Fast computation of the multi-dimensional fractional Laplacian

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ABSTRACT
The paper discusses new cubature formulas for the Riesz potential and the fractional Laplacian $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, in the framework of the method approximate approximations. This approach, combined with separated representations, makes the method successful also in high dimensions. We prove error estimates and report on numerical results illustrating that our formulas provide the predicted convergence rate $2, 4, 6, 8$ up to dimension 10^4.

KEYWORDS
Multidimensional convolution; Riesz potential; Fractional Laplacian; Separated representation.

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1. Introduction

On the entire space $\mathbb{R}^n$, $n \geq 1$, the fractional Laplacian $(-\Delta)^{\alpha/2}$ with $\alpha \in (0,2)$ can be defined in many equivalent ways. It can be represented by means of an hypersingular integral

$$(-\Delta)^{\alpha/2} f(\mathbf{x}) = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{n+\alpha}{2})}{\pi^{n/2}\Gamma(\frac{2-\alpha}{2})} PV \int_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}} d\mathbf{y}, \qquad \text{for} \quad \alpha \in (0,2),$$

where PV stands for the Cauchy principal value of the singular integral, Γ denotes the Gamma function and $|\mathbf{x}-\mathbf{y}|$ denotes the Euclidean distance between $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ (cf., e.g., [10], [27]). An equivalent definition is via a pseudodifferential operator with symbol $(2\pi|\mathbf{y}|)^{\alpha}$, i.e.,

$$(-\Delta)^{\alpha/2} f = \mathcal{F}^{-1}((2\pi|\mathbf{y}|)^{\alpha}\mathcal{F}(f))$$

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Dedicated to Robert P. Gilbert on the occasion of his 90th birthday where $\mathcal{F}$ represents the Fourier transform

$$\mathcal{F}f(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{y}) e^{2\pi i \mathbf{x}\cdot\mathbf{y}} d\mathbf{y}$$

and $\mathcal{F}^{-1}$ its inverse. The fractional Laplacian can also be defined as the inverse of the Riesz potential

$$\mathcal{R}_{n,\alpha}(f) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{n/2} 2^{\alpha} \Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}} d\mathbf{y}$$

that is $(-\Delta)^{\alpha/2}\mathcal{R}_{n,\alpha} f = f$, $0 < \alpha < n$, (cf., e.g., [27], [29]). This representation leads immediately to the representation of the fractional Laplacian directly in terms of the Riesz potential: $(-\Delta)^{\alpha/2} f = -\Delta \mathcal{R}_{n,2-\alpha} f$.

The fractional Laplacian of a function f can be viewed as the divergence of a fractional gradient

$$(-\Delta)^{\alpha/2} f(\mathbf{x}) = -\nabla \cdot \nabla^{\alpha-1} f(\mathbf{x}),$$

where $\nabla = (\partial_{x_1}, ..., \partial_{x_n})$ is the gradient and $\nabla^{\alpha-1}$ is the fractional gradient

$$\nabla^{\alpha-1} f(\mathbf{x}) = \frac{2^{\alpha-1}}{\pi^{n/2}} \frac{\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{2-\alpha}{2})} \int\limits_{\mathbb{R}^n} f(\mathbf{y}) \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{n+\alpha}} d\mathbf{y}$$

(cf.[23, p.245]). When α tends to 2, the fractional Laplacian and the fractional gradient reduce to the usual Laplacian and the ordinary gradient, respectively.

If we introduce the volume potential $\mathcal{N}_\alpha = \mathcal{R}_{n,2-\alpha}$, i.e.,

$$\mathcal{N}_\alpha(f)(\mathbf{x}) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-2+\alpha}} d\mathbf{y}, \quad c_{n,\alpha} = \frac{2^{\alpha-2}}{\pi^{n/2}} \frac{\Gamma(\frac{n-2+\alpha}{2})}{\Gamma(\frac{2-\alpha}{2})}, \tag{1.1}$$

then the fractional gradient and the fractional Laplacian can be represented as ordinary gradient and the ordinary Laplacian of the volume potential $\mathcal{N}_\alpha f$,

$$\nabla^{\alpha-1} f(\mathbf{x}) = -\nabla \mathcal{N}_\alpha(f)(\mathbf{x}), \tag{1.2}$$

$$(-\Delta)^{\alpha/2} f(\mathbf{x}) = -\Delta \mathcal{N}_\alpha(f)(\mathbf{x}), \tag{1.3}$$

respectively (cfr., e.g., [23], [26]).

The fractional Laplacian appears in different fields of mathematics (PDE, harmonic analysis, semigroup theory, probabilistic theory, cf., e.g., [5], [10], [11] and the references therein) as well as in many applications (optimization, finance, materials science, water waves, cf., e.g., [6], [7], [25] and the references therein). For $n = 3$, fractional gradient is associated with a fractional diffusive flux in an isotropic medium given by

$$\mathbf{q}_\alpha(\mathbf{x}) = -\kappa_\alpha \nabla^{\alpha-1} f(\mathbf{x})$$

where κ_α is a fractional diffusivity associated with Lévy flights ([23, p.245]).

The numerical treatment of the fractional Laplacian arises as a computational task. Due to its nonlocality and strong singularity, numerical method introduces considerable challenges in both mathematical analysis and numerical simulations. In recent years several algorithms for the numerical approximation of the multi-dimensional fractional Laplacian have been proposed. They are mainly based on finite elements or finite difference methods (cf., e.g., [4], [8], [22] and the references therein). In this paper we propose a different method of an arbitrary high order for the approximation of $\mathcal{N}_\alpha f$, $\nabla^{\alpha-1} f$ and $(-\Delta)^{\alpha/2} f$, $n \geq 3$, which is based on the approximation of the function f via the basis functions introduced by approximate approximations (cf. [21]), which are product of Gaussians and special polynomials. Then the n-dimensional integral (1.1) applied to the basis functions is represented by means of a one-dimensional integral where the integrand has a separated representation, i.e., it is a product of functions depending only on one of the variables. This construction enables to obtain one-dimensional integral representations with separated integrand also for the fractional gradient (1.2) and the fractional Laplacian (1.3), when applied to the basis functions. An accurate quadrature rule and a separated representation of the density f provide a separated representation for $\mathcal{N}_\alpha f$, $\nabla^{\alpha-1} f$ and $(-\Delta)^{\alpha/2} f$. Thus only one-dimensional operations are used and the resulting approximation procedure is fast and effective also in high dimensional cases, and provides approximations of high order, up to a small saturation error.

The concept of approximate approximations and first related results were introduced by V. Maz'ya in [18], [19]. Various aspects of a general theory of these approximations were further developed and formulas of various integral and pseudo-differential operators have been obtained (cf. [21] and the review paper [28]). By combining cubature formulas for volume potentials based on approximate approximations with the strategy of separated representations (cf., e.g., [3], [9]), it is possible derive a method for approximating volume potentials which is accurate and fast also in the multi-dimensional case and provides approximation formulas of high order. This procedure was applied successfully for the fast integration of the harmonic [12], biharmonic [16], diffraction [15], elastic and hydrodynamic potentials [17]. In [13], [14] this approach was extended to parabolic problems. Here we show that the fast method can be applied to other pseudodifferential operators occurring frequently in applications.

The paper is organized as follows. We start in Section 2 by describing the method and providing error estimates. In Section 3 we consider in detail second order approximations and, for f with separated representation, we derive tensor product representations for $\mathcal{N}_\alpha f$, $\nabla^{\alpha-1} f$ and $(-\Delta)^{\alpha/2} f$ which admit efficient one-dimensional operations. We then consider higher order approximations in Section 4. Finally, in Section 5, we report on numerical results, illustrating that our formulas are accurate and provide the predicted approximation rates $2, 4, 6, 8$ in dimension $n = 3$ and also if the dimension is high $(n = 10^k, k = 1, 2, 3, 4)$.

