Representations and estimates for inverse operators in the harmonic potential theory for polyhedra

Vladimir Maz'ya

Department of Mathematical Sciences
University of Liverpool, M&O Building
Liverpool, L69 3BX, UK

Department of Mathematics
Linköping University
SE-58183 Linköping, Sweden

Dedicated to the memory of Gaetano Fichera

Abstract. The paper mainly concerns the results by N. Grachev and the author in the harmonic potential theory for polyhedra. Pointwise estimates for kernels of inverse operators are presented which imply the invertibility of the integral operator generated by the double layer potential in the space of continuous functions and in L_p. Auxiliary pointwise estimates for Green's kernel of the Neumann problem are proved.

2000 MSC. Primary: 35G15, 45p05, 47b38, 47G10

Keywords: Potential theory, boundary integral equations, polyhedron, Neumann problem, fundamental solutions

1 Survey of results on the invertibility of boundary integral operators

To begin with, I mention some results by N. Grachev and the author on integral equations of the potential theory on smooth surfaces with isolated conic vertices obtained in [GM1], [GM3], [GM4], where, as in [M1] (see also [M2]), the study of integral equations is reduced to the study of certain auxiliary boundary value problems. We found representations for inverse operators of these equations in terms of inverse operators of the interior and exterior Dirichlet and Neumann problems. Using estimates for the fundamental solutions of these boundary value problems, we arrived at estimates for kernels of inverse operators of integral equations. Such estimates lead to theorems on the invertibility of integral equations in various function spaces. In particular, the solvability in the space of continuous functions C for the integral equation associated with the Dirichlet problem could be stated without any assumptions on openings of the cones with smooth generatrices.

For a fairly large class of surfaces, the solvability of the boundary integral equation in the space C was proved by Burago and Maz'ya [BM] and Kral [K], whose approach requires that the essential norm $|T|$ of the double layer potential T is less than 1. This condition can be formulated in geometrical terms. However, it does not hold even for all cones with smooth generatrices. Angell, Kleinman, Kral [AKK] and

Kral and Wendland [KW] succeeded in compelling the inequality $|T| < 1$ to hold by replacing the usual norm in C with some equivalent weighted norm. The polyhedral surfaces considered in [KW] are formed by a finite number of rectangles parallel to the coordinate planes.

Since in the papers [GM1], [GM4] the solvability of the above mentioned integral equation on surfaces with a finite number of conic vertices in the space C was proved without any additional geometric assumptions, it became plausible that the use of the essential norm has been unnecessary and appeared only in the method of proof. We [GM6], and independently Rathsfeld [R], extended the result in [GM4] to arbitrary polyhedra using different methods. In [R], a proof based on the Mellin transform was used. By the same approach, Elschner [E] studied the invertibility and the Fredholm property of a similar integral operator with a complex parameter on a polyhedral surface in certain weighted L_2-Sobolev spaces.

Now I pass to a description of results obtained by Grachev and the author in [GM6]. We denote by Γ the boundary of a compact polyhedron in $\mathbb{R}^3$. By G^+ we denote the interior of the polyhedron and by G^- its exterior. Consider two problems for the Laplace operator

$$\Delta u = 0 \ \text{ on } \ G^+, \qquad u = f \ \text{ on } \ \Gamma, \tag{1.1}$$

$$\Delta v = 0 \ \text{ on } \ G^-, \qquad \partial v/\partial n = g \ \text{ on } \ \Gamma \backslash M. \tag{1.2}$$

Here M is the set of singularities of the polyhedron, i.e. the union of edges and vertices, and $\partial/\partial n$ stands for the derivative in the direction of the outer normal to $\Gamma\backslash M$.

Let $O_1, O_2, \ldots, O_m$ be the vertices of the polyhedron, let $M_1, M_2, \ldots, M_k$ be the edges and let

$$r_i(x) = \text{dist}(x, M_i) =, \quad r(x) = \min_{1 \le i \le k} r_i(x),$$

$$\rho_i(x) = \text{dist}(x, O_i) \quad \rho(x) = \min_{1 \le i \le m} \rho_i(x).$$

By ω_i, $i = 1, 2, \ldots, k$ we denote the opening of the dihedral angle with the edge M_i coinciding with G^+ near M_i and let

$$\lambda_i^+ = \pi/\omega_i, \quad \lambda_i^- = \pi/(2\pi - \omega_i), \quad \lambda = \min\{\lambda_i^+, \lambda_i^-\}.$$

Let K_i, $i = 1, 2, \ldots, m$, be the cone with the vertex O_i which coincides with G^+ near the point O_i. The open set that the cone K_i cuts from the unit sphere S^2 centered at O_i is denoted by Ω^+ and the set $S^2 \backslash \overline{\Omega^+}$ is denoted by Ω^-.

Let δ_i and ν_i be positive numbers such that $\delta_i(\delta_i + 1)$ and $\nu_i(\nu_i + 1)$ are the first positive eigenvalues of the Dirichlet problem in Ω^+ and the Neumann problem in Ω^- for the Laplace-Beltrami operator on S^2. We also set

$$\varkappa_i = \min\{\delta_i, \nu_i, 1\}.$$

Let $W\psi$ denote the classical double layer potential with the density ψ:

$$(W\psi) = \frac{1}{4\pi} \int_\Gamma \frac{\partial}{\partial n_\xi} \Big(\frac{1}{|x - \xi|} \Big) \psi(\xi)\, ds_\xi, \quad x \in G^\pm.$$

We are looking for a solution of the equation (1.1) in the form of a double potential. It is known that the density ψ satisfies the integral equation

$$(1 + T)\, \psi = 2\, f. \tag{1.3}$$

Here T is the operator on Γ defined by the equation

$$(T\psi)(x) = 2W_0\psi(x) + (1 - d(x))\,\psi(x),$$

where $d(x) = 1$ for $x \in \Omega\backslash M$, $d(x) = \omega_i/\pi$ for $x \in M_i$, $d(x) = \text{meas}\,\Omega_i^+/2\pi$ for $x \in O_i$, and $W_0\psi$ is the direct value on Γ of the double layer potential.

Now we formulate the main result for the integral equation associated with the Dirichlet problem.

Theorem 1 *Let* $\varkappa = \min\limits_i \varkappa_i$, $\lambda = \min\limits_i \lambda_i$. *If*

$$p > 2/(1+\varkappa), \quad p > 1/\lambda \tag{1.4}$$

then the integral operator

$$1 + T : \, L_p(\Gamma) \to L_p(\Gamma)$$

and the operator

$$1 + T : \, C(\Gamma) \to C(\Gamma)$$

perform the isomorphisms. The inverse operator admits the representation

$$(1+T)^{-1} f = (1 + L + M)\, f, \tag{1.5}$$

where L *and* M *are integral operators on* Γ *with the kernels* $\mathcal{L}(x,y)$ *and* $\mathcal{M}(x,y)$ *admiting the following estimates:*

If M_j *is the nearest edge to the point* y *and* O_i *is the nearest vertex to* y, *then*

$$|\mathcal{M}(x,y)| \le c\,\rho(y)^{\varkappa_i - 1 - \varepsilon}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda_j - 1 - \varepsilon}. \tag{1.6}$$

The kernel $\mathcal{L}(x,y)$ *is different from zero only if the point* x *lies near the point* y. *Suppose that* x *and* y *lie in a neighbourhood of a vertex* O_i, $i = 1, 2, \dots, m$ *and this neighbourhood contains no vertices of the polyhedron other than* O_i. *If* M_j *and* M_l *are the nearest edges to* y *and* x, *then*

$$\begin{aligned} |\mathcal{L}(x,y)| &\le c\,\rho(y)^{-2}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda_j - 1 - \varepsilon}, \\ &+ c\big(r(y) + |x-y|\big)^{-2}\Big(\frac{r(x)}{r(x) + |x-y|}\Big)^{\lambda_l - \varepsilon}\Big(\frac{r(y)}{r(y) + |x-y|}\Big)^{\lambda_j - 1 - \varepsilon} \end{aligned} \tag{1.7}$$

for $\rho(x)/2 < \rho(y) < 2\rho(x)$, *and*

$$|\mathcal{L}(x,y)| \le c\,\rho(y)^{-1}\big(\rho(x) + \rho(y)\big)^{-1}\Big(\frac{\min\{\rho(x),\rho(y)\}}{\rho(x)+\rho(y)}\Big)^{\varkappa_i - \varepsilon}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda_j - 1 - \varepsilon} \tag{1.8}$$

in the opposite case. Here ε *is any small positive number.*

Remark 1. A similar theorem holds for the integral equation

$$(1 + T^*)\psi = -2\,g, \tag{1.9}$$

associated with the Neumann problem, where T^* is the operator formally adjoint to T. In that case it is sufficient to replace (1.4) by the estimates

$$1 \le p < 2/(1-\varkappa), \quad p < 1/(1-\lambda)$$

and to replace x by y and vice versa in the estimates (1.6) - (1.8). However, the invertibility in the space $C(\Gamma)$ should not be mentioned.

Here is a brief description of our method. First we consider the interior Dirichlet problem and the exterior Neumann problem in some weighted Hölder spaces with the weight $\rho^\beta r^\gamma$, where β and γ are real. It is known (see [MP1]), that there exists a unique solution satisfying (1.1) and that the representation

$$u(x) = \int_\Gamma \mathcal{P}^+(x,\xi)\, f(\xi)\, ds_\xi \tag{1.10}$$

holds with derivatives of the kernel $\mathcal{P}^+(x,\xi)$ admiting the following estimates:

Suppose that the points x and ξ lie in a neighbourhood of a vertex O_i, $i = 1, 2, \ldots, m$, and M_j and M_l are the nearest edges to x and ξ. If either $2\,\rho(\xi) < \rho(x)$ or $\rho(\xi) > 2\,\rho(x)$, then

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{P}^+(x,\xi)| \le c_{\sigma,\tau}\, \rho(x)^{-|\sigma|} \rho(\xi)^{-1-|\tau|} \big(\rho(x)+\rho(\xi)\big)^{-1}$$

$$\times \Big(\frac{\min\{\rho(x),\rho(\xi)\}}{\rho(x)+\rho(\xi)}\Big)^{\delta_i-\varepsilon} \Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda_j-|\sigma|-\varepsilon} \Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda_l-|\tau|-1-\varepsilon}.$$

In the zone $\rho(\xi) < 2\,\rho(x) < 4\,\rho(\xi)$, the estimates have the form

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{P}^+(x,\xi)| \le c_{\sigma,\tau}\, |x-\xi|^{-2-|\sigma|-|\tau|}$$

$$\times \Big(\frac{r(x)}{r(x)+|x-\xi|}\Big)^{\lambda_j-|\sigma|-\varepsilon} \Big(\frac{r(\xi)}{r(\xi)+|x-\xi|}\Big)^{\lambda_l-1-|\tau|-\varepsilon}.$$

In the case $x \in U_i$, $\xi \in U_q$, where U_i and U_q are small neighbourhoods of the vertices O_i and O_q with $i \ne q$, the estimates take the form

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{P}^+(x,\xi)| \le c_{\sigma,\tau}\, \rho(x)^{\delta_i-|\sigma|-\varepsilon} \rho(\xi)^{\delta_q-|\tau|-\varepsilon} \Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda_j-|\sigma|-\varepsilon} \Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda_l-|\tau|-1-\varepsilon}.$$

Here σ and τ are arbitrary multi-indices, ε is a sufficiently small positive number.

