

ESTIMATE FOR A SOLUTION TO THE WATER WAVE PROBLEM IN THE PRESENCE OF A SUBMERGED BODY

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Dedicated to the memory of Mark Vishik

ABSTRACT. We study the two-dimensional problem of propagation of linear water waves in deep water in the presence of a submerged body. Under some geometrical requirements, we derive an explicit bound for the solution depending on the domain and the functions on the right-hand side.

Key words and phrases. Water waves, harmonic oscillations, submerged body, resolvent estimates.

1. Introduction.

1.1. Boundary value problem. We study the linear water waves problem for a two-dimensional domain Ω , which represents water of infinite depth in the presence of a submerged body B .

We fix a Cartesian system $x = (x_1, x_2)$ with the origin O and consider a bounded domain B ,

$$\overline{B} \subset \mathbb{R}_+^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\},$$

with smooth boundary S . The role of the water surface is played by the line $\Gamma = \{x : x_2 = 0\}$ (note that the axis x_2 is directed into water).

Let the velocity potential u be a smooth function in $\overline{\Omega}$ subject to the equations:

$$(1.1) \quad -\Delta u = f \text{ in } \Omega,$$

$$(1.2) \quad \partial_n u = g_1 \text{ on } S,$$

$$(1.3) \quad \partial_n u - \nu u = g_2 \text{ on } \Gamma,$$

where $n = (n_1, n_2)$ is the external normal to Ω , ν is a positive spectral parameter and f, g_1, g_2 are given smooth functions, where f and g_2 vanish at infinity with an appropriate rate of decay.

We are looking for solutions which satisfy the *radiation conditions at infinity* (see (2.1) below, for a precise definition): there exist constants d^+ and d^- , such that:

$$(1.4) \quad u(x) = d^+ e^{-i\nu x_1 - \nu x_2} + o(1), \text{ as } x_1 \rightarrow +\infty,$$

and

$$(1.5) \quad u(x) = d^- e^{i\nu x_1 - \nu x_2} + o(1), \text{ as } x_1 \rightarrow -\infty.$$

Mathematical aspects of this problem have been studied extensively, see e.g. [1]-[20]. In particular, it is well known that the assumption of uniqueness of a solution implies solvability of the problem. We mention the following condition of uniqueness

$$(1.6) \quad x_1(x_1^2 - x_2^2)n_1(x) + 2x_1^2x_2n_2(x) \geq 0, \quad x \in S,$$

which was obtained in [4]. The geometrical interpretation of this inequality is discussed in [5]. Let us consider the one-parametric family of circles passing through the origin, centered at $(0, a)$, with $a > 0$. The condition (1.6) means that while moving upwards from the origin along these circles we intersect S at most once.

1.2. Description of the main result. The motivation for the present paper is the fact that for the time being, there is no quantitative information on the dependence on the data and geometry of the domain for solutions to problem (1.1)-(1.5). Our goal here is to fill this gap.

The main result will be obtained under the following requirements, which together are more stringent than (1.6).

Condition 1. For all $x \in S$

$$(1.7) \quad x_1 n_1(x) \leq 0.$$

Condition 2. For a certain $\varepsilon \in (0, h]$, where h is the distance from Γ to S , there holds:

$$(1.8) \quad x_1(x_1^2 - (x_2 - \varepsilon)^2)n_1(x) + 2x_1^2(x_2 - \varepsilon)n_2(x) \geq 0, \quad x \in S.$$

The geometrical meaning of Condition 1 is obvious, while Condition 2 has the same sense as (1.6), with the only difference that the circles pass through the point $(0, \varepsilon)$ instead of the origin. We show in Section 4 that Conditions 1 and 2 imply (1.6) which in turn implies uniqueness of the solution to problem (1.1)-(1.5).

Our principal result (Theorem (1.1)) is a uniform in ε estimate of the sum

$$\varepsilon^2 (|d^+| + |d^-|) + \varepsilon^4 \nu^2 \left(\int_{\Omega} \frac{|\nabla u|^2 + \nu^2 |u|^2}{1 + \nu^2 |x|^2} dx \right)^{1/2}$$

by certain norms of the data f, g_1, g_2 .

In order to give a complete formulation of the result, we need a number of notations. First, we assume that

$$(1.9) \quad B \subset \{x : |x_1| < L, h < x_2 < H\}$$

for some positive numbers L, h, H and maximum modulus of the curvature of S is less than κ .

To measure the solution u we introduce the norm

$$(1.10) \quad |u| = \left(\|\gamma_0 u\|_{\Omega}^2 + \nu^{-2} \|\gamma_0 \nabla u\|_{\Omega}^2 + \|\gamma_1 u\|_S^2 + \|\gamma_2 u\|_{\Gamma}^2 \right)^{1/2},$$

where $\|w\|_{\Xi}$ stands for the L_2 norm of w on the set Ξ . The weight functions γ_0, γ_1 and γ_2 are defined by

$$(1.11) \quad \gamma_0^2 = \nu^2 (L^2 \nu^2 + 1 + \nu^2 x_1^2 + \nu^2 x_2^2)^{-1},$$

$$(1.12) \quad \gamma_1^2 = (L + \nu^{-1} + H)^{-1}, \quad \gamma_2^2 = \nu(1 + \nu^2 x_1^2)^{-1}.$$

As for the right-hand side

$$(1.13) \quad F = (f, g_1, g_2),$$

we measure it using the norm $|\cdot|_*$, given by

$$(1.14) \quad |F|_* =$$

$$\left(\|\gamma_0^{-1} f\|_{\Omega}^2 + \|\gamma_1^{-1} g_1\|_S^2 + \|\gamma_1^{-1} x_1 \partial_{\sigma} g_1\|_S^2 + \|\gamma_2^{-1} g_2\|_{\Gamma}^2 + \nu^{-1} \|x_1 \partial_{x_1} g_2\|_{\Gamma}^2 \right)^{1/2},$$

where ∂_{σ} is the first derivative with respect to the length of the contour S .

Now we are in a position to formulate the main result.

Theorem 1.1. *Let (1.7) and (1.8) hold. Further, let $(1 + \nu|x|) f \in L_2(\Omega)$, $g_1, x_1 \partial_{\sigma} g_1 \in L_2(S)$ and $(1 + \nu|x_1|) g_2, x_1 \partial_{x_1} g_2 \in L_2(\Gamma)$. Then a unique solution u of (1.1)-(1.3), subject to the radiation conditions (see below (2.1)-(2.3)), satisfies the estimate*

$$(1.15) \quad |u| \leq c(1 + C) |F|_*, \quad |d^+| + |d^-| \leq c(1 + C)^{1/2} |F|_*,$$

where

$$(1.16) \quad C \leq h^{-2} \varepsilon^{-2} (1 + \kappa \tau)^4 (1 + \nu h)^6 \nu^3 \tau^7, \quad \tau = L + \nu^{-1} + H,$$

with $F = (f, g_1, g_2)$ and an absolute positive constant c .

