \to 1$. Applying a very special case of Theorem 1.1 above, we find a sequence of positive functions $\{u_j\}_j \in L^{1,2}_{\text{loc}}(\Omega)$ of:

$$-\text{div}(\mathcal{A}\nabla u_j) = \lambda_j \sigma u_j, \quad \text{with} \fint_B u_j\, dx = 1. \tag{3.43}$$

Here $B \subset\subset \Omega$ is a fixed ball. We may assume that u_j is superharmonic. Next, let us fix a smooth connected subdomain U of Ω, so that $B \subset\subset U$. For a fixed $q < n/(n-1)$, we will prove that for any ball $B(x,r) \subset\subset U$:

$$\int_{B(x,r)} |\nabla u_j|^q\, dx \leq C(m, M, U, q, B(x,r)). \tag{3.44}$$

To this end, note that, by the property (1.9):

$$\int_\Omega \frac{|\nabla u_j|^2}{u_j^2}\, h^2\, dx \leq C(m,M) \int_\Omega |\nabla h|^2 dx \quad \text{for all } h \in C_0^\infty(\Omega). \tag{3.45}$$

Display (3.45) is also a standard property of superharmonic functions, as a consequence of Moser's work [Mos60]. From (3.45) and the Poincaré inequality, one readily deduces as in Proposition 3.8 that $u_j \in BMO(U)$. Therefore, as in Proposition 2.3, from the John-Nirenberg inequality we find a constant $c = c(m, M, n) > 0$, $0 < c < 1$, such that u^c is doubling in U (see (2.3)), with doubling constants depending on m, M, and n. From a Harnack chain argument, we deduce that, for any $B(x,r) \subset\subset U$;

$$\int_{B(x,r)} u^c dx \leq C(B(x,r), U, B, m, M) \int_B u^c dx \leq C(B(x,r), U, B, m, M), \tag{3.46}$$

where in the last equation, we have used normalization of u_j in (3.43). Let us now note that following inequality superharmonic functions, essentially due to Moser [Mos60]: for $1 < q < n/(n-1)$, there exists a constant $C = C(n,q)$ such that:

$$\int_{B(x,r)} |\nabla u_j|^q\, dx \leq C \inf_{B(x,r)} u_j. \tag{3.47}$$

Combining (3.47) and (3.46), the display (3.44) follows. Note that from (3.47) and Rellich's theorem, we deduce that there exists u such that: $u_j \to u$ a.e in Ω, and:

$$\int_B u\, dx = 1.$$

Furthermore, u can be chosen to be superharmonic, by standard convergence properties, see [KM92].

Our aim is to show that $u = \liminf_{j\to} u_j$ q.e. To this end, let $v = \liminf_{j\to\infty} u_j$, and denote by v^* the lower semi-continuous regularization of v. By the fundamental convergence theorem for superharmonic functions, v^* is superharmonic, and $v^* = \liminf_{j\to\infty} u_j$ quasi-everywhere (see Theorem 7.4 of [HKM06]). Since $v^* = u$ a.e. and they are both superharmonic, we have that $u = v^*$ everywhere. The claim follows.

Let us now conclude the argument. Since σ satisfies (1.17), it does not charge sets of capacity zero. Therefore, by Fatou's lemma, it follows that for any $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$:

$$\liminf_{j\to\infty} \int_\Omega u_j \phi\, d\sigma \geq \int_\Omega u\phi\, d\sigma.$$

Combining this last display with the weak convergence of the Riesz measure for superharmonic functions, see e.g. [HKM06], we conclude that (3.42) holds. □

4. Form boundedness

In this section we apply Theorem 1.1 to deduce a new proof of Theorem 4.1 below, which was the primary theorem in [MV02a] (see also [MV06]). Let us consider the case $\Omega = \mathbf{R}^n$, $n \geq 3$, since the case of a general domain, under certain mild restrictions on Ω, can be reduced to the entire space, as was explained in [MV02a].

Theorem 4.1. *Let $\sigma \in D'(\mathbf{R}^n)$, $n \geq 3$. Then the following statements hold.*

(i) *The quadratic form inequality*

$$|\langle \sigma, h^2\rangle| \leq C\,||\nabla h||^2_{L^2(\mathbf{R}^n)}, \quad \text{for all } h \in C_0^\infty(\mathbf{R}^n), \tag{4.1}$$

is valid if and only if σ can be represented in the form

$$\sigma = \operatorname{div} \vec{\Gamma}, \tag{4.2}$$

where $\vec{\Gamma} \in L^2_{\mathrm{loc}}(\mathbf{R}^n)^n$ obeys

$$\int_{\mathbf{R}^n} h^2 |\vec{\Gamma}|^2\, dx \leq C_1\,||\nabla h||^2_{L^2(\mathbf{R}^n)}, \quad \text{for all } h \in C_0^\infty(\mathbf{R}^n). \tag{4.3}$$

(ii) *If σ satisfies* (4.1), *then* (4.2) *holds with $\vec{\Gamma} = \nabla(\Delta^{-1}\sigma)$, where $\Delta^{-1}\sigma \in \mathrm{BMO}(\mathbf{R}^n)$, and*

$$\int_{\mathbf{R}^n} h^2 |\nabla(\Delta^{-1}\sigma)|^2\, dx \leq C_1\,||\nabla h||^2_{L^2(\mathbf{R}^n)}, \quad \text{for all } h \in C_0^\infty(\mathbf{R}^n). \tag{4.4}$$

(iii) *If* (4.3) *holds with $C_1 = \frac{1}{4}$ then* (4.1) *holds with $C = 1$. Conversely, if σ satisfies* (1.6) *with the upper form bound $\lambda < 1$, and* (1.7) *with the lower form bound $\Lambda > 0$, then* (4.4) *holds with a constant C_1 which does not depend on σ.*