A similar representation

$$u(x) = \int_\Gamma \mathcal{Q}^-(x,\xi)\, g(\xi)\, ds_\xi \tag{1.11}$$

holds for the solution of the Neumann problem (1.2) and the kernel $\mathcal{Q}^-(x,\xi)$ obeys the following relations (see [GM4] and Part 2 of the book [MR]):

Suppose that the points x and ξ lie in a neighbourhood of the vertex O_i, $i = 1, 2, \ldots, m$, and M_j and M_l are the nearest edges to x and ξ. If either $2\,\rho(x) < \rho(\xi)$ or $\rho(x) > 2\,\rho(\xi)$, then

$$\mathcal{Q}^-(x,\xi) = \mathcal{Q}^-(0,\xi) + \mathcal{R}^-(x,\xi) \quad \text{for } \ 2\,\rho(x) < \rho(\xi),$$

$$\mathcal{Q}^-(x,\xi) = \mathcal{Q}^-(x,0) + \mathcal{R}^-(\xi,x) \quad \text{for } \ 2\,\rho(\xi) < \rho(x),$$

where

$$\mathcal{Q}^-(0,\xi) = \mathcal{Q}^-(\xi,0) = a_i^-/\rho(\xi) + b_i^- + d_i^-(\xi), \quad a_i^- = 1/\mathrm{meas}\,\Omega_i^-.$$

For $\mathcal{R}(x,\xi)$ and $d_i^-(\xi)$ one has the estimates

$$|\partial_\xi^\sigma\, d_i^-(\xi)| \le c_\sigma\, \rho(x)^{\nu_i-|\sigma|-\varepsilon}\Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^l_{\sigma\varepsilon}},$$

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{R}^-(x,\xi)| \le c_{\sigma,\tau}\, \rho(x)^{\nu_i-|\sigma|-\varepsilon}\rho(\xi)^{-1-\nu_i-|\tau|-\varepsilon}\Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda^j_{\sigma\varepsilon}}\Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^l_{\sigma\varepsilon}}.$$

In the intermediate zone $\rho(x) < 2\,\rho(\xi) < 4\,\rho(x)$, the estimate takes the form

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{Q}^-(x,\xi)| \le \frac{c_{\sigma\tau}}{|x-\xi|^{1+|\sigma|+|\tau|}}\Big(\frac{r(x)}{r(x)+|x-\xi|}\Big)^{\lambda^j_{\sigma\varepsilon}}\Big(\frac{r(\xi)}{r(\xi)+|x-\xi|}\Big)^{\lambda^l_{\tau\varepsilon}}.$$

In the case $x \in U_i$, $\xi \in U_q$, where U_i and U_q are small neighbourhoods of the vertices O_i and O_q with $i \ne q$, we have

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{Q}^-(x,\xi)| \le c_{\sigma,\tau}\, \rho(x)^{\nu_{\sigma\varepsilon}{}^i}\rho(\xi)^{\nu^q_{\tau\varepsilon}}\Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda^j_{\sigma\varepsilon}}\Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^l_{\tau\varepsilon}}.$$

Here we use the notation

$$\lambda^j_{\sigma\varepsilon} = \min\{0, \lambda_j - |\sigma| - \varepsilon\}, \quad \lambda^j_{\tau\varepsilon} = \min\{0, \lambda_l - |\tau| - \varepsilon\},$$

$$\nu^i_{\sigma\varepsilon} = \min\{0, \nu_i - |\sigma| - \varepsilon\}, \quad \nu^q_{\tau\varepsilon} = \min\{0, \nu_q - |\tau| - \varepsilon\}.$$

One can show that the representation

$$(1+T)^{-1} = \frac{1}{2}\Big(1 - Q^- \frac{\partial}{\partial n} P^+\Big) \tag{1.12}$$

for the inverse operator of the integral equation associated with the Dirichlet problem holds in the space of traces on Γ of functions from the weighted Hölder space mentioned above. Here P^+ and Q^- are the integral operators defined by the equalities (1.10) and (1.11).

The estimates for derivatives of the kernels $\mathcal{P}^+(x,\xi)$, $\mathcal{Q}^-(x,\xi)$ and the equality (1.12) allow to establish the representation (1.5) in Hölder spaces and to obtain estimates (1.6) - (1.8) for the kernels $\mathcal{L}(x,y)$ and $\mathcal{M}(x,y)$. With the help of these estimates one can show that the operator $(1+T)^{-1}$ is continuous in the space of continuous functions as well as in an appropriate L_p space and can extend the representation (1.5) to these spaces.

Remark 2. The inverse to the integral operator in (1.9) has the form

$$(1+T^*)^{-1} = \frac{1}{2}\Big(1 - \frac{\partial}{\partial n} P^+ Q^-\Big).$$

2 Properties of the Neumann problem in a polyhedral cone

The results in this section are borrowed from preprint [GM5]. We consider the Neumann problem in a polyhedral cone. Its solvability in certain weighted Hölder and Sobolev spaces is shown and estimates for the fundamental solution are obtained. In Subsection 2.1 the problem is studied in some weighted Hilbert spaces. Subsection 2.2 is devoted to a generalization of previous results to the L_p norm with $p > 2$ which enables one to prove the existence of Green's function and to obtain estimates both for this function and its derivatives (Subsection 2.3). With the help of such estimates, the solvability of the Neumann problem in various function spaces is proved in Subsection 2.4.

Our estimates for Green's function of the Neumann problem are similar to those obtained in the case of the Dirichlet problem in [MP1]. The Neumann and mixed boundary value problems for a class of elliptic systems are treated in Part 2 of the book [MR].

2.1 Solvability of the Neumann problem in a polyhedral cone. The case of weighted Hilbert spaces

1. Function spaces. Let K be an open polyhedral cone in $\mathbb{R}^3$ with the vertex O and the edges M_j, $j = 1, 2, \dots, k$. The faces ∂K_j, $j = 1, 2, \dots, k$, of the cone are plane sectors. By M we denote the set of singularities of K, i.e. $M = \cup_{1\le j\le l} M_j$, and by ω_j the opening of the dihedral angle coinciding with K in a neighbourhood of $\operatorname{int} M_j$. Suppose that O is the origin of some Cartesian system.

Let $\beta \in \mathbb{R}$, $1 < p < \infty$, let l be integer, $l \ge 0$, and let $\boldsymbol{\delta}$ be a vector $(\delta_1, \delta_2, \dots, \delta_k)$, $\delta_j \in \mathbb{R}$. We introduce the space $W^{l,p}_{\beta,\boldsymbol{\delta}}(K)$ of functions u in K with the finite norm

$$\|u\|_{W^{l,p}_{\beta,\boldsymbol{\delta}}(K)} = \Big(\sum_{i=1}^{l} \int_K |x|^{(\beta-l+i)p} \prod_{j=i}^{k} r_j^{p\delta_j} |\nabla_i u|^p dx\Big)^{1/p}.$$

Here $r_j(x) = \operatorname{dist}(x, M_j)$ and $\nabla_j = \{\partial^j/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\}$.

In what follows by $L^p_{\beta,\boldsymbol{\delta}}(K)$ we denote the space $W^{0,p}_{\beta,\boldsymbol{\delta}}(K)$.

We also need the space of traces on ∂K_j for functions from $W^{l,p}_{\beta,\boldsymbol{\delta}}(K)$ denoted by $W^{l-1/p,p}_{\beta,\boldsymbol{\delta}}(\partial K_j)$. Let $W^{l-1/p,p}_{\beta,\boldsymbol{\delta}}(\partial K)$ refer to the space of functions u on ∂K whose restrictions u_j on ∂K_j belong to $W^{l-1/p,p}_{\beta,\boldsymbol{\delta}}(\partial K_j)$ and let

$$\|u\|_{W^{l-1/p,p}_{\beta,\boldsymbol{\delta}}(\partial K)} = \sum_{j=1}^{k} \|u\|_{W^{l-1/p,p}_{\beta,\boldsymbol{\delta}}(\partial K_j)}.$$

2. The model boundary value problem in a plane infinite sector. By Q we denote an infinite sector with opening ω and the vertex O. Let $\partial Q^\pm$ be the sides of this sector and let ∂Q be the boundary of Q, i.e. $\partial Q = \partial Q^+ \cup \partial Q^- \cup \{0\}$.

Given any $\delta \in \mathbb{R}$ and any nonnegative integer l, we introduce the space $W^l_\delta(Q)$ of functions in Q for which the norm

$$\|u\|_{W^l_\delta(Q)} = \Big(\sum_{j=0}^{l} \|r^\delta \nabla_j u\|^2_{L_2(Q)}\Big)^{1/2}$$

is finite. Here $r = r(x)$ is the distance from the point x to the vertex O.

The space of traces on ∂Q^+ of functions from $W^l_\delta(Q)$ will be denoted by $W^{l-1/2}_\delta(\partial Q^\pm)$. It is well-known (see [MP1]) that the norm in $W^{l-1/2}_\delta(\partial Q^\pm)$ can be defined by the formula

$$\|u\|^2_{W^{l-1/2}_\delta(\partial Q^\pm)} = \sum_{j=0}^{l-1} \|r^\delta\, u^{(j)}\|^2_{L_2(\partial Q^\pm)} + \int_0^\infty r^{2\delta} dr \int_0^r \tau^{-2} |u^{(l-1)}(r+\tau) - u^{(l-1)}(r)|^2 d\tau.$$

Furthermore, for any positive t, we introduce the norm $\|u\|_{W^l_\delta(Q,t)}$ defined by

$$\|u\|_{W^l_\delta(Q,t)} = \Big(\sum_{j=0}^{l} t^{2(l-j)}\|r^\delta \nabla_j u\|^2_{L_2(Q)}\Big)^{1/2}.$$

Similarly, one can define the norm depending on a positive parameter t in the space of traces on $\partial Q^\pm$:

$$\|u\|^2_{W^{l-1/2}_\delta(\partial Q^\pm,t)} = \sum_{j=0}^{l-1} t^{2(l-j)-1}\|r^\delta u^{(j)}\|^2_{L_2(\partial Q^\pm)}$$

$$+\int_0^\infty r^{2\delta}dr\int_0^r \tau^{-2}|u^{(l-1)}(r+\tau)-u^{(l-1)}(r)|^2 d\tau.$$

Consider the boundary value problem depending on the complex parameter γ

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\gamma^2 u = f \text{ in } Q, \quad \frac{\partial u}{\partial n} = g \text{ on } \partial Q\backslash 0, \tag{2.1}$$

where $\partial/\partial n$ is the derivative in the direction of outer normal.

Lemma 1 [ZS],[GM2] *Let $\mathcal{A}(\gamma)$ be the operator of the problem* (2.1). *If*

$$0 < 1-\delta < \min\{1, \pi/\omega\},$$

then the operator

$$\mathcal{A}(1):\ W^2_\delta(Q)\to W^0_\delta(Q)\times\prod_\pm W^{1/2}_\delta(\partial Q^\pm)$$

performs an isomorphism.

Lemma 2 *Suppose that $0<1-\delta<\min\{1,\pi/\omega\}$. If γ belons to the line $Re\,\gamma = c_1$, then there exists a posiitive number c_2 such that the problem* (2.1) *has the unique solution $u\in W^2_\delta(Q,|\gamma|)$ for every γ with $|Im\,\gamma|>c_2$, and for any $f\in W^0_\delta(Q)$, $g^\pm\in W^{1/2}_\delta(\partial Q^\pm,|\gamma|)$. The solution admits the estimate*

$$\|u\|_{W^2_\delta(Q,|\gamma|)} \le c\Big(\|f\|_{W^0_\delta(Q)}+\sum_\pm \|g^\pm\|_{W^{1/2}_\delta(\partial Q^\pm,|\gamma|)}\Big), \tag{2.2}$$

where the constant $c>0$ is the same for all γ with $|Im\,\gamma|>c_2$ and $f\in W^0_\delta(Q)$, $g^\pm\in W^{1/2}_\delta(\partial Q^\pm,|\gamma|)$.

Proof. Let $\gamma = a+ib$, $-\infty<b<\infty$. It suffices to consider the case $a=0$. We introduce the function $v(x)=u(|b|\,x)$. Then the existence of the solution of (2.1) and the estimate (2.2) follow from Lemma 1 applied to the function v.

3. The Neumann problem in a cone. Let Ω be a spherical polygon, i.e. $\Omega = K\cap S^2$, where S^2 is the unit sphere centered at O. We introduce the notation

$$E_j = M_j\cap S^2, \qquad \partial\Omega_j = \partial K_j\cap S^2$$

Since K is a polyhedral cone, it follows that in a neighbourhood U_j of each point E_j, $j=1,2,\dots,k$, there exists a diffeomorphism $\varkappa_j$ mapping U_j into a plane sector Q_j.

Suppose that the differential of $\varkappa_j$ is identical at E_j. We define the space $W^l_{\boldsymbol{\delta}}(\Omega)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k)$. We say that a function u on Ω with support in U_j belongs to this space if $\kappa_j\, u \in W^l_{\delta_j}(Q_j)$. If a function u vanishes near all angle points then $u \in W^l_{\boldsymbol{\delta}}(\Omega)$ if and only if $u \in W^l_2(\Omega)$. The case of a function with arbitrary support is considered in a standard manner with the help of partition of unity.