1.3. Plan of the paper. We demonstrate in Section 2 that in order to estimate the scattering coefficients d^+ and d^- and the norm of the solution u it suffices to deal with solutions having the finite Dirichlet integral. In Section 3, a certain weighted L_2 estimate for the tangential derivative on S is obtained. Derivatives in the horizontal and vertical directions are estimated in Sections 4 and 5. One of consequences of the result of Section 5 guarantees the unique solvability of problem (1.1)-(1.5) for bodies sufficiently narrow in the horizontal direction. Estimates for the solution and its boundary traces are given in Section 6. In the last Section 7 we collect the estimates previously obtained to complete the proof of Theorem 1.1.

2. Estimate of scattering coefficients .

We start with making precise the definition of the radiation conditions mentioned in Section 1. In order to formulate (1.4)-(1.5) in more detail we introduce a cut off function $\chi \in C^{\infty}(\mathbb{R})$, such that

$$\chi(t) = 0 \text{ for } t < 1 \text{ and } \chi(t) = 1 \text{ for } t > 3,$$

and $|\chi'| < 1, |\chi''| < 1$. We say that u satisfies radiation conditions if

$$(2.1) \quad u(x) = d^+ U^+(x) + d^- U^-(x) + v(x),$$

where

$$(2.2) \quad U^+(x) = \chi(x_1 \gamma_1^2) e^{-i\nu x_1 - \nu x_2}, \quad U^-(x) = d^- \chi(-x_1 \gamma_1^2) e^{i\nu x_1 - \nu x_2}$$

and d^\pm are constants and v is the remainder such that

$$(2.3) \quad \|\nabla v\|_\Omega + \|v\|_\Gamma < +\infty$$

(compare with (1.5)). If the right-hand side of (1.1)-(1.3) is smooth and compactly supported, then

$$(2.4) \quad |v| = O(r^{-1}) \quad \text{and} \quad |\nabla v| = O(r^{-2}), \quad \text{as} \quad r = (x_1^2 + x_2^2)^{1/2} \rightarrow \infty,$$

see e.g. [12] p. 46.

In this section, we verify that in order to estimate the norm of the solution u it is enough to estimate the remainder v . Let us write down the boundary value problem for v :

$$(2.5) \quad -\Delta v = f_1 \quad \text{in } \Omega,$$

$$(2.6) \quad \partial_\nu v = g_1 \quad \text{on } S,$$

$$(2.7) \quad \partial_n v - \nu v = g_2 \quad \text{on } \Gamma,$$

where

$$(2.8) \quad f_1 = f + \Delta(d^+U^+ + d^-U^-).$$

The function v will be measured by the norm $\|\cdot\|$ given by

$$(2.9) \quad \|v\|^2 = \|\gamma_0 v\|_\Omega^2 + \|\nabla v\|_\Omega^2 + \|\gamma_1 v\|_S^2 + \nu \|v\|_\Gamma^2.$$

Comparison of (2.9) and (1.10) leads to the inequality

$$(2.10) \quad |v| \leq \|v\|.$$

Lemma 2.1. *Let $f \in C_0^\infty(\Omega)$, $g_1 \in C^\infty(S)$, $g_2 \in C_0^\infty(\Gamma)$ and let a solution v of (2.5) - (2.8), be subject to the estimate*

$$(2.11) \quad \|v\| \leq C_0 |F_1|_*,$$

where $F_1 = (f_1, g_1, g_2)$. Then for $F = (f, g_1, g_2)$ the estimate holds

$$(2.12) \quad \|v\| \leq C_1 |F|_*,$$

and the solution u of problem (1.1)-(1.3), (2.1)-(2.3) satisfies the inequalities

$$(2.13) \quad |u| \leq C_1 |F|_*, \quad \|d\| \leq C_2 |F|_*,$$

where $d = (d^+, d^-)$ and $\|d\|^2 = |d^+|^2 + |d^-|^2$ and

$$(2.14) \quad C_1 \leq c(1 + \nu\gamma_1^{-2}C_0^2), \quad C_2 \leq c(1 + \nu\gamma_1^{-2}C_0^2)^{1/2}.$$

Proof. It follows from (2.1), (2.10) and (2.11) that

$$(2.15) \quad \begin{aligned} |u| &\leq |v| + |d^+||U^+| + |d^-||U^-| \\ &\leq \|v\| + \|d\|A \leq C_0|F_1|_* + \|d\|A, \end{aligned}$$

where

$$(2.16) \quad A = (|U^+|^2 + |U^-|^2)^{1/2} \leq 3.$$

We need to estimate u by norm of F not of F_1 . To achieve this we first majorise F_1 in terms of F and d :

$$(2.17) \quad \begin{aligned} |F_1|_* &\leq |F|_* + |F_1 - F|_* = |F|_* + \|\gamma_0^{-1}\Delta(d^+U^+ + d^-U^-)\|_\Omega \\ &\leq |F|_* + \|d\|\mathcal{B}, \end{aligned}$$

where

$$(2.18) \quad \mathcal{B} = (\|\gamma_0^{-1}\Delta U^-\|_\Omega^2 + \|\gamma_0^{-1}\Delta U^+\|_\Omega^2)^{1/2} \leq 2^5\nu^{1/2}\gamma_1^{-1}.$$

By (2.15) and (2.17) we conclude that

$$(2.19) \quad |u| \leq C_0|F|_* + \|d\|(A + C_0\mathcal{B}).$$

It remains to estimate d by F . Green's formula applied to u and \bar{u} implies:

$$(2.20) \quad |d^+|^2 + |d^-|^2 = 2 \operatorname{Im} \left(\int_\Omega f\bar{u}dx + \int_S g_1\bar{u}dS + \int_\Gamma g_2\bar{u}dS \right),$$

where we have used (2.4) (see [12], p. 69, for an analogous formula in the case $f = 0$, $g_2 = 0$). Representation (2.20) combined with (1.10) and (1.14) yields

$$(2.21) \quad \|d\|^2 \leq 2|u||F|_*.$$

Inequalities (2.19) and (2.21) imply

$$(2.22) \quad |u| \leq 2(C_0 + (A + \mathcal{B}C_0)^2)|F|_*,$$

and

$$(2.23) \quad \|d\| \leq 2(C_0 + (A + \mathcal{B}C_0)^2)^{1/2}|F|_*.$$

Finally estimates

$$(2.24) \quad A \leq 3, \quad \mathcal{B} \leq 2^5\nu^{1/2}\gamma_1^{-1},$$

together with (2.22) and (2.23) lead to (2.13) and (2.12). The proof of lemma is complete. \square