Proof of Theorem 4.1. The sufficiency part of statement (i) with $C_1 = C^2/4$ in inequality (4.3) follows using integration by parts and Cauchy's inequality: if $\sigma = \operatorname{div}\vec{\Gamma}$, then

$$\begin{aligned} |\langle \sigma, h^2\rangle| &= 2\left|\int_{\mathbf{R}^n} \vec{\Gamma}\cdot \nabla h\, h\, dx\right| \\ &\leq 2\,||h\vec{\Gamma}||_{L^2(\mathbf{R}^n)}||\nabla h||_{L^2(\mathbf{R}^n)} \leq C\,||\nabla h||^2_{L^2(\mathbf{R}^n)}. \end{aligned}$$

To deduce the remainder of the theorem, we apply part (ii) of Theorem 1.1 with $\tilde{\sigma} := \sigma/(\lambda+\epsilon)$, where λ is the upper form bound of σ, and $\epsilon > 0$, so that the corresponding upper form bound $\tilde{\lambda}$ of $\tilde{\sigma}$ satisfies $\tilde{\lambda} < 1$. This yields the existence of a weak solution $\Psi \in L^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$ of the multi-dimensional Riccati equation:

$$-\Delta\Psi = |\nabla\Psi|^2 + \tilde{\sigma} \quad \text{in} \quad D'(\mathbf{R}^n). \tag{4.5}$$

Using h^2, where $h \in C_0^\infty(\mathbf{R}^n)$, as a test function in this equation, and integrating by parts, we estimate:

$$\begin{aligned} \int_{\mathbf{R}^n} h^2\,|\nabla\Psi|^2 dx &= 2\int_{\mathbf{R}^n} \nabla\Psi\cdot\nabla h\, h\, dx - \langle\tilde{\sigma}, h^2\rangle \\ &\leq 2\,||h\nabla\Psi||_{L^2(\mathbf{R}^n)}||\nabla h||_{L^2(\mathbf{R}^n)} + \tilde{\Lambda}\,||\nabla h||^2_{L^2(\mathbf{R}^n)}, \end{aligned}$$

where $\tilde{\Lambda} = \Lambda/(\lambda+\epsilon)$ is the lower form bound of $\tilde{\sigma}$. From this it follows:

$$\int_{\mathbf{R}^n} h^2\,|\nabla\Psi|^2 dx \leq (1+\sqrt{\tilde{\Lambda}})^2\,||\nabla h||^2_{L^2(\mathbf{R}^n)}, \quad \text{for all } h \in C_0^\infty(\mathbf{R}^n). \tag{4.6}$$

In other words, $|\nabla\Psi| \in M(L^{1,2}_0(\mathbf{R}^n) \to L^2(\mathbf{R}^n))$, where $M(L^{1,2}_0(\mathbf{R}^n) \to L^2(\mathbf{R}^n))$ is the space of pointwise multipliers from the homogeneous Sobolev space $L^{1,2}_0(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$ defined in Sec. 2.1. Hence, $\tilde\sigma$ can be represented in the form

$$\tilde\sigma = -\mathrm{div}\,\nabla\Psi - |\nabla\Psi|^2, \quad |\nabla\Psi| \in M(L^{1,2}_0(\mathbf{R}^n) \to L^2(\mathbf{R}^n)).$$

Moreover, one can deduce from (4.1) that $\Delta^{-1}\,\tilde\sigma$ is well defined in the sense of the weak-$\star$ BMO convergence (see details in [MV06]):

$$\Delta^{-1}(\psi_N\,\tilde\sigma) \overset{weak-\star}{\longrightarrow} \Delta^{-1}\tilde\sigma \in \mathrm{BMO}(\mathbf{R}^n),$$

where $\psi_N(x) = \psi(|x|/N)$, and $\psi \in C^\infty_0(\mathbf{R})$ is a standard cut-off function. It follows from (4.6) that $\Delta^{-1}\,(|\nabla\Psi|^2) \in \mathrm{BMO}(\mathbf{R}^n)$, and hence

$$\Psi = -\Delta^{-1}\,(|\nabla\Psi|^2) - \Delta^{-1}\tilde\sigma \in \mathrm{BMO}(\mathbf{R}^n). \tag{4.7}$$

Thus, $\tilde\sigma$ can be represented in the form

$$\tilde\sigma = \mathrm{div}\,\vec\Gamma, \qquad \vec\Gamma = \nabla\Delta^{-1}\,\tilde\sigma, \qquad \text{in} \quad D'(\mathbf{R}^n), \tag{4.8}$$

where

$$\Delta^{-1}\,\tilde\sigma = -\Psi - \Delta^{-1}\,(|\nabla\Psi|^2) \in \mathrm{BMO}(\mathbf{R}^n). \tag{4.9}$$

To complete the proof of statements (ii) and (iii), it remains to verify

$$\nabla\Delta^{-1}\,\tilde\sigma \in M(L^{1,2}_0(\mathbf{R}^n) \to L^2(\mathbf{R}^n)).$$

Let

$$g = (-\Delta)^{-\frac{1}{2}}\,|\nabla\Psi|^2 \geq 0.$$

In other words, g is the Riesz potential of order 1 of $|\nabla\Psi|^2$, so that

$$|\nabla\Delta^{-1}\,(|\nabla\Psi|^2)(x)| \leq c(n)\,g(x) \quad \text{a.e. on } \mathbf{R}^n.$$

Since $|\nabla\Psi| \in M(L^{1,2}_0(\mathbf{R}^n) \to L^2(\mathbf{R}^n))$, it follows (see Theorem 1.7 in [Ver99]) that $g \in M(L^{1,2}(\mathbf{R}^n) \to L^2(\mathbf{R}^n))$. Hence, $\nabla\Delta^{-1}\,(|\nabla\Psi|^2) \in M(L^{1,2}_0(\mathbf{R}^n) \to L^2(\mathbf{R}^n))$. Thus,

$$\vec\Gamma = \nabla\Delta^{-1}\,\tilde\sigma \in M(L^{1,2}_0(\mathbf{R}^n) \to L^2(\mathbf{R}^n)).$$

This is equivalent to (4.4), which completes the proof of the theorem. □

5. Semi-boundedness

Let $\Omega \subseteq \mathbf{R}^n$ be an open set, and suppose $\mathcal{A}$ is a matrix function satisfying (1.4). In this section, we will consider real-valued distributions $\sigma \in \mathcal{D}'(\Omega)$ which are semi-bounded; that is, the quadratic form of the operator $\mathcal{H} = -\mathrm{div}(\mathcal{A}\nabla\cdot) - \sigma$ is non-negative:

$$\langle\sigma, h^2\rangle \leq \int_\Omega (\mathcal{A}\nabla h)\cdot\nabla h\,dx, \quad \text{for all } h \in C^\infty_0(\Omega). \tag{5.1}$$