The space of traces on $\partial\Omega_i$ of functions from $W^l_{\boldsymbol{\delta}}(\Omega)$, $l \geq 1$, will be denoted by $W^{l-1/2}_{\boldsymbol{\delta}}(\partial\Omega_i)$. We say that $u \in W^{l-1/2}_{\boldsymbol{\delta}}(\partial\Omega)$ if the restriction u_i of u to every component $\partial\Omega_i$ is in $W^{l-1/2}_{\boldsymbol{\delta}}(\partial\Omega_i)$ and

$$\|u\|_{W^{l-1/2}_{\boldsymbol{\delta}}(\partial\Omega)} = \sum_{i=1}^{k} \|u\|_{W^{l-1/2}_{\boldsymbol{\delta}}(\partial\Omega_i)}.$$

Further, replacing $W^l_{\delta_j}(Q_j)$ by $W^l_{\delta_j}(Q_j, t)$ in the definition of the $W^l_{\boldsymbol{\delta}}(\Omega)$-norm, we introduce the norm $\|u\|_{W^l_{\boldsymbol{\delta}}(\Omega,t)}$ for any positive t. Similarly, one can define the norm $\|u\|_{W^{l-1/2}_{\boldsymbol{\delta}}(\partial\Omega,t)}$, also depending on the parameter t.

Consider the Neumann problem

$$\Delta u = f \ \text{ in } \ K, \qquad \frac{\partial u}{\partial n} = \varphi \ \text{ on } \ \partial K \backslash M. \tag{2.3}$$

We assume that $f \in L^2_{\beta,\boldsymbol{\delta}}(K)$, $\varphi \in W^{1/2}_{\beta,\boldsymbol{\delta}}(\partial K)$. We are looking for a set of indices β, $\boldsymbol{\delta}$ for which the problem (2.1) is solvable in $W^{2,2}_{\beta,\boldsymbol{\delta}}(K)$. Let $\rho = |x|$ and let Δ' be the Lapalace-Beltrami operator on S^2. We rewrite the problem (2.3) in the form

$$\Big(\Big(\frac{\rho\,\partial}{\partial\rho}\Big)^2 + \rho\frac{\partial}{\rho} + \Delta'\Big)u = \rho^2 f \ \text{ in } K, \quad \frac{\partial u}{\partial n} = \rho\varphi \ \text{ on } \partial\Omega\backslash E, \tag{2.4}$$

where $E = \cup_j U_j$. Using the Mellin transform

$$\tilde{u}(\gamma, \cdot) = (2\pi)^{-1/2} \int_0^\infty \rho^{-\gamma-1} u(\rho, \cdot)\, d\rho,$$

we can formally write the system (2.4) as the following system with the complex parameter γ

$$(\gamma^2 + \gamma + \Delta')\tilde{u} = \tilde{F} \ \text{ in } \Omega, \quad \frac{\partial \tilde{u}}{\partial n} = \tilde{\Phi} \ \text{ on } \partial\Omega, \tag{2.5}$$

where $\tilde{F} = \tilde{f}(\gamma - 2)$, $\tilde{\Phi}(\gamma) = \tilde{\varphi}(\gamma - 1)$.

Let $\mathfrak{A}(\gamma)$ stand for the operator of the problem (2.5).

Theorem 2 *Suppose that*

$$\gamma = \frac{1}{2} - \beta - \sum_{j=1}^{k} \delta_j$$

is not an eigenvalue of the operator pencil $\mathfrak{A}(\gamma)$. Let $0 < \delta_j < 1$ and $1 - \delta_j < \pi/\omega_j$. Then

(i) Given $f \in L^2_{\beta,\boldsymbol{\delta}}(K)$ and $\varphi \in W^{1/2}_{\beta,\boldsymbol{\delta}}(\partial K)$, there exists a unique solution $u \in W^{2,2}_{\beta,\boldsymbol{\delta}}(K)$ satisfying the problem (2.3) and there is a positive constant c depending only on K such that

$$\|u\|_{W^{2,2}_{\beta,\boldsymbol{\delta}}(K)} \leq c\left(\|f\|_{L^2_{\beta,\boldsymbol{\delta}}(K)} + \|\varphi\|_{W^{1/2}_{\beta,\boldsymbol{\delta}}(\partial K)}\right). \tag{2.6}$$

(ii) Let

$$f \in L^2_{\beta,\boldsymbol{\delta}}(K) \cap L^2_{\beta',\boldsymbol{\delta}'}(K), \qquad \varphi \in W^{1/2}_{\beta,\boldsymbol{\delta}}(\partial K) \cap W^{1/2}_{\beta',\boldsymbol{\delta}'}(\partial K),$$

where $\beta' \in \mathbb{R}$ *and* $\boldsymbol{\delta}'$ *is a vector with components* δ'_j, $0 < 1 - \delta'_j < \min\{1, \pi/\omega\}$, $j = 1, 2, \ldots, k$. *Suppose that the numbers*

$$\frac{1}{2} - \beta - \sum_{j=1}^{k} \delta_j \quad \text{and} \quad \frac{1}{2} - \beta' - \sum_{j=1}^{k} \delta'_j$$

are not eigenvalues of $\mathfrak{A}(\gamma)$ *and that the eigenvalues* $\gamma_1, \ldots, \gamma_s$ *of the operator pencil* $\mathfrak{A}(\gamma)$ *lie in the interval*

$$\frac{1}{2} - \beta - \sum_{j=1}^{k} \delta_j < \gamma_j < \frac{1}{2} - \beta' - \sum_{j=1}^{k} \delta'_j.$$

If u *is a solution of the problem (2.3) from the space* $W^{2,2}_{\beta,\boldsymbol{\delta}}(K)$, *then*

$$u(x) = \sum_{i=1}^{s} \sum_{j=1}^{k_j} c_{ij} r^{\gamma_j} \varphi_{ij}(\omega) + R(x). \tag{2.7}$$

Here φ_{ij}, $j = 1, \ldots, k_i$ *are eigenfunctions of the Laplace-Beltrami operator* Δ' *corresponding to the eigenvalues* $\gamma_i^2 + \gamma_j$, c_{ij} *are certain constants, and* R *is a solution of the problem (2.3) from the space* $W^{2,2}_{\beta,\boldsymbol{\delta}}(K)$.

Proof. Let $\mathcal{A}_j(\gamma)$ denote the operator of the problem (2.1) in the sector Q_j and let $\mathfrak{A}_j(\gamma)$ stand for the transformation of $\mathfrak{A}(\gamma)$ under the diffeomorphism $\varkappa_j$. Let η_ε be a smooth function identically equal to 1 in the ball B_ε of radius ε with center at the vertex of the sector Q_j and vanishing outside the ball $B_{2\varepsilon}$. From (2.5) it follows that the norm of the operator

$$\eta_\varepsilon\big(\mathfrak{A}_j(\gamma) - \mathcal{A}_j(\gamma)\big): \; W^2_\beta(Q_j, |\gamma|) \;\rightarrow\; W^0_\beta(Q_j) \times \prod_{\pm} W^{1/2}_\delta(\partial Q_j^\pm, |\gamma|)$$

is small for small ε and large $|\gamma|$. Hence, by Lemma 2, the solvability of the problem (2.5) in $W^2_\beta(Q_j, |\gamma|)$ and the estimate

$$\|\tilde{u}\|_{W^2_{\boldsymbol{\beta}}(\Omega,|\gamma|)} \le c\left(\|\tilde{F}\|_{W^0_{\boldsymbol{\beta}}(\Omega)} + \|\tilde{\Phi}\|_{W^{1/2}_{\boldsymbol{\beta}}(\partial\Omega,|\gamma|)}\right)$$

are established by a standard argument (see [AV]).

The properties of the Mellin transform imply the equalities

$$\int_{Re\gamma = l-\beta-\sum_{j=1}^{k}\delta_j - 3/2} \|\tilde{u}\|^2_{W^l_{\boldsymbol{\delta}}(\Omega,|\gamma|)} d\gamma = \|u\|^2_{W^{l,2}_{\beta,\boldsymbol{\delta}}(K)},$$

$$\int_{Re\gamma = l-\beta-\sum_{j=1}^{k}\delta_j - 3/2} \|\tilde{\varphi}\|^2_{W^{l-1/2}_{\boldsymbol{\delta}}(\partial\Omega,|\gamma|)} d\gamma = \|u\|^2_{W^{l-1/2,2}_{\beta,\boldsymbol{\delta}}(\partial K)}.$$

Hence the function

$$u(\rho, \cdot) = (2\pi)^{-1/2} \int_{Re\gamma = 1/2-\beta-\sum_{j=1}^{k}\delta_j} \rho^\gamma \mathfrak{A}^{-1}(\gamma)[\tilde{F}, \tilde{\varphi}] d\gamma \tag{2.8}$$

belongs to $W^{2,2}_{\beta,\boldsymbol{\delta}}(K)$, satisfies the problem (2.3) and obeys the estimate (2.6).

Replacing the line of integration in (2.8) by the line $Re\,\gamma = 1/2 - \beta' - \sum_{j=1}^{k} \delta'_j$, we arrive at (2.7) (see [Ko]).

2.2 Solvability of the Neumann problem in a polyhedral cone. The case of weighted Sobolev spaces

1. The Neumann problem in a dihedral angle. Let D be an open dihedral angle in $\mathbb{R}^3$ with opening $\omega \in (0, 2\pi)$, $\omega \neq \pi$, and let $\partial D^{\pm}$ be its sides. By $L^l_{p,\beta}(D)$ we denote the completion of $C_0^\infty(\overline{D})$ in the norm

$$\Big(\|r^\delta \nabla_l u\|^p_{L_p(D)} + \|u\|^p_{L_p(B)}\Big)^{1/p}.$$

Here $r(x)$ is the distance from the point x to the edge M of D, B is a ball with radius 1, $\overline{B} \subset D$.

Let $L^{l-1/p}_{p,\delta}(\partial D^{\pm})$ stand for the space of traces on $\partial D^{\pm}$ of functions from $L^l_{p,\beta}(D)$ and $L^{l-1/p}_{p,\beta}(\partial D)$ for the space of functions u whose restrictions $u^{\pm}$ to $\partial D^{\pm}$ are in $L^{l-1/p}_{p,\delta}(\partial D^{\pm})$. We set

$$\|u\|^{l-1/p}_{p,\delta}(\partial D) = \sum_{\pm} \|u^{\pm}\|^{l-1/p}_{p,\delta}(\partial D^{\pm}).$$

Consider the Neumann problem

$$\Delta u = f \ \text{ in } D, \quad \frac{\partial u}{\partial n} = \varphi \ \text{ on } \partial D \backslash M. \tag{2.9}$$

The following result is well-known (see [GM2], [ZS]).

Theorem 3 *Let $0 < 1 - \beta < \min\{1/2, \pi/\omega\}$. Then*

(i) The operator of the Neumann problem (2.9) performs an isomorphism

$$L^2_{2,\beta}(D) \approx L^0_{2,\beta}(D) \times L^{1/2}_{2,\beta}(\partial D).$$

(ii) Let $p > 1$, $\delta > -2/p$, let l be an integer, $l > 0$, $0 < l - \delta + 2 - 2/p < \pi/\omega$, and let $f \in L^l_{2,\delta}(D)$, $\varphi \in L^{l+1-1/p}_{p,\delta}(\partial D)$. If $u \in L^l_{2,\beta}(D)$ is a solution of the problem (2.9), then $u \in L^{l+2}_{p,\delta}(D)$ and the exists a positive constant c depending only on D such that

$$\|u\|_{L^{l+2}_{p,\delta}(D)} \leq c\big(\|f\|_{L^l_{2,\delta}(D)} + \|\varphi\|_{L^{l+1-1/p}_{p,\delta}(\partial D)} + \|u\|_{L^2_{2,\beta}(D)}\big).$$

By $W^l_{p,\beta}(D)$ we denote the space of functions u with the finite norm

$$\|u\|_{W^l_{p,\beta}(D)} = \Big(\sum_{j=0}^{l} \|r^\delta \nabla_j u\|^p_{L_p(D)}\Big)^{1/p}.$$

We also introduce the space $W^{l-1/p}_{p,\beta}(\partial D)$ whose definition is obtained from the definition of the space $L^{l-1/p}_{p,\beta}(\partial D)$ after replacing L by W.