2. Description of the method and error estimates

We replace f in (1.1) by the approximate quasi-interpolant

$$\mathcal{M}_h f(\mathbf{x}) := \mathcal{D}^{-n/2} \sum_{\mathbf{m} \in \mathbb{Z}^n} f(h\mathbf{m}) \eta \left(\frac{\mathbf{x} - h\mathbf{m}}{h\sqrt{\mathcal{D}}} \right) \tag{2.1}$$

with fixed positive parameters h and $\mathcal{D}$ and with some generating function η sufficiently smooth and of rapid decay. Then the sum

$$\mathcal{N}_\alpha(\mathcal{M}_h f)(\mathbf{x}) = \frac{(h\sqrt{\mathcal{D}})^{2-\alpha}}{\mathcal{D}^{n/2}} \sum_{\mathbf{m}\in\mathbb{Z}^3} f(h\mathbf{m})\mathcal{N}_\alpha(\eta)\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) \tag{2.2}$$

provides a cubature formula for (1.1), provided $\mathcal{N}_\alpha(\eta)$ can be computed analytically or at least efficiently. Due to the semi-analytic cubature nature of the formula (2.2), the fractional gradient $\nabla^{\alpha-1}f = -\nabla\mathcal{N}_\alpha f$ and the fractional Laplacian $(-\Delta)^{\alpha/2}f = -\Delta\mathcal{N}_\alpha f$ can be approximated by $-\nabla\mathcal{N}_\alpha(\mathcal{M}_h f)$ and $-\Delta\mathcal{N}_\alpha(\mathcal{M}_h f)$, respectively. We deduce the following approximating formulas

$$\frac{\partial}{\partial x_j}(\mathcal{N}_\alpha f)(\mathbf{x}) \approx \frac{(h\sqrt{\mathcal{D}})^{1-\alpha}}{\mathcal{D}^{n/2}} \sum_{\mathbf{m}\in\mathbb{Z}^n} f(h\mathbf{m})\frac{\partial}{\partial x_j}(\mathcal{N}_\alpha\eta)\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right), j=1,...,n, \tag{2.3}$$

$$\Delta(\mathcal{N}_\alpha f)(\mathbf{x}) \approx \frac{(h\sqrt{\mathcal{D}})^{-\alpha}}{\mathcal{D}^{n/2}} \sum_{\mathbf{m}\in\mathbb{Z}^n} f(h\mathbf{m})\Delta(\mathcal{N}_\alpha\eta)\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right). \tag{2.4}$$

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwarz space of smooth and rapidly decaying functions and by $W_p^L(\mathbb{R}^n)$, $L \in \mathbf{N}$, the Sobolev space of the $L_p(\mathbb{R}^n)$ functions whose generalized derivatives up to the order L also belong to $L_p(\mathbb{R}^n)$. In the following, $\nabla_k f$ denotes the vector of partial derivatives $\{\partial^\beta f\}_{|\beta|=k}$. The norm in $W_p^L(\mathbb{R}^n)$ is defined by

$$||f||_{W_p^L} = \sum_{k=0}^{L} ||\nabla_k f||_{L_p}, \qquad ||\nabla_k f||_{L_p} = \sum_{|\beta|=k} ||\partial^\beta f||_{L_p}.$$

If $\eta \in \mathcal{S}(\mathbb{R}^n)$ satisfies the moment condition of order N,

$$\int_{\mathbb{R}^n} \eta(\mathbf{x})\,\mathbf{x}^\alpha d\mathbf{x} = \delta_{0,\alpha}, \quad 0 \le |\alpha| < N \tag{2.5}$$

then for any $f \in W_\infty^N(\mathbb{R}^n)$ the approximation error of the quasi-interpolation can be estimated pointwise by

$$|f(\mathbf{x}) - \mathcal{M}_{h,\mathcal{D}}f(\mathbf{x})| \le c(\sqrt{\mathcal{D}}h)^N \|\nabla_N f\|_{L_\infty} + \sum_{k=0}^{N-1} \varepsilon_k(\mathcal{D})(\sqrt{\mathcal{D}}h)^k \left|\nabla_k f(\mathbf{x})\right|$$

with

$$0 < \varepsilon_k(\mathcal{D}) \le \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{0\}} \left|\nabla_k \mathcal{F}\eta(\sqrt{\mathcal{D}}\mathbf{m})\right|; \lim_{\mathcal{D}\to\infty} \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{0\}} \left|\nabla_k \mathcal{F}\eta(\sqrt{\mathcal{D}}\mathbf{m})\right| = 0$$

([21, p.34]). Similar approximation properties in integral norms are valid. The following theorem was proved in [21, p.42].

Theorem 2.1. *Suppose that* $\eta \in \mathcal{S}(\mathbb{R}^n)$ *satisfies* (2.5). *Then for any* $f \in W_p^L(\mathbb{R}^n)$, $1 \le p \le \infty$ *and* $L > n/p$, $L \ge N$, *the quasi-interpolant* (2.1) *satisfies*

$$||f - \mathcal{M}_h f||_{L_p} \le c_\eta (\sqrt{\mathcal{D}} h)^N ||\nabla_N f||_{L_p} + \sum_{k=0}^{N-1} \frac{\varepsilon_k(\mathcal{D})}{(2\pi)^k} (\sqrt{\mathcal{D}} h)^k ||\nabla_k f||_{L_p}.$$

Let us estimate the approximation error of the cubature formula (2.2) for the volume potential $\mathcal{N}_\alpha f$. By construction the cubature error equals to

$$\mathcal{N}_\alpha(\mathcal{M}_h f)(\mathbf{x}) - \mathcal{N}_\alpha f(\mathbf{x}) = \mathcal{N}_\alpha(\mathcal{M}_h f - f)(\mathbf{x}).$$

From Sobolev's theorem, the operator $\mathcal{N}_\alpha$ is a bounded mapping from $L_p(\mathbb{R}^n)$ into $L_q(\mathbb{R}^n)$, $1 < p < n/(2-\alpha)$, $1/q = 1/p - (2-\alpha)/n$ ([29, p.119]). Then,

$$||\mathcal{N}_\alpha f - \mathcal{N}_\alpha(\mathcal{M}_h f)||_{L_q} \le A_{pq}^{(\alpha)} ||f - \mathcal{M}_h f||_{L_p} \tag{2.6}$$

where $A_{pq}^{(\alpha)}$ denotes the norm of $\mathcal{N}_\alpha : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$. Theorem 2.1 and (2.6) immediately give the following error estimate.

Theorem 2.2. *Suppose that* $\eta \in \mathcal{S}(\mathbb{R}^n)$ *satisfies* (2.5). *Let* $n \ge 3$, $0 < \alpha < 2$, $1 < p < n/(2-\alpha)$, $1/q = 1/p - (2-\alpha)/n$ *and let* $f \in W_p^L(\mathbb{R}^n)$ *with* $L > n/p$, $L \ge N$. *Then*

$$\begin{aligned} &||\mathcal{N}_\alpha f - \mathcal{N}_\alpha(\mathcal{M}_h f)||_{L_q} \\ &\qquad \le A_{pq}^{(\alpha)} \left(c_\eta (\sqrt{\mathcal{D}} h)^N ||\nabla_N f||_{L_p} + \sum_{k=0}^{N-1} \frac{\varepsilon_k(\mathcal{D})}{(2\pi)^k} (\sqrt{\mathcal{D}} h)^k ||\nabla_k f||_{L_p} \right). \end{aligned}$$

The previous theorem shows that if the function η and the parameter $\mathcal{D}$ are chosen such that the values $\varepsilon_k(\mathcal{D})$ are sufficiently small, then the cubature of $\mathcal{N}_\alpha$ approximates with order h^N up to the prescribed accuracy. However, by the smoothing properties of $\mathcal{N}_\alpha$ ([29, p.131]), also the saturation error converges to zero with rate $h^{2-\alpha}$. Indeed, quasi-interpolation has the remarkable property that it converges in certain weak norms since the saturation error, which is caused by fast oscillating functions, converges weakly to zero. We denote by $H_p^s(\mathbb{R}^n)$ the Bessel potential space, equipped with the norm

$$||f||_{H_p^s} = ||\mathcal{F}^{-1}((1 + 4\pi^2 |\cdot|^2)^{s/2} \mathcal{F} f)||_{L_p} = ||(I - \Delta)^{s/2} f||_{L_p}$$

(cf., e.g., [29, p.130], [20, p.516]). Instead of Theorem 2.1 we use the following result, a direct consequence of [21, Theorem 4.6].