It had been conjectured that a necessary and sufficient condition for (5.1) to hold is the following condition: *there exist* $\vec\Gamma \in L^2_{loc}(\Omega)^n$ *and a constant* $C > 0$ *so that*

$$\sigma \leq \mathrm{div}(\vec\Gamma), \quad \text{and} \quad \int_\Omega |h|^2|\vec\Gamma|^2\,dx \leq C\int_\Omega (\mathcal{A}\nabla h)\cdot\nabla h\,dx \quad \text{for all } h \in C^\infty_0(\Omega). \tag{5.2}$$

A simple estimate using integration by parts and Cauchy's inequality in the form (3.38) shows that condition (5.2) with $C = \frac{1}{4}$ is sufficient for (5.1) to hold. However, it is *not necessary*, with any $C > 0$, for (5.1). We defer the proof of this fact to Proposition 7.1 below. On the other hand, the following theorem provides a characterization of semi-bounded distributions.

Theorem 5.1. *Let Ω be an open set, and let $\sigma \in \mathcal{D}'(\Omega)$ be a real valued distribution. In addition, let $\mathcal{A}$ be a symmetric matrix function defined on Ω satisfying (1.4). Then (5.1) holds if and only if there exists a vector field $\vec{\Gamma} \in L^2_{loc}(\Omega)$ so that:*

$$\sigma \leq \operatorname{div}(\mathcal{A}\vec{\Gamma}) - (\mathcal{A}\vec{\Gamma}) \cdot \vec{\Gamma} \qquad \text{in } \mathcal{D}'(\Omega). \tag{5.3}$$

There is an extension of Theorem 5.1 to non-symmetric matrices $\mathcal{A}$. Indeed the necessity of the condition (5.3) extends to the non-symmetric case, see Proposition 5.3 below. On the other hand, a repetition of the proof of Lemma 3.12 shows: if (5.3) holds, then (5.1) holds with a constant $(M/m)^2$ introduced in the right hand side. Here m and M are the ellipticity constants from (1.4).

Remark 5.2. The proof of Theorem 5.1 shows that (5.1) holds if and only if there exist solutions to the differential inequality:

$$-\operatorname{div}(\mathcal{A}\nabla u) - (\mathcal{A}\nabla u) \cdot \nabla u \geq \sigma \qquad \text{in } \Omega. \tag{5.4}$$

The inequality in (5.4) cannot be strengthened to an equality for general distributions $\sigma \in \mathcal{D}'(\Omega)$. Indeed, if there exists a solution $v \in L^{1,2}_{\text{loc}}(\Omega)$ of the equation

$$-\operatorname{div}\mathcal{A}\nabla u = (\mathcal{A}\nabla u) \cdot \nabla u + \sigma \qquad \text{in } \Omega,$$

then it follows that $\sigma \in L^{-1,2}_{\text{loc}}(\Omega) + L^1_{\text{loc}}(\Omega)$. For instance, when $n \geq 3$, one can pick $\sigma = -\delta_{x_0}$ for $x_0 \in \Omega$, then obviously (5.1) holds but σ does not lie in the aforementioned class. In fact, in the special case when σ is a measure, it is known that $\sigma \in L^{-1,2}_{\text{loc}}(\Omega) + L^1_{\text{loc}}(\Omega)$ if and only if σ does not charge sets of capacity zero (see Theorem 2.1 of [BGO96]).

Let us now move onto proving the Theorem:

Proof of Theorem 5.1. The sufficiency of (5.3) for (5.1) is a repetition of the proof of Lemma 3.11. The necessity of (5.3) is somewhat more involved, and follows from Proposition 5.3 below. □

Proposition 5.3. *Let Ω be an open set, with $\mathcal{A}$ a (possibly non-symmetric) matrix function defined on Ω satisfying (1.4). Let $\sigma \in \mathcal{D}'(\Omega)$ satisfying (5.1). Then there exists $\vec{\Gamma} \in L^2_{loc}(\Omega)^n$ so that:*

$$\sigma \leq \operatorname{div}(\mathcal{A}\vec{\Gamma}) - (\mathcal{A}\vec{\Gamma}) \cdot \vec{\Gamma} \qquad \text{in } \mathcal{D}'(\Omega). \tag{5.5}$$

Proof of Proposition 5.3. Without loss of generality, we may assume that Ω is connected. Otherwise, we simply repeat the argument which follows in each component.

Step 1 (*Approximation*). Let Ω_j be an exhaustion of Ω by bounded smooth connected domains. Let $\lambda_j \in (0,1)$ be any sequence so that $\lambda_j \to 1$, and:

$$\epsilon_j < 1/2 \min(d(\Omega_j, \partial\Omega_{j+1}), 2^{-j}).$$

Consider $\sigma_j = \lambda_j \phi_{\epsilon_j} * \sigma$, and denote by $\mathcal{A}_j = \phi_{\epsilon_j} * \mathcal{A}$. Then from Lemma 2.7, it follows that σ_j satisfies:

$$\int_{\Omega_j} |h|^2 d\sigma_j \leq \lambda_j \int_{\Omega_j} \mathcal{A}_j(\nabla h) \cdot \nabla h \, dx, \quad \text{for all } h \in C_0^\infty(\Omega_j). \tag{5.6}$$

In addition to (5.6), note that we can also estimate, for any $h \in C_0^\infty(\Omega_j)$:

$$\int_{\Omega_j} |h|^2 d\sigma_j \geq -||\sigma_j||_{L^\infty(\Omega_j)} \int_{\Omega_j} |h|^2 dx \geq -C||\sigma_j||_{L^\infty(\Omega_j)} |\Omega_j|^{2/n} ||\nabla h||_2^2, \tag{5.7}$$

the last inequality here follows from Sobolev's inequality. By (1.4), we conclude that:

$$\int_{\Omega_j} |h|^2 d\sigma_j \geq -C||\sigma_j||_{L^\infty(\Omega_j)} |\Omega_j|^{2/n} \int_{\Omega_j} \mathcal{A}_j \nabla h \cdot \nabla h\, dx. \tag{5.8}$$

