# ACCRETIVITY OF THE GENERAL SECOND ORDER LINEAR DIFFERENTIAL OPERATOR

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*Dedicated to Carlos Kenig with admiration and deep respect*

ABSTRACT. For the general second order linear differential operator

$$\mathcal{L}_0 = \sum_{j,\,k=1}^{n} a_{jk}\,\partial_j \partial_k + \sum_{j=1}^{n} b_j\,\partial_j + c$$

with complex-valued distributional coefficients $a_{jk}$, $b_j$, and $c$ in an open set $\Omega \subseteq \mathbb{R}^n$ ($n \geq 1$), we present conditions which ensure that $-\mathcal{L}_0$ is accretive, i.e., $\operatorname{Re}\langle -\mathcal{L}_0 \phi, \phi\rangle \geq 0$ for all $\phi \in C_0^\infty(\Omega)$.

CONTENTS



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## 1. Introduction

Let $\mathcal{L}_0$ be the general second order differential operator in an open set $\Omega \subseteq \mathbb{R}^n$,

$$\mathcal{L}_0 = \sum_{j,\,k=1}^{n} a_{jk}\,\partial_j\partial_k + \sum_{j=1}^{n} b_j\,\partial_j + c, \tag{1.1}$$

where $a_{jk}$, $b_j$, and $c$ are complex-valued distributions in $D'(\Omega)$. In this paper, we are concerned with the *accretivity* of $-\mathcal{L}_0$ defined in terms of the real part of its quadratic form:

$$\operatorname{Re}\langle -\mathcal{L}_0\, u,\, u\rangle \ge 0, \tag{1.2}$$

for all complex-valued functions $u \in C_0^\infty(\Omega)$. In other words, we study the dissipativity property associated with $\mathcal{L}_0$.

If the principal part $\mathcal{A}u$ of the differential operator is given in the divergence form,

$$\mathcal{A}u = \operatorname{div}(A\nabla u), \quad u \in C_0^\infty(\Omega), \tag{1.3}$$

then we consider the operator

$$\mathcal{L}u = \operatorname{div}(A\nabla u) + \mathbf{b}\cdot\nabla u + c\,u, \tag{1.4}$$

with distributional coefficients $A = (a_{jk})$, $\mathbf{b} = (b_j)$, and $c$. The corresponding sesquilinear form $\langle \mathcal{L}u, v\rangle$ is given by

$$\langle \mathcal{L}u, v\rangle = -\langle A\nabla u, \nabla v\rangle + \langle \mathbf{b}\cdot\nabla u, v\rangle + \langle c\,u, v\rangle. \tag{1.5}$$

We observe that $\mathcal{L}_0 = \mathcal{L} - \operatorname{Div} A\cdot\nabla$ (see, for instance, [15], [23]), where Div: $D'(\Omega)^{n\times n} \to D'(\Omega)^n$ is the row divergence operator defined in Sec. 6. Hence, we can always express $\langle \mathcal{L}_0 u, v\rangle$ in the form (1.5), with $\mathbf{b} - \operatorname{Div} A$ in place of $\mathbf{b}$, for distributional coefficients $A$ and $\mathbf{b}$.

If the differential operator is given in a more general divergence form,

$$\mathcal{L}_1 u = \operatorname{div}(A\nabla u) + \mathbf{b_1}\cdot\nabla u + \operatorname{div}(\mathbf{b_2}\,u) + c_1\,u, \tag{1.6}$$

then obviously it is reduced to (1.4) with $\mathbf{b} = \mathbf{b_1} + \mathbf{b_1}$ and $c = c_1 + \operatorname{div}\mathbf{b_2}$.

From now on, without loss of generality we will treat the accretivity property

$$\operatorname{Re}\langle -\mathcal{L}\,u,\, u\rangle \ge 0, \quad \text{for all } u \in C_0^\infty(\Omega), \tag{1.7}$$

associated with the divergence form operator (1.4).

Assuming that $A = (a_{jk})$, $\mathbf{b} = (b_j)$ and $c$ are locally integrable in $\Omega$, we write the sesquilinear form of $\mathcal{L}$ as

$$\langle \mathcal{L}u, v\rangle = \int_\Omega (-(A\,\nabla u)\cdot\nabla\overline{v} + \mathbf{b}\cdot\nabla u\;\overline{v} + c\,u\,\overline{v})\,dx, \tag{1.8}$$

where $u,\, v \in C_0^\infty(\Omega)$. Sometimes it will be convenient to write (1.5) in this form even for distributional coefficients $A$, $\mathbf{b}$, and $c$.

Our main results on the accretivity problem are stated in Sec. 2 below, in particular, Proposition 2.1 and Theorem 2.7 in higher dimensions $n \geq 2$, and Theorem 2.3 in the one-dimensional case.

This problem is of substantial interest in the real-variable case as well, where the goal is to characterize operators $-\mathcal{L}$ with real-valued coefficients whose quadratic form is nonnegative definite,

$$\langle -\mathcal{L}h, h\rangle \geq 0, \quad \text{for all real-valued } h \in C_0^\infty(\Omega). \tag{1.9}$$

Such operators $-\mathcal{L}$ are called nonnegative definite.

In the special case of Schrödinger operators

$$\mathcal{H}u = \text{div}\,(P\nabla u) + \sigma\, u, \tag{1.10}$$

with real-valued $P \in D'(\Omega)^{n\times n}$ and $\sigma \in D'(\Omega)$, a characterization of this property was obtained earlier in [12, Proposition 5.1] under the assumption that $P$ is uniformly elliptic, i.e.,

$$m\,||\xi||^2 \leq P(x)\xi\cdot\xi \leq M\,||\xi||^2, \quad \text{for all } \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega, \tag{1.11}$$

with the ellipticity constants $m > 0$ and $M < \infty$.

An analogous characterization of (1.9) for more general operators which include drift terms, $\mathcal{L} = \text{div}(P\nabla\cdot) + \mathbf{b}\cdot\nabla + c$, with real-valued coefficients and $P$ satisfying (1.11), is given in Theorem 2.2 below.

Returning to the accretivity problem (1.7) for $\mathcal{L} = \text{div}(A\nabla\cdot) + \mathbf{b}\cdot\nabla + c$ in the complex-valued case, define the symmetric component $A^s$ and its skew-symmetric counterpart $A^c$ respectively by

$$A^s = \frac{1}{2}(A + A^\perp), \quad A^c = \frac{1}{2}(A - A^\perp), \tag{1.12}$$

where $A = (a_{jk}) \in D'(\Omega)^{n\times n}$, and $A^\perp = (a_{kj})$ is the transposed matrix.

As we will see below, in order that $\mathcal{L}$ be accretive, the matrix $A^s$ must have a nonnegative definite real part: $P = \text{Re}\, A^s$ should satisfy

$$P\xi\cdot\xi \geq 0 \quad \text{for all } \xi \in R^n, \quad \text{in } D'(\Omega). \tag{1.13}$$

Moreover, the corresponding Schrödinger operator $\mathcal{H}$ defined by (1.10) with

$$P = \text{Re}\, A^s, \quad \sigma = \text{Re}\, c - \frac{1}{2}\text{div}\,(\text{Re}\,\mathbf{b}),$$

must be nonnegative definite:

$$[h]^2_{\mathcal{H}} = \langle -\mathcal{H}h, h\rangle = \langle P\nabla h, \nabla h\rangle - \langle \sigma h, h\rangle \geq 0, \tag{1.14}$$

for all real-valued (or complex-valued) $h \in C_0^\infty(\mathbb{R}^n)$.

The rest of the accretivity problem for $\mathcal{L}$ (see Sec. 2.1) boils down to the commutator inequality involving these quadratic forms,

$$\Big|\langle \tilde{\mathbf{b}}, u\nabla v - v\nabla u\rangle\Big| \leq [u]_{\mathcal{H}}\,[v]_{\mathcal{H}}, \tag{1.15}$$

for all real-valued $u, v \in C_0^\infty(\mathbb{R}^n)$, where the real-valued vector field $\tilde{\mathbf{b}}$ is given by

$$\tilde{\mathbf{b}} = \frac{1}{2} \left[ \operatorname{Im} \mathbf{b} - \operatorname{Div}(\operatorname{Im} A^c) \right].$$

Under some mild restrictions on $\mathcal{H}$, the "norms" $[u]_{\mathcal{H}}$ and $[v]_{\mathcal{H}}$ on the right-hand side of (1.15) can be replaced, up to a constant multiple, with the corresponding Dirichlet norms $||\nabla \cdot ||_{L^2(\Omega)}$. This leads to explicit criteria of accretivity in terms of $\mathrm{BMO}^{-1}$ estimates, as in Theorem 2.7 below.

Similar commutator inequalities related to compensated compactness theory [4] were studied earlier [23] in the context of the *form boundedness* problem for $\mathcal{L}$,

$$|\langle \mathcal{L}\, u,\, v\rangle| \le C\, ||\nabla u||_{L^2(\Omega)}\, ||\nabla v||_{L^2(\Omega)}, \quad u,\, v \in C_0^\infty(\Omega), \tag{1.16}$$

where the constant $C$ does not depend on $u$, $v$.

If (1.16) holds, then $\langle \mathcal{L}\, u,\, v\rangle$ can be extended by continuity to $u, v \in L^{1,\,2}(\Omega)$. Here $L^{1,\,2}(\Omega)$ is the completion of (complex-valued) $C_0^\infty(\Omega)$ functions with respect to the norm $||u||_{L^{1,\,2}(\Omega)} = ||\nabla u||_{L^2(\Omega)}$. Equivalently,

$$\mathcal{L} : \; L^{1,\,2}(\Omega) \to L^{-1,\,2}(\Omega) \tag{1.17}$$

is a bounded operator, where $L^{-1,\,2}(\Omega) = L^{1,\,2}(\Omega)^*$ is a dual Sobolev space. Analogous problems have been studied for the inhomogeneous Sobolev space $W^{1,\,2}(\Omega) = L^{1,\,2}(\Omega) \cap L^2(\Omega)$, fractional Sobolev spaces, infinitesimal form boundedness, and other related questions ([22]–[25]).

The form boundedness problem (1.16) for the general second order differential operator $\mathcal{L}$ in the case $\Omega = \mathbb{R}^n$ was characterized completely by the authors in [23] using harmonic analysis and potential theory methods. We observe that no ellipticity assumptions were imposed in [23] on the principal part $\mathcal{A}$ of $\mathcal{L}$.

For the Schrödinger operator $\mathcal{H} = \Delta + \sigma$ with $\sigma \in D'(\Omega)$, where either $\Omega = \mathbb{R}^n$, or $\Omega$ is a bounded domain that supports Hardy's inequality (see [2]), a characterization of form boundedness was obtained earlier in [21]. A different approach for $\mathcal{H} = \operatorname{div}(P\nabla\cdot) + \sigma$ in general open sets $\Omega \subseteq \mathbb{R}^n$ based on PDE and real analysis methods, under the uniform ellipticity assumptions on $P$, was developed in [12]. A quasilinear version for operators of the $p$-Laplace type can be found in [13].