Theorem 3 leads directly to the following assertion.

Lemma 3 *Let θ and ζ be functions from $C_0^\infty(\overline{D})$ such that $\theta\zeta = \theta$. Suppose that $0 < 1 - \beta < \min\{1, \pi/\omega\}$ and $\delta > -2/p$, $0 < l - \delta + 2 - 2/p < \pi/\omega$.*

If u is a solution of the problem (2.9) and $\zeta f \in W^l_{p,\delta}(D)$, $\zeta\varphi \in W^{l+1-1/p}_{p,\delta}(\partial D)$, and $\zeta u \in W^2_{2,\beta}(D)$, then $\theta u \in W^{l+2}_{p,\delta}(D)$ and there exists a positive constant c independent of f and φ such that

$$\|\theta u\|_{W^{l+2}_{p,\delta}(D)} \leq c\,\big(\|\zeta f\|_{W^l_{p,\delta}(D)} + \|\zeta\varphi\|_{W^{l+1-1/p}_{p,\delta}(\partial D)} + \|\zeta u\|_{W^2_{2,\beta}(D)}\big).$$

2. The Neumann problem in a cone. We prove the following theorem.

Theorem 4 *Let l be a nonnegative integer, $p \ge 2$, and let the components δ_j of a vector $\boldsymbol{\delta}$ satisfy the inequalities*

$$(a) \quad \delta_j > 1 - 2/p, \quad 0 < l + 2 - \delta_j - 2/p < \pi/\omega_j.$$

(b) Assume that the number

$$l + 2 - 3/p - \beta - \sum_{j=1}^{k} \delta_j$$

is not an eigenvalue of $\mathfrak{A}(\gamma)$. Then

(i) The operator of the problem (2.9) performs an isomorphism

$$W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K) \approx W^{l,p}_{\beta,\boldsymbol{\delta}}(K) \times W^{l+1-1/p,p}_{\beta,\boldsymbol{\delta}}(\partial K).$$

(ii) Suppose that

$$f \in W^{l,p}_{\beta,\boldsymbol{\delta}}(K) \cap W^{l',p'}_{\beta',\boldsymbol{\delta}'}(K), \quad \varphi \in W^{l+1-1/p,p}_{\beta,\boldsymbol{\delta}}(\partial K) \cap W^{l'+1-1/p',p'}_{\beta',\boldsymbol{\delta}'}(\partial K),$$

where β, $\boldsymbol{\delta}'$, l', and p' satisfy the conditions (a) (b), and suppose that the closed interval with end points

$$l + 2 - 3/p - \beta - \sum_{j=1}^{k} \delta_j \quad \text{and} \quad l' + 2 - 3/p' - \beta' - \sum_{j=1}^{k} \delta'_j$$

contains no eigenvalues of $\mathfrak{A}(\gamma)$.

If u is a solution of the problem (2.3) from the space $W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)$, then $u \in W^{l'+2,p'}_{\beta',\boldsymbol{\delta}'}(K)$.

First we prove an auxiliary assertion. We introduce the sets $U_j = \{x \in K : 2^{j-1} < |x| < 2^{j+1}\}$, $j = \pm 1, \ldots$, and by χ_j, ψ_j we denote the functions in the class $C_0^\infty(\mathbb{R}^3)$ such that

$$(1) \;\; \operatorname{supp} \chi_j \subset \{x : 2^{j-1} < |x| < 2^{j+1}\}, \quad \operatorname{supp} \psi_j \subset \{x : 2^{j-2} < |x| < 2^{j+2}\}$$

$$(2) \;\; \chi_j(x)\psi_j(x) = \chi_j(x), \qquad \sum_j \chi_j(x) = 1 \;\text{ for all } x \in \overline{K} \backslash 0$$

$$(3) \;\; |x|^{|\alpha|} |\partial^\alpha \chi_j(x)| \le c_\alpha, \quad |x|^{|\alpha|} \partial^\alpha \psi_j(x)| \le c_\alpha \;\text{ for all multiindices } \alpha.$$

Lemma 4 *Let $p \ge 2$ and let β, $\boldsymbol{\delta}$ satisfy the conditions (a), (b). If $\varphi = 0$, $f \in W^{l,p}_{\beta,\boldsymbol{\delta}}(K)$ and $\operatorname{supp} f \subset U_n$, then there exists a solution of the problem (2.3) such that $\chi_j u \in W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)$ for all $j = 0, \pm 1, \ldots$ and the estimate*

$$\|\chi_j u\|_{W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)} \le c\, 2^{-\varepsilon|n-j|} \|f\|_{W^{l,p}_{\beta,\boldsymbol{\delta}}(K)}$$

holds, where ε and c are positive constants.

Proof. By $\boldsymbol{\gamma}$ we denote the vector with components $\gamma_j = \{\delta_j - 1 + 2/p\} + 1 - 2/p$, where $\{x\}$ stands for the fractional part of x. We set

$$\delta_j' = \gamma_j - 1 + 2/p + \nu, \;\; \beta' = \beta + \sum_{j=1}^{k} \delta_j - l - 2 + 3/p - \sum_{j=1}^{k} \delta_j' + 1/2 + \mu.$$

Here ν is a positive number such that $\delta_j' < 1$ and μ is so small that the closed interval with end points

$$l + 2 - 3/p - \beta - \sum_{j=1}^{k} \delta_j \mp \mu$$

contains no poles of the operator-function $\mathfrak{A}^{-1}(\gamma)$. We assume that μ is positive for $n < j$ and negative otherwise.

We introduce the function h defined by $h(x) = f(2^n x)$. By Hardy's inequality, there is a positive constant c such that

$$\|h\|_{W^{0,p}_{\beta,\boldsymbol{\gamma}}(K)} \le c\,\|h\|_{W^{l,p}_{\beta,\boldsymbol{\delta}}(K)}$$

for all h supported in U_1. Returning to the function f, we obtain

$$\|f\|_{W^{0,p}_{\beta,\boldsymbol{\gamma}}(K)} \le c\,2^{n\sigma}\|f\|_{W^{l,p}_{\beta,\boldsymbol{\delta}}(K)}, \quad \sigma = l - \sum_{j=1}^{k}(\delta_j - \gamma_j). \tag{2.10}$$

By Hölder's inequality,

$$\|f\|_{W^{0,2}_{\beta',\boldsymbol{\delta}'}(K)} \le \|\rho^{\beta}\prod_{j=1}^{k} r_j^{\gamma_j} f\|_{L_p(K)}\,\|\rho^{\beta'-\beta}\prod_{j=1}^{k} r_j^{\beta_j'-\gamma}\|_{L_{2p/(p-2)}(U_n)}.$$

The choice of the indices δ_j' shows that the second factor on the right-hand side is bounded. Besides, clearly, this norm is equal to $c' 2^{n(\mu-\sigma)}$. From this and (2.10) we conclude that $f \in W^{0,2}_{\beta',\boldsymbol{\delta}}(K)$ and

$$\|f\|_{W^{0,2}_{\beta',\boldsymbol{\delta}}(K)} \le c\,2^{\mu n}\|f\|_{W^{l,p}_{\beta,\delta}(K)}. \tag{2.11}$$

By Lemma 4, there is a constant c such that

$$\|\chi_j u\|_{W^{l+2,p}_{\beta,\delta}(K)} \le c\big(\|\psi_j f\|_{W^{l,p}_{\beta,\delta}(K)} + 2^{-\mu j}\|\psi_j u\|_{W^{2,2}_{\beta'\boldsymbol{\delta}'}(K)}\big) \tag{2.12}$$

for all $j = 0, \pm 2, \ldots$.

Theorem 2 implies

$$\|\psi_j u\|_{W^{2,2}_{\beta'\boldsymbol{\delta}'}(K)} \le c\,\|u\|_{W^{2,2}_{\beta'\boldsymbol{\delta}'}(K)} \le c\,\|f\|_{W^{0,2}_{\beta'\boldsymbol{\delta}'}(K)}. \tag{2.13}$$

By inequalities (2.11) - (2.13),

$$\|\chi_j u\|_{W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)} \le c\,2^{\mu(n-j)}\|f\|_{W^{l,p}_{\beta,\delta}(K)}.$$

We made use of the fact that the first term on the right-hand side of (2.12) vanishes for $|n-j| > 2$. The lemma is proved.

Proof of Theorem 4. Let $\varphi = 0$. 1) The existence of a solution $u \in W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)$ of the problem (2.3) and the estimate

$$\|u\|_{W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)} \le c\,\|f\|_{W^{l,p}_{\beta,\delta}(K)}$$

follow directly from Lemma 4 and from Lemma 1.1. in [MP2].

Now we prove the uniqueness of the solution. Let $v \in W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)$ be a solution of (2.3) for $f = 0$. Consider the function $v_{\varepsilon,R}$ defined by $v_{\varepsilon,R} = (1-\eta_\varepsilon)\eta_R v$, where $\eta_\varepsilon(x) = \eta(|x|/\varepsilon)$ and η is a function of the class $C_0^\infty(\mathbb{R}^+)$ which is equal to 1 on $\{t : 0 < t \le 1/2\}$ and vanishing on $\{t : t \ge 1\}$. The argument used in the proof of estimate (2.11) shows that $v_{\varepsilon,R} \in W^{2,2}_{\beta',\boldsymbol{\delta}'}(K)$ for

$$\delta'_j = (\delta_j - 1 + 2/p) + \nu, \quad \beta' = \beta + \sum_{j=1}^{k} (\delta_j - \delta'_j) - l - 3/2 + 3/p$$

and that there is a positive constant c, independent of ε and R, such that

$$\|v_{\varepsilon,R}\|_{W^{2,2}_{\beta',\boldsymbol{\delta}'}(K)} \le c\,\|v\|_{W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)}.$$

We pass to the limit in the last inequality as $\varepsilon \to 0$ and $R \to \infty$. Thus, $v \in W^{2,2}_{\beta,\boldsymbol{\delta}}(K)$ and hence $v = 0$ by Theorem 3.

2) The arguments in the part 1) show that the inclusions

$$W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K) \subset W^{2,2}_{s,\boldsymbol{\gamma}}(K), \quad W^{l'+2,p'}_{\beta',\boldsymbol{\delta}}(K) \subset W^{2,2}_{s',\boldsymbol{\gamma}}(K) \tag{2.14}$$

hold for

$$\gamma_j = \{\delta_j - 1 + 2/p\} + \nu, \quad s = \beta + \sum_{j=1}^{k} (\delta_j - \gamma_j) - l - 3/2 + 3/p,$$

$$\gamma'_j = \{\delta'_j - 1 + 2/p\} + \nu', \quad s' = \beta' + \sum_{j=1}^{k} (\delta'_j - \gamma'_j) - l' - 3/2 + 3/p'.$$

From the first inclusion in (2.14) we have $u \in W^{2,2}_{s,\boldsymbol{\gamma}}(K)$. Hence, by Theorem 2, $u \in W^{2,2}_{s',\boldsymbol{\gamma}'}(K)$. By part 1) of this theorem, the problem (2.3) is uniquely solvable in the spaces $W^{l'+2,p'}_{\beta',\boldsymbol{\delta}'}(K)$ and $W^{2,2}_{s',\boldsymbol{\gamma}}(K)$. Thus, the second inclusion in (2.14) leads to the second assertion of the theorem.

To obtain the result for any φ it is sufficient to refer to the following theorem.

Theorem 5 *Let l and δ_i, $i = 1,\dots,k$, satisfy the conditions (a) in Theorem 4. If $\varphi \in W^{l+1-1/p,p}_{\beta,\boldsymbol{\delta}}(\partial K)$, then there exists a function $u \in W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)$ such that $\partial u/\partial n = \varphi$ on $\partial K\backslash M$ and*

$$\|u\|_{W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)} \le c\,\|\varphi\|_{W^{l+1-1/p,p}_{\beta,\boldsymbol{\delta}}(\partial K)}.$$

One can choose the operator $\varphi \to u$ independent of ρ, $\boldsymbol{\delta}$, and l.

Proof. Let $\operatorname{supp}\varphi \subset \{x \in \partial K : 1/2 < |x| < 2\}$. By Theorem 3, there exists a function

$$u \in W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K), \quad \operatorname{supp} u \subset \{x \in K : 1/4 < |x| < 4\}$$

satisfying the conditions of the present theorem.