From (5.6) and (5.8), we see that the hypothesis of Theorem 1.1 are satisfied in Ω_j with potential σ_j. It therefore follows from Theorem 1.1 that there exists $v_j \in L^{1,2}_{\text{loc}}(\Omega_j)$ so that:

$$-\text{div}(\mathcal{A}_j \nabla v_j) = \mathcal{A}_j \nabla v_j \cdot \nabla v_j + \sigma_j, \qquad \text{in } \mathcal{D}'(\Omega_j). \tag{5.9}$$

By addition of a suitable constant, we may assume that, for a fixed ball $B \subset\subset \Omega_1$:

$$\Big| \int_B v_j dx \Big| = 1, \qquad \text{for all } j. \tag{5.10}$$

Step 2 (*A uniform bound*). Fix $1 \leq j \leq k$. Our aim is to show that $v_k \in L^{1,2}_{\text{loc}}(\Omega_j)$, with constants independent of k. Let $h \in C_0^\infty(\Omega_j)$, by testing the weak formulation of v_k in (5.9) with h^2, we deduce from (1.4) that:

$$m \int_{\Omega_k} |\nabla v_k|^2 h^2 dx \leq M \int_{\Omega_k} 2h |\nabla v_k| |\nabla h| - \int_{\Omega_k} h^2 d\sigma_k.$$

Applying Cauchy's inequality in the first term on the right hand side:

$$m \int_{\Omega_k} |\nabla v_k|^2 h^2 dx \leq \frac{m}{2} \int_{\Omega_k} |\nabla v_k|^2 h^2 dx + 2\frac{M^2}{m} \int_{\Omega_k} |\nabla h|^2 dx - \int_{\Omega_k} h^2 d\sigma_k,$$

and hence, as $\lambda_k \in (0,1)$:

$$\int_{\Omega_k} |\nabla v_k|^2 h^2 dx \leq C \int_{\Omega_k} |\nabla h|^2 dx + C |\langle \phi_{\epsilon_k} * h^2, \sigma \rangle|. \tag{5.11}$$

Next from standard distribution theory (see e.g. [Str03], Chapter 8), it follows that:

$$|\langle \phi_{\epsilon_k} * h^2, \sigma \rangle| \leq C,$$

for a constant C depending on σ, the support of h and $||\partial^{\alpha_\ell} h||_{L^\infty}$ for some collection of multi-indices $\alpha_1, \ldots, \alpha_N$. (One can see this from either the structure theorem, or by the definition of continuity). In conclusion, for any $h \in C_0^\infty(\Omega_j)$:

$$\int_{\Omega_k} |\nabla v_k|^2 h^2 dx \leq C(\sigma, h). \tag{5.12}$$

This proves the claim that $v_k \in L^{1,2}_{\text{loc}}(\Omega_j)$, with constants independent of k.

Step 3 (*Conclusion*). This will be quite similar to Section 3.4. Indeed, consider first Ω_1. Then from (5.12) and weak compactness, we find a subsequence $v_{j,1}$ of v_j, and $v^1 \in L^{1,2}_{\text{loc}}(\Omega_1)$ so that $v_{j,1} \to v^1$ weakly in $L^{1,2}_{\text{loc}}(\Omega_1)$. From (5.10) and an application of Rellich's theorem, the limit function v_1 is not identically infinite.

Let $\vec{G} \in (L^2(\Omega_1))^n \cap L^\infty$, with compact support in Ω_1. Note:

$$\int_\Omega \mathcal{A}_{j,1} \nabla v_{j,1} \cdot \vec{G}\, dx = \int_\Omega ((\mathcal{A}_{j,1} - \mathcal{A}) \nabla v_{j,1}) \cdot \vec{G}\, dx + \int_\Omega \mathcal{A} \nabla v_{j,1} \cdot \vec{G}\, dx. \tag{5.13}$$

The first term on the right of (5.13) converges to zero as $k \to \infty$. Indeed, one can estimate:

$$\Big|\int_\Omega ((\mathcal{A}_{j,1} - \mathcal{A})\nabla v_{j,1}) \cdot \vec{G}\, dx\Big|$$
$$\leq ||\vec{G}||_\infty \Big(\int_{\text{supp}(\vec{G})} |\nabla v_{j,1}|^2 dx\Big)^{1/2} \Big(\int_{\text{supp}(\vec{G})} |\mathcal{A}_{j,1} - \mathcal{A}|^2 dx\Big)^{1/2},$$

and the right hand side converges to zero by (5.12) and standard properties of approximate identities. For the second term on the right hand side of (5.13), note that by weak convergence:

$$\int_\Omega \mathcal{A}\nabla v_{j,1} \cdot \vec{G}\, dx \to \int_\Omega \mathcal{A}\nabla v^1 \cdot \vec{G}\, dx, \quad \text{ as } j \to \infty.$$

It follows that, for all $\vec{G} \in (L^2(\Omega_1))^n \cap L^\infty$, with compact support in Ω_1,

$$\int_\Omega \mathcal{A}_{j,1}\nabla v_{j,1} \cdot \vec{G}\, dx \to \int_\Omega \mathcal{A}\nabla v^1 \cdot \vec{G}\, dx, \quad \text{ as } j \to \infty. \tag{5.14}$$

It is not difficult to see that one can extend (5.14) for all $\vec{G} \in (L^2(\Omega_1))^n$, with compact support in Ω_1. In other words, that $\mathcal{A}_{j,1}\nabla v_{j,1} \to \mathcal{A}\nabla v^1$ weakly in $L^2_{\text{loc}}(\Omega_1)^n$.