Both the accretivity and form boundedness problems have numerous applications, including mathematical quantum mechanics ([29], [30]), elliptic and parabolic PDE with singular coefficients ([5], [7], [14], [15], [20], [26], [9], [27]), fluid mechanics and Navier-Stokes equations ([8], [16], [31], [33]), semigroups and Markov processes ([17]), homogenization theory ([34]), harmonic analysis ([4], [6]), etc.

We remark that, for the form boundedness, the assumption that the coefficients are complex-valued is not essential. It is easily reduced to the real-valued case.

The situation is quite different for the accretivity problem, where the presence of complex-valued coefficients leads to additional complications, especially in higher dimensions ($n \geq 2$) when the matrix $\operatorname{Im} A$ is not symmetric, and/or the imaginary part of $\mathbf{b}$ is nontrivial. Then commutator inequalities of the type (1.15) with sharp constants, BMO estimates, and other tools of harmonic analysis come into play.

These phenomena, along with some examples demonstrating possible interaction between the principal part, drift term and zero-order term of the operator $\mathcal{L}$, are discussed in the next section.

## 2. Main results

**2.1. General accretivity criterion.** Let $\Omega \subseteq \mathbb{R}^n$ ($n \geq 1$) be an open set, and let $\mathcal{L}$ be a divergence form second order linear differential operator with complex-valued distributional coefficients defined by (1.4).

For $A = (a_{jk}) \in D'(\Omega)^{n\times n}$, define its symmetric part $A^s$ and skew-symmetric part $A^c$ respectively by (1.12). The accretivity property for $-\mathcal{L}$ can be characterized in terms of the following real-valued expressions:

$$P = \operatorname{Re} A^s, \quad \tilde{\mathbf{b}} = \frac{1}{2}\left[\operatorname{Im} \mathbf{b} - \operatorname{Div}(\operatorname{Im} A^c)\right], \quad \sigma = \operatorname{Re} c - \frac{1}{2}\operatorname{div}(\operatorname{Re} \mathbf{b}), \tag{2.1}$$

where $P = (p_{jk}) \in D'(\Omega)^{n\times n}$, $\tilde{\mathbf{b}} = (\tilde{b_j}) \in D'(\Omega)^n$, and $\sigma \in D'(\Omega)$. This is a consequence of the relation (see Sec. 3)

$$\operatorname{Re}\langle -\mathcal{L}u, u\rangle = \operatorname{Re}\langle -\mathcal{L}_2 u, u\rangle, \quad u \in C_0^\infty(\Omega), \tag{2.2}$$

where

$$\mathcal{L}_2 = \operatorname{div}(P\nabla\cdot) + 2i\,\tilde{\mathbf{b}}\cdot\nabla + \sigma. \tag{2.3}$$

Moreover, in order that $-\mathcal{L}$ be accretive, the matrix $P$ must be nonnegative definite, i.e., $P\xi\cdot\xi \geq 0$ in $D'(\Omega)$ for all $\xi \in \mathbb{R}^n$. In particular, each $p_{jj}$ ($j = 1, \ldots, n$) is a nonnegative Radon measure.

A comprehensive characterization of accretive operators $-\mathcal{L}$ is given in the following proposition.

**Proposition 2.1.** *Let $\mathcal{L} = \operatorname{div}(A\nabla\cdot) + \mathbf{b}\cdot\nabla + c$, where $A \in D'(\Omega)^{n\times n}$, $\mathbf{b} \in D'(\Omega)^n$ and $c \in D'(\Omega)$ are complex-valued. Suppose that $P$, $\tilde{\mathbf{b}}$, and $\sigma$ are defined by (2.1).*

*The operator $-\mathcal{L}$ is accretive if and only if $P$ is a nonnegative definite matrix, and the following two conditions hold:*

$$[h]^2_{\mathcal{H}} = \langle P\nabla h, \nabla h\rangle - \langle \sigma\, h, h\rangle \geq 0, \tag{2.4}$$

*for all real-valued $h \in C_0^\infty(\Omega)$, and*

$$\left|\langle \tilde{\mathbf{b}}, u\nabla v - v\nabla u\rangle\right| \leq [u]_{\mathcal{H}}\,[v]_{\mathcal{H}}, \tag{2.5}$$

*for all real-valued $u, v \in C_0^\infty(\Omega)$.*

In equation (2.4), the expression $[h]^2_{\mathcal{H}} = \langle -\mathcal{H}h, h\rangle$ stands for the quadratic form associated with the Schrödinger operator $\mathcal{H} = \mathrm{div}\,(P\nabla h)+\sigma$, discussed in Sec. 2.3.

In Theorem 5.1 below, we show that it is possible to replace $\tilde{\mathbf{b}}$ in Proposition 2.1 by $\tilde{\mathbf{b}} - P\nabla\lambda$, with an appropriate change in $\sigma$. In particular, the commutator condition (2.5) trivializes if $\tilde{\mathbf{b}} = P\nabla\lambda$. This reduction is used in Sec. 4 in the one-dimensional case.

2.2. **Real-valued coefficients.** As a consequence of Proposition 2.1, we see that, for operators with real-valued coefficients, the sole condition (2.4) characterizes nonnegative definite operators $-\mathcal{L}$ in an open set $\Omega \subseteq \mathbb{R}^n$ $(n \geq 1)$. We next state a more explicit characterization of this property, under the assumption that $P = A^s \in L^1_{\mathrm{loc}}(\Omega)^{n\times n}$ in the sufficiency part, and that $P$ is uniformly elliptic in the necessity part.

**Theorem 2.2.** *Let $\mathcal{L} = \mathrm{div}(A\nabla\cdot) + \mathbf{b}\cdot\nabla + c$, where $A \in D'(\Omega)^{n\times n}$, $\mathbf{b} \in D'(\Omega)^n$ and $c \in D'(\Omega)$ are real-valued. Suppose that $P = A^s \in L^1_{\mathrm{loc}}(\Omega)^{n\times n}$ is a nonnegative definite matrix a.e.*

(i) *If there exists a measurable vector field $\mathbf{g}$ in $\Omega$ such that $(P\mathbf{g})\cdot\mathbf{g} \in L^1_{\mathrm{loc}}(\Omega)$, and*

$$\sigma = c - \frac{1}{2}\mathrm{div}\,(\mathbf{b}) \leq \mathrm{div}\,(P\mathbf{g}) - (P\mathbf{g})\cdot\mathbf{g} \quad \textit{in } D'(\Omega), \tag{2.6}$$

*then the operator $-\mathcal{L}$ is nonnegative definite.*

(ii) *Conversely, if $-\mathcal{L}$ is nonnegative definite, then there exists a vector field $\mathbf{g} \in L^2_{\mathrm{loc}}(\Omega)^n$ so that $(P\mathbf{g})\cdot\mathbf{g} \in L^1_{\mathrm{loc}}(\Omega)$, and (2.6) holds, provided $P$ is uniformly elliptic.*

Conditions similar to (2.6) are well known in ordinary differential equations, in relation to disconjugate Sturm-Liouville equations and Riccati equations with continuous coefficients ([10, Sec. XI.7], Corollary 6.1, Theorems 6.2 and 7.2). See also [9], [22], as well as the discussion in Sec. 2.4 and Sec. 4 below in the one-dimensional case.

2.3. **Schrödinger operators.** As was mentioned above, in the special case of Schrödinger operators $\mathcal{H} = \mathrm{div}\,(P\nabla h)+\sigma$, with real-valued $\sigma \in D'(\Omega)$ and uniformly elliptic $P$, Theorem 2.2 was obtained originally in [12, Proposition 5.1]. Under these assumptions, $-\mathcal{H}$ is nonnegative definite, i.e.,

$$[h]^2_{\mathcal{H}} = \langle -\mathcal{H}h, h\rangle \geq 0, \quad \text{for all } h \in C_0^\infty(\Omega),$$

if and only if there exists a vector field $\mathbf{g} \in L^2_{\mathrm{loc}}(\Omega)^n$ such that

$$\sigma \leq \mathrm{div}\,(P\mathbf{g}) - (P\mathbf{g})\cdot\mathbf{g} \quad \text{in } D'(\Omega). \tag{2.7}$$

A "linear" sufficient condition for $-\mathcal{H}$ to be nonnegative definite is given by $\sigma \leq \mathrm{div}\,(P\mathbf{g})$, where $\mathbf{g} \in L^2_{\mathrm{loc}}(\Omega)^n$ satisfies the inequality

$$\int_\Omega (P\mathbf{g}\cdot\mathbf{g})\, h^2\, dx \leq \frac{1}{4}\int_\Omega (P\nabla h\cdot P\nabla h)\, dx, \quad \text{for all } h \in C_0^\infty(\Omega).$$

Here $P\mathbf{g}\cdot\mathbf{g}$ is an *admissible* measure (Sec. 6). However, such conditions are not necessary, with any constant in place of $\frac{1}{4}$, even when $P = I$; see [12].

We recall that in Proposition 2.1 above, the nonnegative definite quadratic form $[h]^2_{\mathcal{H}}$ is associated with the Schrödinger operator $\mathcal{H}$ with real-valued coefficients $P = \operatorname{Re} A^s, \quad \sigma = \operatorname{Re} c - \frac{1}{2}\operatorname{div}(\operatorname{Re}\mathbf{b})$.

Hence, (2.7) characterizes the first condition of Proposition 2.1 given by (2.4). The second one, the commutator condition (2.5), will be discussed further in Sections 2.5 and 2.6; see also an example in Sec. 2.7.

Notice that, even for form bounded $\sigma$ such that

$$|\langle \sigma, h^2\rangle| \le C\,||\nabla h||^2_{L^2(\Omega)}, \quad \text{for all } h \in C_0^\infty(\Omega), \tag{2.8}$$

the fact that $-\mathcal{H}$ is nonnegative definite is not equivalent to the existence of a positive solution $u$ to the Schrödinger equation $\mathcal{H}u = 0$. In other words, in our setup, the Allegretto-Piepenbrink theorem is generally not true. See [12], [21], [22], and the literature cited there for further discussion.

2.4. **The one-dimensional case.** In the one-dimensional case, it is possible to avoid problems with commutator estimates using methods of ordinary differential equations ([10], [11]). In particular, the following theorem gives a generalization of Theorem 2.2 for complex-valued coefficients in the one-dimensional case. In the statements below we will make use of the standard convention $\frac{0}{0} = 0$.