3. Estimate of tangential derivative on S

Let T denote a constant and let $Z = (Z_1, Z_2)$ be a smooth real vector field in Ω with bounded derivatives and $Z_2(x_1, 0) = 0$ for all x_1 . Without loss of generality we suppose that v , a solution of (2.5)-(2.8), is real. The following identity is stated in [4], see also [12], p.71:

$$(3.1) \quad 2\{(Z \cdot \nabla v + Tv)\Delta v\} = 2\nabla \cdot \{(Z \cdot \nabla v + Tv)\nabla v\} \\ + (Q\nabla v) \cdot \nabla v - \nabla \cdot (|\nabla v|^2 Z) .$$

Here Q is a 2×2 matrix with the components $Q_{ij} = (\nabla \cdot Z - 2T)\delta_{ij} - (\partial_i Z_j + \partial_j Z_i)$, $i, j = 1, 2$. Let us integrate (3.1) over Ω and using (2.4) integrate by parts:

$$(3.2) \quad \int_{\Omega} 2\{(Z \cdot \nabla v + Tv)\Delta v\} dx = 2 \int_{\partial\Omega} (Z \cdot \nabla v + Tv)\partial_n v ds + \\ \int_{\Omega} (Q\nabla v) \cdot \nabla v dx - \int_{\partial\Omega} |\nabla v|^2 (Z \cdot n) ds = \\ 2 \int_{\Gamma} (Z \cdot \nabla v + Tv)(\partial_n - \nu)v dx_1 + 2\nu \int_{\Gamma} (Z \cdot \nabla v + Tv)v dx_1 + \\ 2 \int_S (Z \cdot \nabla v + Tv)\partial_n v ds + \int_{\Omega} (Q\nabla v) \cdot \nabla v dx - \int_S |\nabla v|^2 (Z \cdot n) ds.$$

Since $Z_2 = 0$ on Γ the right-hand side can be written as:

$$(3.3) \quad 2 \int_{\Gamma} (Z_1 \partial_{x_1} v + Tv)(\partial_n - \nu)v dx_1 + 2\nu \int_{\Gamma} (Z_1 \partial_{x_1} v + Tv)v dx_1 + \\ 2 \int_S (Z \cdot \nabla v + Tv)\partial_n v ds + \int_{\Omega} (Q\nabla v) \cdot \nabla v dx - \int_S |\nabla v|^2 (Z \cdot n) ds.$$

We have

$$Z\nabla v = (Z \cdot n)v_n + (Z \cdot \sigma)v_{\sigma},$$

where σ is the unit tangential vector to S . Then (3.3) can be expressed in the form

$$2 \int_{\Gamma} (Z_1 \partial_{x_1} v + Tv)(\partial_{x_2} - \nu)v dx_1 + 2\nu \int_{\Gamma} (Z_1 \partial_{x_1} v + Tv)v dx_1 + \\ 2 \int_S ((Z \cdot n)\partial_n v + (Z \cdot \sigma)\partial_{\sigma} v + Tv)\partial_n v ds + \int_{\Omega} (Q\nabla v) \cdot \nabla v dx - \int_S |\nabla v|^2 (Z \cdot n) ds.$$

Which can be written as follows integrating by parts

$$2 \int_{\Gamma} v(T - \frac{\partial}{\partial x_1} Z_1)(\partial_{x_2} - \nu)v dx_1 + \nu \int_{\Gamma} (2T - \partial_{x_1} Z_1)|v|^2 dx_1 +$$

$$2 \int_S ((Z \cdot n) \partial_n v + (Z \cdot \sigma) \partial_\sigma v + Tv) \partial_n v ds + \int_\Omega (Q \nabla v) \cdot \nabla v dx - \int_S |\nabla v|^2 (Z \cdot n) ds.$$

Noting that

$$|\nabla v|^2 = v_\sigma^2 + v_n^2,$$

we arrive at the identity.

$$\begin{aligned} & \int_\Omega 2\{(Z \cdot \nabla v + Tv) \Delta v\} dx = \\ & 2 \int_\Gamma v (T - \frac{\partial}{\partial x_1} Z_1) (\partial_{x_2} - \nu) v dx_1 + \nu \int_\Gamma (2T - \partial_{x_1} Z_1) |v|^2 dx_1 + \\ & \int_S ((Z \cdot n) \partial_n v + 2(Z \cdot \sigma) \partial_\sigma v + 2Tv) \partial_n v ds + \int_\Omega (Q \nabla v) \cdot \nabla v dx - \int_S |\partial_\sigma v|^2 (Z \cdot n) ds. \end{aligned}$$

Now using (2.5)-(2.8), we obtain

$$(3.4) \quad \nu \int_\Gamma (\frac{\partial Z_1}{\partial x_1} - 2T) |v|^2 dx_1 - \int_\Omega (Q \nabla v) \cdot \nabla v dx + \int_S |\partial_\sigma v|^2 (Z \cdot n) ds = I(v, Z, T),$$

where

$$\begin{aligned} I(v, Z, T) &= 2 \int_\Gamma v (T - \frac{\partial}{\partial x_1} Z_1) g_2 dx_1 + \\ &+ \int_S g_1 ((Z \cdot n) g_1 + 2(Z \cdot \sigma) \partial_\sigma v + 2Tv) ds + \int_\Omega 2f_1 (Z \cdot \nabla v + Tv) dx. \end{aligned}$$

We shall use (3.4) in the next Lemma to estimate the tangential derivative of the solution v .

Lemma 3.1. *Let $f \in C_0^\infty(\Omega)$, $g_1 \in C^\infty(S)$ and $g_2 \in C_0^\infty(\Gamma)$. Then a solution v of (2.5)-(2.8) such that $\nabla v \in L_2(\Omega)$ and $v|_\Gamma \in L_2(\Gamma)$, satisfies the estimate*

$$(3.5) \quad \int_S |\partial_\sigma v|^2 (W \cdot n) ds \leq C_3 \|v\| \|F_1\|_* + |F_1|_*^2,$$

where

$$(3.6) \quad F_1 = (f_1, g_1, g_2),$$

$$(3.7) \quad W(x_1, x_2) = \left(x_1 \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}, \frac{2x_1^2 x_2}{x_1^2 + x_2^2} \right),$$

$$(3.8) \quad C_3 = (3 + 2(\kappa L + 10)),$$

and κ is the maximum modulus of the curvature of S .