Let $h \in C_0^\infty(\Omega_1)$ so that $h \geq 0$. We next claim that:

$$\liminf_{j\to\infty} \int_\Omega (\mathcal{A}_{j,1}\nabla v_{j,1}) \cdot \nabla v_{j,1} h\, dx \geq \int_\Omega (\mathcal{A}\nabla v^1) \cdot \nabla v^1 h\, dx. \tag{5.15}$$

To see this, denote by $\mathcal{A}^s$ the symmetric part of $\mathcal{A}$, i.e. $2\mathcal{A}^s = \mathcal{A} + \mathcal{A}^t$. Then as before it follows $(\mathcal{A}_{j,1})^s\nabla v_{j,1} \to \mathcal{A}^s\nabla v^1$ weakly in $L^2_{\text{loc}}(\Omega_1)$. In addition, by standard properties of mollification, $(\mathcal{A}_{j,1})^s \to \mathcal{A}^s$ in the weak-$\star$ topology of $L^\infty(\Omega_1)$. We will repeatedly use the observation that the non-symmetric part does not contribute toward the quadratic form. First note by weak convergence:

$$\int_{\Omega_1} (\mathcal{A}\nabla v^1) \cdot \nabla v^1 h dx = \int_{\Omega_1} (\mathcal{A}^s\nabla v^1) \cdot \nabla v^1 h dx = \liminf_{j\to\infty} \int_{\Omega_1} (\mathcal{A}_{j,1})^s\nabla v_{j,1} \cdot \nabla v^1 h dx.$$

By symmetry of the matrix (see (3.38)), we estimate for each j:

$$\int_{\Omega_1} (\mathcal{A}_{j,1})^s\nabla v_{j,1} \cdot \nabla v^1 h dx \leq \Big(\int_{\Omega_1} (\mathcal{A}_{j,1})^s\nabla v_{j,1} \cdot \nabla v_{j,1} h dx\Big)^{1/2} \cdot \Big(\int_{\Omega_1} (\mathcal{A}_{j,1})^s\nabla v^1 \cdot \nabla v^1 h dx\Big)^{1/2}$$

By taking the limit infimum of both sides, using the weak-$\star$ convergence of the convolution, one obtains (5.15). Here we are using the following elementary fact for two bounded sequences (a_j) and (b_j):

$$\text{If } \liminf_{j\to\infty} a_j = a \geq 0 \text{ and } \lim_{j\to\infty} b_j = b \geq 0, \text{ then } \liminf_{j\to\infty} a_j b_j \leq ab.$$

On the other hand, by standard properties of the convolution, and since $\lambda_j \to 1$:

$$\lambda_j \langle \sigma, \phi_{\epsilon_j} * h\rangle \to \langle \sigma, h\rangle, \text{ as } j \to \infty.$$

Keeping (5.9) and (5.14) in mind, we conclude that

$$-\text{div}(\mathcal{A}\nabla v^1) \geq (\mathcal{A}\nabla v^1) \cdot \nabla v^1 + \sigma \qquad \text{in } \mathcal{D}'(\Omega_1). \tag{5.16}$$

For $k \geq 1$, and given the sequence $\{v_{j,k-1}\}$, a repetition of the above argument yields a subsequence $v_{j,k}$ of $v_{j,k-1}$ so that $v_{j,k}$ converges to $v^k \in L^{1,2}_{\text{loc}}(\Omega_k)$ with:

$$-\text{div}(\mathcal{A}\nabla v^k) \geq (\mathcal{A}\nabla v^k) \cdot \nabla v^k + \sigma \qquad \text{in } \mathcal{D}'(\Omega_k). \tag{5.17}$$

Furthermore, as in Section 3.4, we may assert that $v^k = v^{k-1}$ in Ω_{k-1}. One can therefore define a function $v \in L^{1,2}_{\text{loc}}(\Omega)$ so that

$$-\text{div}(\mathcal{A}\nabla v) - (\mathcal{A}\nabla v) \cdot \nabla v \geq \sigma \qquad \text{in } \mathcal{D}'(\Omega).$$

To complete the proof, it suffices to let $\vec{\Gamma} = -\nabla v$. □

6. The local regularity of solutions to the Schrödinger equation with prescribed boundary values

The goal of this section is to apply the regularity techniques developed in this paper to the recent work of Frazier, Nazarov and Verbitsky [FNV10]. The point here is to prove regularity of a given solution with prescribed boundary values, when we already know there exists a majorant of the given solution.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain so that the boundary Harnack inequality is valid (for instance a Lipschitz, or more generally a NTA domain). Let us now fix $x_0 \in \Omega$, and let $G(x, x_0)$ be the Green's function for the Laplace operator relative to Ω. Then we define $m(x) = \min(1, G(x, x_0))$. If Ω is a $C^{1,1}$ domain then $m(x)$ is pointwise comparable to $\text{dist}(x, \partial\Omega)$.

Let σ be a locally finite Borel measure in Ω. Then the following theorem is proved:

Theorem 6.1. [FNV10] *Suppose that σ satisfies the following embedding inequality:*

$$\int_\Omega h^2 d\sigma \leq \lambda \int_\Omega |\nabla h|^2 dx, \ \textit{for all } h \in C_0^\infty(\Omega), \tag{6.1}$$

with $0 < \lambda < 1$. In addition, suppose that there is a constant $c > 0$ so that:

$$\int_\Omega m(x) \exp\Big(\frac{c}{m(x)} \int_\Omega m(y) d\sigma(y)\Big) d\sigma(x) < \infty. \tag{6.2}$$

Then there is a solution u_1 of the equation:

$$\begin{cases} -\Delta u_1 = \sigma u_1 \ \textit{in } \Omega, \\ u_1 = 1 \ \textit{on } \partial\Omega. \end{cases} \tag{6.3}$$

Conversely, if there is a solution of (6.3), then (6.2) holds for a positive constant $c = c(\Omega)$, and (6.1) holds with $\lambda = 1$.

The solution constructed in Theorem 6.1 is interpreted in the potential theoretic sense, i.e., $u_1 \in L^1(\Omega, m d\sigma)$, and

$$u_1(x) = \int_\Omega G(x, y)\, u_1(y)\, d\sigma(y) + 1.$$

If Ω is a bounded $C^{1,1}$ domain then $u_1 \in L^1(\Omega, dx) \cap L^1(\Omega, m d\sigma)$, and is a solution to (6.3) in the very weak sense (see [FNV10]).