The case $\operatorname{supp}\varphi \subset \{x \in \partial K : 2^{j-1} < |x| < 2^{j+1}\}$ can be reduces to the case considered above using the transformation $x \to 2^j x$.

Let φ be an arbitrary function from $W_{\beta,\boldsymbol{\delta}}^{l+1-1/p,p}(\partial K)$. By u_j we denote functions from $W_{\beta,\boldsymbol{\delta}}^{l+2,p}(K)$ with $\operatorname{supp} u_j \subset \{x \in K : 2^{j-2} < |x| < 2^{j+2}\}$ and such that $\partial u_j/\partial n = \chi_j \varphi$ and

$$\|u_j\|_{W_{\beta,\boldsymbol{\delta}}^{l+2,p}(K)} \le c\,\|\chi_j\varphi\|_{W_{\beta,\boldsymbol{\delta}}^{l+1-1/p,p}(\partial K)}.$$

Here χ_j are the functions introduced before Lemma 4. Thus, the function $u = \sum_{-\infty}^{\infty} u_j$ is the required one.

2.3 Estimates for Green's function of the Neumann problem

1. Auxiliary assertions. From Theorem 4, in a standard manner (see [MP2], [GM2]), we obtain the following local estimates.

Lemma 5 *Let $\rho \ge 0$ and let l be integer, $l \ge 0$. Suppose that β and $\boldsymbol{\delta}$ satisfy the conditions (a) and (b) of Theorem 4. By θ and ζ we denote functions with compact support in $\mathbb{R}^3$ such that $\theta\zeta = \theta$. If u is a function satisfying the homogeneous equation (2.3) on $K \cap \operatorname{supp}\zeta$ and $\zeta u \in L_2(K)$, then $\theta u \in W_{\beta,\boldsymbol{\delta}}^{l+2,p}(K)$ and the estimate*

$$\|\theta u\|_{W_{\beta,\boldsymbol{\delta}}^{l+2,p}(K)} \le c\,\|\zeta u\|_{L_2}$$

holds.

Lemma 6 *Let $u \in W_{\beta,\boldsymbol{\delta}}^{l+2,p}(K)$. The derivatives of u of order $|\alpha| < l - 3/p$ admit the poitwise estimates*

$$|x|^t \prod_{j=1}^{k} r_j(x)^{\mu_j} |\partial^\alpha u(x)| \le c\,\|u\|_{W_{\beta,\boldsymbol{\delta}}^{l,p}(K)},$$

where

$$t + \sum_{j=1}^{k} \mu_j = \beta + \sum_{j=1}^{k} \delta_j + |\alpha| - l + 3/p,$$

$$\mu_j \ge \max\{0, \delta_j + |\alpha| - l + 3/p\}, \quad \delta_j + |\alpha| - l + 3/p \ne 0,$$

and $r(x) = \operatorname{dist}(x, M)$.

Proof. First we consider a function u defined in the interior D of a dihedral angle. Let $u \in W_\delta^{l,p}(D)$ and let $\operatorname{supp} u \subset \{x : r(x) < 1/2\}$. It is well-known that for any u there exists a function $\tilde{u} \in W_{\beta,\boldsymbol{\delta}}^{l+2,p}(\mathbb{R}^3 \backslash M)$ such that $\tilde{u} = u$ on D, $\operatorname{supp} \tilde{u} \subset \{x : r(x) < 1\}$ and

$$\|\tilde{u}\|_{W_\beta^{l,p}(\mathbb{R}^3 \backslash M)} \le c\,\|u\|_{W_\delta^{l,p}(D)}.$$

In what follows we assume that the function u is defined on $\mathbb{R}^3 \backslash M$ and that $\operatorname{supp} u \subset \{x : r(x) < 1\}$. By K_x we denote the rotational cone with the opening $\pi/2$ whose axis is orthogonal to the edge M.

By the Sobolev integral representation,

$$|\partial^\alpha u(x)| \le c \int_{K_x} \frac{|\nabla_l u(y)|\,dy}{|x-y|^{3-l+|\alpha|}}$$

$$\le c\Big(\int_{K_x} r(y)^{p\delta} |\nabla_l u|^p dy\Big)^{1/p} \Big(\int_{\{y \in K_x : |x-y|<1\}} \frac{dy}{r(y)^{q\delta} |x-y|^{(3+|\alpha|-l)q}}\Big)^{1/q}. \qquad (2.15)$$

Using spherical coordinates with center at x, we obtain that the second factor in the right-hand side of (2.15) does not exceed

$$c\Big(\int_0^1 \frac{\rho^2 d\rho}{\rho^{(3+|\alpha|-l)q}(r(x)+\rho)^{q\delta}}\Big)^{1/q} \le c\Big(\frac{1}{r^{q\delta}(x)}\int_0^{r(x)} \frac{\rho^2 d\rho}{\rho^{(3-l+|\alpha|-l)q}} + \int_{r(x)}^1 \frac{\rho^2 d\rho}{\rho^{(3-l+|\alpha|+\delta)q}}\Big)^{1/q}$$

$$\le \begin{cases} c\, r(x)^{l-|\alpha|-\delta-3/p}, & l-|\alpha|-\delta-3/p<0 \\ c, & l-|\alpha|-\delta-3/p>0 \\ c\,|\log r(x)|, & l-|\alpha|-\delta-3/p=0. \end{cases} \tag{2.16}$$

Suppose that $u \in W^{l,p}_{\beta,\boldsymbol{\delta}}(K)$. By B_x we denote the ball of radius $|x|/2$ with center at x. The estimates (2.15) and (2.16) imply

$$\prod_{j=1}^k r^{\mu_j}(x)|\partial^\alpha u(x)| \le c\sum_{i=0}^l \int_{K\cap B_x} \prod_{j=1}^k r_j^{p\delta_j}(y)|\nabla_i u(y)|^p dy$$

for all x with $|x|=1$.

Let $|x|=\rho$. We introduce the function v by $v(y)=u(\rho y)$, $y=x/\rho$. Applying the last inequality to v and returning to the function u, we arrive at the desired estimate. The lemma is proved.

Consider the Neumann problem in the dihedral angle D of opening ω with the edge M

$$\Delta u = f \ \text{ in } D, \qquad \frac{\partial u}{\partial n} = 0 \ \text{ on } \partial D\backslash M. \tag{2.17}$$

The next assertion is borrowed from [ZS].

Theorem 6 *There exists Green's function $\mathcal{G}(x,y)$ of the problem (2.17), i.e. a unique solution of the boundary value problem*

$$\begin{aligned} &\Delta_x \mathcal{G}(x,y) = \delta(x-y), \qquad x,y\in D, \\ &\frac{\partial}{\partial n_x}\mathcal{G}(x,y) = 0, \qquad x\in\partial D\backslash M,\ y\in D \end{aligned} \tag{2.18}$$

such that the function $x \to (1-\eta(|x-y|/r(y))\mathcal{G}(x,y)$ is in the space $L^2_{2,\beta}(D)$, $0 < 1-\beta < \min\{1/2, \pi/\omega\}$ for every fixed $y\in D$. Here $\eta \in C^\infty[0,\infty)$, $\eta(t)=1$ for $0\le t\le 1/2$, $\eta(t)=0$ for $t\ge 1$, and $r(x)=\mathrm{dist}(x,M)$.

Equation (2.18) is understood in the sense that

$$v(y) = \int_D \mathcal{G}(x,y)\Delta v(x)ds - \int_{\partial D}\mathcal{G}(x,y)\frac{\partial v}{\partial n}(x)ds$$

for all $v\in C_0^\infty(\overline{D})$. Green's function $\mathcal{G}$ admits the estimates

$$|\partial_x^\alpha \partial^{\sigma_y}\mathcal{G}(x,y)| \le c_{\alpha\sigma}|x-y|^{-1-|\alpha|-|\sigma|}$$

if $|x-y|<r(x)/2$ and

$$|\partial_x^\alpha \partial^{\sigma_y}\mathcal{G}(x,y)| \le c_{\alpha\sigma}|x-y|^{-1-|\alpha|-|\sigma|}\Big(\frac{r(x)}{|x-y|}\Big)^{\nu_{\alpha\varepsilon}}\Big(\frac{r(y)}{|x-y|}\Big)^{\nu_{\sigma\varepsilon}}$$

in the opposite case. Here $\nu_{\alpha\varepsilon}=\min\{0,\pi/\omega-\varepsilon-|\alpha|\}$, $\nu_{\sigma\varepsilon}=\min\{0,\pi/\omega-\varepsilon-|\sigma|\}$, and ε is a sufficiently small positive number.

2. Green's function of the problem (2.3). Let η be the function from Theorem 6.

Theorem 7 *If the interval (c_1, c_2) contains no points of the spectrum of the pencil $\mathfrak{A}(\gamma)$ corresponding to the problem (2.5), then*

(i) There exists a unique solution $G(x, y)$ of the boundary problem (2.17), i.e. a solution of the problem

$$\Delta_x G(x,y) = \delta(x,y) \;\; x, y \in K, \quad \frac{\partial G}{\partial n_x}(x,y) = 0 \;\; x \in \partial K \backslash M, \;\; y \in K \tag{2.19}$$

such that the function $x \to (1 - \eta(|x-y|/r(y)))G(x,y)$, for any fixed $y \in K$, belongs to the space $W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)$ with $l = 0, 1, \ldots,$

$$\delta_j > 1 - 2/p, \quad 0 < l + 2 - \delta_j - 2/p < \pi/\omega_j, \quad c_1 < l + 2 - \sum_{j=1}^{k} \delta_j - \beta - 3/p < c_2.$$

(ii) The function G is infinitely differentiable with respect to $x, y \in \overline{K} \backslash M$, $x \neq y$. If $|x| < |y| < 2|x|$, then

$$|\partial_x^\alpha \partial^{\sigma_y} G(x,y)| \leq c_{\alpha\sigma} |x-y|^{-1-|\alpha|-|\sigma|} \tag{2.20}$$

for $|x - y| < r(x)/2$ and

$$|\partial_x^\alpha \partial^{\sigma_y} G(x,y)| \leq c_{\alpha\sigma} |x-y|^{-1-|\alpha|-|\sigma|} \prod_{j=1}^{k} \Big(\frac{r_j(x)}{|x-y|} \Big)^{\nu^j_{\alpha\varepsilon}} \Big(\frac{r_j(y)}{|x-y|} \Big)^{\nu^j_{\sigma\varepsilon}} \tag{2.21}$$

in the opposite case. Here $r_j(x) = \mathrm{dist}(x, M_j)$, $r(x) = \mathrm{dist}(x, M)$,

$$\nu^j_{\alpha\varepsilon} = \min\{0, \pi/\omega_j - \varepsilon - |\alpha|\}, \quad \nu^j_{\sigma\varepsilon} = \min\{0, \pi/\omega_j - \varepsilon - |\sigma|\}$$

and ε is a sufficiently small positive number.

(iii) The function G is a unique solution of the boundary value problem

$$\Delta_y G(x,y) = \delta(x,y), \;\; x, y \in K, \quad \frac{\partial G}{\partial n_y}(x,y) = 0 \;\; y \in \partial K \backslash M, \;\; x \in K \tag{2.22}$$

such that the function $x \to (1-\eta(|x-y|/r(y)))G(x,y)$ belongs to the space $W^{l'+2,p'}_{\beta',\boldsymbol{\delta}'}(K)$,

$$\delta'_j > 1-2/p', \quad 0 < l'+2-\delta'_j-2/p' < \pi/\omega_j, \quad 1+c_1 < \beta'+\sum_{j=1}^{k} \delta'_j - l' - 2 + 3/p' < 1+c_2,$$

$l = 0, 1, \ldots,$ for any fixed $x \in K$.

Equations (2.19) and (2.22) should be understood in the sense that

$$v(y) = \int_K G(x,y) \Delta v(x) dx - \int_{\partial K} G(x,y) \frac{\partial v}{\partial n}(x)\, ds_x; \tag{2.23}$$

$$w(x) = \int_K G(x,y) \Delta w(y) dy - \int_{\partial K} G(x,y) \frac{\partial w}{\partial n}(y)\, ds_y; \tag{2.24}$$

for any $v, w \in C_0^\infty(\overline{K} \backslash 0)$.