**Theorem 2.3.** *Let $I \subseteq \mathbb{R}$ be an open interval (possibly unbounded). Let $a, b, c \in D'(I)$, and $\mathcal{L}u = (a\,u')' + bu' + c$. Suppose that $p = \operatorname{Re} a \in L^1_{\text{loc}}(I)$, and $\operatorname{Im} b \in L^1_{\text{loc}}(I)$.*

(i) *The operator $-\mathcal{L}$ is accretive if and only if $\frac{(\operatorname{Im} b)^2}{p} \in L^1_{\text{loc}}(I)$, where $p \ge 0$ a.e., and the following quadratic form inequality holds:*

$$\int_I p(h')^2 dx - \langle \operatorname{Re} c - \frac{1}{2}(\operatorname{Re} b)', h^2\rangle - \int_I \frac{(\operatorname{Im} b)^2}{4p}\, h^2\, dx \ge 0, \tag{2.9}$$

*for all real-valued $h \in C_0^\infty(I)$.*

(ii) *If there exists a function $f \in L^1_{\text{loc}}(I)$ such that $\frac{f^2}{p} \in L^1_{\text{loc}}(I)$, and*

$$\operatorname{Re} c - \frac{1}{2}(\operatorname{Re} b)' - \frac{(\operatorname{Im} b)^2}{4p} \le f' - \frac{f^2}{p} \quad \text{in } D'(I), \tag{2.10}$$

*then the operator $-\mathcal{L}$ is accretive.*

*Conversely, if $-\mathcal{L}$ is accretive, and $m \le p(x) \le M$ a.e. for some constants $M, m > 0$, then there exists a function $f \in L^2_{\text{loc}}(I)$ such that (2.10) holds.*

*Remark* 2.4. Clearly, the functions $f$ in Theorem 2.3 and $g$ in Theorem 2.2 are related through $f = p\,g$.

*Remark* 2.5. In general, $p = \operatorname{Re} a \ge 0$ is a Radon measure in $I$. It is easy to see that condition (2.9) with $\rho$ in place of $p$, where $\rho = \frac{dp}{dx}$ is the absolutely continuous part of the measure $p$, is sufficient for $-\mathcal{L}$ to be accretive.

On the other hand, $-\mathcal{L}$ is accretive if, for instance, $a = 2\delta_{x_0}$, $c = -2\delta_{x_0}$, and $b = i\delta_{x_0}$, where $x_0 \in I$. This example is immediate from Proposition 2.1. Operators with measure-valued $A$ in the principal part $\mathcal{A} = \operatorname{div}(A\nabla\cdot)$ are treated in [3] in the context of $L^p$-dissipativity.

The characterization of accretivity obtained in Theorem 2.3 in the one-dimensional case does not involve $\operatorname{Im} a$ and $\operatorname{Im} c$. However, $\operatorname{Im} b$ plays an important role. In higher dimensions, the situation is more complicated. The term $\operatorname{Im} b$ may contain both the irrotational and divergence-free components, the latter in combination with $\operatorname{Im} A^c$. (See Theorem 2.7, and Example in Sec. 2.7 below.)

There is an analogue of Theorem 2.3 in higher dimensions for operators with complex-valued coefficients, but only in the case where $\tilde{\mathbf{b}}$ has a specific form, for instance, if $\tilde{\mathbf{b}} = P\nabla\lambda$ for some $\lambda \in D'(\Omega)$. More general vector fields are treated in Theorem 5.1 below.

2.5. **Upper and lower bounds of quadratic forms.** Returning now to general operators with complex-valued coefficients in the case $n \geq 2$, we recall that the first condition of Proposition 2.1 is necessary for the accretivity of $-\mathcal{L}$, namely,

$$\langle \sigma\, h, h\rangle \leq \int_\Omega (P\nabla h \cdot \nabla h)\, dx, \tag{2.11}$$

for all real-valued $h \in C_0^\infty(\Omega)$, where $\sigma = \operatorname{Re} c - \frac{1}{2}\operatorname{div}(\operatorname{Re}\mathbf{b}) \in D'(\Omega)$, and $\operatorname{Re} A^s = P \in D'(\Omega)^{n\times n}$ is a nonnegative definite matrix.

Suppose now that $\sigma$ has a slightly smaller upper form bound, that is,

$$\langle \sigma\, h, h\rangle \leq (1-\epsilon^2) \int_\Omega (P\nabla h \cdot \nabla h)\, dx, \quad h \in C_0^\infty(\Omega), \tag{2.12}$$

for some $\epsilon \in (0, 1]$. We also consider the corresponding lower bound,

$$\langle \sigma\, h, h\rangle \geq -K \int_\Omega (P\nabla h \cdot \nabla h)\, dx, \quad h \in C_0^\infty(\Omega), \tag{2.13}$$

for some constant $K \geq 0$.

Such restrictions on real-valued $\sigma \in D'(\Omega)$ were invoked in [12], for uniformly elliptic $P$.

*Remark* 2.6. Notice that (2.12) is obviously satisfied for any $\epsilon \in (0, 1)$, up to an extra term $C\,||h||^2_{L^2(\Omega)}$, if $\sigma$ is *infinitesimally form bounded* ([29]), i.e.,

$$\left|\langle \sigma, h^2\rangle\right| \leq \epsilon\, ||\nabla h||^2_{L^2(\Omega)} + C(\epsilon)\, ||h||^2_{L^2(\Omega)}, \quad h \in C_0^\infty(\Omega),$$

for any $\epsilon \in (0, 1)$. This property was characterized in [25]. The second term on the right is sometimes included in the definition of accretivity of the operator $-\mathcal{L}$. We can always incorporate it as a constant term in $\sigma - C(\epsilon)$. The same is true with regards to the lower bound where we can use $\sigma + C(\epsilon)$.

If both bounds (2.12) and (2.13) hold for some $\epsilon \in (0,1]$ and $K \geq 0$, then obviously

$$\epsilon \int_\Omega (P\nabla h \cdot \nabla h)\, dx \leq [h]^2_{\mathcal{H}} \leq (K+1)^{\frac{1}{2}} \int_\Omega (P\nabla h \cdot \nabla h)\, dx, \quad h \in C_0^\infty(\Omega).$$

Assuming that $P$ satisfies the uniform ellipticity assumptions (1.11), we see that in this case condition (2.5) is equivalent, up to a constant multiple, to

$$\left| \langle \tilde{\mathbf{b}}, u\nabla v - v\nabla u \rangle \right| \leq C\, ||\nabla u||_{L^2(\Omega)}\, ||\nabla v||_{L^2(\Omega)} \tag{2.14}$$

where $C > 0$ is a constant which does not depend on real-valued $u, v \in C_0^\infty(\Omega)$.

This commutator inequality was characterized completely in the case $\Omega = \mathbb{R}^n$ in [23, Lemma 4.8] for complex-valued $u, v$. Clearly, that characterization works also in the case of real-valued $u, v$ as well (with a change of the constant $C$ up to a factor of $\sqrt{2}$).

2.6. **Main theorem on the entire space.** Combining the characterization of the commutator inequality (2.14) obtained in [23] with Proposition 2.1, we deduce our main theorem in the case $\Omega = \mathbb{R}^n$. We employ separately the lower bound (2.13) in the necessity part, and the upper bound (2.12) in the sufficiency part.

**Theorem 2.7.** *Let $\mathcal{L}$ be a second order differential operator in divergence form (1.4) with complex-valued coefficients $A \in D'(\mathbb{R}^n)^{n\times n}$, $\mathbf{b} \in D'(\mathbb{R}^n)^n$ and $c \in D'(\mathbb{R}^n)$ ($n \geq 2$). Let $P$, $\tilde{\mathbf{b}}$ and $\sigma$ be given by (2.1), where $P$ is uniformly elliptic.*

(i) *Suppose that $-\mathcal{L}$ is accretive, i.e., (1.7) holds, and suppose that (2.13) holds for some $K \geq 0$.*

(a) *If $n \geq 3$, then $\tilde{\mathbf{b}}$ can be represented in the form*

$$\tilde{\mathbf{b}} = \nabla f + \operatorname{Div} G, \tag{2.15}$$

*where $f \in D'(\mathbb{R}^n)$ is real-valued, and there exists a positive constant $C$ so that*

$$\int_{\mathbb{R}^n} |\nabla f|^2 h^2 dx \leq C \int_{\mathbb{R}^n} |\nabla h|^2 dx, \quad \text{for all } h \in C_0^\infty(\mathbb{R}^n), \tag{2.16}$$

*and $G \in \mathrm{BMO}(\mathbb{R}^n)^{n\times n}$ is a real-valued skew-symmetric matrix field.*

*Moreover, $f$ and $G$ above can be defined explicitly as*

$$f = \Delta^{-1}(\operatorname{div} \tilde{\mathbf{b}}), \quad G = \Delta^{-1}(\operatorname{Curl} \tilde{\mathbf{b}}), \tag{2.17}$$

*where the constant $C$ in (2.16) and the BMO-norm of $G$ may depend on $K$.*

(b) *If $n = 2$, then $\tilde{\mathbf{b}} = (-\partial_2 g, \partial_1 g)$, where $g \in \mathrm{BMO}(\mathbb{R}^2)$ is a real-valued function so that $\operatorname{div}(\tilde{\mathbf{b}}) = 0$.*

(ii) *Conversely, suppose that (2.12) holds for some $\epsilon \in (0,1]$. Then $-\mathcal{L}$ is accretive if representation (2.15) holds when $n \geq 3$, or $f = 0$ and*

$\tilde{\mathbf{b}} = (-\partial_2 g, \partial_1 g)$ *when* $n = 2$*, so that both the constant* $C$ *in (2.16) and the* BMO*-norm of* $G$ *(or* $g$ *when* $n = 2$*) are small enough, depending only on* $\epsilon$*.*

*Remark* 2.8. Notice that the condition imposed on the divergence-free component $\operatorname{Div} G$ in the Hodge decomposition (2.15) is much weaker that the condition on the irrotational component $\nabla f$.

In particular, (2.16) means that $|\nabla f|^2 dx \in \mathfrak{M}^{1,2}(\mathbb{R}^n)$ is an *admissible measure*. Several equivalent characterizations of the class $\mathfrak{M}^{1,2}(\mathbb{R}^n)$ are discussed in Sec. 6 below.

*Remark* 2.9. Under the assumptions of Theorem 2.7, $\tilde{\mathbf{b}} \in \mathrm{BMO}^{-1}(\mathbb{R}^n)^n$ (see Sec. 6). A thorough discussion of the space $\mathrm{BMO}^{-1}(\mathbb{R}^n)$ and its applications is given in [16].

*Remark* 2.10. In (2.17), the Newtonian potential $\Delta^{-1}$ is understood in terms of the weak-$*$ BMO convergence (see [23], [32]), and Div and Curl are the usual matrix operators defined in Sec. 6.

In the case $n = 3$, we can use the usual vector-valued $\mathrm{curl}(\mathbf{g}) \in D'(\mathbb{R}^3)^3$ in place of $\operatorname{Div} G$ in decomposition (2.15), with $\mathbf{g} = \Delta^{-1}(\mathrm{curl}\, \tilde{\mathbf{b}})$ in (2.17).