Theorem 2.3. *Suppose that* $\eta \in \mathcal{S}(\mathbb{R}^n)$ *satisfies* (2.5). *For any* $\varepsilon > 0$, *there exists* $\mathcal{D} > 0$ *such that for any* $f \in H_p^L(\mathbb{R}^n)$, $1 < p < \infty$, $L > n/p$, $L \ge N \ge 2$ *and* $\beta \in (0, 2)$, *the quasi-interpolant* (2.1) *satisfies*

$$||f - \mathcal{M}_h f||_{H_p^{-\beta}} \le c_\eta (h\sqrt{\mathcal{D}})^N ||f||_{H_p^L} + \varepsilon h^\beta c_{p,\beta} \sum_{k=0}^{N-1-[\beta]} \frac{(\sqrt{\mathcal{D}} h)^k}{k!} ||\nabla_k f||_{H_p^\beta} \tag{2.7}$$

where $[\beta]$ *denotes the integer part of* β*, the constants* c_η *and* $c_{p,\beta}$ *do not depend on* f*,* h *and* $\mathcal{D}$*.*

An estimate similar to (2.7) is valid for $\beta < 0$, which implies that the saturation term increases with the factor h^β ([21, Remark 4.7, p.84]).

We can formulate the following.

Theorem 2.4. *Suppose that* $\eta \in \mathcal{S}(\mathbb{R}^n)$ *satisfies* (2.5)*. Let* $0 < \alpha < 2$*,* $1 < p < n/(2-\alpha)$*,* $1/q = 1/p - (2-\alpha)/n$*. For any* $f \in W_p^L(\mathbb{R}^n)$ *with* $L \geq N \geq 2$ *and* $L > n/p$*, there exist positive constants* $C, C_{p,\alpha}, C_{q,\alpha}$ *not depending on* f*,* h *and* $\mathcal{D}$ *such that*

$$
\begin{aligned}
&||\mathcal{N}_\alpha f - \mathcal{N}_\alpha(\mathcal{M}_h f)||_{L_q} \leq C(h\sqrt{\mathcal{D}})^N ||f||_{W_p^L} + \\
&\qquad\qquad \varepsilon h^{2-\alpha} \sum_{k=0}^{N-2+[\alpha]} \frac{(h\sqrt{\mathcal{D}})^k}{k!} \left(C_{p,\alpha} A_{p,q}^{(\alpha)} ||\nabla_k f||_{H_p^{2-\alpha}} + C_{q,\alpha} ||\nabla_k f||_{H_q^{2-\alpha}} \right).
\end{aligned}
$$

Proof. We set $\beta = 2 - \alpha > 0$. We have ([29, p.117])

$$
||\mathcal{N}_\alpha f||_{L_q} = ||(-\Delta)^{-\beta/2}(I-\Delta)^{\beta/2}(I-\Delta)^{-\beta/2} f)||_{L_q} = ||(-\Delta)^{-\beta/2}(I-\Delta)^{-\beta/2} f||_{H_q^\beta}.
$$

The norm $||u||_{H_q^\beta}$ is equivalent to $||u||_{L_q} + ||(-\Delta)^{\beta/2} u||_{L_q}$ ([27, Theorem 7.16]). Hence, keeping in mind (2.6), we get

$$
\begin{aligned}
&||\mathcal{N}_\alpha f - \mathcal{N}_\alpha(\mathcal{M}_h f)||_{L_q} \\
&\quad \leq c_1(||(-\Delta)^{-\beta/2}(I-\Delta)^{-\beta/2}(f - \mathcal{M}_h f)||_{L_q} + ||(I-\Delta)^{-\beta/2}(f - \mathcal{M}_h f)||_{L_q}) \\
&\qquad \leq c_1(A_{pq}^{(\alpha)} ||(I-\Delta)^{-\beta/2}(f - \mathcal{M}_h f)||_{L_p} + ||(I-\Delta)^{-\beta/2}(f - \mathcal{M}_h f)||_{L_q}) \\
&\qquad\qquad = c_1(A_{pq}^{(\alpha)} ||(f - \mathcal{M}_h f)||_{H_p^{-(2-\alpha)}} + ||(f - \mathcal{M}_h f)||_{H_q^{-(2-\alpha)}}).
\end{aligned}
$$

We use Theorem 2.3, the fact that $H_p^s(\mathbb{R}^n)$ are interpolation spaces that coincide with the Sobolev spaces for $s \in \mathbf{N}$, $H_p^s(\mathbb{R}^n) = W_p^s(\mathbb{R}^n)$([20, p. 458]) and the continuous embedding $W_p^L(\mathbb{R}^n) \subset W_q^{L-2}(\mathbb{R}^n)$ ([2, p.97]) to obtain the required estimate. $\square$

The cubature formulas (2.3) and (2.4) approximate $\frac{\partial}{\partial x_j}(\mathcal{N}_\alpha f)$ and $\Delta(\mathcal{N}_\alpha f)$, respectively, with order h^N up to the prescribed accuracy. If $0 < \alpha < 1$ then (2.3) approximates $\frac{\partial}{\partial x_j}(\mathcal{N}_\alpha f)$ with order N and the saturation error goes to zero with order $\varepsilon h^{1-\alpha}$. Indeed,

Theorem 2.5. *Suppose that* $\eta \in \mathcal{S}(\mathbb{R}^n)$ *satisfies* (2.5)*. Let* $0 < \alpha < 1$ *,* $1 < p < n/(1-\alpha)$*,* $1/q = 1/p - (1-\alpha)/n > 0$*. For any* $f \in W_p^L(\mathbb{R}^n)$ *with* $L \geq N \geq 2$ *and* $L > n/p$ *there exist positive constants* $C, C_{p,\alpha}, C_{q,\alpha}$ *not depending on* f*,* h *and* $\mathcal{D}$ *such that*

$$
\begin{aligned}
&||\nabla \mathcal{N}_\alpha f - \nabla \mathcal{N}_\alpha(\mathcal{M}_h f)||_{L_q} \leq C(h\sqrt{\mathcal{D}})^N ||f||_{W_p^L} + \\
&\qquad\qquad \varepsilon h^{1-\alpha} \sum_{k=0}^{N-1+[\alpha]} \frac{(h\sqrt{\mathcal{D}})^k}{k!} \left(C_{p,\alpha} B_{p,q}^{(\alpha)} ||\nabla_k f||_{H_p^{1-\alpha}} + C_{q,\alpha} ||\nabla_k f||_{H_q^{1-\alpha}} \right).
\end{aligned}
$$

Proof. The norm $||\nabla f||_{L_q}$ is equivalent to $||(-\Delta)^{1/2} f||_{L_q}$ (cf., e.g., [20, p. 458], [27, p. 193]). Acting as in the proof of Theorem 2.4 we deduce that

$$||(-\Delta)^{1/2}(\mathcal{N}_\alpha f - \mathcal{N}_\alpha(\mathcal{M}_h f))||_{L_q} \leq c_1 \Big(B^{(\alpha)}_{p,q} ||f - \mathcal{M}_h f||_{H^{\alpha-1}_p} + ||f - \mathcal{M}_h f||_{H^{\alpha-1}_q} \Big)$$

where $B^{(\alpha)}_{p,q}$ denotes the norm of the bounded mapping $(-\Delta)^{(\alpha-1)/2} : L_p \to L_q$. Then the theorem is a direct consequence of Theorem 2.3. □

If $1 \leq \alpha < 2$ then

$$||(-\Delta)^{1/2}(\mathcal{N}_\alpha - \mathcal{N}_\alpha(\mathcal{M}_h f))||_{L_p} = ||(-\Delta)^{(\alpha-1)/2}(f - \mathcal{M}_h f))||_{L_p} \leq c||f - \mathcal{M}_h f||_{H^{\alpha-1}_p}.$$

From Theorem 2.3 and [21, Remark 4.7] we deduce $\nabla \mathcal{N}_\alpha(\mathcal{M}_h f)$ approximates $\nabla \mathcal{N}_\alpha f$ with order h^N plus the corresponding small saturation error which increases with $h^{1-\alpha}$. Similarly, since $(-\Delta)^{\alpha/2}$ is a pseudodifferential operator of order α then

$$||(-\Delta)^{\alpha/2}(f - \mathcal{M}_h f)||_{L_p} = \mathcal{O}((h\sqrt{\mathcal{D}})^N + \varepsilon h^{-\alpha})$$

and the corresponding cubatures approximate with order N for $h \geq h_0$ but they do not converge.