Proof. We put

$$(3.9) \quad Z = W, \quad T = 1/2.$$

in (3.4). This is exactly the choice from [4] (see also [12], p.76.), where it was verified that the quadratic form $(Q\nabla v) \cdot \nabla v$ is non-positive. Moreover the first term in (3.4) vanishes. Then it follows from (3.4) that,

$$(3.10) \quad \int_S |\partial_\sigma v|^2 (W \cdot n) ds \leq I(v, W, 1/2),$$

where

$$(3.11) \quad \begin{aligned} I(v, W, 1/2) = & - \int_\Gamma v(g_2 + 2x_1 \partial_{x_1} g_2) dx_1 + \\ & + \int_S g_1 ((W \cdot n)g_1 + 2(W \cdot \sigma) \partial_\sigma v + v) ds + \int_\Omega f_1 (2W \cdot \nabla v + v) dx. \end{aligned}$$

Let us estimate $I(v, W, 1/2)$. For the last term in (3.11) we have

$$(3.12) \quad \begin{aligned} \left| \int_\Omega f_1 (2W \cdot \nabla v + v) dx \right| & \leq 2 \|f_1 W\|_\Omega \|\nabla v\|_\Omega + \|\gamma_0^{-1} f_1\|_\Omega \|\gamma_0 v\|_\Omega \\ & \leq 2 \|\gamma_0^{-1} f_1\|_\Omega \|\nabla v\|_\Omega + \|\gamma_0^{-1} f_1\|_\Omega \|\gamma_0 v\|_\Omega, \end{aligned}$$

where we have used (1.11) and (2.4). For the first term on the right-hand side of (3.11) we use (1.12) and obtain,

$$(3.13) \quad \begin{aligned} \left| \int_\Gamma v(g_2 + 2x_1 \partial_{x_1} g_2) dx_1 \right| & \leq \|g_2\|_\Gamma \|v\|_\Gamma + 2 \|x_1 \partial_{x_1} g_2\|_\Gamma \|v\|_\Gamma \\ & \leq \nu^{1/2} \|v\|_\Gamma \|\gamma_2^{-1} g_2\|_\Gamma + 2\nu^{1/2} \|v\|_\Gamma \nu^{-1/2} \|x_1 \partial_{x_1} g_2\|_\Gamma. \end{aligned}$$

For the remaining term, on the right-hand side of (3.11), we have

$$(3.14) \quad \begin{aligned} \left| \int_S g_1 ((W \cdot n)g_1 + 2(W \cdot \sigma) \partial_\sigma v + v) ds \right| & \leq L \|g_1\|_S^2 + \\ & \|\gamma_1^{-1} g_1\|_S \|\gamma_1 v\|_S + 2 \left| \int_S v \partial_\sigma ((W \cdot \sigma)g_1) ds \right| \leq \\ & \|\gamma_1^{-1} g_1\|_S^2 + \|\gamma_1^{-1} g_1\|_S \|\gamma_1 v\|_S + 2 \|\gamma_1 v\|_S \|\gamma_1^{-1} (W \cdot \sigma) \partial_\sigma g_1\|_S \\ & + 2 \left| \int_S v g_1 \partial_\sigma (W \cdot \sigma) ds \right| \\ & \leq \|\gamma_1^{-1} g_1\|_S^2 + \|\gamma_1^{-1} g_1\|_S \|\gamma_1 v\|_S + 2 \|\gamma_1 v\|_S \|\gamma_1^{-1} x_1 \partial_\sigma g_1\|_S \end{aligned}$$

$$+2(\kappa L + 10) \|\gamma_1^{-1} g_1\|_S \|\gamma_1 v\|_S.$$

Combining (3.11)-(3.14) we complete the proof. \square

4. Estimate of the horizontal derivative.

In this section we estimate $\partial_{x_1} v$ in the domain Ω .

Lemma 4.1. *Under the assumptions of Lemma 3.1 on f, g_1, g_2 and v , the estimate holds*

$$(4.1) \quad \|v_{x_1}\|_{\Omega}^2 \leq C_4 \|v\| \|F_1\|_* + 2^{-1} |F_1|_*^2 - 2^{-1} \int_S |\partial_{\sigma} v|^2 x_1 n_1(x) ds.$$

where

$$(4.2) \quad C_4 = \sqrt{2} + \kappa L \leq 2^{-1} C_3.$$

Proof. We put in (3.4) $T = 1/2, Z = V$, where

$$V(x) = (x_1, 0).$$

Then

$$(4.3) \quad 2\|v_{x_1}\|_{\Omega}^2 + \int_S x_1 n_1(x) |\partial_{\sigma} v|^2 ds = I(v, V, 1/2),$$

where

$$(4.4) \quad \begin{aligned} I(v, V, 1/2) = & - \int_{\Gamma} v(g_2 + 2x_1 \partial_{x_1} g_2) dx_1 + \\ & + \int_S g_1 ((V \cdot n)g_1 + 2(V \cdot \sigma) \partial_{\sigma} v + v) ds + \int_{\Omega} f_1 (2V \cdot \nabla v + v) dx. \end{aligned}$$

Next we estimate $I(v, V, 1/2)$. For the last term in (4.4) we have

$$(4.5) \quad \begin{aligned} \left| \int_{\Omega} f_1 (2V \cdot \nabla v + v) dx \right| & \leq 2\|f_1 V\|_{\Omega} \|\nabla v\|_{\Omega} + \|\gamma_0^{-1} f_1\|_{\Omega} \|\gamma_0 v\|_{\Omega} \\ & \leq 2\|\gamma_0^{-1} f_1\|_{\Omega} \|\nabla v\|_{\Omega} + \|\gamma_0^{-1} f_1\|_{\Omega} \|\gamma_0 v\|_{\Omega}, \end{aligned}$$

where we have used (1.11). In order to estimate the first term on the right-hand side of (4.4) we use (1.12) and obtain,

$$(4.6) \quad \begin{aligned} \left| \int_{\Gamma} v(g_2 + 2x_1 \partial_{x_1} g_2) dx_1 \right| & \leq \|g_2\|_{\Gamma} \|v\|_{\Gamma} + 2\|x_1 \partial_{x_1} g_2\|_{\Gamma} \|v\|_{\Gamma} \\ & \leq \nu^{1/2} \|v\|_{\Gamma} \|\gamma_2^{-1} g_2\|_{\Gamma} + 2\nu^{1/2} \|v\|_{\Gamma} \nu^{-1/2} \|x_1 \partial_{x_1} g_2\|_{\Gamma}. \end{aligned}$$

Finally for the second integral on the right-hand side of (4.4) we have

$$\begin{aligned}
(4.7) \quad & \left| \int_S g_1 ((V \cdot n)g_1 + 2(V \cdot \sigma)\partial_\sigma v + v) ds \right| \leq L\|g_1\|_S^2 + \\
& \|\gamma_1^{-1}g_1\|_S\|\gamma_1v\|_S + 2 \left| \int_S v\partial_\sigma ((V \cdot \sigma)g_1) ds \right| \leq \\
& \|\gamma_1^{-1}g_1\|_S^2 + \|\gamma_1^{-1}g_1\|_S\|\gamma_1v\|_S + 2\|\gamma_1v\|_S\|\gamma_1^{-1}(V \cdot \sigma)\partial_\sigma g_1\|_S \\
& + 2 \left| \int_S vg_1\partial_\sigma (V \cdot \sigma) ds \right| \\
& \leq \|\gamma_1^{-1}g_1\|_S^2 + \|\gamma_1^{-1}g_1\|_S\|\gamma_1v\|_S + 2\|\gamma_1v\|_S\|\gamma_1^{-1}x_1\partial_\sigma g_1\|_S \\
& + 2(\kappa L + 1)\|\gamma_1^{-1}g_1\|_S\|\gamma_1v\|_S.
\end{aligned}$$