Our primary result in this section is the following:

Theorem 6.2. *Consider the solution u_1 of (6.3) constructed in Theorem 6.1, then:*

$$u_1 \in L^{1,2}_{loc}(\Omega). \tag{6.4}$$

Proof. Let Ω_j be a exhaustion of Ω by smooth domains. Let $\phi_j \in C_0^\infty(\Omega)$ be such that $\phi_j \equiv 1$ on Ω_j and $0 \le \phi_j \le 1$. Note that (as in Lemma 2.6) $\phi_j\sigma \in L^{-1,2}(\Omega)$. Since (6.1) holds with constant $0 < \lambda < 1$, we follow a similar argument to Section 3.1, using the Lax-Milgram lemma to obtain a unique $v_j \in L^{1,2}(\Omega)$ satisfying:

$$\begin{cases} -\Delta v_j = \phi_j \sigma v_j \in \Omega, \\ v_j - 1 \in L_0^{1,\,2}(\Omega). \end{cases} \tag{6.5}$$

Note here that $v_{j+1} \ge v_j$.

Let $B(x, 2r) \subset\subset \Omega$. By repeating the proof of Lemma 3.3, we deduce that there is a constant $C = C(n)$ so that:

$$\int_{B(x,r)} |\nabla v_j|^2\, dx \le \frac{C}{r^2} \int_{B(x,2r)} v_j^2\, dx. \tag{6.6}$$

From (6.6), one asserts, as in displays (3.15) through (3.17):

$$\Big(\fint_{B(x,r)} v_j^{2n/(n-2)}\, dx\Big)^{(n-2)/2} \le C \fint_{B(x,2r)} v_j^2\, dx. \tag{6.7}$$

Since (6.7) holds for all balls $B(x, 2r) \subset\subset \Omega$, we see that the hypothesis of Lemma 2.4 are valid. Applying the lemma with $t < n/(n-2)$, one finds a constant $C(t) > 0$ so that:

$$\Big(\fint_{B(x,r)} v_j^2 dx\Big)^{1/t} \le C(t) \Big(\fint_{B(x,2r)} v_j^t dx\Big)^{1/t}. \tag{6.8}$$

Combining (6.8) with (6.6), we conclude:

$$\int_{B(x,r)} |\nabla v_j|^2 dx \le C(t) r^{n-2} \Big(\fint_{B(x,2r)} v_j^t\, dx\Big)^{1/t}. \tag{6.9}$$

We now wish to show that $v_j \le u_1$. For a locally finite measure σ, denote by $\mathcal{G}^\sigma(x, y)$ the minimal Green's function of $-\Delta - \sigma$ (see [FV10]), i.e. the minimal positive solution $u(\cdot, y)$ of the equation

$$-\Delta u(\cdot, y) - \sigma\, u(\cdot, y) = \delta_y, \quad y \in \Omega.$$

Then $u_1 = 1 + \int_\Omega \mathcal{G}^\sigma(x, y) d\sigma(y)$, and $v_j = 1 + \int_\Omega \mathcal{G}^{\phi_j \sigma}(x, y)\phi_j(y) d\sigma(y)$ (recall $\phi_j\sigma \in W^{-1,p'}(\Omega)$, so this representation coincides with the unique solution). By construction $\mathcal{G}^\sigma$ is monotone in σ (it can be represented by a Neumann series), and since $\phi_j\sigma \le \sigma$, it therefore follows that $v_j \le u_1$, for each j. Here u_1 is as defined in (6.3). Thus, from (6.9):

$$\int_{B(x,r)} |\nabla v_j|^2 dx \le C r^{n-2} \Big(\frac{1}{|B(x, 2r)|} \int_{B(x,2r)} u_1^t dx\Big)^{2/t}. \tag{6.10}$$

Letting $t < n/(n-2)$, and recalling the weak Harnack inequality (valid since $\sigma \ge 0$), we deduce the estimate:

$$\int_{B(x,r)} |\nabla v_j|^2\, dx \le C r^{n-2} \Big(\inf_{B(x,r)} u_1\Big)^2. \tag{6.11}$$

Using (6.11), we readily deduce that there exists $v \in L^{1,2}_{\mathrm{loc}}(\Omega)$, so that v_j increases to v, and $v_j \to v$ weakly in $L^{1,2}_{\mathrm{loc}}$. Furthermore, as in the proof of Proposition 3.2:

$$-\Delta v = \sigma\, v \quad \text{in } \mathcal{D}'(\Omega). \tag{6.12}$$

Since $v_j \le u_1$, it follows $v \le u_1$. On the other hand, we see that $v = 1$ on $\partial\Omega$ in the potential theoretic sense. Indeed, with $G(\mu)$ denoting the Green's function of the Laplacian relative to Ω applied to μ, we see by (6.12) that:

$$v = G(\sigma v) + h, \tag{6.13}$$

where h is the greatest harmonic minorant of v. Clearly, $h \ge 1$. On the other hand, since $v \le u_1$,

$$h - 1 \le v - 1 \le u_1 - 1 = G(\sigma u_1).$$

Thus $h-1$ is a nonnegative harmonic minorant of $G(\sigma u_1)$. However, by the Riesz decomposition theorem, the greatest harmonic minorant of $G(\sigma u_1)$ is zero. Thus $h \equiv 1$, and $v = 1$ on $\partial\Omega$ in the potential theoretic sense.

By minimality of u_1, it thus follows that $v = u_1$, and $u_1 \in L^{1,2}_{\text{loc}}(\Omega)$. □

7. Examples

Our first result in this section completes our discussion of the condition (5.2), and that it does not provide a characterization of distributions satisfying (5.1).

Proposition 7.1. *Let $\Omega = \mathbf{R}^n$, and $\mathcal{A}$ be the $n \times n$ identity matrix. Let σ be the radial potential defined by:*

$$\sigma(x) = \cos r + \frac{n-1}{r} \sin r - \sin^2 r, \quad r = |x|, \quad x \in \mathbf{R}^n.$$

Then σ satisfies (5.1), but cannot be represented in the form (5.2).