2.7. **Example.** We conclude Sec. 2 with an example in two dimensions that demonstrates possible interaction between the principal part and lower order terms in the accretivity problem for operators with complex-valued coefficients.

Consider the operator $\mathcal{L} = \operatorname{div}(A\nabla\cdot) + \mathbf{b}\cdot\nabla + c$ in $\mathbb{R}^2$ with $A = (a_{jk})$, where $a_{11} = a_{22} = 1$, $a_{12} = -a_{21} = i\,\lambda\,\log|x|$, $\mathbf{b} = -x|x|^2$ and $c = -2|x|^2$, where $x \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

If $|\lambda| \le C$, where $C$ is an absolute constant, then by statement (ii) of Theorem 2.7, the operator $-\mathcal{L}$ is accretive due to the interaction between the principal part, the drift term, and the zero-order term (harmonic oscillator).

In this example $P = I$, $\sigma = 0$, $\tilde{\mathbf{b}} = (-\partial_2 g, \partial_1 g)$, where $g = \frac{\lambda}{2}\log|x| \in \mathrm{BMO}(\mathbb{R}^2)$. The upper bound (2.12) obviously holds for any $\epsilon \in (0, 1]$, but the lower bound (2.12) fails.

We note in passing that, by Proposition 2.1, the optimal value of the constant $|\lambda|$ in this example is found from the inequality

$$\left|\int_{\mathbb{R}^2} g(x)\, J[u, v]\, dx\right| \le ||\nabla u||_{L^2(\mathbb{R}^2)}||\nabla u||_{L^2(\mathbb{R}^2)}, \tag{2.18}$$

for all real-valued $u, v \in C_0^\infty(\mathbb{R}^2)$, where $J[u, v] = \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_2} - \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_1}$ is the determinant of the Jacobian matrix $\mathbf{D}(u, v)$ (see [4], [23], and Sec. 6 below).

## 3. Proofs of Proposition 2.1 and Theorem 2.2

*Proof of Proposition 2.1.* Clearly, the principal part of $\operatorname{Re}\langle\mathcal{L}u, u\rangle$ depends only on $\operatorname{Re} A^s$ and $\operatorname{Im} A^c$, since for $u = f + ig \in D'(\Omega)$ ($f, g$ are real-valued), we have

$$
\begin{aligned}
-\mathrm{Re}\,\langle \mathcal{A}\, u, u\rangle &= \langle \mathrm{Re} A^s\, \nabla f,\, \nabla f\rangle + \langle \mathrm{Re} A^s\, \nabla g,\, \nabla g\rangle - 2\,\langle \mathrm{Im} A^c\, \nabla f,\, \nabla g\rangle \\
&= \langle \mathrm{Re} A^s\, \nabla f,\, \nabla f\rangle + \langle \mathrm{Re} A^s\, \nabla g,\, \nabla g\rangle + 2\,\langle \mathrm{div}\,(\mathrm{Im} A^c\, \nabla f),\, g\rangle \\
&= \langle \mathrm{Re} A^s\, \nabla f,\, \nabla f\rangle + \langle \mathrm{Re} A^s\, \nabla g,\, \nabla g\rangle \\
&\quad + \langle \mathrm{div}\,(\mathrm{Im} A^c\, \nabla f),\, g\rangle - \langle \mathrm{div}\,(\mathrm{Im} A^c\, \nabla g),\, f\rangle.
\end{aligned}
$$

Since $A^c$ is skew-symmetric, it follows that

$$
\mathrm{div}\,(A^c \nabla u) = -\mathrm{Div}(A^c)\cdot \nabla u \quad \text{in } D'(\Omega),
$$

where the vector field $\mathrm{Div}(A^c)$ is solenoidal (divergence free). In particular,

$$
\mathrm{div}\,(\mathrm{Im} A^c\, \nabla f) = -\mathrm{Div}(\mathrm{Im}\, A^c)\nabla f,
$$

for real-valued $f$.

Letting $\mathbf{b_1} = \mathbf{b} -\, \mathrm{Div}(A^c)$, we see that the skew symmetric-part $A^c$ can always be included in the first-order term $\mathbf{b}_1 \cdot \nabla$, and hence does not affect the principal part of $\mathcal{L}$. Consequently, we have

$$
\begin{aligned}
\langle -\mathcal{L}u, u\rangle &= \langle P\nabla f, \nabla f\rangle + \langle P\nabla g, \nabla g\rangle - \langle \mathbf{b_1}, f\nabla f + g\nabla g\rangle \\
&\quad - i\langle \mathbf{b_1}, f\nabla g - g\nabla f\rangle - \langle c, f^2 + g^2\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Re}\,\langle -\mathcal{L}u, u\rangle &= \langle P\nabla f, \nabla f\rangle + \langle P\nabla g, \nabla g\rangle - \langle \mathrm{Re}\,\mathbf{b_1}, f\nabla f + g\nabla g\rangle \\
&\quad + \langle \mathrm{Im}\,\mathbf{b_1}, f\nabla g - g\nabla f\rangle - \langle \mathrm{Re}\, c, f^2 + g^2\rangle.
\end{aligned}
$$

Integrating by parts, and using the fact that $\mathrm{div}\,(\mathrm{Re}\,\mathbf{b_1}) = \mathrm{div}\,(\mathrm{Re}\,\mathbf{b})$, we deduce

$$
\langle \mathrm{Re}\,\mathbf{b_1}, f\nabla f + g\nabla g\rangle = -\frac{1}{2}\langle \mathrm{div}\,(\mathrm{Re}\,\mathbf{b}), f^2 + g^2\rangle.
$$

It follows that

$$
\begin{aligned}
\mathrm{Re}\,\langle -\mathcal{L}u, u\rangle &= \langle P\nabla f, \nabla f\rangle + \langle P\nabla g, \nabla g\rangle \\
&\quad + \langle \mathrm{Im}\,\mathbf{b_1}, f\nabla g - g\nabla f\rangle - \langle \sigma, f^2 + g^2\rangle,
\end{aligned}
$$

where $\sigma = \mathrm{Re}\, c - \frac{1}{2}\mathrm{div}\,(\mathrm{Re}\,\mathbf{b})$.

This proves that $\mathrm{Re}\,\langle \mathcal{L}u, u\rangle = \mathrm{Re}\,\langle \mathcal{L}_2 u, u\rangle$, where $\mathcal{L}_2$ is defined by (2.3). Thus, (2.2) holds.

Interchanging the roles of $f$ and $g$ we deduce that $\mathrm{Re}\,\langle \mathcal{L}u, u\rangle \geq 0$ if and only if

$$
\langle P\nabla f, \nabla f\rangle + \langle P\nabla g, \nabla g\rangle - \langle \sigma, f^2 + g^2\rangle \geq \left|\langle \mathrm{Im}\,\mathbf{b_1}, f\nabla g - g\nabla f\rangle\right|.
$$

Using the quadratic form $[f]^2_{\mathcal{H}}$ defined by (2.4), we rearrange the preceding inequality as follows,

$$
|\langle \mathrm{Im}\,\mathbf{b_1}, f\nabla g - g\nabla f\rangle| \leq [f]^2_{\mathcal{H}} + [g]^2_{\mathcal{H}}. \tag{3.1}
$$

for all are real-valued $f, g \in C_0^\infty(\Omega)$. Clearly, the right-hand side of this inequality equals $[f]^2_{\mathcal{H}} + [g]^2_{\mathcal{H}} = [u]^2_{\mathcal{H}}$, where $[u]^2_{\mathcal{H}} = \langle -\mathcal{H}u, u\rangle \geq 0$ for every complex-valued $u \in C_0^\infty(\Omega)$. In particular, $-\mathcal{H}$ is nonnegative definite.

Replacing $f, g$ in (3.1) with $\alpha f, \frac{1}{\alpha} g$ respectively, and minimizing over all real $\alpha \neq 0$, we deduce that $\mathrm{Re}\,\langle \mathcal{L}u, u\rangle \geq 0$ if and only if

$$|\langle \mathrm{Im}\,\mathbf{b_1}, f\nabla g - g\nabla f\rangle| \leq 2\,[f]_{\mathcal{H}}[g]_{\mathcal{H}}, \tag{3.2}$$

where $f, g \in C_0^\infty(\Omega)$ are real-valued, provided $[u]^2_{\mathcal{H}} \geq 0$ for every complex-valued (or equivalently real-valued) $u \in C_0^\infty(\Omega)$. Clearly, $\frac{1}{2}\mathrm{Im}\,\mathbf{b_1} = \tilde{\mathbf{b}}$, where $\tilde{\mathbf{b}}$ is defined by (2.1), so that (3.2) coincides with (2.5).

It remains to show that if $-\mathcal{L}$ is an accretive operator, then $P = \mathrm{Re}\,A^s$ is a non-negative definite matrix. Let $u = e^{it\,x\cdot\xi}\,v$, where $v \in C_0^\infty(\Omega)$ is real-valued, $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Then clearly,

$$\begin{aligned}\langle -\mathcal{L}u, u\rangle =& t^2\langle (A\xi\cdot\xi)\,v, v\rangle + \langle A\nabla v, \nabla v\rangle + it\langle (A\xi)v, \nabla v\rangle\\ &- it\langle A\nabla v, v\xi\rangle - it\langle (\mathbf{b}\cdot\xi)v, v\rangle - \langle \mathbf{b}\cdot\nabla v, v\rangle - \langle c\,v, v\rangle.\end{aligned}$$

It follows,

$$\begin{aligned}\mathrm{Re}\,\langle -\mathcal{L}u, u\rangle =& t^2\langle (P\xi\cdot\xi)\,v, v\rangle + \langle P\nabla v, \nabla v\rangle - t\langle (\mathrm{Im}\,A)\xi v, \nabla v\rangle\\ &+ t\langle (\mathrm{Im}\,A)\nabla v, v\xi\rangle + t\langle (\mathrm{Im}\mathbf{b}\cdot\xi)v, v\rangle - \langle \sigma\,v, v\rangle \geq 0.\end{aligned}$$

Dividing both sides by $t^2$ and letting $t \to \infty$, we immediately get that $\langle (P\xi\cdot\xi)\,v, v\rangle \geq 0$ for every real-valued $v \in C_0^\infty(\Omega)$. Then, for any $h \in C_0^\infty(\Omega)$, $h \geq 0$, denote by $\eta \in C_0^\infty(\Omega)$ a cut-off function such that $\eta\,h = h$. Setting $v = \eta(h+\delta)^{\frac{1}{2}} \in C_0^\infty(\Omega)$, for $\delta > 0$, we see that $\langle P\xi\cdot\xi, h + \delta\,\eta^2\rangle \geq 0$. Letting $\delta \to 0$ yields $P\xi\cdot\xi \geq 0$ in $D'(\Omega)$. This completes the proof of Proposition 2.1. □

*Proof of Theorem 2.2.* We recall some estimates for non-negative definite, symmetric matrices $P = (p_{jk})$, starting with the Schwarz inequality

$$|P\xi\cdot\eta| \leq \Big(P\xi\cdot\xi\Big)^{\frac{1}{2}}\Big(P\eta\cdot\eta\Big)^{\frac{1}{2}}, \quad \text{for all } \xi, \eta \in \mathbb{R}^n. \tag{3.3}$$

From (3.3) with $\eta = P\xi$, we deduce the estimate

$$|P\xi|^2 \leq ||P||\,(P\xi\cdot\xi), \quad \text{for all } \xi \in \mathbb{R}^n,$$

where $||P||$ is the operator norm of $P$. Since $||P|| \leq \sum_{j,k=1}^n |p_{jk}|$, using the preceding inequality with $\xi = \mathbf{g}$, we deduce, for any $h \in C_0^\infty(\Omega)$,

$$\int_\Omega |P\mathbf{g}|\,h^2\,dx \leq \left(\int_\Omega \Big(P\mathbf{g}\cdot\mathbf{g}\Big)\,h^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega (\sum_{j,k=1}^n |p_{jk}(x)|)\,h^2\,dx\right)^{\frac{1}{2}}.$$

We now prove statement (i) of Theorem 2.2. From the preceding estimate it follows that $P \in L^1_{\mathrm{loc}}(\Omega)^{n\times n}$ and $(P\mathbf{g})\cdot\mathbf{g} \in L^1_{\mathrm{loc}}(\Omega)^n$ yield $P\mathbf{g} \in L^1_{\mathrm{loc}}(\Omega)^n$.