Combining (4.4)-(4.6) we arrive at

$$(4.8) \quad 2\|v_{x_1}\|_\Omega^2 + \int_S |\partial_\sigma v|^2 x_1 n_1(x) ds \leq (2\sqrt{2} + 2\kappa L)\|v\| \|F_1\|_* + \|F_1\|_*^2.$$

□

Lemma 4.2. *Under Conditions 1 and 2 (see (1.7),(1.8)) we have the estimate:*

$$(4.9) \quad -x_1 n_1 \leq \frac{H}{\varepsilon} W(x) \cdot n(x), \quad x \in S.$$

Proof. Let us notice that if for some point $x \in S$ we have $x_2 = h$, then inequality (4.9) is obvious since then $x_1 n_1(x) = 0$, (see (1.9)) and $W(x) \cdot n(x) \geq 0$ by the same reason. As a result we can assume that

$$(4.10) \quad x_2 > h \geq \varepsilon.$$

Condition (1.8) implies,

$$\begin{aligned}
(4.11) \quad & 0 \leq \|x\|^{-2} (x_1(x_1^2 - (x_2 - \varepsilon)^2)n_1 + 2x_1^2(x_2 - \varepsilon)n_2) \\
& = W(x) \cdot n(x) + \varepsilon\|x\|^{-2} (2x_1x_2n_1 - 2x_1^2n_2) - \varepsilon^2\|x\|^{-2}x_1n_1.
\end{aligned}$$

We see that the term in brackets in (4.11) can be written as,

$$\begin{aligned}
(4.12) \quad & \|x\|^{-2} (2x_1x_2n_1 - 2x_1^2n_2) = \|x\|^{-2}x_2^{-1} (x_1n_1\|x\|^2 - x_1(x_1^2 - x_2^2)n_1 - 2x_1^2x_2n_2) \\
& = x_2^{-1}x_1n_1 - x_2^{-1}W(x) \cdot n(x).
\end{aligned}$$

Then (4.11) implies:

$$(4.13) \quad 0 \leq W(x) \cdot n(x) \left(1 - \frac{\varepsilon}{x_2}\right) + \frac{\varepsilon}{x_2}x_1n_1 \left(1 - \frac{\varepsilon x_2}{\|x\|^2}\right) \leq$$

$$W(x) \cdot n(x) \left(1 - \frac{\varepsilon}{x_2}\right) + \frac{\varepsilon}{x_2} x_1 n_1 \left(1 - \frac{\varepsilon}{x_2}\right),$$

where we have used Condition 1. By (1.9) and (4.10) we arrive at (4.9). \square

Combining Lemma 3.1, Lemma 4.1 and Lemma 4.2 we arrive at the following assertion.

Theorem 4.3. *Under Conditions 1 and 2 (see (1.7), (1.8)) and the assumptions of Lemma 3.1 on f, g_1, g_2 and v we have the estimate:*

$$(4.14) \quad \|v_{x_1}\|_{\Omega}^2 \leq C_5 \|v\| |F_1|_* + C_6 |F_1|_*^2,$$

where

$$(4.15) \quad C_5 \leq C_4 + \frac{HC_3}{2\varepsilon} \leq C_3 \frac{H}{\varepsilon},$$

and

$$(4.16) \quad C_6 \leq \frac{H}{2\varepsilon} + \frac{1}{2} \leq \frac{H}{\varepsilon}.$$

5. Estimate of the vertical derivative

In this section we do not use Conditions 1 and 2. We only assume that,

$$(5.1) \quad B \subset \{x : |x_1| < L, h < x_2\}$$

for some positive numbers L, h .

Let us write the identity

$$(5.2) \quad \int_{\Omega} f_1 v dx + \int_S g_1 v dS + \int_{\Gamma} g_2 v dx_1 = \int_{\Omega} |\nabla v|^2 dx - \nu \int_{\Gamma} v^2 dx_1.$$

and put

$$(5.3) \quad J := \int_{\Omega} f_1 v dx + \int_S g_1 v dS + \int_{\Gamma} g_2 v dx_1.$$

The next lemma will be used in the proof of Theorem 1.1. But it is also of independent interest.

Lemma 5.1. *Under conditions (5.1) and the assumptions of Lemma 3.1 on f, g_1, g_2 and v , the estimate holds*

$$(5.4) \quad \|\nabla v\|_{\Omega}^2 \leq 4J + 18 (\|r f_1\|_{\Omega}^2 + \nu \|r g_2\|_{\Gamma}^2) + (C_7 - 3) \|v_{x_1}\|_{\Omega}^2,$$

where

$$(5.5) \quad C_7 \leq 72 \frac{\nu L^2}{h} (1 + \nu h)^3.$$

Proof. We divide the right-hand side of (5.2) in three parts:

$$(5.6) \quad \int_{\Omega} |\nabla v|^2 dx - \nu \int_{\Gamma} v^2 dx_1 = I_1 + I_2 + I_3,$$

where

$$(5.7) \quad I_j = \int_{\Omega_j} |\nabla v|^2 dx - \nu \int_{\Gamma_j} v^2 dx_1, \quad j = 1, 2, 3,$$

with

$$\Omega_1 = \Omega \cap \{x_1 < -L\}, \quad \Omega_2 = \Omega \cap \{-L < x_1 < L\},$$

$$\Omega_3 = \Omega \cap \{L < x_1\}$$

and

$$\Gamma_1 = \Gamma \cap \{x_1 < -L\}, \quad \Gamma_2 = \Gamma \cap \{-L < x_1 < L\},$$

$$\Gamma_3 = \Gamma \cap \{L < x_1\}.$$

Let us treat I_1 . We consider the orthogonal decomposition of v in Ω_1 , see [1]:

$$(5.8) \quad v(x) = \Theta(x) + a(x_1)\Phi(x_2),$$

where

$$(5.9) \quad \Phi(x_2) = \sqrt{2\nu} e^{-\nu x_2}, \quad a(x_1) = \int_0^{+\infty} v(x_1, x_2)\Phi(x_2)dx_2.$$

Substituting (5.8) into the definition of I_1 we get

$$(5.10) \quad \begin{aligned} I_1 = & \int_{\Omega_1} v_{x_1}^2 dx + \frac{1}{4} \int_{\Omega_1} v_{x_2}^2 dx + \frac{3}{4} \int_{\Omega_1} \Theta_{x_2}^2 dx + \frac{3}{4} \nu^2 \int_{\Omega_1} a^2(x_1)\Phi^2(x_2)dx \\ & - \nu \int_{\Gamma_1} \Theta^2 dx_1 - \nu \int_{\Gamma_1} a^2(x_1)\Phi^2(0)dx_1 = \\ & \int_{\Omega_1} v_{x_1}^2 dx + \frac{1}{4} \int_{\Omega_1} v_{x_2}^2 dx + \left(\frac{3}{4} \int_{\Omega_1} \Theta_{x_2}^2 dx - \nu \int_{\Gamma_1} \Theta^2 dx_1 \right) - \nu^2 \frac{5}{4} \int_{\Gamma_1} a^2(x_1)dx_1. \end{aligned}$$