After having estimated the cubature error, to construct high-order cubature formulas for (1.1) it remains to choose η which satisfies the moment condition (2.5), such that the values $\varepsilon_k(\mathcal{D})$ can be made arbitrarily small by a proper choice of $\mathcal{D}$ and the integral $\mathcal{N}_\alpha \eta$ can be computed analytically or at least efficiently. If the integral $\mathcal{N}_\alpha \eta$ is expressed analytically then (2.2) is a semi-analytic cubature formula. This allows to obtain semi-analytic cubature formulas also for (2.3) and (2.4).

3. Second order approximation

For second order approximations we choose $\eta_2(\mathbf{x}) = \pi^{-n/2} \mathrm{e}^{-|\mathbf{x}|^2}$. It satisfies (2.5) with $N = 2$. The convolution of the Gaussian with a radial function can be obtained from the formula ([21, (5.15)])

$$\int_{\mathbb{R}^n} Q(|\mathbf{x} - \mathbf{y}|) \mathrm{e}^{-|\mathbf{y}|^2} d\mathbf{y} = \frac{2\pi^{n/2} \mathrm{e}^{-|\mathbf{x}|^2}}{|\mathbf{x}|^{n/2-1}} \int_0^\infty Q(r) I_{n/2-1}(2|\mathbf{x}|r) r^{n/2} \mathrm{e}^{-r^2} dr$$

with the modified Bessel function of the first kind I_s ([1, p.374]). In our case we get

$$\mathcal{N}_\alpha(\mathrm{e}^{-|\cdot|^2})(\mathbf{x}) = \frac{2c_{n,\alpha} \pi^{n/2} \mathrm{e}^{-|\mathbf{x}|^2}}{|\mathbf{x}|^{n/2-1}} \int_0^\infty r^{2-n/2-\alpha} \mathrm{e}^{-r^2} I_{n/2-1}(2|\mathbf{x}|r) dr. \tag{3.1}$$

The integrand in the last integral is summable in $(0, +\infty)$ because $\alpha \in (0, 2)$, $I_{n/2-1}(r) \approx \frac{r^{n/2-1}}{2\Gamma(n/2)}$, $r \to 0^+$ ([1, 9.6.7]) and $I_{n/2-1}(r) \approx \frac{\mathrm{e}^r}{\sqrt{2\pi}\sqrt{r}}$, $r \to \infty$ ([1, 9.7.1]).

The one-dimensional integral in (3.1) can be expressed by means of the confluent hypergeometric functions ${}_1F_1$ ([24, 2.15.5.4])

$$\mathcal{N}_\alpha(\mathrm{e}^{-|\cdot|^2})(\mathbf{x}) = \frac{\Gamma(\frac{n-2+\alpha}{2})}{2^{2-\alpha}\Gamma(\frac{n}{2})}\mathrm{e}^{-|\mathbf{x}|^2}{}_1F_1(\frac{2-\alpha}{2}, \frac{n}{2}, |\mathbf{x}|^2)\,. \tag{3.2}$$

By using Kummer transformation ([1, 13.1.27]) we can also write

$$\mathcal{N}_\alpha(\mathrm{e}^{-|\cdot|^2})(\mathbf{x}) = \frac{\Gamma(\frac{n-2+\alpha}{2})}{2^{2-\alpha}\Gamma(\frac{n}{2})}{}_1F_1(\frac{n-2+\alpha}{2}, \frac{n}{2}, -|\mathbf{x}|^2). \tag{3.3}$$

We deduce that

$$\frac{\partial}{\partial x_j}\mathcal{N}_\alpha(\mathrm{e}^{-|\cdot|^2})(\mathbf{x}) = -2^{\alpha-1}\frac{\Gamma(\frac{\alpha+n}{2})}{\Gamma(\frac{n+2}{2})}x_j{}_1F_1\left(\frac{\alpha+n}{2}, \frac{n+2}{2}, -|\mathbf{x}|^2\right), j = 1, ..., n, \tag{3.4}$$

$$\begin{aligned}\Delta\mathcal{N}_\alpha(\mathrm{e}^{-|\cdot|^2})(\mathbf{x}) = 2^\alpha\frac{\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n}{2})}\Big(2|\mathbf{x}|^2\frac{\alpha+n}{n(n+2)}{}_1F_1\left(\frac{\alpha+n+2}{2}, \frac{n+4}{2}, -|\mathbf{x}|^2\right)\\ - {}_1F_1\left(\frac{\alpha+n}{2}, \frac{n+2}{2}, -|\mathbf{x}|^2\right)\Big)\end{aligned} \tag{3.5}$$

where we used formulas for the derivatives of ${}_1F_1$ in [1, 13.4.9]. Formulas (2.2), (2.3) and (2.4), together with (3.2), (3.4) and (3.5), give rise to second order semi-analytic cubature formulas for $\mathcal{N}_\alpha$, $\nabla^{\alpha-1}$ and $(-\Delta)^{\alpha/2}$, respectively, up to the saturation error.

The second order cubature formula for the approximation of $\mathcal{N}_\alpha$ on the uniform grid $\{h\mathbf{k}\}$ leads to the convolutional sum

$$\mathcal{N}_\alpha(\mathcal{M}_h f)(h\mathbf{k}) = \frac{(h\sqrt{\mathcal{D}})^{2-\alpha}}{\mathcal{D}^{n/2}}\sum_{\mathbf{m}\in\mathbb{Z}^n} f(h\mathbf{m})\mathcal{N}_\alpha(\eta_2)\left(\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathcal{D}}}\right). \tag{3.6}$$

The computation of the multidimensional convolutional sum (3.6) is very time consuming and requires a lot of memory. We propose a method which reduces the computational effort and gives rise to fast formulas. We write (3.2) in a different way by using the integral representation of the hypergeometric functions ${}_1F_1$ ([1, 13.2.1])

$${}_1F_1(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int\limits_0^1 \mathrm{e}^{z\tau}\tau^{a-1}(1-\tau)^{-a+c-1}d\tau, \quad \mathrm{Re}\,(c) > \mathrm{Re}\,(a) > 0\,.$$

With the substitution $\tau = t/(1+t)$ we get

$${}_1F_1(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int\limits_0^\infty \mathrm{e}^{\frac{t}{1+t}z}\frac{t^{a-1}}{(1+t)^c}dt, \quad \mathrm{Re}\,(c) > \mathrm{Re}\,(a) > 0\,.$$

Hence, from (3.2) we obtain

$$\Phi_2^{(0)}(\mathbf{x}) := (\mathcal{N}_\alpha \eta_2)(\mathbf{x}) = \frac{\pi^{-n/2}}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})} \int_0^\infty \frac{t^{-\frac{\alpha}{2}} \mathrm{e}^{-\frac{|\mathbf{x}|^2}{1+t}}}{(1+t)^{n/2}} dt. \tag{3.7}$$

From the relations

$$\frac{\partial}{\partial x_j} \mathrm{e}^{-a|\mathbf{x}|^2} = -2ax_j \mathrm{e}^{-a|\mathbf{x}|^2}, j = 1, ..., n, \qquad \Delta \mathrm{e}^{-a|\mathbf{x}|^2} = (4a^2|\mathbf{x}|^2 - 2\,a\,n)\mathrm{e}^{-a|\mathbf{x}|^2}$$

we get the one-dimensional integral representations with separated integrands

$$\Phi_{2,j}^{(1)}(\mathbf{x}) := \frac{\partial}{\partial x_j}(\mathcal{N}_\alpha \eta_2)(\mathbf{x}) = -\frac{2\pi^{-n/2} x_j}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})} \int_0^\infty \frac{t^{-\frac{\alpha}{2}} \mathrm{e}^{-\frac{|\mathbf{x}|^2}{1+t}}}{(1+t)^{n/2+1}} dt\,, j = 1, ..., n, \tag{3.8}$$

$$\Phi_2^{(2)}(\mathbf{x}) := \Delta(\mathcal{N}_\alpha \eta_2)(\mathbf{x}) = \frac{2\pi^{-n/2}}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})} \int_0^\infty \left(\frac{2|\mathbf{x}|^2}{(1+t)^2} - \frac{n}{1+t} \right) \frac{t^{-\frac{\alpha}{2}} \mathrm{e}^{-\frac{|\mathbf{x}|^2}{1+t}}}{(1+t)^{n/2}} dt. \tag{3.9}$$