Applying (3.3) with $\xi = \mathbf{g}(\cdot)$ and $\eta = \nabla h(\cdot)$, we obtain

$$\begin{aligned}\left|\int_\Omega [(P\mathbf{g})\cdot\nabla h]\,h\,dx\right| &\leq \int_\Omega [(P\mathbf{g})\cdot\mathbf{g}]^{\frac{1}{2}}\,[(P\nabla h)\cdot\nabla h]^{\frac{1}{2}}\,h\,dx\\ &\leq \left(\int_\Omega [(P\mathbf{g})\cdot\mathbf{g}]\,h^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega [(P\nabla h)\cdot\nabla h]\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Using (2.6), along with the preceding inequality, and integrating by parts, we estimate,

$$\begin{aligned}\langle \sigma, h^2\rangle &\le \langle \operatorname{div}(P\mathbf{g}), h^2\rangle - \int_\Omega [(P\mathbf{g})\cdot \mathbf{g}]\, h^2 dx\\ &= -2\int_\Omega [(P\mathbf{g})\cdot \nabla h]\, h\, dx - \int_\Omega [(P\mathbf{g})\cdot \mathbf{g}]\, h^2 dx\\ &\le 2\left(\int_\Omega [(P\mathbf{g})\cdot \mathbf{g}]\, h^2 dx\right)^{\frac12}\left(\int_\Omega [(P\nabla h)\cdot \nabla h]\, dx\right)^{\frac12} - \int_\Omega [(P\mathbf{g})\cdot \mathbf{g}]\, h^2 dx\\ &\le \int_\Omega [(P\nabla h)\cdot \nabla h]\, dx.\end{aligned}$$

In other words, (2.4) holds. Since $\tilde{\mathbf{b}} = 0$, and hence the commutator condition (2.5) is vacuous, $-\mathcal{L}$ is nonnegative definite by Proposition 2.1. This proves statement (i) of Theorem 2.2.

Statement (ii) of Theorem 2.2 is immediate from Proposition 2.1, and the corresponding result for the Schrödinger operator $\mathcal{H}$ in Sec. 2.3, which yields the existence of $\mathbf{g} \in L^2_{\rm loc}(\Omega)$ such that (2.4) holds. The proof of Theorem 2.3 is complete. □

## 4. Proof of Theorem 2.3

Let $\mathcal{L}u = (a\,u')' + bu' + c$ be a second order linear differential operator with complex-valued distributional coefficients $a$, $b$, $c$ on an open interval $I \subseteq \mathbb{R}$ (possibly unbounded). As in (2.1), we define the associated real-valued distributions $p$, $\tilde b$, and $\sigma$ by

$$p = \operatorname{Re} a, \quad \tilde b = \frac12 \operatorname{Im} b, \quad \sigma = \operatorname{Re} c - \frac12 \operatorname{Re} b'.$$

Proposition 2.1 gives a criterion of accretivity for $-\mathcal{L}$ in terms of the quadratic form inequality for $-\mathcal{H}$,

$$[h]^2_{\mathcal{H}} = \langle ph', h'\rangle - \langle \sigma\, h, h\rangle \ge 0, \quad \text{for all } h \in C_0^\infty(I),$$

where $\mathcal{H}h = (ph')' + \sigma\, h$ is the Sturm-Liouville operator on $I$, together with the commutator inequality

$$\left|\langle \tilde b, uv' - vu'\rangle\right| \le\, [u]^2_{\mathcal{H}}\, [v]^2_{\mathcal{H}},$$

for all real-valued $u, v \in C_0^\infty(I)$.

In the case where $p$, $\tilde b \in L^1_{\rm loc}(I)$, it is possible to avoid the commutator inequality by including $\tilde b$ in the stronger quadratic form inequality (2.9), i.e.,

$$[h]^2_{\mathcal{N}} = \int_I p\,(h')^2\, dt - \langle \sigma, h^2\rangle - \int_I \frac{\tilde b^2}{p}\, h^2\, dt \ge 0, \tag{4.1}$$

for all real-valued $h \in C_0^\infty(I)$. Here $\mathcal{N}h = (ph')' + q\,h$, with

$$q = \operatorname{Re} c - \frac12 (\operatorname{Re} b)' - \frac{\tilde b^2}{p}.$$

This means that $-\mathcal{L}$ is accretive if and only if the Sturm-Liouville operator $-\mathcal{N}$ is nonnegative definite. In this case, the condition $\frac{\tilde{b}^2}{p} \in L^1_{\rm loc}(I)$ is necessary for accretivity. We recall that throughout the paper, we are using the convention $\frac{0}{0} = 0$.

The reduction of the accretivity of $-\mathcal{L}$ to the property that the Schrödinger operator $-\mathcal{N}$ is nonnegative definite is performed via an exponential substitution $u = ze^{-i\lambda}$, where $\lambda \in D'(\Omega)$ is a real-valued vector field. It actually works in the general case $n \geq 1$, provided $\mathbf{b}$ has a specific form, so that $\tilde{\mathbf{b}} = P\nabla\lambda$, under certain restrictions on $P$ and $\lambda$ (see Sec. 5 below). If $n = 1$, we will show that this is always possible with $\lambda' = \frac{\tilde{b}}{p}$ in $I$.

We first prove statement (i) of Theorem 2.3. Our first step is to show that, in the general case $\Omega \subseteq \mathbb{R}^n$, for $n \geq 1$, we can replace the operator $\mathcal{L}$ in the corresponding quadratic form inequalities with the operators $\mathcal{L}_\epsilon$ with mollified coefficients $A_\epsilon$, $\mathbf{b}_\epsilon$, and $c_\epsilon$ ($\epsilon > 0$).

More precisely, if $u \in C_0^\infty(\Omega)$, then we can pick a smooth real-valued cut-off function $\eta \in C_0^\infty(\Omega)$ so that $\eta\, u = u$. Then, applying the inequality $\mathrm{Re}\,\langle -\mathcal{L}u, u\rangle \geq 0$ with $\eta\, u = u$, we can clearly replace $A$, $\mathbf{b}$, and $c$ by $\eta^2 A$, $\eta^2\mathbf{b}$, and $\eta^2 c$, respectively, so that we may assume that the coefficients of $\mathcal{L}$ are compactly supported.

Notice that we can also replace $u(x)$ with a shifted test function $u_y(x) = u(x+y)$, where $|y| < \epsilon$, for a small enough $\epsilon$ depending on the support of $u$ and $\eta$. Let $\phi_\epsilon(y) = \epsilon^{-n}\phi(\frac{t}{\epsilon})$ be a mollifier, so that $\phi \geq 0$, $\phi \in C^\infty(B(0,1))$ and $\int_{B(0,1)} \phi(y)\, dy = 1$. Then $\mathrm{supp}\,(\phi_\epsilon) \subset B(0,\epsilon)$. Integrating both sides of the inequality $\mathrm{Re}\,\langle -\mathcal{L}u_y, u_y\rangle \geq 0$ against $\phi_\epsilon(t)\, dt$, we obtain

$$\mathrm{Re}\,\langle -\mathcal{L}_\epsilon u, u\rangle \geq 0,$$

where the coefficients $A_\epsilon$, $\mathbf{b}_\epsilon$, and $c_\epsilon$ are the mollifications of the distributions $\eta^2 A$, $\eta^2\mathbf{b}$, and $\eta^2 c$, respectively. Conversely, if this inequality holds for small enough $\epsilon > 0$, then passing to the limit as $\epsilon \to 0$, we recover the inequality $\mathrm{Re}\,\langle -\mathcal{L}u, u\rangle \geq 0$. In other words, we can assume without loss of generality that the coefficients of $\mathcal{L}$ in the inequality $\mathrm{Re}\,\langle -\mathcal{L}u, u\rangle \geq 0$ are $C_0^\infty(\Omega)$ functions.

Returning to the one-dimensional case, we fix $u \in C_0^\infty(I)$, and define $p_\epsilon$, $\tilde{b}_\epsilon$, $\sigma_\epsilon$ as the mollifications of $p$, $\tilde{b}$, and $\sigma$, respectively. Obviously, we can always replace $p$ in the inequality $\mathrm{Re}\,\langle -\mathcal{L}u, u\rangle \geq 0$ by $p + \delta$, for some $\delta > 0$, and eventually set $\delta \downarrow 0$.

Let $\eta \in C_0^\infty(I)$ be a real-valued function such that $\eta\, u = u$. We set

$$\lambda(t) = \eta(t) \int_{t_0}^t \frac{\tilde{b}_\epsilon(\tau)}{p_\epsilon(\tau) + \delta}, \quad t \in I, \tag{4.2}$$

where $t_0 \in I$, and $\delta > 0$. Then clearly $\lambda \in C_0^\infty(I)$, and

$$\lambda'(t) = \eta(t)\, \frac{\tilde{b}_\epsilon(t)}{p_\epsilon(t) + \delta} + \eta'(t)\, \lambda(t), \quad t \in I.$$

Notice that, on the support of $u$, we have

$$\tilde{b}_\epsilon(t)-\lambda'(t)\,(p_\epsilon(t)+\delta)(t)=0, \quad 2\,\tilde{b}_\epsilon(t)\,\lambda'(t)-(p_\epsilon(t)+\delta)\,[\lambda'(t)]^2=\frac{\tilde{b}_\epsilon(t)^2}{p_\epsilon(t)+\delta}.$$

Using the exponential substitution $u = ze^{-i\lambda}$ discussed in Sec. 5 below, we deduce from (5.7) and (5.8) that $\mathrm{Re}\,\langle -\mathcal{L}_\epsilon u, u\rangle \geq 0$ holds if and only if (4.1) holds with $p_\epsilon + \delta$, $\tilde{b}_\epsilon$ and $\sigma_\epsilon$ in place of $p$, $\tilde{b}$ and $\sigma$, for all small enough $\epsilon$ and $\delta > 0$, that is

$$\int_I (p_\epsilon(t)+\delta)\,h'(t)^2\,dt - \langle\sigma_\epsilon, h^2\rangle - \int_I \frac{\tilde{b}_\epsilon(t)^2}{p_\epsilon(t)+\delta}\,h(t)^2\,dt \geq 0, \tag{4.3}$$

where $h \in C_0^\infty(I)$ has the same support as $u$, since $h$ is a linear combination of $f$ and $g$, the real and imaginary parts of $u$ (see Sec. 3).