To estimate the third term in the right-hand side of (5.10) we use the F. John's estimate, see [1],

$$(5.11) \quad \int_{\Omega_1} \Theta_{x_2}^2 dx / 2 \geq \nu \int_{\Gamma_1} \Theta^2 dx_1,$$

which follows from orthogonality condition

$$(5.12) \quad \int_0^{+\infty} \Theta(x_1, x_2)e^{-\nu x_2} dx_2 = 0, \quad x_1 \in (\infty, -L),$$

see (5.8) and (5.9). Then

$$(5.13) \quad I_1 \geq \int_{\Omega_1} v_{x_1}^2 dx + \frac{1}{4} \int_{\Omega_1} v_{x_2}^2 dx + \frac{1}{4} \int_{\Omega_1} \Theta_{x_2}^2 dx - \frac{5\nu^2}{4} \int_{\Gamma_1} a^2(x_1) dx_1.$$

We wish to check that the third term on the right-hand side of (5.13) controls the L_2 norm of Θ in the horizontal semi-strip

$$\Omega'_1 := (-\infty, -L) \times (0, h).$$

In fact, for any $w \in H^1(0, N)$ and positive α and N we have the obvious inequality

$$(5.14) \quad M_N(\alpha) \int_0^N w^2 dt \leq \alpha w^2(0) + \int_0^N |w'|^2 dt,$$

where

$$M_N(\alpha) \geq \frac{\pi^2 \alpha}{N(4\alpha N + \pi^2)} \geq \frac{\alpha}{N(\alpha N + 1)}.$$

Applying (5.14) (with $\alpha = \nu$ and $N = h$) to the third term on the right-hand side of (5.13) and using (5.11) we derive,

$$(5.15) \quad M_h(\nu) \|\Theta\|_{\Omega'_1}^2 \leq \nu \|\Theta\|_{\Gamma_1}^2 + \|\Theta_{x_2}\|_{\Omega'_1}^2 \leq 2\|\Theta_{x_2}\|_{\Omega_1}^2.$$

On the other hand, we have

$$(5.16) \quad \begin{aligned} \|v\|_{\Omega'_1}^2 &\leq 2\|\Theta\|_{\Omega'_1}^2 + 2\|a\Phi\|_{\Omega'_1}^2 = \\ &2\|\Theta\|_{\Omega'_1}^2 + 2\|a\|_{\Gamma_1}^2 (1 - e^{-2\nu h}) \leq 2\|\Theta\|_{\Omega'_1}^2 + 4\nu h \|a\|_{\Gamma_1}^2. \end{aligned}$$

Combining (5.15) and (5.16) we obtain,

$$(5.17) \quad \frac{M_h(\nu)}{4} \|v\|_{\Omega'_1}^2 - M_h(\nu) \nu h \|a\|_{\Gamma_1}^2 \leq \|\Theta_{x_2}\|_{\Omega_1}^2.$$

By (5.17) and (5.13) we have

$$(5.18) \quad \begin{aligned} I_1 &\geq \|v_{x_1}\|_{\Omega_1}^2 + \frac{1}{4} \|v_{x_2}\|_{\Omega_1}^2 + \frac{M_h(\nu)}{16} \|v\|_{\Omega'_1}^2 - \left(\frac{M_h(\nu) \nu h}{4} + \frac{5\nu^2}{4} \right) \|a\|_{\Gamma_1}^2 \\ &\geq \|v_{x_1}\|_{\Omega_1}^2 + \frac{1}{4} \|v_{x_2}\|_{\Omega_1}^2 + \frac{M_h(\nu)}{16} \|v\|_{\Omega'_1}^2 - \frac{3\nu^2}{2} \|a\|_{\Gamma_1}^2. \end{aligned}$$

In order to estimate the last term on the right-hand side of (5.18), we need to return to equation (2.5) and boundary conditions (2.7). It follows from the orthogonal decomposition (5.8) that the function a satisfies the equation

$$(5.19) \quad \partial_{x_1}^2 a(x_1) + \nu^2 a(x_1) = f_2(x_1), \text{ for a.e. } x_1 \in (\infty, -L),$$

where

$$(5.20) \quad f_2(x_1) = \int_0^{+\infty} f_1(x_1, x_2) \Phi(x_2) dx_2 - \sqrt{2\nu} g_2(x_1).$$

Multiplying (5.19) by $2x_1 \partial_{x_1} a(x_1)$, integrating over $(\infty, -L)$ and integrating by parts, we obtain

$$(5.21) \quad 2 \int_{-\infty}^{-L} f_2 x_1 a_{x_1} dx_1 = -L \left((a_{x_1}(-L))^2 + \nu^2 (a(-L))^2 \right) - \int_{-\infty}^{-L} a_{x_1}^2 dx_1 - \nu^2 \int_{-\infty}^{-L} a^2 dx_1$$

Therefore,

$$(5.22) \quad \nu^2 \int_{-\infty}^0 a^2 dx_1 \leq \int_{-\infty}^{-L} x_1^2 f_2^2 dx_1 \leq 3 \|x_1 f_1\|_{\Omega_1}^2 + 3\nu \|x_1 g_2\|_{\Gamma_1}^2.$$

By (5.22) and (5.18)

$$(5.23) \quad I_1 \geq \|v_{x_1}\|_{\Omega_1}^2 + \frac{1}{4} \|v_{x_2}\|_{\Omega_1}^2 + \frac{M_h(\nu)}{16} \|v\|_{\Omega'_1}^2 - \frac{9}{2} \left(\|x_1 f_1\|_{\Omega_1}^2 + \nu \|x_1 g_2\|_{\Gamma_1}^2 \right).$$

The estimate of the same type holds for I_3 :

$$(5.24) \quad I_3 \geq \|v_{x_1}\|_{\Omega_3}^2 + \frac{1}{4} \|v_{x_2}\|_{\Omega_3}^2 + \frac{M_h(\nu)}{16} \|v\|_{\Omega'_3}^2 - \frac{9}{2} \left(\|x_1 f_1\|_{\Omega_3}^2 + \nu \|x_1 g_2\|_{\Gamma_3}^2 \right),$$

where

$$\Omega'_3 = (L, +\infty) \times (0, h).$$

Let us treat I_2 . We have the following auxiliary estimate,

$$(5.25) \quad -\frac{\alpha(1 + \alpha N)}{N} \int_0^N w^2 dt \leq -\alpha w^2(0) + \int_0^N |w'|^2 dt,$$