Then a separated representation of the integrals in (3.7), (3.8) and (3.9) is obtained by applying an accurate quadrature rule with nodes $\{\tau_s\}$ and weights $\{\omega_s\}$; for example

$$\Phi_2^{(0)} \left(\frac{\mathbf{k} - \mathbf{m}}{\sqrt{\mathcal{D}}} \right) \approx \frac{\pi^{-n/2}}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})} \sum_s \omega_s \frac{\tau_s^{-\frac{\alpha}{2}}}{(1+\tau_s)^{n/2}} \mathrm{e}^{-\frac{|\mathbf{k}-\mathbf{m}|^2}{\mathcal{D}(1+\tau_s)}}. \tag{3.10}$$

The computation of the sum in (3.6) with $\mathcal{N}_\alpha \eta_2$ in (3.10) is very efficient for densities which allow a separated representation, i.e., for given accuracy ε, they can be represented as a sum of products of vectors in dimension 1

$$f(\mathbf{x}) = \sum_{p=1}^{R} r_p \prod_{j=1}^{n} f_j^{(p)}(x_j) + \mathcal{O}(\epsilon), \qquad \mathbf{x} = (x_1, ..., x_n).$$

We infer that an approximation of $\mathcal{N}_\alpha f(h\mathbf{k})$ can be computed by the sum of products of one-dimensional convolutions

$$(\mathcal{N}_\alpha f)(h\mathbf{k}) \approx$$
$$\frac{1}{(\pi\mathcal{D})^{n/2}} \frac{(h\sqrt{\mathcal{D}})^{2-\alpha}}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})} \prod_{p=1}^{P} r_p \sum_s \frac{\omega_s \tau_s^{-\frac{\alpha}{2}}}{(1+\tau_s)^{n/2}} \prod_{j=1}^{n} \sum_{m_j \in \mathbb{Z}} f_j^{(p)}(hm_j) \mathrm{e}^{-\frac{(k_j - m_j)^2}{\mathcal{D}(1+\tau_s)}}.$$

In the same way we get the fast formulas

$$\nabla^{\alpha-1} f(h\mathbf{k}) \approx \frac{(h\sqrt{\mathcal{D}})^{1-\alpha}}{(\pi\mathcal{D})^{n/2}} \frac{2}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})}\times$$

$$\prod_{p=1}^{P} r_p \sum_{s} \frac{\omega_s \tau_s^{-\frac{\alpha}{2}}}{(1+\tau_s)^{n/2+1}} \prod_{j=1}^{n} \sum_{m_j\in\mathbb{Z}} f_j^{(p)}(hm_j)\frac{k_j-m_j}{\sqrt{\mathcal{D}}}\mathrm{e}^{-\frac{(k_j-m_j)^2}{\mathcal{D}(1+\tau_s)}}, j=1,...,n;$$

$$(-\Delta)^{\alpha/2} f(h\mathbf{k}) \approx \frac{(h\sqrt{\mathcal{D}})^{-\alpha}}{(\pi\mathcal{D})^{n/2}} \frac{-2}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})} \sum_{p=1}^{P} r_p \sum_{s} \frac{\omega_s \tau_s^{-\frac{\alpha}{2}}}{(1+\tau_s)^{n/2}}\times$$

$$\Big(\frac{2\tau_s^2}{(1+\tau_s)^2} \sum_{i=1}^{n} \sum_{m_i\in\mathbb{Z}} f_i^{(p)}(hm_i)\frac{(k_i-m_i)^2}{D}\mathrm{e}^{-\frac{(k_i-m_i)^2}{\mathcal{D}(1+\tau_s)}} \prod_{\substack{j=1\\ j\neq i}}^{n} \sum_{m_j\in\mathbb{Z}} f_j^{(p)}(hm_j)\mathrm{e}^{-\frac{(k_j-m_j)^2}{\mathcal{D}(1+\tau_s)}}$$

$$-\, n \prod_{j=1}^{n} \sum_{m_j\in\mathbb{Z}} f_j^{(p)}(hm_j)\mathrm{e}^{-\frac{(k_j-m_j)^2}{\mathcal{D}(1+\tau_s)}}\Big).$$

4. High order approximations

In order to derive high order approximation formulas, we assume as basis functions the tensor product of univariate functions

$$\eta_{2M}(\mathbf{x}) = \prod_{j=1}^{n} \widetilde{\eta}_{2M}(x_j); \quad \widetilde{\eta}_{2M}(x_j) = \frac{(-1)^{M-1}}{2^{2M-1}\sqrt{\pi}\,(M-1)!} \frac{H_{2M-1}(x_j)\mathrm{e}^{-x_j^2}}{x_j} \tag{4.1}$$

where H_k are the Hermite polynomials

$$H_k(x) = (-1)^k \mathrm{e}^{x^2} \left(\frac{d}{dx}\right)^k \mathrm{e}^{-x^2}.$$

The function η_{2M} satisfies the moment conditions of order $2M$ and the values $\varepsilon_k(\mathcal{D}) = \mathcal{O}(\mathrm{e}^{-\pi^2\mathcal{D}})$ ([21, Theorem 3.5]). Then (4.1) gives rise to approximation formulas of order $2M$ plus the saturation error.

In this section we provide one-dimensional integral representations for $\mathcal{N}_\alpha \eta_{2M}$ and $\Delta\mathcal{N}_\alpha \eta_{2M}$. Using the relation ([21, p.55])

$$\widetilde{\eta}_{2M}(x) = A\left(\frac{d}{dx}\right)\mathrm{e}^{-x^2}, \quad A\left(\frac{d}{dx}\right) = \frac{1}{\sqrt{\pi}} \sum_{s=0}^{M-1} \frac{(-1)^s}{s!4^s}\frac{d^{2s}}{dx^{2s}},$$

integrating by parts and making use of (3.7), we get

$$\mathcal{N}_\alpha(\eta_{2M})(\mathbf{x}) = \frac{c_{n,\alpha}}{\pi^{n/2}} \prod_{j=1}^{n} \sum_{s_j=0}^{M-1} \frac{(-1)^{s_j}}{s_j!4^{s_j}} \frac{d^{2s_j}}{dx_j^{2s_j}} \int_{\mathbb{R}^n} \frac{\mathrm{e}^{-|\mathbf{y}|^2}}{|\mathbf{x}-\mathbf{y}|^{n-2+\alpha}} d\mathbf{y} =$$

$$\prod_{j=1}^{n} \sum_{s_j=0}^{M-1} \frac{(-1)^{s_j}}{s_j!4^{s_j}} \frac{d^{2s_j}}{dx_j^{2s_j}} (\mathcal{N}_\alpha \eta_2)(\mathbf{x}) = \frac{\pi^{-n/2}}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})} \prod_{j=1}^{n} \sum_{s_j=0}^{M-1} \frac{(-1)^{s_j}}{s_j!4^{s_j}} \frac{d^{2s_j}}{dx_j^{2s_j}} \int_0^\infty \frac{t^{-\frac{\alpha}{2}} \mathrm{e}^{-\frac{|\mathbf{x}|^2}{1+t}}}{(1+t)^{n/2}} dt\,.$$

Keeping in mind that

$$\frac{d^{2s}}{dx^{2s}} \mathrm{e}^{-ax^2} = a^s H_{2s}(\sqrt{a}x)\mathrm{e}^{-ax^2}, \qquad a > 0, \quad s \geq 0,$$

we obtain the following one-dimensional integral representation with separated integrand

$$\Phi^{(0)}_{2M}(\mathbf{x}) := \mathcal{N}_\alpha(\eta_{2M})(\mathbf{x}) = \frac{\pi^{-n/2}}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})} \int_0^\infty \prod_{j=1}^{n} S_M(\frac{1}{t+1}, x_j) \frac{\mathrm{e}^{-\frac{x_j^2}{1+t}}}{(1+t)^{1/2}} \quad t^{-\frac{\alpha}{2}} dt, \quad (4.2)$$

where we introduced the polynomials

$$S_M(a,x) = \sum_{s=0}^{M-1} \frac{(-1)^s a^s}{s!4^s} H_{2s}(\sqrt{a}x), \quad a > 0.$$