Clearly, $p_{\epsilon_k} \to p$ in $L^1_{\mathrm{loc}}(I)$, and $\sigma_{\epsilon_k} \to \sigma$ in $D'(I)$ as $\epsilon_k \to 0$. Since $\tilde{b} \in L^1_{\mathrm{loc}}(I)$ and $p \in L^1_{\mathrm{loc}}(I)$, there exists a subsequence $k \to \infty$ so that $\tilde{b}_{\epsilon_k} \to \tilde{b}$, and $p_{\epsilon_k} \to p$ a.e. Passing to the limit as $k \to \infty$, and using Fatou's lemma, we deduce the inequality

$$\int_I (p(t)+\delta)\,h'(t)^2\,dt - \langle\sigma, h^2\rangle - \int_I \frac{\tilde{b}(t)^2}{p(t)+\delta}\,h(t)^2\,dt \geq 0. \tag{4.4}$$

Letting $\delta \downarrow 0$ and using the dominated convergence theorem and the monotone convergence theorem, we see that $\frac{\tilde{b}^2}{p} \in L^1_{\mathrm{loc}}(I)$, and (4.1) holds, provided $-\mathcal{L}$ is accretive. This proves the necessity of condition (4.1) for the accretivity of the operator $-\mathcal{L}$.

To prove the sufficiency of condition (4.1), assuming $p \in L^1_{\mathrm{loc}}(I)$, notice that it obviously yields (4.4) for every $\delta > 0$. Using the same mollification process as above we deduce

$$\int_I (p_\epsilon(t)+\delta)\,h'(t)^2\,dt - \langle\sigma_\epsilon, h^2\rangle - \int_I \Big(\frac{\tilde{b}^2}{p+\delta}\Big)_\epsilon(t)\,h(t)^2\,dt \geq 0. \tag{4.5}$$

By Jensen's inequality, we have

$$(\tilde{b}_\epsilon(t))^2 \leq (p_\epsilon(t)+\delta)\Big(\frac{\tilde{b}^2}{p+\delta}\Big)_\epsilon(t), \quad t \in I.$$

Consequently, (4.5) yields (4.3). As was mentioned above, inequality (4.3), via the exponential substitution $u = ze^{-i\lambda}$ with $\lambda$ defined by (4.2), is equivalent to the inequality $\mathrm{Re}\,\langle -\mathcal{L}_\epsilon u, u\rangle \geq 0$. Letting $\epsilon \to 0$, we conclude that $-\mathcal{L}$ is accretive. This proves statement (i) of Theorem 2.3.

To prove statement (ii), suppose that (2.10) holds for some $f \in L^1_{\mathrm{loc}}(I)$ such that $\frac{f^2}{p} \in L^2_{\mathrm{loc}}(I)$. Letting $g = \frac{f}{p}$, we see that $p\,g^2 \in L^1_{\mathrm{loc}}(I)$, and condition (2.6) holds with $q$ in place of $\sigma$. Hence, by Theorem 2.2, the operator $-\mathcal{N}$ is nonnegative definite, i.e., (2.9) holds, which yields that $-\mathcal{L}$ is accretive by statement (i) of Theorem 2.3.

In the converse direction, if $-\mathcal{L}$ is accretive, then (2.9) holds by statement (i) of Theorem 2.3. In other words, the operator $-\mathcal{N}$ is nonnegative definite.

Thus, by Theorem 2.2, there exists a function $g \in L^2_{\mathrm{loc}}(I)$ such that (2.10) holds with $f = p\, g$. Here $p$ is uniformly bounded above and below by positive constants, so that $f \in L^2_{\mathrm{loc}}(I)$, and the right-hand side of (2.10) is well-defined. The proof of Theorem 2.3 is complete. □

## 5. Decomposition of the drift term

In this section, we deduce a version of Proposition 2.1 for vector fields

$$\tilde{\mathbf{b}} = P\nabla\lambda + \mathbf{d}, \tag{5.1}$$

where $\lambda$ and $\mathbf{d}$ are real-valued. In particular, it yields more explicit criteria of accretivity in the special cases where $\mathbf{d} = 0$, i.e., $\tilde{\mathbf{b}} = P\nabla\lambda$, or $P = I$ and $\operatorname{div} \mathbf{d} = 0$, so that (5.1) is the Hodge decomposition.

This is a consequence of the following theorem, which in a sense represents a higher dimensional analogue of Theorem 2.3 in the case $n = 1$, with $\lambda' = \frac{\operatorname{Im} b}{2p}$.

**Theorem 5.1.** *Let $P$, $\tilde{\mathbf{b}}$, and $\sigma$ be defined by (2.1), where $P \in L^\infty_{\mathrm{loc}}(\Omega)^{n\times n}$ is a nonnegative definite matrix, and $\tilde{\mathbf{b}} \in L^2_{\mathrm{loc}}(\Omega)^n$. Suppose $\nabla\lambda \in L^2_{\mathrm{loc}}(\Omega)^n$, where $\lambda \in D'(\Omega)$ is real-valued. Then $-\mathcal{L}$ is an accretive operator if and only if the following two conditions hold:*

$$\begin{aligned} [h]^2_{\mathcal{N}} = & \int_\Omega (P\nabla h \cdot \nabla h)\, dx - \langle \sigma\, h, h\rangle \\ & - \int_\Omega (2\tilde{\mathbf{b}} - P\nabla\lambda)\cdot\nabla\lambda\, |h|^2 dx \ge 0, \end{aligned} \tag{5.2}$$

*for all $h \in C_0^\infty(\Omega)$, and*

$$\left| \langle \tilde{\mathbf{b}} - P\nabla\lambda, u\nabla v - v\nabla u\rangle \right| \le [u]_{\mathcal{N}}\, [v]_{\mathcal{N}}, \tag{5.3}$$

*for all real-valued $u, v \in C_0^\infty(\Omega)$.*

*Remark* 5.2. In the special case where $P$ is invertible (for instance, uniformly elliptic), and $P^{-1}\tilde{\mathbf{b}} = \nabla\lambda$ is a gradient field, the sole condition (5.2), namely,

$$[h]^2_{\mathcal{N}} = \int_\Omega (P\nabla h \cdot \nabla h)\, dx - \langle \sigma\, h, h\rangle - \int_\Omega (P^{-1}\tilde{\mathbf{b}}\cdot\tilde{\mathbf{b}})\, |h|^2 dx \ge 0, \tag{5.4}$$

for all $h \in C_0^\infty(\Omega)$, characterizes accretive operators $-\mathcal{L}$.

This is an analogue of condition (2.9) in the one-dimensional case.

*Remark* 5.3. If $P = I$, then in decomposition (5.1) we can pick the irrotational component of $\tilde{\mathbf{b}}$ as $\nabla\lambda$. In this case, Theorem 5.1 is clearly equivalent to the inequality

$$[h]^2_{\mathcal{N}} = ||\nabla h||^2_{L^2(\Omega)} - \langle \sigma\, h, h\rangle - \int_\Omega \left( |\tilde{\mathbf{b}}|^2 - |\mathbf{d}|^2 \right) |h|^2 dx \ge 0, \tag{5.5}$$

for all $h \in C_0^\infty(\Omega)$, where $\mathbf{d} = \tilde{\mathbf{b}} - \nabla\lambda$ is the divergence-free component of $\tilde{\mathbf{b}}$, combined with the commutator inequality

$$|\langle \mathbf{d}, u\nabla v - v\nabla u\rangle| \le [u]_{\mathcal{N}}\,[v]_{\mathcal{N}}, \tag{5.6}$$

for all real-valued $u, v \in C_0^\infty(\Omega)$.

Notice that here the condition $\tilde{\mathbf{b}} \in L^2_{\rm loc}(\Omega)$ is necessary for $-\mathcal{L}$ to be accretive, provided $\mathbf{d} \in L^2_{\rm loc}(\Omega)$, as in the one-dimensional case where we can set $\mathbf{d} = 0$.

*Proof.* It follows from (2.2), (2.3) that, without loss of generality, we may assume

$$\mathcal{L}u = \operatorname{div}(P\nabla u) + 2i\,\tilde{\mathbf{b}}\cdot\nabla + \sigma,$$

where $P$, $\tilde{\mathbf{b}}$, and $\sigma$ are given by (2.1).

Let us assume for simplicity that $P \in L^\infty_{\rm loc}(\Omega)^{n\times n}$, where $P$ is a nonnegative definite, symmetric, real-valued $n \times n$ matrix, and $\nabla\lambda \in L^2_{\rm loc}(\Omega)$, where $\lambda \in D'(\Omega)$ is real-valued.

We use the substitution $u = z\,e^{-i\lambda}$, where $u \in C_0^\infty(\Omega)$ is complex-valued, to replace $\tilde{\mathbf{b}}$ with $\tilde{\mathbf{b}} - \nabla\lambda$.