Proof. (i) The uniqueness of $G(x,y)$ follows from Theorem 4, Since the operators in (1.3.5) are homogeneous, we have the relation $G(tx, ty) = t^{-1} G(x,y)$ for every positive t. Therefore, without loss of generality we may assume that $|y| = 1$.

We prove the existence of $G(x,y)$. let y be a fixed point in K with $|y|=1$. By M_y we denote the edge nearest to the point y and by D_y the dihedral angle which coincides with K in a neighbourhood of $\mathrm{int} M_y$. By Theorem 6, there exists Green's function $\mathcal{G}(x,y)$ of the problem (2.17) for D_y.

We define the function G by the equality

$$G(x,y) = \eta(|x-y|/\tau)\mathcal{G}(x,y) - R(x,y), \tag{2.25}$$

where η is the same as in Theorem 6, τ is a small positive number such that $\mathrm{supp}\,\eta(|x-y|/\tau) \cap (M \backslash M_y) = \emptyset$ and R is a solution of the Dirichlet problem

$$\Delta_x R = 2\nabla_x \eta \nabla \mathcal{G} + \mathcal{G}\Delta_x \eta \quad \text{in } K, \; \frac{\partial R}{\partial n_x} = \mathcal{G}\frac{\partial \eta}{\partial n_x} \;\text{ on } \partial K \backslash M \tag{2.26}$$

from the space $W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)$. The existence of G is proved.

(ii) The smoothness of $G(x,y)$ for $x \neq y$, $x,y \in \overline{K}\backslash M$ and the estimates (2.20), (2.21) follow directly from the construction of $G(x,y)$, Theeorem 4 and the homogeneity of $G(x,y)$.

(iii) Since the space $C_0^\infty(\overline{K}\backslash 0)$ is dense in $W^{2,2}_{\beta',\boldsymbol{\delta}'}(K)$, it follows that the equation (2.23) holds for all $v \in W^{2,2}_{\beta',\boldsymbol{\delta}'}(K)$.

By (2.5) we have that both γ and $1-\gamma$ belong to the spectrum of the pencil $\mathcal{G}(\gamma)$. Let β, $\boldsymbol{\delta}$ and β', $\boldsymbol{\delta}'$ satisfy the conditions (i) and (iii) of the theorem for $l=0$, $p=2$ and for $l'=0$, $p'=2$. Let

$$\beta' + \beta + \sum_{j=1}^k \delta'_j + \sum_{j=1}^k \delta_j = 2.$$

Let $H(x,y)$ stand for the solution of the problem (2.22) which exists by the first part of the theorem and let $H_\tau(x,z)$ be the mean value of $H(x,z)$ with respect to the variable z over the ball of radius τ. We substitute the function

$$v(z) = \big(\eta(|x-y|/R) - \eta(|x-z|/\varepsilon)\big)H(x,z) + H_\tau(x,z)\eta(|x-z|/\varepsilon)$$

into (2.23) and then pass to the limit as $R \to \infty$ and $\tau \to 0$. We have $\varepsilon > 0$ so small that the ball $\{\xi : |\xi - x| \le 2\varepsilon\}$ lies in K and does not contain the point y. Hence we arrive at the equality $H(x,y) = G(x,y)$. The theorem is proved.

Corollary 1 *The solution $u \in W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)$ of the problem (2.3) admits the representation*

$$u(x) = \int_K G(x,y)\, f(y) dy - \int_{\partial K} G(x,y)\,\varphi(y)\, ds_y.$$

Theorem 8 *Let the interval (c_1, c_2) contain no points of the spectrum of the operator pencil $\mathcal{G}(\gamma)$ corresponding to the problem (2.5).*

(i) If $|x| < |y|/2$, then

$$|\partial_x^\alpha \partial^{\sigma_y} G(x,y)| \le c_{\alpha\sigma} \frac{|x|^{c_2 - |\alpha| - \varepsilon}}{|y|^{c_2 + 1 + |\sigma| - \varepsilon}} \prod_{j=1}^k \Big(\frac{r_j(x)}{|x|}\Big)^{\nu^j_{\alpha\varepsilon}} \Big(\frac{r_j(y)}{|y|}\Big)^{\nu^j_{\sigma\varepsilon}},$$

where

$$\nu^j_{\alpha\varepsilon} = \min\{0, \pi/\omega_j - \varepsilon - |\alpha|\}, \quad \nu^j_{\sigma\varepsilon} = \min\{0, \pi/\omega_j - \varepsilon - |\sigma|\},$$

and ε is a sufficiently small positive number.

(ii) If $|x| > 2|y|$, then

$$|\partial_x^\alpha \partial^{\sigma_y} G(x,y)| \le c_{\alpha\sigma} \frac{|y|^{-c_1-1+|\alpha|-\varepsilon}}{|x|^{-c_1+|\sigma|-\varepsilon}} \prod_{j=1}^k \Big(\frac{r_j(x)}{|x|}\Big)^{\nu^j_{\alpha\varepsilon}} \Big(\frac{r_j(y)}{|y|}\Big)^{\nu^j_{\sigma\varepsilon}}.$$

Proof. (i) Let $|y| = 2$, $|x| < 1$. Consider the function

$$v(\xi) = \eta(4|\xi - x|)\partial_y G(\xi, y).$$

Lemmas 5 and 6 imply

$$|x|^t \prod_{j=1}^k r_j^{\mu_j}(x) |\partial_x^\alpha \partial_y^\sigma G(x,y)| \le c\Big(\int_K \eta^2(4|\xi - x|)|\partial_y^\sigma G(\xi,y)|^2 d\xi\Big)^{1/2}, \tag{2.27}$$

where

$$t + \sum_{j=1}^k \mu_j = \beta + \sum_{j=1}^k \delta_j + |\alpha| - l - 2 + 3/p, \quad \mu_j > \delta_j + |\alpha| - l - 2 + 3/p, \;\; \mu_j \ge 0.$$

We introduce the solution $w \in W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)$ of the Neumann problem

$$\Delta w(z) = \partial_y^\sigma G(x,y)\eta(2|x - z|), \;\; z \in K, \quad \frac{\partial w}{\partial n}(z) = 0, \;\; z \in \partial K\backslash M.$$

By Lemmas 5 and 6,

$$\prod_{j=1}^k r_j^{\mu'_j}(y)|\partial_z w(z)|_{z=y}| \le c\Big(\int_K \eta^2(2|z-y|)|w(z)|^2 dz\Big)^{1/2}, \tag{2.28}$$

where $\mu'_j > \delta_j + |\sigma| - l - 2 + 3/p$, $\mu'_j \ge 0$. The right-hand side of (2.28) does not exceed

$$c\,\|w\|_{W^{2,2}_{s,\boldsymbol{q}}(K)} \le c\Big(\int_K \eta^2(|z-y|)|\partial_y^\sigma G(z,y)|^2 dz\Big)^{1/2}.$$

Here

$$1/2 < q_j < 1, \quad c_1 < 1/2 - \sum_{j=1}^k q_j - s < c_2.$$

The left-hand side is equal to

$$\prod_{j=1}^k r_j^{\mu'_j}(y) \int_K |\partial_y^\sigma G(\xi,y)|^2 \eta(2\,|\xi - x|)\,d\xi.$$

Therefore, (2.28) leads to the estimate

$$\prod_{j=1}^k r_j^{\mu'_j}(y)\Big(\int_K |\partial_y^\sigma G(\xi,y)|^2\eta(2\,|\xi - x|)d\xi\Big)^{1/2} \le \text{const}.$$

From this and (2.27) we get

$$|x|^t \prod_{j=1}^k r_j^{\mu_j}(x) r_j^{\mu'_j}(y)|\partial_x^\alpha \partial_y^\sigma G(x,y)| \le \text{const}.$$

Setting

$$\delta_j = l+2-2/p-\pi/\omega_j+\varepsilon/2, \quad \beta+\sum_{j=1}^{k}\delta_j = l+2-3/p+c_2+\varepsilon, \quad p>2/\varepsilon,$$

we have

$$|\partial_x^\alpha \partial_y^\sigma G(x,y)| \le c_{\alpha\sigma}|x|^{c_2-|\alpha|-\varepsilon}\prod_{j=1}^{k}\Big(\frac{r_j(x)}{|x|}\Big)^{\nu^j_{\alpha\varepsilon}} r_j^{\nu^j_{\sigma\varepsilon}}(y).$$

Using the homogeneity of $\partial_x^\alpha \partial_y^\sigma G(x,y)$, we arrive at the desired estimate for all $y\in K$ in the case $|x|<|y|/2$.

(2) Considering the problem (2.22) instead of (2.19), we arrive at the estimate for $\partial_x^\alpha \partial_y^\sigma G(x,y)$ in the case $|x|>|y|/2$. The theorem is proved.

Corollary 2 *Let γ_1 be the first positive eigenvalue of $\mathfrak{A}(\gamma)$, i.e. $\gamma_1 = (-1+\sqrt{1+4\lambda_1}/2$, where λ_1 is the first positive eigenvalue of the Neumann problem on Ω for the spherical part of the Laplace operator.*

Let $G(x,y)$ denote Green's function from Theorem 8, where $c_1=-1$, $c_2=0$. Then $G(x,y)=G(y,x)$ and

$$G(x,y) = -\frac{1}{\operatorname{meas}\Omega}\frac{1}{|y|}+R(x,y)$$

for $|x|<|y|/2$, where R is a function satisfying the estimate

$$|\partial_x^\alpha \partial_y^\sigma R(x,y)| \le c_{\alpha\sigma}\frac{|x|^{\gamma_1-|\alpha|-\varepsilon}}{|y|^{1+\gamma_1+|\sigma|-\varepsilon}}\prod_{j=1}^{k}\Big(\frac{r_j(x)}{|x|}\Big)^{\nu^j_{\alpha\varepsilon}}\Big(\frac{r_j(y)}{|y|}\Big)^{\nu^j_{\sigma\varepsilon}}. \tag{2.29}$$

Here

$$\nu^j_{\alpha\varepsilon} = \min\{0,\pi/\omega_j-|\alpha|-\varepsilon\}, \qquad \nu^j_{\sigma\varepsilon} = \min\{0,\pi/\omega_j-|\sigma|-\varepsilon\},$$

and ε is a sufficiently small positive number.

Proof. The equality $G(x,y)=G(y,x)$ is an immediate corollary of part (*iii*) of Theorem 7 and the uniqueness of $G(x,y)$.

Let $G_1(x,y)$ be Green's function from Theorem 8 with $c_1=0$, $c_2=\gamma_1$. By (2.25) and (2.7) for solutions of (2.26),

$$G(x,y) = G(0,y)+G_1(x,y).$$

Since by Theorem 8 the estimate (2.29) for $G_1(x,y)$ holds for $|x|<|y|/2$, it remains to prove the equality

$$G(0,y) = -\big(\operatorname{meas}\Omega\,|y|\big)^{-1}.$$

We fix $y\in K$ and set the function

$$v(x) = \eta(x)w_\tau(x)+(1-\eta(x))|x|^{-1}$$

into (2.23). Here $w_\tau(x)$ is the mean value of $|x|^{-1}$ over the ball of radius τ, η is a cut-off function such that $\eta=1$ near the origin and $\eta(y)=0$. Then

$$\frac{1}{|y|} = \int_K G(x,y)\Delta v(x)dx - \int_{\partial K} G(x,y)\frac{\partial v(x)}{\partial n}ds_x. \tag{2.30}$$

Let B_ε be the ball of radius ε centered at 0 and let $\Omega_\varepsilon = \partial B_\varepsilon \cap K$, $\Gamma_\varepsilon = B_\varepsilon \cap \partial K$. Using Green's formula, we rewrite (2.29) in the form

$$\frac{1}{|y|} = \int_{\Omega_\varepsilon} G(x,y)\frac{\partial v(x)}{\partial n}(x)ds_x - \int_{\Omega_\varepsilon} \frac{\partial G}{\partial n_x}(x,y)v(x)ds_x$$

$$-\int_{\partial K\setminus\Gamma_\varepsilon} G(x,y)\frac{\partial v(x)}{\partial n}(x)ds_x + \int_{K\setminus B_\varepsilon} G(x,y)\Delta v(x)dx.$$

Passing to the limit as $\tau \to 0$, we get

$$\frac{1}{|y|} = -\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} G(x,y)ds_x - \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \frac{\partial G}{\partial n_x}(x,y)ds_x.$$

To complete the proof we pass to the limit as $\varepsilon \to 0$.