From (4.2), we deduce the following one-dimensional integral representation with separated integrand

$$\Phi^{(2)}_{2M}(\mathbf{x}) := \Delta(\mathcal{N}_\alpha \eta_{2M})(\mathbf{x}) = \frac{\pi^{-n/2}}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})} \int_0^\infty \sum_{i=1}^{n} R_M(\frac{1}{t+1}, x_i) \prod_{j \neq i}^{1,n} S_M(\frac{1}{t+1}, x_j) \frac{\mathrm{e}^{-\frac{|\mathbf{x}|^2}{1+t}} t^{-\frac{\alpha}{2}}}{(1+t)^{n/2}} dt \quad (4.3)$$

where

$$R_M(a,x) = \mathrm{e}^{ax^2} \frac{d^2}{dx^2} \Big(S_M(a,x) \mathrm{e}^{-ax^2} \Big).$$

It is easy to obtain

$$\begin{aligned} R_M(a,x) &= \frac{d^2}{dx^2} S_M(a,x) - 4ax \frac{d}{dx} S_M(a,x) + 2a(2ax-1) S_M(a,x) \\ &= 2a\Big(\sum_{s=0}^{M-2} \frac{(-a)^s}{s!4^s} \Big(\big(2ax^2 - 2as - a - 1\big) H_{2s}(\sqrt{a}x) + 2a\sqrt{a}x H_{2s+1}(\sqrt{a}x) \Big) \\ &\qquad + (2ax^2 - 1) \frac{(-a)^{M-1}}{(M-1)!4^{M-1}} H_{2M-2}(\sqrt{a}x) \Big). \end{aligned}$$

For example, we have

$$S_1(a,x)=1, \qquad R_1(a,x)=2a(2ax^2-1);$$
$$S_2(a,x)=1+\frac{a}{2}-a^2x^2;$$
$$R_2(a,x)=R_1(a,x)+2a(-2a^3x^4+6a^2x^2-\frac{3a}{2});$$
$$S_3(a,x)=S_2(a,x)+\frac{a^2}{8}\left(4a^2x^4-12ax^2+3\right);$$
$$R_3(a,x)=R_2(a,x)+2a(a^5x^6-\frac{15a^4x^4}{2}+\frac{45a^3x^2}{4}-\frac{15a^2}{8});$$
$$S_4(a,x)=S_3(a,x)+\frac{a^3}{48}\left(-8a^3x^6+60a^2x^4-90ax^2+15\right);$$
$$R_4(a,x)=R_3(a,x)+2a(-\frac{1}{3}a^7x^8+\frac{14a^6x^6}{3}-\frac{35a^5x^4}{2}+\frac{35a^4x^2}{2}-\frac{35a^3}{16}).$$

5. Implementation and Numerical Results

From representations (2.2),(4.2) and (2.4),(4.3) we derive the approximating formulas

$$\mathcal{N}_\alpha f(\mathbf{x})\approx\mathcal{N}_{\alpha,h}^{(M)}f(\mathbf{x}):=\frac{(h\sqrt{\mathcal{D}})^{2-\alpha}}{\mathcal{D}^{n/2}}\sum_{\mathbf{m}\in\mathbb{Z}^n}f(h\mathbf{m})\Phi_{2M}^{(0)}\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right), \tag{5.1}$$

$$(-\Delta)^{\alpha/2}f(\mathbf{x})\approx\mathcal{L}_{\alpha,h}^{(M)}f(\mathbf{x}):=-\frac{(h\sqrt{\mathcal{D}})^{-\alpha}}{\mathcal{D}^{n/2}}\sum_{\mathbf{m}\in\mathbb{Z}^n}f(h\mathbf{m})\Phi_{2M}^{(2)}\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right). \tag{5.2}$$

At the point of a uniform grid $h\mathbf{k}=(hk_1,...,hk_n)$ we obtain the convolutional sums

$$\mathcal{N}_\alpha f(h\mathbf{k})\approx\mathcal{N}_{\alpha,h}^{(M)}f(h\mathbf{k})=\frac{(h\sqrt{\mathcal{D}})^{2-\alpha}}{\mathcal{D}^{n/2}}\sum_{\mathbf{m}\in\mathbb{Z}^n}f(h\mathbf{m})a_{\mathbf{k}-\mathbf{m}}^{(M)},$$
$$(-\Delta)^{\alpha/2}f(h\mathbf{k})\approx\mathcal{L}_{\alpha,h}^{(M)}f(h\mathbf{k})=-\frac{(h\sqrt{\mathcal{D}})^{-\alpha}}{\mathcal{D}^{n/2}}\sum_{\mathbf{m}\in\mathbb{Z}^n}f(h\mathbf{m})b_{\mathbf{k}-\mathbf{m}}^{(M)}$$

where

$$a_{\mathbf{k}}^{(M)}=\frac{\pi^{-n/2}}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})}\int_0^\infty\prod_{j=1}^n S_M(\frac{1}{t+1},k_j)\frac{\mathrm{e}^{-\frac{k_j^2}{1+t}}}{(1+t)^{1/2}}\quad t^{-\frac{\alpha}{2}}dt,$$

$$b_{\mathbf{k}}^{(M)}=\frac{\pi^{-n/2}}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})}\int_0^\infty\sum_{i=1}^n R_M(\frac{1}{t+1},k_i)\frac{\mathrm{e}^{-\frac{k_i^2}{1+t}}}{(1+t)^{1/2}}\prod_{j\neq i}^{1,n}S_M(\frac{1}{t+1},k_j)\,\frac{\mathrm{e}^{-\frac{k_j^2}{1+t}}}{(1+t)^{1/2}}\,t^{-\frac{\alpha}{2}}dt.$$

It is known that the double exponential formulas for numerical integration, proposed by Takahasi and Mori ([30], see also [31]), are highly efficient. The idea is to transform a given integral to an integral over the real line through a change of variable $t=\phi(u)$

such that the integrand has a double exponential decay, and then apply the trapezoidal formula to the transformed integral.

For the transformation function $\phi(u)$, we make the substitution proposed in [30]

$$t=\phi(u) \text{ with } \phi(u)=\mathrm{e}^{\psi(u)}, \quad \psi(u)=a(b(u-\mathrm{e}^{-u})+\mathrm{e}^{b(u-\mathrm{e}^{-u})})$$

with positive constants a and b. After the substitution we have

$$a_{\mathbf{k}}^{(M)}=\frac{\pi^{-n/2}}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})}\int\limits_{-\infty}^{\infty}\prod_{j=1}^{n}S_M(\frac{1}{\phi(u)+1},k_j)\frac{\mathrm{e}^{-\frac{k_j^2}{1+\phi(u)}}}{(1+\phi(u))^{1/2}}\,(\phi(u))^{\frac{2-\alpha}{2}}\psi'(u)du;$$

$$b_{\mathbf{k}}^{(M)}=\frac{\pi^{-n/2}}{2^{2-\alpha}\Gamma(\frac{2-\alpha}{2})}\int\limits_{-\infty}^{\infty}\sum_{i=1}^{n}R_M(\frac{1}{\phi(u)+1},k_i)\frac{\mathrm{e}^{-\frac{k_i^2}{1+\phi(u)}}}{(1+\phi(u))^{1/2}}\times$$
$$\prod_{j\neq i}^{1,n}S_M(\frac{1}{\phi(u)+1},k_j)\,\frac{\mathrm{e}^{-\frac{k_j^2}{1+\phi(u)}}}{(1+\phi(u))^{1/2}}\,(\phi(u))^{\frac{2-\alpha}{2}}\psi'(u)du.$$

We provide results of some experiments which show the accuracy and the convergence orders of the method.

The sum $\mathcal{N}_{\alpha,h}^{(M)}f$ in (5.1) approximates $\mathcal{N}_\alpha f$ with order $\mathcal{O}(h^{2M}+h^{2-\alpha}\mathrm{e}^{-\pi^2\mathcal{D}})$. Therefore, if $\mathcal{D}$ is large enough, then $\mathcal{N}_{\alpha,h}^{(M)}f$ behaves in numerical computations like a high order formula. We compute the volume potential $\mathcal{N}_\alpha$ of $f(\mathbf{x})=\mathrm{e}^{-|\mathbf{x}|^2}$ which has the exact value in (3.2) or (3.3), by using the approximating formulas (5.1). In Tables 1-2 we report on absolute errors, relative errors and the approximation rates

$$(\log|\mathcal{N}_\alpha f(\mathbf{x})-\mathcal{N}_{\alpha,2h}^{(M)}f(\mathbf{x})|-\log|\mathcal{N}_\alpha f(\mathbf{x})-\mathcal{N}_{\alpha,h}^{(M)}f(\mathbf{x})|)/\log 2$$

for the computations of the 3-dimensional volume potential $\mathcal{N}_\alpha(\mathrm{e}^{-|\cdot|^2})$ at a fixed point by assuming $\alpha=0.5$ (Table 1) and $\alpha=1.5$ (Table 2). The numerical results confirm the h^{2M} convergence of the approximating formula when $M=1,2,3,4$. For small h, the 8th-order formula reaches the saturation error.