Applying this inequality to I_2 we arrive at:

$$(5.26) \quad \begin{aligned} I_2 &\geq \|v_{x_1}\|_{\Omega_2}^2 + \frac{1}{4} \|v_{x_2}\|_{\Omega_2}^2 + \frac{3}{4} \|v_{x_2}\|_{\Omega'_2}^2 - \nu \|v\|_{\Gamma_2}^2 \\ &\geq \|v_{x_1}\|_{\Omega_2}^2 + \frac{1}{4} \|v_{x_2}\|_{\Omega_2}^2 - \frac{\nu(3 + 4\nu h)}{3h} \|v\|_{\Omega'_2}^2, \end{aligned}$$

where

$$\Omega'_2 = (-L, L) \times (0, h).$$

Combining (5.23), (5.24) and (5.26) we obtain

$$(5.27) \quad \begin{aligned} I &\geq \|v_{x_1}\|_{\Omega}^2 + \frac{1}{4} \|v_{x_2}\|_{\Omega}^2 + R \left(\|v\|_{\Omega'_1}^2 + \|v\|_{\Omega'_3}^2 \right) \\ &\quad - K \|v\|_{\Omega'_2}^2 - \frac{9}{2} \left(\|x_1 f\|_{\Omega}^2 + \nu \|x_1 g_2\|_{\Gamma}^2 \right), \end{aligned}$$

where

$$(5.28) \quad R \geq \frac{M_h(\nu)}{16}, \quad K \leq \frac{\nu(1 + \nu h)}{h}.$$

We shall use elementary inequality

$$(5.29) \quad M_L[\alpha] \int_0^L w^2 dt \leq \int_0^{+\infty} |w'|^2 dt + \alpha^2 \int_L^{+\infty} w^2 dt.$$

Applying (5.29) twice we get

$$(5.30) \quad I \geq \|v_{x_1}\|_{\Omega}^2 + \frac{1}{4}\|v_{x_2}\|_{\Omega}^2 - \frac{9}{2} (\|x_1 f_1\|_{\Omega}^2 + \nu \|x_1 g_2\|_{\Gamma}^2) - A \|v_{x_1}\|_{\Omega}^2 \\ + (AM_L [R^{1/2} A^{-1/2}] - K) \|v\|_{\Omega_2'}^2,$$

with an arbitrary positive A . It remains to choose A so that the last term in (5.30) is positive, i.e

$$(5.31) \quad AR^{1/2} - A^{1/2}KL - KL^2R^{1/2} > 0.$$

Solving this inequality, quadratic with respect to $A^{1/2}$, we see that (5.31) follows from

$$(5.32) \quad A \geq L^2 \frac{(K + \sqrt{K^2 + 4KR})^2}{4R}.$$

Since

$$L^2 \frac{(K + \sqrt{K^2 + 4KR})^2}{4R} \leq L^2 K (KR^{-1} + 2) \leq L^2 \frac{\nu(1 + \nu h)}{h} (16(1 + \nu h)^2 + 2),$$

we conclude that the last term in (5.30) is positive if

$$A \geq 18 \frac{\nu L^2}{h} (1 + \nu h)^3.$$

Now, (5.4) follows from (5.30) and (5.32). \square

We state a uniqueness result for problem (1.1)-(1.5) which follows directly from (5.4).

Corollary 5.2. *If*

$$(5.33) \quad 24 \frac{\nu L^2}{h} (1 + \nu h)^3 < 1,$$

then problem (1.1)-(1.5) is uniquely solvable.

Note that here we did not use Conditions 1 and 2. In particular the unique solvability holds either the body B is sufficiently narrow in the horizontal direction or ν is small.

Remark 5.1. Using notations from Section 1 we write (5.4) as

$$(5.34) \quad \|\nabla v\|_{\Omega}^2 \leq 4|F_1|_* \|v\| + 18|F_1|_*^2 + C_7 \|v_{x_1}\|_{\Omega}^2,$$

where

$$(5.35) \quad C_7 = 72 \frac{\nu L^2}{h} (1 + \nu h)^3.$$

6. Estimates of the function and its traces

In this section we estimate the solution v and its boundary traces in weighted L_2 spaces.

Lemma 6.1. *Let (1.9) and the assumptions of Lemma 3.1 on f, g_1, g_2 and v hold. Then one has the estimates*

$$(6.1) \quad \nu \|v\|_{\Gamma}^2 \leq 3|F_1|_* \|v\| + 18|F_1|_*^2 + C_7 \|v_{x_1}\|_{\Omega}^2,$$

$$(6.2) \quad \|\gamma_0 v\|_{\Omega}^2 \leq 16 \left(\|\nabla v\|_{\Omega}^2 + \frac{\nu}{2} \|v\|_{\Gamma}^2 \right),$$

$$(6.3) \quad \|\gamma_1 v\|_S^2 \leq \|\nabla v\|_{\Omega}^2 + \nu \|v\|_{\Gamma}^2 + C_8 \|\gamma_0 v\|_{\Omega}^2,$$

where

$$(6.4) \quad C_7 = 72 \frac{\nu L^2}{h} (1 + \nu h)^3,$$

and

$$(6.5) \quad C_8 = 24 + 18\kappa\gamma_1^{-2}.$$

Proof. 1. Estimate (6.1) follows from (5.4) and (5.2).

2. We majorise L_2 norm of v in Ω . First we have auxiliary Hardy type inequality

$$(6.6) \quad \frac{\nu}{2} w^2(0) + \int_0^{+\infty} |w'|^2 dt \geq \frac{1}{4} \int_0^{+\infty} \frac{\nu^2}{(\nu t + 1)^2} w^2 dt.$$

Hence

$$(6.7) \quad \|\nabla v\|_{\Omega}^2 + \frac{\nu}{2} \|v\|_{\Gamma}^2 \geq \int_{\Omega_1 \cup \Omega_3} \mu^2(x_2) v^2 dx + \|v_{x_1}\|_{\Omega \setminus \Omega_2}^2 + \|\nabla v\|_{\Omega_2}^2,$$

where

$$(6.8) \quad \mu(x_2) = \frac{\nu}{2(1 + \nu x_2)}.$$

In order to estimate v in Ω_2 we use (6.7) and (5.29) and obtain

$$(6.9) \quad \|\nabla v\|_{\Omega}^2 + \frac{\nu}{2} \|v\|_{\Gamma}^2 \geq \int_{\Omega_2} M_L[\mu(x_2)] v^2 dx.$$

Adding (6.7) and (6.9), we arrive at

$$(6.10) \quad 2\|\nabla v\|_{\Omega}^2 + \nu \|v\|_{\Gamma}^2 \geq \frac{1}{8} \int_{\Omega_2} \rho^2(x_2) v^2 dx,$$

where

$$\rho^2(x_2) = 8 \min \{ \mu^2(x_2), M_L[\mu(x_2)] \} \geq \frac{8\mu^2}{2L^2\mu^2 + 1} = \frac{8}{2L^2 + 4(\nu^{-1} + x_2)^2} \geq \gamma_0^2,$$

from which (6.2) follows.