2.4 Solvability in weighted Hölder and Sobolev spaces

Here we prove the solvability of the problem (2.3) in certain weighted Hölder spaces and formulate a similar result wor weighted Sobolev spaces.

1. Function spaces in the cone. Consider the cone K as the union of sets $\cup_{1\le j\le k} K_j$, where $K_j = \{x \in K : r_j(x) < 2r(x)\}$, $r(x) = \min_{1\le j\le k} r_j(x)$. Let l be a nonnegative integer, let $\alpha \in (0,1)$, $\beta \in \mathbb{R}$, and let $\boldsymbol{\gamma}$ be a vector $(\gamma_1, \gamma_2, \dots, , \gamma_k)$, $\gamma_j \in \mathbb{R}$. We introduce the space $C^{l,\alpha}_{\beta,\boldsymbol{\gamma}}(K)$ of functions u in K with the finite norm

$$\|u\|_{C^{l,\alpha}_{\beta,\boldsymbol{\gamma}}(K)} = \sup_{x\in K}\prod_{j=1}^{k} r_j(x)^{\gamma_j}\left[u\right]^{l+\alpha}_{K\cap B(r/2,x)}$$

$$+ \max_{\substack{\{j:1\le j\le k,\\ l+\alpha-\gamma_j>0\}}} \sup_{x\in K_j} |x|^{\beta+\|\gamma\|-\gamma_j}\left[u\right]^{l+\alpha-\gamma}_{K_j\cap B(\rho/2,x)} + \max_{1\le j\le k}\sup_{x\in K_j} |x|^{\beta+\|\gamma\|-l-\alpha}\left[\frac{r_j(x)}{|x|}\right]^{\sigma_j}|u(x)|.$$

Here

$$[u]^\rho_K = \sup_{x,y\in K}\sum_{|\sigma|=[\rho]} |x-y|^{[\rho]-\rho}|\partial^\sigma_x u(x) - \partial^\sigma_y u(y)|,$$

$[\rho]$ is the integer part of ρ, $B(r,x)$ is the open ball in $\mathbb{R}^3$ of radius r with center at x, $\|\gamma\| = \sum\gamma_j$, $\sigma = \max\{0, \gamma_j - l - \alpha\}$.

By $C^{l,\alpha}_{\beta,\boldsymbol{\gamma}}(\partial K_j)$ we mean the space of traces on ∂K_j of functions from $C^{l,\alpha}_{\beta,\boldsymbol{\gamma}}(K)$. We say that u belongs to $C^{l,\alpha}_{\beta,\boldsymbol{\gamma}}(\partial K)$ if and only if the restriction u_j to each component ∂K_j belongs to $C^{l,\alpha}_{\beta,\boldsymbol{\gamma}}(\partial K_j)$. We introduce the norm

$$\|u\|_{C^{l,\alpha}_{\beta,\boldsymbol{\gamma}}(K)} = \sum_j \|u_j\|_{C^{l,\alpha}_{\beta,\boldsymbol{\gamma}}(\partial K_j)}.$$

2. The Neumann problem in the dihedral angle. let D be the inetrior of a dihedral angle of opening ω. By ∂D^+ and ∂D^- we denote the sides of the dihedral angle. Let M stand for the edge and ∂D for the boundary of D, i.e. $\partial D = \partial D^+ \cup \partial D^- \cup M$.

We introduce the space $N^{l,\alpha}_\gamma(D)$ with the norm

$$\|u\|_{N^{l,\alpha}_\gamma(D)} = \sup_{x\in D} r(x)^\gamma\left[u\right]^{l+\alpha}_{D\cap B(r/2,x)} + \sup_{x\in D} r(x)^{\gamma-l-\alpha}|u(x)|$$

and the space $C^{l,\alpha}_\gamma(D)$ with the norm

$$\|u\|_{C^{l,\alpha}_\gamma(D)} = \sup_{x\in D} r(x)^\gamma \big[u\big]^{l+\alpha}_{D\cap B(r/2,x)} + \|u\|_{C^{l+\alpha-\beta}(\overline{D})}.$$

Here $C^s(\overline{D})$ is the Hölder space and $r(x) = \mathrm{dist}(x, M)$.

For $\gamma > l+\alpha$ we denote $N^{l,\alpha}_\gamma(D)$ by $C^{l,\alpha}_\gamma(D)$.

In addition, let $C^{l,\alpha}_\gamma(\partial D^\pm)$ refer to the space of traces on $D^\pm$ of functions in $C^{l,\alpha}_\gamma(D)$, that is a function u belongs to $C^{l,\alpha}_\gamma(\partial D)$ if and only if the restriction $u^\pm$ to each side $\partial D^\pm$ is in $C^{l,\alpha}_\gamma(\partial D^\pm)$. We shall use the norm

$$\|u\|_{C^{l,\alpha}_\gamma(\partial D)} = \sum_\pm \|u^\pm\|_{C^{l,\alpha}_\gamma(\partial D^\pm)}.$$

The following assertion was proved in [ZS].

Theorem 9 *Let θ and ζ denote functions from $C^\infty_0(\overline{D})$ such that $\theta\zeta = \theta$. Suppose that $\gamma > 0$, $0 < l+2+\alpha-\gamma < \pi/\omega$, and $\alpha - \gamma$ is not integer.*

If u is a solution of the Neumann problem (2.9) and $\zeta f \in C^{l,\alpha}_\gamma(D)$, $\zeta\varphi \in C^{l+1,\alpha}_\gamma(\partial D)$, then $\theta u \in C^{l+2,\alpha}_\gamma(D)$ and there is a positive constant c, independent of f and φ, such that

$$\|\theta u\|_{C^{l+2,\alpha}_\gamma(D)} \le c\big(\|\zeta f\|_{C^{l,\alpha}_\gamma(D)} + \|\zeta\varphi\|_{C^{l+1,\alpha}_\gamma(\partial D)} + \sup_{x\in D}|\zeta(x)u(x)|\big).$$

3. The Neumann problem in a cone.

Theorem 10 *Let l be a nonnegative integer and let the components δ_j of the vector $\boldsymbol{\delta}$ satisfy the conditions*

$$(a) \quad \delta_j > 0,\ \ 0 < l+2+\alpha-\delta_j < \pi/\omega_j,$$

$$(b) \quad l+2+\alpha-\beta-\sum_{j=1}^k \delta_j \ \text{ is not an eigenvalue of the pencil } \mathfrak{A}(\gamma).$$

Then

(i) The operator of the problem (2.9) performs an isomorphism

$$C^{l+2,\alpha}_{\beta,\boldsymbol{\delta}}(K) \approx C^{l,\alpha}_{\beta,\boldsymbol{\delta}}(K) \times C^{l+1,\alpha}_{\beta,\boldsymbol{\delta}}(\partial K).$$

(ii) Suppose that $f \in C^{l,\alpha}_{\beta,\boldsymbol{\delta}}(K) \cap C^{l',\alpha'}_{\beta',\boldsymbol{\delta}'}(K)$ and $\varphi \in C^{l+1,\alpha}_{\beta,\boldsymbol{\delta}}(\partial K) \cap C^{l'+1,\alpha'}_{\beta',\boldsymbol{\delta}'}(\partial K)$, where β', $\boldsymbol{\delta}'$, l', and α' satisfy the conditions (a) and (b). Suppose also that the closed interval with endpoints

$$l'+\alpha'+2-\beta'-\sum_{j=1}^k \delta'_j \quad \text{and} \quad l+\alpha+2-\beta-\sum_{j=1}^k \delta_j$$

contains no poles of the holomorphic operator function $\mathfrak{A}^{-1}(\gamma)$. If u is a solution of the problem (2.3) from the space $C^{l+2,\alpha}_{\beta,\boldsymbol{\delta}}(K)$, then $u \in C^{l'+2,\alpha'}_{\beta',\boldsymbol{\delta}'}(K)$.

First we prove auxiliary assertions. Let $G(x,y)$ be Green's function in Theorem 8, where c_1 and c_2 are numbers such that

$$c_1 < l+\alpha+2-\beta-\sum_{j=1}^k \delta_j, \qquad l'+\alpha'+2-\beta'-\sum_{j=1}^k \delta'_j < c_2$$

and the interval (c_1, c_2) contains no points of the spectrum of the pencil $\mathfrak{A}(\gamma)$ of the problem (2.5). By u we denote the same function as in Corollary 1, i.e.

$$u(x) = \int_K G(x,y) f(y) dy - \int_{\partial K} G(x,y)\, \varphi(y) ds_y. \tag{2.31}$$

Lemma 7 *If u is a function defined by (2.31), then*

$$\sup_{x\in K} |x|^{\mu} |u(x)| < c\Big(\sup_{x\in K} |x|^{\mu+2} \prod_{j=1}^{k} \Big[\frac{r_j(x)}{|x|}\Big]^{1+\lambda_j} |f(x)| + \sup_{x\in\partial K} |x|^{\mu+1} \prod_{j=1}^{k} \Big[\frac{r_j(x)}{|x|}\Big]^{\lambda_j} |\varphi(x)|\Big)$$

with $0 < \lambda_j < 1$, $c_1 < -\mu < c_2$.

Proof. Clearly, the function

$$w_1 = \int_K G(x,y) f(y) dy.$$

satisfies

$$|w_1(x)| \le \sup_{z\in K} |z|^{\mu+2} \prod_{j=1}^{k} \Big[\frac{r_j(z)}{|z|}\Big]^{1+\lambda_j} |f(z)|$$

$$\times \int_K |G(x,y)| |y|^{-\mu-2} \prod_{j=1}^{k} \Big[\frac{r_j(y)}{|y|}\Big]^{-1-\lambda_j} dy. \tag{2.32}$$

To estimate the integral on the right-hand side of (2.32), we represent it as the sum of three integrals I_i over the sets K_i, $i = 1,2,3$, where $K_1 = \{y \in K : |x| < |y|/2\}$, $K_2 = \{y \in K : |y|/2 < |x|\}$, $K_3 = \{y \in K : |x| > 2|y|\}$. By Theorem 8,

$$I_1 < c \int_{K_1} \frac{|x|^{c_2-\varepsilon}}{|y|^{c_2+1-\varepsilon}} |y|^{-\mu-2} \prod_{j=1}^{k} \Big[\frac{r_j(y)}{|y|}\Big]^{-1-\lambda_j} dy < c|x|^{-\mu}$$

for all $0 < \lambda_j < 1$, $\mu > -c_2 + \varepsilon$ and

$$I_3 < c \int_{K_3} \frac{|y|^{-c_1-1-\varepsilon}}{|x|^{c_1-\varepsilon}} |y|^{-\mu-2} \prod_{j=1}^{k} \Big[\frac{r_j(y)}{|y|}\Big]^{-1-\lambda_j} dy < c|x|^{-\mu}$$

for all $0 < \lambda_j < 1$, $\mu < -c_1 - \varepsilon$.

Similarly, the estimate of $G(x,y)$ in the intermediate zone given by Theorem 7 leads to the same inaquality for I_2 for all $0 < \lambda_j < 1$.

The function

$$w_2 = \int_{\partial K} G(x,y) \varphi(y) ds_y$$

can be treated in a similar way. The lemma is proved.