Suppose first that $\lambda \in C_0^\infty(\Omega)$. Since $z = u\,e^{i\lambda} \in C_0^\infty(\Omega)$, we have

$$P\nabla u = (P\nabla z - iz\,P\nabla\lambda)\,e^{-i\lambda}, \quad \nabla\bar{u} = (\nabla\bar{z} + i\bar{z}\,\nabla\lambda)\,e^{i\lambda}.$$

Hence,

$$P\nabla u\cdot\nabla\bar{u} = P\nabla z\cdot\nabla\bar{z} + (P\nabla\lambda\cdot\nabla\lambda)\,|z|^2 - i\,(P\nabla\lambda)\cdot(z\,\nabla\bar{z} - \bar{z}\nabla z).$$

We deduce

$$\int_\Omega (P\nabla u\cdot\nabla\bar{u})\,dx = \int_\Omega (P\nabla z\cdot\nabla\bar{z})\,dx + \int_\Omega (P\nabla\lambda\cdot\nabla\lambda)\,|z|^2\,dx + 2\int_\Omega P\nabla\lambda\cdot\operatorname{Im}(z\nabla\bar{z})\,dx.$$

Since $\tilde{\mathbf{b}} \in L^1_{\rm loc}(\Omega)$, we have

$$\operatorname{Re}\langle -\mathcal{L}u, u\rangle = \int_\Omega (P\nabla z\cdot\nabla\bar{z})\,dx - \int_\Omega \Big[2(\tilde{\mathbf{b}}\cdot\nabla\lambda) - (P\nabla\lambda\cdot\nabla\lambda)\Big]\,|z|^2\,dx - \langle\sigma,\,|z|^2\rangle - 2\langle\tilde{\mathbf{b}} - P\nabla\lambda, \operatorname{Im}(z\,\nabla\bar{z})\rangle.$$

It follows that

$$\operatorname{Re}\langle -\mathcal{L}u, u\rangle \ge 0 \Longleftrightarrow \operatorname{Re}\langle -\mathcal{M}z, z\rangle \ge 0,$$

where

$$\mathcal{M}z = \operatorname{div}(P\nabla z) + 2i\,(\tilde{\mathbf{b}} - P\nabla\lambda)\cdot\nabla z + \Big(\sigma + 2(\tilde{\mathbf{b}}\cdot\nabla\lambda) - (P\nabla\lambda\cdot\nabla\lambda)\Big).$$

Thus, $\operatorname{Re}\langle -\mathcal{L}u, u\rangle \ge 0$ if and only if

$$[h]^2_{\mathcal{M}} = \int_\Omega (P\nabla h\cdot\nabla h)\,dx - \langle\sigma\,h, h\rangle - \int_\Omega \Big[2(\tilde{\mathbf{b}}\cdot\nabla\lambda) - (P\nabla\lambda\cdot\nabla\lambda)\Big]\,|h|^2\,dx \ge 0, \tag{5.7}$$

for all real-valued $h \in C_0^\infty(\Omega)$, and

$$\Big|\langle \tilde{\mathbf{b}} - P\nabla\lambda, u\nabla v - v\nabla u\rangle\Big| \le [u]_{\mathcal{M}}\,[v]_{\mathcal{M}}, \tag{5.8}$$

for all real-valued $u, v \in C_0^\infty(\Omega)$.

In the case $\nabla\lambda \in L^2_{\rm loc}(\Omega)$, we notice that, without loss of generality we may assume that $\lambda$ is compactly supported in $\Omega$. Otherwise, we consider $\lambda\eta$, where $\eta \in C_0^\infty(\Omega)$ is a cut-off function such that $\eta\, u = u$, and apply the subsequent estimates to $\lambda\eta$. We next replace $\lambda$ with its mollification $\lambda_\epsilon = \lambda \star \phi_\epsilon$, for $\epsilon > 0$, where as usual $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$, for some $\phi \in C_0^\infty(\Omega)$.

Using the same substitution as above, for $z \in C_0^\infty(\Omega)$, we set

$$u_\epsilon = z\, e^{-i\lambda_\epsilon} \in C_0^\infty(\Omega), \quad \nabla u_\epsilon = (\nabla z - i\nabla\lambda_\epsilon)\, e^{-i\lambda_\epsilon}.$$

Notice that, as above,

$$\begin{aligned}\mathrm{Re}\,\langle \mathcal{L}u_\epsilon, u_\epsilon\rangle = \int_\Omega (P\nabla z\cdot\nabla z)\,dx - \int_\Omega \Big[2(\tilde{\mathbf{b}}\cdot\nabla\lambda_\epsilon) - (P\nabla\lambda_\epsilon\cdot\nabla\lambda_\epsilon)\Big]\,|z|^2\,dx \\ - \langle\sigma, |z|^2\rangle - 2\langle\tilde{\mathbf{b}} - P\nabla\lambda_\epsilon, \mathrm{Im}(z\,\nabla\bar{z})\rangle \ge 0,\end{aligned}$$

which yields the following two conditions:

$$\begin{aligned}[z]^2_{\mathcal{M}} = \int_\Omega (P\nabla z\cdot\nabla z)\,dx - \int_\Omega \Big[2(\tilde{\mathbf{b}}\cdot\nabla\lambda_\epsilon) - (P\nabla\lambda_\epsilon\cdot\nabla\lambda_\epsilon)\Big]\,|z|^2\,dx \\ - \langle\sigma, |z|^2\rangle \ge 0, \quad \Big|\langle\tilde{\mathbf{b}} - P\nabla\lambda_\epsilon, u\nabla v - v\nabla u\Big| \le [u]_{\mathcal{M}}[v]_{\mathcal{M}},\end{aligned}$$

for all $z \in C_0^\infty(\Omega)$ (real- or complex-valued) and real-valued $u, v \in C_0^\infty(\Omega)$.

We have $||(\nabla\lambda_\epsilon - \nabla\lambda)\eta||_{L^2(\Omega)} \to 0$ as $\epsilon \to 0$. Consequently, it follows $||(\tilde{\mathbf{b}}\cdot\nabla\lambda_\epsilon - \tilde{\mathbf{b}}\cdot\nabla\lambda)\eta||_{L^2(\Omega)} \to 0$ for $\tilde{\mathbf{b}} \in L^2_{\rm loc}(\Omega)$. Since $P \in L^\infty_{\rm loc}(\Omega)^{n\times n}$, we have $||(P\nabla\lambda_\epsilon - P\nabla\lambda)\eta||_{L^2(\Omega)} \to 0$ as well. Passing to the limit as $\epsilon \to 0$ completes the proof of Theorem 5.1. □

## 6. BMO estimates, trace inequalities, and admissible measures

In this section, we discuss BMO estimates, trace inequalities and admissible measures used in Theorem 2, which gives necessary and sufficient conditions on $A$, $\mathbf{b}$ and $c$ for the accretivity of $-\mathcal{L}$ on $\mathbb{R}^n$, under some additional assumptions on the upper and lower bounds of the quadratic forms $[\cdot]_{\mathcal{H}}$ imposed in Sec. 2.5.

By $L^{1,2}(\Omega)$ we denote the energy space (homogeneous Sobolev space) defined as the completion of the complex-valued $C_0^\infty(\Omega)$ functions in the Dirichlet norm $||\nabla\cdot||_{L^2(\Omega)}$.

For $f \in L^1_{\rm loc}(\mathbb{R}^n)$, we set $m_B(f) = \frac{1}{|B|}\int_B f(x)\,dx$, where $B$ is a ball in $\mathbb{R}^n$, and denote by $\mathrm{BMO}(\mathbb{R}^n)$ the class of functions $f \in L^r_{\rm loc}(\mathbb{R}^n)$ for which

$$\sup_{x_0\in\mathbb{R}^n,\,\delta>0} \frac{1}{|B_\delta(x_0)|}\int_{B_\delta(x_0)} |f(x) - m_{B_\delta(x_0)}(f)|^r\,dx < +\infty,$$

for any (or, equivalently, all) $1 \le r < +\infty$.

The corresponding vector- and matrix-valued function spaces are introduced in a similar way. In particular, $\mathrm{BMO}(\mathbb{R}^n)^n$ stands for the class of vector fields $\mathbf{f} = \{f_j\}_{j=1}^n : \mathbb{R}^n \to \mathbb{C}^n$, such that $f_j \in \mathrm{BMO}(\mathbb{R}^n)$, $j = 1, 2, \ldots, n$. The matrix-valued analogue is denoted by $\mathrm{BMO}(\mathbb{R}^n)^{n\times n}$, etc. The notion of the weak-$*$ BMO-convergence is discussed based on the $H^1 - \mathrm{BMO}$ duality in [32, Ch. IV].

The matrix divergence operator $\mathrm{Div} : D'(\Omega)^{n\times n} \to D'(\Omega)^n$ is defined on matrix fields $F = (f_{ij})_{i,j=1}^n \in D'(\Omega)^{n\times n}$ by $\mathrm{Div}\, F = \left(\sum_{j=1}^n \partial_j\, f_{ij}\right)_{i=1}^n \in D'(\Omega)^n$. If $F$ is skew-symmetric, i.e., $f_{ij} = -f_{ji}$, then we obviously have $\mathrm{div}\,(\mathrm{Div}\, F) = 0$.

The Jacobian, $\mathbf{D} : D'(\Omega)^n \to D'(\Omega)^{n\times n}$, is the formal adjoint of $-\mathrm{Div}$,

$$\langle \mathrm{Div}\, F,\, \mathbf{v}\rangle = -\,\langle F,\, \mathbf{D}\, \mathbf{v}\rangle, \qquad \mathbf{v} \in C_0^\infty(\Omega)^n.$$

Here the scalar product of matrix fields $F = (f_{ij})_{i,j=1}^n$ and $G = (g_{ij})_{i,j=1}^n$ is defined by $\langle F,\, G\rangle = \sum_{i,j=1}^n \langle f_{ij},\, g_{ij}\rangle$. If $F,\, G \in L^2(\Omega)^{n\times n}$, then

$$\langle F,\, G\rangle = \int_\Omega \mathrm{trace}\,(F^t \cdot \bar{G})\, dx,$$

where $F^t = (f_{ji})_{i,j=1}^n$ is the transposed matrix, and $\bar{G} = (\bar{g}_{ij})_{i,j=1}^n$.

The matrix curl operator $\mathrm{Curl} : D'(\Omega)^n \to D'(\Omega)^{n\times n}$ is defined on vector fields $\mathbf{f} = (f_k)_{k=1}^n$ by $\mathrm{Curl}\, \mathbf{f} = (\partial_j f_k - \partial_j f_k)_{j,k=1}^n$. Clearly, $\mathrm{Curl}\, \mathbf{f}$ is always a skew-symmetric matrix field.

Notice that in the case $n = 3$ we can use the usual vector-valued curl operator which maps $D'(\Omega)^3 \to D'(\Omega)^3$. For instance, if a vector field is represented as $\mathbf{b} = \mathrm{curl}(\mathbf{g})$, then we can write commutator inequalities of the type $|\langle \mathbf{b}, u\nabla\, v - v\, \nabla u\rangle| \le C\, ||\nabla u||_{L^2(\mathbb{R}^3)}\, ||\nabla v||_{L^2(\mathbb{R}^3)}$, in the equivalent form

$$|\langle \mathbf{g}, \nabla u \times \nabla v\rangle| \le C\, ||\nabla u||_{L^2(\mathbb{R}^3)}\, ||\nabla v||_{L^2(\mathbb{R}^3)},$$

where $\mathbf{g} \in \mathrm{BMO}(\mathbb{R}^3)^3$. This is an analogue of the Jacobian determinant inequality (2.18) in two dimensions. Such inequalities are studied in compensated compactness theory [4].