Proof. We first consider the case $n = 1$ and $\Omega = \mathbf{R}_+ = (0, +\infty)$. Note that a criterion of form boundedness takes the form [MV02a], [MV02b]: $\sigma = \Gamma'$ where

$$\int_a^\infty |\Gamma|^2 dx \le \frac{C}{a}, \qquad a > 0. \tag{7.1}$$

Let $\sigma = \cos x - \sin^2 x$. Then σ is semibounded by (5.3), i.e.,

$$\int_{\mathbf{R}_+} h^2 \, \sigma \, dx \le \int_{\mathbf{R}_+} |h'|^2 \, dx, \quad h \in C_0^\infty(\mathbf{R}_+),$$

but σ cannot be represented in the form

$$\sigma = \Gamma' - \mu, \quad \mu \ge 0, \tag{7.2}$$

where Γ satisfies (7.1) with any $C > 0$.

In fact, even a weaker condition

$$\int_a^{a+1} |\Gamma(x)| \, dx = o(1) \quad \textit{as} \quad a \to +\infty \tag{7.3}$$

cannot be satisfied if σ is of the form (7.2).

Indeed, suppose

$$\cos x - \sin^2 x = \Gamma' - \mu, \quad \mu \ge 0, \tag{7.4}$$

Then

$$\Gamma(x) = \sin x \left(1 + \frac{1}{2} \cos x\right) - \frac{x}{2} + \phi(x),$$

where $\phi(x)$ is nondecreasing on $\mathbf{R}_+$.

Let $\alpha_0 = \arccos \frac{\sqrt{5}-1}{2} \approx .904 < 1$ so that $\cos\alpha - \sin^2\alpha \geq 0$ for $-\alpha_0 \leq \alpha \leq \alpha_0$, and consequently Γ is nondecreasing in the interval $[2\pi n - \alpha_0,\, 2\pi n + \alpha_0]$. Hence, for $a = \frac{\alpha_0}{2}$, it follows that

$$\begin{aligned} &\Gamma(a + 2\pi n) - \Gamma(2\pi n) \\ &= \sin a \left(1 + \frac{1}{2}\cos a\right) - \frac{a}{2} + \phi(a + 2\pi n) - \phi(2\pi n) \\ &\geq \sin a \left(1 + \frac{1}{2}\cos a\right) - \frac{a}{2} = C, \end{aligned}$$

where C is independent of n. Here $C > 0$ since $\sin x \left(1 + \frac{1}{2}\cos x\right) - \frac{x}{2}$ is increasing on $(-\alpha_0, \alpha_0)$ and equals zero at the origin.

On the other hand, for $\alpha = a + 2\pi n$ we have:

$$\Gamma(\alpha) \leq \frac{2}{\alpha_0} \int_\alpha^{\alpha+a} \Gamma(x)\, dx \leq \frac{2}{\alpha_0} \int_\alpha^{\alpha+1} |\Gamma(x)|\, dx.$$

Similarly,

$$\Gamma(2\pi n) \geq \frac{2}{\alpha_0} \int_{-a+2\pi n}^{2\pi n} \Gamma(x)\, dx.$$

Hence,

$$\Gamma(a + 2\pi n) - \Gamma(2\pi n) \leq \frac{2}{\alpha_0} \int_{a+2\pi n}^{a+2\pi n+1} |\Gamma(x)|\, dx + \frac{2}{\alpha_0} \int_{-a+2\pi n}^{-a+2\pi n+1} |\Gamma(x)|\, dx.$$

By (7.3) the right-hand side of the preceding inequality tends to zero as $n \to +\infty$. This contradicts the estimate

$$\Gamma(a + 2\pi n) - \Gamma(2\pi n) \geq C > 0$$

obtained above.

We now are in a position to consider the multi-dimensional case $\Omega = \mathbf{R}^n$, $n \geq 3$. Let

$$\sigma = \cos r + \frac{n-1}{r} \sin r - \sin^2 r, \quad r = |x|, \quad x \in \mathbf{R}^n. \tag{7.5}$$

Then by (5.3) with $\Gamma = \frac{x}{r} \sin r$ it follows that (5.1) holds.

Note that $\frac{\sin r}{r}$ satisfies the inequality

$$\left| \int_{\mathbf{R}^n} h^2 \, \frac{\sin r}{r} \, dx \right| \leq C\, ||\nabla h||^2_{L^2(\mathbf{R}^n)}, \quad h \in C_0^\infty(\mathbf{R}^n). \tag{7.6}$$

Indeed, using polar coordinates and integration by parts, we obtain

$$\begin{aligned} &\left| \int_{S^{n-1}} \int_{\mathbf{R}_+} h(r\xi)^2 \, \frac{\sin r}{r} \, r^{n-1} dr d\xi \right| \leq \left| \int_{S^{n-1}} \int_{\mathbf{R}_+} 2h(r\xi)\, \nabla h(r\xi) \cdot \xi \, \frac{\cos r}{r} \, r^{n-1}\, dr d\xi \right| \\ &+(n-2) \left| \int_{S^{n-1}} \int_{\mathbf{R}_+} \cos r\, h^2(r\xi)\, r^{n-3}\, dr d\xi \right| \leq 2 \left(\int_{\mathbf{R}^n} \frac{h(x)^2}{r^2}\, dx \right)^{\frac{1}{2}} ||\nabla h||_{L^2(\mathbf{R}^n)} \\ &+(n-2) \int_{\mathbf{R}^n} \frac{h(x)^2}{r^2}\, dx \leq C\, ||\nabla h||^2_{L^2(\mathbf{R}^n)}, \end{aligned}$$

by Cauchy's inequality and Hardy's inequality. Thus, (7.6) holds.

It remains to show that

$$\sigma_1 = \cos r - \sin^2 r$$

does not satisfy the inequality

$$\sigma_1 \le \operatorname{div} \Gamma,$$

whenever Γ obeys (5.2), or equivalently (see [MV02a]), $\operatorname{div} \Gamma$ is form bounded. We note that as was proved in [MV02a], we may always pick $\Gamma = \nabla \Phi$ where $\operatorname{div} \Gamma = \Delta \Phi$, and

$$\int_{\mathbf{R}^n} h^2 \, |\nabla \Phi|^2 \, dx \le C \, ||\nabla h||^2_{L^2(\mathbf{R}^n)}, \qquad h \in C_0^\infty(\mathbf{R}^n), \tag{7.7}$$

for some C independent of h.