Table 3 shows that the method is effective also for higher space dimensions. We assumed $n=10^k$, $k=1,2,3,4$ and $\alpha=0.5$. The approximate values are computed by the formulas $\mathcal{N}_{\alpha,h}^{(M)}$ for $M=1,2,3,4$. We use uniform grid size $h=0.1\times 2^{-k}$, $k=0,...,4$. For high dimensional cases the second order formula fails whereas the 8th-order formula $\mathcal{N}_{\alpha,h}^{(4)}$ approximates with the predicted approximation rates.

In all the experiments we choose $\mathcal{D}=5$ to have the saturation error comparable with the double precision rounding errors.

In Tables 4 we report on the absolute errors and the convergence rate of the 3-dimensional volume potential $\mathcal{N}_\alpha\mathrm{e}^{-|\cdot|^2}(0.8,0,0)$ with $\alpha=0.5$ and $\alpha=1.5$ using the cubature formulas $\mathcal{N}_{\alpha,h}^{(3)}$, for different values of h and $\mathcal{D}$. The results indicate approximations of order $2-\alpha$ if $\mathcal{D}=1$ or $\mathcal{D}=2$ and h is small, caused by the relatively large saturation error $\mathcal{O}(h^{2-\alpha}\,\mathrm{e}^{-\pi^2\mathcal{D}})$. If $\mathcal{D}=3$ the rate of convergence 6 is obtained because the saturation error is negligible compared to the first term of the approximation error.

Table 1. Absolute errors, relative errors and approximation rates for the 3-dimensional volume potential $\mathcal{N}_\alpha(\mathrm{e}^{-|\cdot|^2})$ at $(.6, .6, .6)$ using $\mathcal{N}_{\alpha,h}^{(M)}(\mathrm{e}^{-|\cdot|^2})$, with $\alpha = 0.5$.

	$M=4$			$M=3$		
h^{-1}	absolute error	relative error	rate	absolute error	relative error	rate
10	0.137D-06	0.454D-06		0.214D-05	0.709D-05	
20	0.620D-09	0.205D-08	7.79	0.386D-07	0.128D-06	5.79
40	0.251D-11	0.832D-11	7.95	0.625D-09	0.207D-08	5.95
80	0.977D-14	0.324D-13	8.01	0.986D-11	0.327D-10	5.99
160	0.278D-15	0.920D-15	5.14	0.154D-12	0.509D-12	6.00

	$M=2$			$M=1$		
h^{-1}	absolute error	relative error	rate	absolute error	relative error	rate
10	0.123D-04	0.409D-04		0.484D-02	0.160D-01	
20	0.556D-06	0.184D-05	4.47	0.122D-02	0.404D-02	1.99
40	0.311D-07	0.103D-06	4.16	0.305D-03	0.101D-02	2.00
80	0.188D-08	0.624D-08	4.04	0.763D-04	0.253D-03	2.00
160	0.117D-09	0.387D-09	4.00	0.191D-04	0.632D-04	2.00

Table 2. Absolute errors, relative errors and approximation rates for the 3-dimensional volume potential $\mathcal{N}_\alpha(\mathrm{e}^{-|\cdot|^2})$ at $(1, 1, 1)$ using $\mathcal{N}_{\alpha,h}^{(M)}(\mathrm{e}^{-|\cdot|^2})$, with $\alpha = 1.5$.

	$M=4$			$M=3$		
h^{-1}	absolute error	relative error	rate	absolute error	relative error	rate
10	0.146D-06	0.173D-05		0.502D-05	0.594D-04	
20	0.602D-09	0.714D-08	7.92	0.837D-07	0.991D-06	5.91
40	0.238D-11	0.283D-10	7.98	0.133D-08	0.157D-07	5.98
80	0.930D-14	0.110D-12	8.00	0.208D-10	0.247D-09	5.99
160	0.971D-16	0.115D-14	6.58	0.326D-12	0.386D-11	6.00

	$M=2$			$M=1$		
h^{-1}	absolute error	relative error	rate	absolute error	relative error	rate
10	0.134D-04	0.158D-03		0.182D-03	0.215D-02	
20	0.883D-06	0.105D-04	3.92	0.473D-04	0.560D-03	1.94
40	0.560D-07	0.663D-06	3.98	0.119D-04	0.141D-03	1.99
80	0.351D-08	0.416D-07	3.99	0.299D-05	0.354D-04	2.00
160	0.220D-09	0.260D-08	4.00	0.748D-06	0.886D-05	2.00

Table 3. Absolute errors and approximation rates for $(\mathcal{N}_\alpha \mathrm{e}^{-|\cdot|^2})(1,1,0,\ldots,0)$ using $\mathcal{N}_{\alpha,h}^{(M)}$ with $\alpha = 0.5$, $n = 10^k, k = 1,2,3,4$ and $M = 1,2,3,4$.

	n	10		100		1000		10 000	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
$M=4$	10	0.964D-07		0.344D-06		0.645D-06		0.114D-05	
	20	0.434D-09	7.80	0.153D-08	7.81	0.287D-08	7.81	0.513D-08	7.80
	40	0.176D-11	7.95	0.619D-11	7.95	0.116D-10	7.95	0.207D-10	7.95
	80	0.696D-14	7.98	0.244D-13	7.99	0.457D-13	7.99	0.817D-13	7.99
	160	0.486D-16	8.00	0.118D-15	7.69	0.170D-15	8.07	0.313D-15	8.03
$M=3$	10	0.231D-05		0.827D-05		0.152D-04		0.234D-04	
	20	0.397D-07	5.86	0.142D-06	5.86	0.266D-06	5.84	0.474D-06	5.63
	40	0.636D-09	5.96	0.228D-08	5.96	0.426D-08	5.96	0.761D-08	5.96
	80	0.100D-10	5.99	0.358D-10	5.99	0.669D-10	5.99	0.120D-09	5.99
	160	0.157D-12	6.00	0.561D-12	6.00	0.105D-11	6.00	0.187D-11	6.00
$M=2$	10	0.701D-04		0.203D-03		0.262D-03		0.805D-04	
	20	0.461D-05	3.93	0.140D-04	3.86	0.251D-04	3.38	0.351D-04	1.20
	40	0.292D-06	3.98	0.889D-06	3.97	0.164D-05	3.94	0.288D-05	3.61
	80	0.183D-07	4.00	0.558D-07	3.99	0.103D-06	3.99	0.184D-06	3.97
	160	0.115D-08	4.00	0.349D-08	4.00	0.645D-08	4.00	0.115D-07	4.00
$M=1$	10	0.288D-02		0.239D-02					
	20	0.762D-03	1.92	0.118D-02	1.02				
	40	0.193D-03	1.98	0.365D-03	1.70	0.358D-03		0.805D-04	
	80	0.485D-04	1.99	0.964D-04	1.92	0.146D-03	1.29	0.789D-04	0.03
	160	0.121D-04	2.00	0.244D-04	1.98	0.421D-04	1.80	0.502D-04	0.65

On the other hand, due to the rapid decay of the functions $\eta_{2M}(\mathbf{x})$, one has to take into account only a finite number of terms in the sum (5.1) to compute the value of $\mathcal{N}_\alpha f$ at a given point within a given accuracy, and the number of summands for fixed h increases in $\mathcal{D}$ ([21, p.65]).

In the remainder of the paper we show numerical results for the approximations of the fractional Laplacian $(-\Delta)^{\alpha/2} f$ by $\mathcal{L}_{\alpha,h}^{(M)} f$ in (5.2), with $|(-\Delta)^{\alpha/2} f - \mathcal{L}_{\alpha,h}^{(M)} f| = \mathcal{O}(h^{2M} + h^{-\alpha} \mathrm{e}^{-\pi^2 \mathcal{D}})$. Hence the related cubatures approximate with the order $2M$ for $h \geq h_0$ but they do not converge. In Tables 5-6 we report on absolute errors, relative errors and the approximation rates for the computations of the 3-dimensional fractional Laplacian $(-\Delta)^{\alpha/2}$ at a fixed point by assuming $\alpha = 0.5$ (Table 5) and $\alpha = 1.5$ (Table 6). The numerical results confirm the h^{2M} convergence of the approximating formula when $M = 1,2,3$. For $M = 4$, if $\alpha = 0.5$ the predicted order 8 is obtained; if $\alpha = 1.5$ and small h, the 8th-order rate is not obtained due to the relatively large saturation error.