Similarly, when $n = 3$, in Theorem 2.7 we can use the Hodge decomposition in $\mathbb{R}^3$,

$$\tilde{\mathbf{b}} = \nabla f + \mathrm{curl}(\mathbf{g}),$$

where $\mathbf{g} = \Delta^{-1}(\mathrm{curl}\, \tilde{\mathbf{b}}) \in \mathrm{BMO}(\mathbb{R}^3)^3$. Here the operator $\Delta^{-1}$ is understood in the sense of the weak-$*$ BMO-convergence, as explained in [23]. Notice that this decomposition does not contain any harmonic vector fields $\mathbf{h}$ such that both $\mathrm{div}\,(\mathbf{h}) = 0$ and $\mathrm{curl}(\mathbf{h}) = 0$. This is, of course, true in $\mathbb{R}^n$ for higher dimensions $n \ge 4$ as well.

The capacity of a compact set $e \subset \mathbb{R}^n$ is defined by ([20], Sec. 2.2):

$$\mathrm{cap}\,(e) = \inf \left\{ ||u||^2_{L^{1,2}(\mathbb{R}^n)} : \quad u \in C_0^\infty(\mathbb{R}^n), \quad u(x) \ge 1 \text{ on } e \right\}. \tag{6.1}$$

For a cube or ball $Q$ in $\mathbb{R}^n$,

$$\text{cap}\,(Q) \simeq |Q|^{1-\frac{2}{n}} \quad \text{if } n \geq 3; \quad \text{cap}\,(Q) = 0 \quad \text{if } n = 2. \tag{6.2}$$

The capacity $\text{Cap}\,(\cdot)$ associated with the inhomogeneous Sobolev space $W^{1,\,2}(\mathbb{R}^n)$ defined by

$$\text{Cap}\,(e) = \inf \left\{ ||u||^2_{W^{1,2}(\mathbb{R}^n)} : \quad u \in C_0^\infty(\mathbb{R}^n), \quad u(x) \geq 1 \text{ on } e \right\}, \tag{6.3}$$

for compact sets $e \subset \mathbb{R}^n$. Note that $\text{Cap}\,(e) \simeq \text{cap}\,(e)$ if $\text{diam}\,(e) \leq 1$, and $n \geq 3$. For a cube or ball $Q$ in $\mathbb{R}^n$,

$$\text{Cap}\,(Q) \simeq |Q|^{1-\frac{2}{n}} \quad \text{if } n \geq 3; \quad \text{Cap}\,(Q) \simeq \left(\log \tfrac{2}{|Q|}\right)^{-1} \quad \text{if } n = 2, \tag{6.4}$$

provided $|Q| \leq 1$. For these and other properties of capacities, as well as related notions of potential theory we refer to [1], [20].

By $\mathcal{M}^+(\Omega)$ we denote the calls of all nonnegative Radon measures (locally finite) in an open set $\Omega \subseteq \mathbb{R}^n$. We discuss in this section several equivalent characterizations of the class of *admissible measures* $\mu \in \mathfrak{M}^{1,\,2}_+(\Omega)$ which obey the so-called trace inequality (6.11) (see [1], [20], and the extensive literature cited there).

We start with the case $\Omega = \mathbb{R}^n$. A measure $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ is said to be admissible, i.e., $\mu \in \mathfrak{M}^{1,\,2}_+(\mathbb{R}^n)$, if it obeys the trace inequality:

$$\left( \int_{\mathbb{R}^n} |u|^2 \, d\mu \right)^{\frac{1}{2}} \leq C \, ||\nabla u||_{L^2(\mathbb{R}^n)}, \qquad u \in C_0^\infty(\mathbb{R}^n), \tag{6.5}$$

where $C$ is a positive constant which does not depend on $u$.

For $\mu \in \mathcal{M}^+(\mathbb{R}^n)$, we denote by $I_1\mu = (-\Delta)^{-\frac{1}{2}}\mu$ the Riesz potential of order 1,

$$I_1\mu(x) = (-\Delta)^{-\frac{1}{2}}\mu(x) = c(n) \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-1}}, \quad x \in \mathbb{R}^n.$$

Here $c(n)$ is a normalization constant which depends only on $n$.

We have the following equivalent characterizations of admissible measures in $\mathbb{R}^n$ (see [20, Ch. 11]).

**Theorem 6.1.** *Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$. Then $\mu \in \mathfrak{M}^{1,\,2}_+(\mathbb{R}^n)$ if and only if any one of the following statements holds.*

(i) *The Riesz potential $I_1\mu \in L^2_{\text{loc}}(\mathbb{R}^n)$, and $(I_1\mu)^2 \in \mathfrak{M}^{1,\,2}_+(\mathbb{R}^n)$, i.e.,*

$$\left( \int_{\mathbb{R}^n} |u|^2 \, (I_1\mu)^2 \, dx \right)^{\frac{1}{2}} \leq c_1 \, ||\nabla u||_{L^2(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{6.6}$$

*where $c_1 > 0$ does not depend on $u$.*

(ii) *For every compact set $e \subset \mathbb{R}^n$,*

$$\mu(e) \leq c_2 \, \text{cap}\,(e), \tag{6.7}$$

*where $c_2$ does not depend on $e$.*

(iii) *For every ball $B$ in $\mathbb{R}^n$,*

$$\int_B (I_1\mu_B)^2\, dx \le c_3\, \mu(B), \tag{6.8}$$

*where $d\mu_B = \chi_B\, d\mu$, and $c_3$ does not depend on $B$.*

(iv) *The pointwise inequality*

$$I_1[(I_1\mu)^2(x)] \le c_4\, I_1\mu(x) < \infty \tag{6.9}$$

*holds a.e., where $c_4$ does not depend on $x \in \mathbb{R}^n$.*

(v) *For every dyadic cube $P$ in $\mathbb{R}^n$,*

$$\sum_{Q\subseteq P} \frac{\mu(Q)^2}{|Q|^{1-\frac{2}{n}}} \le c_5\, \mu(P), \tag{6.10}$$

*where the sum is taken over all dyadic cubes $Q$ contained in $P$, and $c_5$ does not depend on $P$.*

*Moreover, the least constants $c_i$, $i = 1, \ldots, 5$, are equivalent to the least constant $c$ in* (6.5).

*Remark* 6.2. It follows from Poincaré's inequality and the formula for the capacity of a ball, $\mathrm{cap}\,(B(x,r)) = c_n\, r^{n-2}$, for $n \ge 3$, that if $d\mu = |\nabla v|^2 dx \in \mathfrak{M}^{1,2}_+(\mathbb{R}^n)$, where $v \in L^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$, then $v \in \mathrm{BMO}(\mathbb{R}^n)$.

*Remark* 6.3. An analogous characterization holds for admissible measures on the Sobolev space $W^{1,2}(\mathbb{R}^n)$ in place of $L^{1,2}(\mathbb{R}^n)$. One only needs to replace Riesz potentials $I_1\mu = (-\Delta)^{-\frac{1}{2}}\mu$ in statements (i), (iii), and (iv) by Bessel potentials $J_1\mu = (1-\Delta)^{-\frac{1}{2}}\mu$, the capacity $\mathrm{cap}\,(\cdot)$ in (ii) by $\mathrm{Cap}\,(\cdot)$, and restrict oneself to cubes $P$ such that $|P| \le 1$ in (v).

Originally, the trace inequality in the Sobolev space $L^{1,2}_0(\Omega)$, for an arbitrary open set $\Omega \subseteq \mathbb{R}^n$, was characterized by the first author in [18], [19] in capacity terms as follows. A measure $\mu \in \mathcal{M}^+(\Omega)$ is said to be admissible if the inequality

$$\left( \int_\Omega |u|^2\, d\mu \right)^{\frac{1}{2}} \le C\, ||\nabla u||_{L^2(\Omega)} \tag{6.11}$$

holds for all $u \in C_0^\infty(\Omega)$, where $C$ is a positive constant which does not depend on $u$. The class of admissible measures for (6.11) is denoted by $\mathfrak{M}^{1,2}(\Omega)$.

The capacity $\mathrm{cap}\,(e,\Omega)$ of a compact subset $e \subset \Omega$ is defined by (see [20], Sec. 2.2):

$$\mathrm{cap}\,(e,\Omega) = \inf \left\{ ||\nabla u||^2_{L^2(\Omega)} : \quad u \in C_0^\infty(\Omega), \quad u(x) \ge 1 \text{ on } e \right\}. \tag{6.12}$$

Then $\mu \in \mathfrak{M}^{1,2}(\Omega)$ if and only if ([18], [19]; see also [20, Sec. 2.3])

$$\mu(e) \le c\, \mathrm{cap}\,(e,\Omega), \tag{6.13}$$

where the constant $c$ does not depend on $e$.

Moreover, condition (6.13) with $c = \frac{1}{4}$ is sufficient for (6.11) to hold with $C = 1$. Conversely, (6.13) with $c = 1$ is necessary in order that (6.11) hold with $C = 1$. Both constants $c = \frac{1}{4}$ and $c = 1$ and in these statements are sharp (see [20, Sec. 2.5.2]).

There is a dual characterization of the trace inequality which does not use capacities. Let us assume that $G$ is a nontrivial nonnegative Green's function associated with the Dirichlet Laplacian in $\Omega$. Then $\mu \in \mathfrak{M}^{1,2}(\Omega)$ if and only if the inequality

$$\int_{e\times e} G(x,y)\, d\mu(x)\, d\mu(y) \le c\, \mu(e) \tag{6.14}$$

holds for all measurable sets $e \subset \Omega$.

Moreover, inequality (6.14) is equivalent to the weighted norm inequality

$$||G(f d\mu)||_{L^2(\Omega,\mu)} \le C\, ||f||_{L^2(\Omega,\mu)}, \quad \text{for all } f \in L^2(\Omega,\mu).$$

It is also equivalent to the weak-type $(1,1)$ inequality

$$||G(f d\mu)||_{L^{1,\infty}(\Omega,\mu)} \le C\, ||f||_{L^1(\Omega,\mu)}, \quad \text{for all } f \in L^1(\Omega,\mu).$$

Here $G(f d\mu)(x) = \int_\Omega G(x,y)\, f(y)\, d\mu(y)$ is Green's potential of $f\, d\mu$.

In [28, Theorem 6.5], similar results are proved for nonnegative kernels $G$ satisfying a weak form of the *maximum principle*:

$$\sup\{G\nu(x):\ x \in \Omega\} \le \mathfrak{b}\, \sup\{G\nu(x):\ x \in \operatorname{supp} \nu\},$$

where $\mathfrak{b} \ge 1$ is a constant which does not depend on $\nu \in \mathcal{M}^+(\Omega)$.

In particular, if $G$ is a *quasi-metric* kernel, i.e., $d(x,y) = \frac{1}{G(x,y)}$ is symmetric and satisfies a quasi-triangle inequality, then it suffices to verify (6.14) on quasi-metric balls $B(x,r) = \{y \in \Omega:\ d(x,y) \le r\}$ in place of arbitrary sets $e$ (see [6], [28]).

Analogous results hold ([6]) for a more general class of quasi-metrically modifiable kernels $G$. This is important since the Green kernel $G$ is known to be quasi-metrically modifiable if $\Omega$ satisfies the boundary Harnack principle, for instance, if $\Omega$ is a bounded NTA domain ([14]).

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