Lemma 8 *Let $\delta_j > 0$ and let*

$$0 < l + 2 + \alpha - \delta_j < \pi/\omega_j, \qquad c_1 < l + 2 + \alpha - \sum_{j=1}^{k} \delta_j < c_2.$$

If u admits the representation (2.31) and $f \in C^{l,\alpha}_{\beta,\boldsymbol{\delta}}(K)$, $\varphi \in C^{l+1,\alpha}_{\beta,\boldsymbol{\delta}}(\partial K)$, then $u \in C^{l+2,\alpha}_{\beta,\boldsymbol{\delta}}(K)$ and there exists a positive constant c, independent of f and φ, such that

$$\|u\|_{C^{l+2,\alpha}_{\beta,\boldsymbol{\delta}}(K)} \le c\big(\|f\|_{C^{l,\alpha}_{\beta,\boldsymbol{\delta}}(K)} + \|\varphi\|_{C^{l+1,\alpha}_{\beta,\boldsymbol{\delta}}(\partial K)}\big).$$

Proof. Let χ_j and ψ_j be the functions defined before Lemma 4. By Theorem 9,

$$\|\chi_j u\|_{C^{l+2,\alpha}_{\beta,\boldsymbol{\delta}}(K)} \le c\big(\|\psi_1 f\|_{C^{l,\alpha}_{\beta,\boldsymbol{\delta}}(K)} + \|\psi_1\varphi\|_{C^{l+1,\alpha}_{\beta,\boldsymbol{\delta}}(\partial K)} + \sup_{x\in K}|\psi_1(x)u(x)|\big).$$

Using the dilation $x \to 2^j x$, we arrive at

$$\|\chi_j u\|_{C^{l+2,\alpha}_{\beta,\boldsymbol{\delta}}(K)} \le c\big(\|\psi_j f\|_{C^{l,\alpha}_{\beta,\boldsymbol{\delta}}(K)} + \|\psi_j\varphi\|_{C^{l+1,\alpha}_{\beta,\boldsymbol{\delta}}(\partial K)} + 2^{js}\sup_{x\in K}|\psi_1(x)u(x)|\big),$$

where $s = \beta + \sum_{j=1}^k \delta_j - l - 2 - \alpha$. Thus,

$$\|u\|_{C^{l+2,\alpha}_{\beta,\boldsymbol{\delta}}(K)} \le c\big(\|f\|_{C^{l,\alpha}_{\beta,\boldsymbol{\delta}}(K)} + \|\varphi\|_{C^{l+1,\alpha}_{\beta,\boldsymbol{\delta}}(\partial K)} + \sup_{x\in K}|x|^2|u(x)|\big).$$

By Corollary 1, the function $u \in W^{2,2}_{\beta,\boldsymbol{\delta}}(K)$ admits the representation (2.31). To complete the proof, we refer Lemma 7.

Proof of Theorem 10. It is clear that the operator of the problem (2.3) is a continuous mapping:

$$C^{l+2,\alpha}_{\beta,\boldsymbol{\delta}}(K) \to C^{l,\alpha}_{\beta,\boldsymbol{\delta}}(K) \times C^{l+1,\alpha}_{\beta,\boldsymbol{\delta}}(\partial K).$$

We prove the existence of a solution of the Neumann problem (2.3). Let $f \in C^{l,\alpha}_{\beta,\boldsymbol{\delta}}(K)$, $\varphi \in C^{l+1,\alpha}_{\beta,\boldsymbol{\delta}}(\partial K)$. Consider the functions $f^\pm$ and $\varphi^\pm$ defined by the equalities $f^+ = f\eta$, $f^- = \varphi\eta$, $f^- = f - f^+$, $\varphi^- = \varphi - \varphi^+$, where η is a function introduced in Theorem 6.

By $\boldsymbol{\lambda}$ we denote a vector with components $(\lambda_1, \lambda_2, \dots, \lambda_k)$, where $\lambda_j = \delta_j - \alpha - l - 1 + \varepsilon$, and $\varepsilon > 0$ is so small that $\lambda_j < 1$. Further, let

$$\beta^\pm = \beta + \sum_{j=1}^k (\delta_j - \lambda_j) - 3/2 - \alpha \pm \varepsilon',$$

where $\varepsilon' > 0$ is such that

$$c_1 < 1/2 - \beta^\pm - \sum_{j=1}^k \lambda_j < c_2.$$

One verifies directly that

$$f^\pm \in W^{0,2}_{\beta^\pm,\boldsymbol{\lambda}}(K), \qquad \varphi^\pm \in W^{1/2,2}_{\beta^\pm,\boldsymbol{\lambda}}(\partial K).$$

Theorem 2 and Corollary 1 imply the existence of the solutions $u^\pm \in W^{2,2}_{\beta^\pm,\boldsymbol{\lambda}}(K)$ to the problem (2.3) with the data $f^\pm$ and $\varphi^\pm$:

$$u^\pm(x) = \int_K G(x,y) f^\pm(y)dy - \int_{\partial K} G(x,y)\,\varphi^\pm(y)ds_y.$$

By Lemma 8, $u \in C^{l+2,\alpha}_{\beta,\boldsymbol{\delta}}(K)$ and

$$\|u\|_{C^{l+2,\alpha}_{\beta,\boldsymbol{\delta}}(K)} \le c\big(\|f\|_{C^{2,\alpha}_{\beta,\boldsymbol{\delta}}(K)} + \|\varphi\|_{C^{l+1,\alpha}_{\beta,\boldsymbol{\delta}}(\partial K)}\big).$$

It remains to prove the uniqueness of the solution of (2.3).

Let $u \in C^{l+2,\alpha}_{\beta,\boldsymbol{\delta}}(K)$ be a solution of (2.3) with $f = 0$, $\varphi = 0$. We introduce two functions $u^+ = u\,\eta$ and $u^- = u - u^+$. Clearly, $u^\pm \in W^{2,2}_{\beta^\pm,\boldsymbol{\lambda}}(K)$. By Corollary 1,

$$u^\pm(x) = \int_K G(x,y)\Delta u^\pm(y)dy - \int_{\partial K} G(x,y)\,\frac{\partial u^\pm}{\partial n}(y)ds_y.$$

Thus,

$$u(x) = \int_K G(x,y)\Delta u(y)dy - \int_{\partial K} G(x,y)\,\frac{\partial u}{\partial n}(y)ds_y = 0.$$

(ii) The second part of the theorem follows from Lemma 8 and the fact that the solution admits the representation (2.31). The theorem is proved.

Applying the argument similar to that used in the proof of Theorem 10, we arrive at the following assertion on the solvability of (2.3) in weighted Sobolev spaces.

Theorem 11 *Let $p > 1$, l be a nonnegative integer, and let the components δ_j of a vector $\boldsymbol{\delta}$ satisfy the conditions*

$$(a) \quad \delta_j > 0, \;\; 0 < l + 2 + \alpha - \delta_j < \pi/\omega_j,$$

$$(b) \quad l + 2 + \alpha - \beta - \sum_{j=1}^{k} \delta_j \;\; \textit{is not an eigenvalue of the pencil } \mathfrak{A}(\gamma).$$

Then

(i) The operator of the problem (2.9) performs the isomorphism

$$W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K) \approx W^{l,\alpha}_{\beta,\boldsymbol{\delta}}(K) \times W^{l+1-1/p,p}_{\beta,\boldsymbol{\delta}}(\partial K).$$

(ii) Suppose that

$$f \in W^{l,p}_{\beta,\boldsymbol{\delta}}(K) \cap W^{l',p'}_{\beta',\boldsymbol{\delta}'}(K), \quad \varphi \in W^{l+1-1/p,p}_{\beta,\boldsymbol{\delta}}(\partial K) \cap C^{l'+1-1/p',p'}_{\beta',\boldsymbol{\delta}'}(\partial K),$$

where β', $\boldsymbol{\delta}'$, l', and p' satisfy the conditions (a) and (b). Suppose also that the closed interval with the endpoints

$$l + 2 - 3/p - \beta - \sum_{j=1}^{k} \delta_j \quad \text{and} \quad l' + 2 - 3/p' - \beta' - \sum_{j=1}^{k} \delta'_j$$

contains no poles of the operator holomorphic function $\mathfrak{A}^{-1}(\gamma)$. If u is a solution of the problem (2.3) from the space $W^{l+2,p}_{\beta,\boldsymbol{\delta}}(K)$, then $u \in W^{l'+2,p'}_{\beta',\boldsymbol{\delta}'}(K)$.

References

[AV] Agranovich, M.S., Vishik, M.I. *Elliptic boundary value problems with a parameter and parabolic problems of general type*, Usp. Mat. Nauk **19**: 3 (1964), 53–161.

[AKK] Angell, T.S., Kleinman, R.E, Kral, J. *Layer potentials on boundaries with corners and edges*, Časopis Pěst. Mat., **113** (1988), no. 4, 387–402

[BM] Burago, Yu.D. *Multi-didmensional potential theory and the solution of boundary value problems in domains with non-regular boundaries*, Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova **3** (1967), 1-152; English translatioin: Sem. Math. V.A.Steklov Math. Inst. Leningr. **3** (1969), 1–68.

[E] Elschner, J. *The double layer potential operator over polyhedral domains. I. Solvability in weighted Sobolev spaces*, Appl. Anal. **45** (1992), no. 1–4, 117–134.

[GM1] Grachev, N.V., Maz'ya, V.G. *Representations and estimates for inverse operators of the potential theory integral equations on surfaces with conic points*, Soobsch. Akad. Nauk Gruzin. SSR **32**:1 (1988), 21-24.

[GM2] Grachev, N.V., Maz'ya, V.G. *On the Fredholm radius of integral equations of the potential theory*, In *Differential Equations. Spectral Theory. Wave Propogation*, pp. 109–133, Leningrad University Press, 1990.

[GM3] Grachev, N.V., Maz'ya, V.G. *Estimates for kernels of the inverse operators of the integral equations of elasticity on surfaces with conic points*, Preprint of Linköping University LiTH-MAT-R-91-06, 1–59.

[GM4] Grachev, N.V., Maz'ya, V.G. *On invertibility of the boundary integral operators of elasticity on surfaces with conic points in spaces generated by notms C, C^α, L_p*, Preprint of Linköping University LiTH-MAT-R-91-06, 1–59.

[GM5] Grachev, N.V., Maz'ya, V.G. *Estimates of the fundamental solutions of the Neumann problem in polyhedral cone*, Preprint of Linköping University LiTH-MAT-R-91-28, 1–45.

[GM6] Grachev, N.V., Maz'ya, V.G. *Solvability of boundary integral equtions on a polyhedron*, Preprint of Linköping University LiTH-MAT-R-91-50, 1–45.

[Ko] Kondratiev, V.A. *Boundary value problems for elliptic equations in regions with conical or angular points*, Tr. Mosk. Mat. Ob. **16** (1967), 209–292.

[K] Kral, J. *Integral Operators in Potential Theory*, Lect. Notes Nath. **823**, Springer, 1980.

[KW] Kral, J. , Wendland, W. *Some examples concerning applicability of the Fredholm-Radon method in potential theory*, Apl. Mat. **31** (1986), 293–318.

[M1] Maz'ya, V. *Boundary integral equations of elasticity in domains with piecewise smooth boundaries*, Equadiff 6, Proc. Int. Conf. Brno 1985, Lect. Notes Math. **1192**, 1986, 235–242.

[M2] Maz'ya, V. *Boundary integral equations*, Encyclopaedia of Math. Sciences, **27** (1991), 127–223.

[MP1] Maz'ya, V., Plamenevsky, B. *The first boundary value problem for classical equations of mathematical physics in domains with piecewise smooth boundaries*, ZAA, 1: **24** (1983), 335–339; 2: **26** (1983), 523–551.

[MP2] Maz'ya, V., Plamenevsky, B. *Estimates in L_p and in Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary*, Math. Nachr. **81** (1978), 25–82; English translatioin: Amer. Math. Soc. Transl. **123** (1984),

[MP3] Maz'ya, V., Plamenevsky, B. *Asymptotic behavior of the fundamental solutions of elliptic boundary value problems in domains with conic points*, Problemy Mat. Anal. **1** (1979), 100–145.

[MR] Maz'ya, V., Rossmann, J. *Elliptic Equations in Polyhedral Domains*, AMS, Providence, Rhode Island, 2010.

[R] Rathsfeld, A. *The invertibility of the double layer potential operator in the space of continuous functions defined on a polyhedron: the panel method*, Appl. Anal. **45** (1992), no. 1-4, 135–177.

[N] Nazarov, S.A. *Estimates near edges for solutions of the Neumann problem for elliptic systems*, Vest. Leningrad. Univ. **1** (1988), 37–42.

[S] Solonnikov, V.A. *Solvability of the* 3*-dimensional problem with free boundary for the stationary Navier-Stokes system*, Zap. Nauchn. Sem. LOMI **84** (1979), 252–285.

[ZS] Zaĭonchkovskiĭ, V., Solonnikov, V. A. *The Neumann problem for second-order elliptic equations in domains with ribs on the boundary*, Boundary value problems of mathematical physics and related questions in the theory of functions, 15, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **127** (1983), 7–48.