We applied formulas (5.2) also in higher dimensions (Table 7). We assumed $n = 10^k, k = 1,2,3,4$ and $\alpha = 0.5$. The approximate values are computed by the formulas $\mathcal{L}_{\alpha,h}^{(M)}$ for $M = 1,2,3,4$. We used uniform grid size $h = 0.1 \times 2^{-k}$, $k = 0, ..., 4$. For high dimensional cases the second order formula fails whereas the 4th, 6th and 8th formulas approximates with the predicted approximation rates.

For all the calculations the same quadrature rule is used for computing the one-dimensional integrals, the parameters are $a = 6$, $b = 5$, $\tau = 0.004$ and 600 summands in the quadrature sum.

Table 4. Absolute error and approximation rate for $(\mathcal{N}_\alpha \mathrm{e}^{-|\cdot|^2})(0.8, 0, 0)$ using $\mathcal{N}^{(3)}_{\alpha,h}$ with $\alpha = 0.5$ and $\alpha = 1.5$, for different h and $\mathcal{D}$.

		$\mathcal{D} = 1$		$\mathcal{D} = 2$		$\mathcal{D} = 3$	
	h^{-1}	error	rate	error	rate	error	rate
$\alpha = 0.5$	5	0.641D-04		0.102D-04		0.327D-04	
	10	0.205D-04	1.65	0.172D-06	5.90	0.585D-06	5.80
	20	0.702D-05	1.54	0.146D-08	6.87	0.949D-08	5.95
	40	0.246D-05	1.51	0.419D-09	1.81	0.150D-09	5.99
	80	0.869D-06	1.50	0.162D-09	1.37	0.233D-11	6.01
	160	0.307D-06	1.50	0.573D-10	1.50	0.311D-13	6.23
$\alpha = 1.5$	5	0.202D-02		0.271D-04		0.871D-04	
	10	0.128D-02	0.65	0.212D-06	7.00	0.161D-05	5.75
	20	0.882D-03	0.54	0.163D-06	0.39	0.264D-07	5.93
	40	0.620D-03	0.51	0.116D-06	0.48	0.405D-09	6.03
	80	0.437D-03	0.50	0.816D-07	0.51	0.267D-11	7.25
	160	0.309D-03	0.50	0.576D-07	0.50	0.158D-11	0.63

Table 5. Relative errors, absolute errors and approximation rates for the 3-dimensional fractional Laplacian $(-\Delta)^{\alpha/2}(\mathrm{e}^{-|\cdot|^2})$ at $(.6, .6, .6)$ using $\mathcal{L}^{(M)}_{\alpha,h}(\mathrm{e}^{-|\cdot|^2})$, with $\alpha = 0.5$.

	$M = 4$			$M = 3$		
h^{-1}	absolute error	relative error	rate	absolute error	relative error	rate
10	0.285D-05	0.730D-05		0.471D-04	0.121D-03	
20	0.131D-07	0.334D-07	7.77	0.845D-06	0.216D-05	5.80
40	0.306D-10	0.260D-09	7.96	0.137D-07	0.350D-07	5.95
80	0.210D-12	0.538D-12	7.98	0.215D-09	0.551D-09	5.99
160	0.944D-15	0.241D-14	7.80	0.337D-11	0.863D-11	6.00

	$M = 2$			$M = 1$		
h^{-1}	absolute error	relative error	rate	absolute error	relative error	rate
10	0.512D-03	0.131D-02		0.718D-02	0.184D-01	
20	0.368D-04	0.941D-04	3.80	0.170D-02	0.435D-02	2.08
40	0.238D-05	0.609D-05	3.95	0.418D-03	0.107D-02	2.02
80	0.150D-06	0.384D-06	3.99	0.104D-03	0.267D-03	2.01
160	0.939D-08	0.240D-07	4.00	0.260D-04	0.666D-04	2.00

Table 6. Relative errors, absolute errors and approximation rates for the 3-dimensional fractional Laplacian $(-\Delta)^{\alpha/2}(\mathrm{e}^{-|\cdot|^2})$ at $(1, 1, 1)$ using $\mathcal{L}^{(M)}_{\alpha,h}(\mathrm{e}^{-|\cdot|^2})$, with $\alpha = 1.5$.

	$M = 4$			$M = 3$		
h^{-1}	absolute error	relative error	rate	absolute error	relative error	rate
10	0.690D-06	0.450D-05		0.310D-04	0.203D-03	
20	0.243D-08	0.158D-07	8.15	0.508D-06	0.332D-05	5.93
40	0.913D-11	0.596D-10	8.06	0.802D-08	0.524D-07	5.98
80	0.820D-13	0.535D-12	6.80	0.126D-09	0.821D-09	6.00
160	0.679D-13	0.443D-12		0.199D-11	0.130D-10	5.98

	$M = 2$			$M = 1$		
h^{-1}	absolute error	relative error	rate	absolute error	relative error	rate
10	0.800D-03	0.522D-02		0.157D-01	0.103D+00	
20	0.528D-04	0.345D-03	3.92	0.395D-02	0.258D-01	1.99
40	0.335D-05	0.219D-04	3.98	0.989D-03	0.645D-02	2.00
80	0.210D-06	0.137D-05	3.99	0.247D-03	0.161D-02	2.00
160	0.131D-07	0.858D-07	4.00	0.618D-04	0.403D-03	2.00

Table 7. Absolute errors and approximation rates for $((-\Delta)^{\alpha/2}\mathrm{e}^{-|\cdot|^2})(1,1,0,\dots,0)$ using $\mathcal{L}_{\alpha,h}^{(M)}$ with $\alpha=0.5$, $n=10^k, k=1,2,3,4$ and $M=1,2,3,4$.

	n	10		100		1000		10 000	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
$M=4$	10	0.260D-05		0.704D-04		0.129D-02		0.229D-01	
	20	0.120D-07	7.76	0.314D-06	7.81	0.576D-05	7.81	0.103D-03	7.80
	40	0.489D-10	7.94	0.127D-08	7.95	0.233D-07	7.95	0.415D-06	7.95
	80	0.193D-12	7.99	0.500D-11	7.99	0.917D-10	7.99	0.164D-08	7.98
	160	0.111D-15	9.76	0.119D-13	8.72	0.343D-12	8.06	0.157D-10	6.71
$M=3$	10	0.485D-04		0.166D-02		0.304D-01		0.468D+00	
	20	0.857D-06	5.82	0.286D-04	5.86	0.532D-03	5.84	0.948D-02	5.63
	40	0.138D-07	5.95	0.458D-06	5.96	0.852D-05	5.96	0.152D-03	5.96
	80	0.218D-09	5.99	0.721D-08	5.99	0.134D-06	5.99	0.239D-05	5.99
	160	0.341D-11	6.00	0.113D-09	6.00	0.210D-08	6.00	0.374D-07	6.00
$M=2$	10	0.109D-02		0.399D-01		0.522D+00		0.161D+01	
	20	0.732D-04	3.90	0.275D-02	3.86	0.502D-01	3.38	0.703D+00	1.20
	40	0.466D-05	3.97	0.175D-03	3.97	0.327D-02	3.94	0.575D-01	3.61
	80	0.292D-06	3.99	0.110D-04	3.99	0.206D-03	3.99	0.367D-02	3.97
	160	0.183D-07	4.00	0.688D-06	4.00	0.129D-04	4.00	0.230D-03	4.00
$M=1$	10	0.354D-01		0.455D+00					
	20	0.943D-02	1.91	0.227D+00	1.01				
	40	0.239D-02	1.98	0.703D-01	1.69	0.713D+00		0.161D+01	
	80	0.601D-03	1.99	0.186D-01	1.92	0.291D+00	1.29	0.158D+01	0.03
	160	0.150D-03	2.00	0.471D-02	1.98	0.838D-01	1.80	0.100D+01	0.65

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