3. Let us introduce curvilinear coordinates in a neighborhood of S : (s, ρ) , where ρ is the distance to S and s is the coordinate on S . We choose a smooth cut off function θ such that

$$\theta(0) = 1 \text{ for } t=0 \text{ and } \theta(t) = 0 \text{ for } t > 2.$$

and $|\theta| \leq 1, |\theta'| \leq 1$. Then the function $\theta(4(\kappa + \gamma_1^2)\rho) \nabla \rho$ is well defined in Ω .

Consider the integral

$$(6.11) \quad T = \int_{\Omega} \theta(4(\kappa + \gamma_1^2)\rho) \nabla \rho \nabla v^2 dx.$$

Obviously, we have

$$(6.12) \quad |T| \leq 2 \|v\|_{\tilde{\Omega}} \|\nabla v\|_{\tilde{\Omega}},$$

where

$$(6.13) \quad \tilde{\Omega} = \Omega \cap \{x : 0 < \rho < 2^{-1}(\kappa + \gamma_1^2)^{-1}\}.$$

On the other hand, integrating by parts, we get

$$(6.14) \quad \begin{aligned} T = & - \int_{\Omega} v^2 \theta(4(\kappa + \gamma_1^2)\rho) \Delta \rho dx - \int_{\Omega} v^2 \nabla \theta(4(\kappa + \gamma_1^2)\rho) \cdot \nabla \rho dx \\ & + \int_{\partial \Omega} v^2 \theta(4(\kappa + \gamma_1^2)\rho) \partial_n \rho ds = \\ & - \int_{\Omega} v^2 \theta(4(\kappa + \gamma_1^2)\rho) \Delta \rho dx - 4(\kappa + \gamma_1^2) \int_{\Omega} v^2 \theta'(4(\kappa + \gamma_1^2)\rho) \nabla \rho \cdot \nabla \rho dx - \int_S v^2 ds \\ & + \int_{\Gamma} v^2 \theta(4(\kappa + \gamma_1^2)\rho) \partial_n \rho ds. \end{aligned}$$

Let k be the curvature of S . Noting that

$$|\Delta \rho| = |k(1 + k\rho)^{-1}| \leq 2\kappa, \text{ in } \tilde{\Omega},$$

and

$$\nabla \rho \cdot \nabla \rho = 1,$$

we conclude from (6.14) that

$$(6.15) \quad \int_S v^2 ds \leq |T| + \|v\|_{\Gamma}^2 + (6\kappa + 4\gamma_1^2) \|v\|_{\tilde{\Omega}}^2.$$

Combining (6.12) and (6.15) we arrive at

$$(6.16) \quad \|\gamma_1 v\|_S^2 \leq \|\nabla v\|_{\tilde{\Omega}}^2 + \nu \|v\|_{\Gamma}^2 + 6(\gamma_1^4 + \gamma_1^2 \kappa) \|v\|_{\tilde{\Omega}}^2.$$

It remains to estimate the last term in (6.16). Using (6.13) and (1.11), we obtain

$$(6.17) \quad \|v\|_{\tilde{\Omega}}^2 \leq \left(\max_{x \in \tilde{\Omega}} \gamma_0^{-2} \right) \|\gamma_0 v\|_{\tilde{\Omega}}^2 \leq (3\gamma_1^{-4} + (\kappa + \gamma_1^2)^{-2}) \|\gamma_0 v\|_{\tilde{\Omega}}^2.$$

Inequality (6.3) follows from (6.17) and (3.11). This concludes the proof. \square

7. Proof of the main result

Theorem 7.1. *Let $f \in C_0^\infty(\Omega)$, $g_1 \in C^\infty(S)$ and $g_2 \in C_0^\infty(\Gamma)$. Then a unique solution v of (2.5)-(2.8) such that $\nabla v \in L_2(\Omega)$ and $v|_\Gamma \in L_2(\Gamma)$, satisfies the estimate*

$$(7.1) \quad \|v\| \leq C_0 |F_1|_*,$$

where

$$(7.2) \quad \begin{aligned} C_0 &\leq 2^{30} H \varepsilon^{-1} (1 + \kappa L) (1 + \kappa \gamma_1^{-2}) (1 + \nu L^2 h^{-1} (1 + \nu h)^3) \\ &\leq c h^{-1} \varepsilon^{-1} (1 + \kappa \tau)^2 (1 + \nu h)^3 \nu \tau^3, \end{aligned}$$

and

$$\tau = L + H + \nu^{-1}.$$

Proof. According to Lemma 6.1 and Remark 5.1 the inequality holds

$$(7.3) \quad \|v\|^2 \leq 36(1 + C_8) (4\|v\| |F_1|_* + 18|F_1|_*^2 + C_7 \|v_{x_1}\| \Omega).$$

The right-hand side of (7.3) is dominated by

$$36(1 + C_8) ((4 + C_5 C_7) \|v\| |F_1|_* + (18 + C_6 C_7) |F_1|_*^2),$$

see Theorem 4.3, which is not greater than

$$(7.4) \quad 2^{11} C_8 ((1 + C_7 C_5) \|v\| |F_1|_* + (1 + C_7 C_6) |F_1|_*^2).$$

Using (4.15) and (4.16), we majorise (7.4) by

$$2^{11} C_8 ((1 + C_7 C_3 H \varepsilon^{-1}) \|v\| |F_1|_* + (1 + C_7 H \varepsilon^{-1}) |F_1|_*^2).$$

Noting that $C_3 H \varepsilon^{-1} \geq 1$, because (3.8), we arrive at the inequality

$$(7.5) \quad \|v\|^2 \leq 2^{11} H \varepsilon^{-1} C_8 C_3 (1 + C_7) (\|v\| |F_1|_* + |F_1|_*^2).$$

Since $H \varepsilon^{-1} C_8 C_3 (1 + C_7) \geq 1$, we conclude

$$(7.6) \quad \|v\| \leq 2^{12} H \varepsilon^{-1} C_8 C_3 (1 + C_7) |F_1|_*.$$

It remains to apply (3.8), (5.35) and (6.5) to (7.6) to conclude the proof. \square

Now, we can finish the proof of Theorem 1.1.

Proof. Let $f \in C_0^\infty(\Omega)$, $g_1 \in C^\infty(S)$ and $g_2 \in C_0^\infty(\Gamma)$. Then (1.15) and (2.12) follow from Theorem 7.1 and Lemma 2.1. We conclude the proof by approximating the right-hand side of (1.1)-(1.3) by C_0^∞ functions in the norm $|\cdot|_*$ and passing to the limit in (2.1), (1.15) and (2.12). \square

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