Suppose now that

$$\sigma_1 \le \Delta \Phi, \tag{7.8}$$

where Φ satisfies (7.7). Since σ_1 is radially symmetric, it follows by using the average of Φ over the unit sphere that (7.8) holds with a radially symmetric $\Phi_0(x) = \phi(r)$ so that

$$\sigma_1(r) \le \phi''(r) + \frac{n-1}{r} \phi'(r), \quad r > 0.$$

Moreover,

$$\phi'(r) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \sum_{k=1}^n \frac{\partial \Phi}{\partial x_k}(r\xi_k) \, \xi_k \, d\xi,$$

and hence

$$|\phi'(r)|^2 \le \frac{1}{|S^{n-1}|} \int_{S^{n-1}} |\nabla \Phi(r\xi)|^2 \, d\xi.$$

From this and (7.7) with a radially symmetric test function h, we deduce

$$\begin{aligned} \int_{\mathbf{R}_+} h^2(r) \, |\phi'(r)|^2 \, r^{n-1} dr &\le \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \int_{\mathbf{R}_+} |\nabla \Phi(r\xi)|^2 \, h^2(r) \, r^{n-1} dr d\xi \\ &\le C \int_{\mathbf{R}_+} |h'(r)|^2 \, r^{n-1} \, dr. \end{aligned}$$

Consequently,

$$\int_0^a |\phi'(r)|^2 \, r^{n-1} \, dr \le C \, a^{n-2}, \quad a > 0.$$

We now let $\psi(r) = \phi'(r)$. It follows from the preceding estimate that

$$\int_a^{a+1} |\psi(r)| \, dr \le C \, a^{-\frac{1}{2}}, \quad a > 0.$$

This implies

$$\int_a^{+\infty} \frac{|\psi(r)|}{r} dr \le C \, a^{-\frac{1}{2}}, \quad a > 0.$$

Next, denote by $g(r)$ the function:

$$g(r) = \psi(r) - (n-1) \int_r^{+\infty} \frac{\psi(r)}{r} dr.$$

It is easy to see that g satisfies the same inequality as ψ:

$$\int_a^{a+1} |g(r)| \, dr \le C \, a^{-\frac{1}{2}}, \quad a > 0.$$

Furthermore,

$$g'(r) = \psi'(r) + \frac{n-1}{r} \, \psi(r) = \phi''(r) + \frac{n-1}{r} \, \phi(r) = \Delta \Phi(x).$$

Thus,

$$\sigma_1(r) = \cos r - \sin^2 r \le g'(r), \quad r > 0,$$

where g satisfies the condition

$$\int_a^{a+1} |g(r)|\, dr = o(1), \quad a \to +\infty.$$

Hence, by the one-dimensional example on $\mathbf{R}_+$ considered above with $g = \Gamma$, we arrive at a contradiction. □

In the following example we consider non-symmetric operators, with the aim to show that the non-symmetric part can effect the constant appearing in the form bound. In Theorem 1.1, it was shown that when one has a solution to the Schrödinger equation (1.1), or Riccati equation (1.3), whose operator $\mathcal{A}$ is non-symmetric, then σ satisfies (1.6) with λ depending on the ellipticity constants (1.4). This example shows that such a conclusion is not artificial:

Example 7.2. Let $n = 3$, and suppose $\mathcal{A} = I + B$, where B has zero entries except for $b_{1,2} = Ca(x_1)$ for a constant C, and $a(x_1)$ a Lipschitz continuous function. Suppose in addition $b_{2,1} = -b_{1,2}$. If $u(x) = 1 + |x|^2$, then u solves:

$$-\mathrm{div}(\mathcal{A}\nabla u) = \sigma u(x), \text{ with } \sigma = \frac{-6 + 2x_2 Ca'(x_1)}{1 + |x|^2}$$

On the other hand:

$$\mathcal{A}(\xi) \cdot \xi = |\xi|^2.$$

It follows that in the case of non-symmetric matrices $\mathcal{A}$, the constant in statement (iii) of Theorem 1.1 depends on the constant C, and hence the operator $\mathcal{A}$.

The next example (which is well known) demonstrates the sharpness of our primary theorem for Schrödinger type equations. In particular we confirm the assertions made in the introduction.

Example 7.3. Consider positive solutions u of:

$$\begin{cases} -\Delta u = \frac{c}{|x|^2} u \text{ in } \mathbf{R}^n, \\ \inf_{\mathbf{R}^n} u = 0, \end{cases} \tag{7.9}$$

with $c \le (n-2)^2/4$. It is well known that (7.9) has positive solutions (up to constant multiple) of the form $u_\pm(x) = |x|^{\alpha_\pm}$, where:

$$\alpha_\pm = \frac{2-n}{2} \pm \frac{1}{2}\sqrt{(n-2)^2 - 4c}. \tag{7.10}$$

If $c < (n-2)^2/4$, then by Hardy's inequality it follows that (1.6) holds with $0 < \lambda < 1$. For $0 < c < (n-2)^2/4$ we see that by choosing α_+, there exists a solution $u_+ \in L^{1,2}_{\mathrm{loc}}(\Omega)$ of (7.9). Taking c arbitrarily close to $(n-2)^2/4$, we see that the existence of a solution $u_+ \in L^{1,2}_{\mathrm{loc}}(\Omega)$ of (7.9) is the optimal local regularity. The same example shows that solutions need not be locally bounded, and therefore positive solutions of (7.9) do not satisfy the Harnack inequality.

Choosing α_- in (7.10), it follows that there exist positive solutions $u_- \in L^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$ of (7.9), which are not in $L^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$.

Finally, let $c = (n-2)^2/4$ in (7.9). The resulting unique positive solution does not lie in $L^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$. This latter point shows the assumption $0 < \lambda < 1$ in (1.6) is necessary in order to prove statement (i) of Theorem 1.1. We remark that the uniqueness of the positive superharmonic solution in this case is known even for quasilinear generalizations of (7.9), see [PS05].

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