Uniform asymptotic formulae for Green's functions in singularly perturbed domains

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Dedicated to Professor W.D. Evans
on the occasion of his sixty fifth birthday

Abstract

Asymptotic formulae for Green's functions for the operator $-\Delta$ in domains with small holes are obtained. A new feature of these formulae is their uniformity with respect to the independent variables. The cases of multi-dimensional and planar domains are considered.

Key words: Hadamard's variational formula, Green's function, singular perturbations

1 Introduction

Hadamard's paper [1] contains, among much else, asymptotic formulae for Green's kernels of classical boundary value problems under small variations of a domain. In [1], the perturbed domain Ω_ε, depending on a small parameter $\varepsilon > 0$, approximates the limit domain Ω in such a way that the angle between the two outward normals at nearby points of $\partial\Omega$ and $\partial\Omega_\varepsilon$ is small.

In short, Hadamard's formulae are related to the case of *a regularly perturbed* domain. A drawback of these formulae is their non-uniformity with respect to the independent variables. A uniform version of one of Hadamard's formulae containing a boundary layer was formulated in [2]. Besides, uniform asymptotic representations of Green's functions for several types of *singularly perturbed* domains were given in [2] without proofs.

The objective of the present article is to prove two theorems announced in [2]. Namely, we derive uniform asymptotic formulae for Green's functions of the Dirichlet problem for the operator $-\Delta$ in n-dimensional domains with small holes, first for $n > 2$ in Section 2 and then for $n = 2$ in Section 3. Corollaries, presented in Section 4, show that these formulae can be simplified under certain constraints on the independent variables.

We make use of the version of the method of compound asymptotic expansions of solutions to boundary value problems in singularly perturbed domains developed in [3].

Now, we list several notations adopted in the text of the paper. Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with compact closure $\overline{\Omega}$ and boundary $\partial\Omega$. By F we denote a compact set of positive harmonic capacity in $\mathbb{R}^n$; its complement is $F^c = \mathbb{R}^n \setminus F$. We suppose that both Ω and F contain the origin $\mathbf{O}$ as an interior point. Without loss of generality, it is assumed that the minimum distance between $\mathbf{O}$ and the points of $\partial\Omega$ is equal to 1. Also, the maximum distance between $\mathbf{O}$ and the points of ∂F^c will be taken as 1. We introduce the set $F_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in F\}$, where ε is a small positive parameter, and the open set $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$. The notation B_ρ stands for the open ball centered at $\mathbf{O}$ with radius ρ.

The main object of our study, Green's function for the operator $-\Delta$ in Ω_ε, will be denoted by G_ε. In the sequel, along with $\mathbf{x}$ and $\mathbf{y}$, we use the scaled variables $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$.

By Const we always mean different positive constants depending only on n. Finally, the notation $f = O(g)$ is equivalent to the inequality $|f| \leq \text{Const}\, g$.

2 Green's function for a multi-dimensional domain with a small hole

We assume here that $n > 2$. Let G and g denote Green's functions of the Dirichlet problem for the operator $-\Delta$ in the sets Ω and $F^c = \mathbb{R}^n \setminus F$. We make use of the regular parts of G and g, respectively:

$$H(\mathbf{x}, \mathbf{y}) = (n-2)^{-1}|S^{n-1}|^{-1}|\mathbf{x} - \mathbf{y}|^{2-n} - G(\mathbf{x}, \mathbf{y}), \tag{1}$$

and

$$h(\boldsymbol{\xi},\boldsymbol{\eta}) = (n-2)^{-1}|S^{n-1}|^{-1}|\boldsymbol{\xi}-\boldsymbol{\eta}|^{2-n} - g(\boldsymbol{\xi},\boldsymbol{\eta}), \tag{2}$$

where $|S^{n-1}|$ denotes the $(n-1)$-dimensional measure of the unit sphere S^{n-1}.

By $P(\boldsymbol{\xi})$ we mean the equilibrium potential of F defined as a unique solution of the following Dirichlet problem in F^c

$$\Delta_\xi P(\boldsymbol{\xi}) = 0 \quad \text{in } F^c, \tag{3}$$

$$P(\boldsymbol{\xi}) = 1 \quad \text{on } \partial F^c, \tag{4}$$

$$P(\boldsymbol{\xi}) \to 0 \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty, \tag{5}$$

where the boundary condition (4) is interpreted in the sense of the Sobolev space H^1.

The following auxiliary assertion is classical.

Lemma 1.

(i) The potential P satisfies the estimate

$$0 < P(\boldsymbol{\xi}) \le \min\left\{1, |\boldsymbol{\xi}|^{2-n}\right\}. \tag{6}$$

(ii) If $|\boldsymbol{\xi}| \ge 2$, then

$$\left|P(\boldsymbol{\xi}) - \frac{\operatorname{cap}(F)}{(n-2)|S^{n-1}|}|\boldsymbol{\xi}|^{2-n}\right| \le \text{Const } |\boldsymbol{\xi}|^{1-n} \tag{7}$$

Proof. *(i)* Inequalities (6) follow from the maximum principle for variational solutions of Laplace's equation.

(ii) Inequality (7) results from the expansion of P in spherical harmonics. □

Lemma 2. *For all $\boldsymbol{\eta} \in F^c$ and for $\boldsymbol{\xi}$ with $|\boldsymbol{\xi}| > 2$ the estimate holds:*

$$|h(\boldsymbol{\xi},\boldsymbol{\eta}) - P(\boldsymbol{\eta})(n-2)^{-1}|S^{n-1}|^{-1}|\boldsymbol{\xi}|^{2-n}| \le \text{Const } |\boldsymbol{\xi}|^{1-n}P(\boldsymbol{\eta}). \tag{8}$$

Proof. By (2), h satisfies the Dirichlet problem

$$\Delta_\xi h(\boldsymbol{\xi},\boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi},\boldsymbol{\eta} \in F^c, \tag{9}$$

$$h(\boldsymbol{\xi}, \boldsymbol{\eta}) = (n-2)^{-1}|S^{n-1}|^{-1}|\boldsymbol{\xi} - \boldsymbol{\eta}|^{2-n}, \quad \boldsymbol{\xi} \in \partial F^c \text{ and } \boldsymbol{\eta} \in F^c, \tag{10}$$

$$h(\boldsymbol{\xi}, \boldsymbol{\eta}) \to 0 \quad \text{as } |\boldsymbol{\xi}| \to \infty \text{ and } \boldsymbol{\eta} \in F^c. \tag{11}$$

We fix $\boldsymbol{\eta} \in F^c$. By the series expansion of g in spherical harmonics,

$$|\boldsymbol{\xi}|^{n-2}\Big(g(\boldsymbol{\xi}, \boldsymbol{\eta}) - \frac{C(\boldsymbol{\eta})}{(n-2)|S^{n-1}||\boldsymbol{\xi}|^{n-2}}\Big) \to 0 \quad \text{as } |\boldsymbol{\xi}| \to \infty. \tag{12}$$

We apply Green's formula to the functions $g(\boldsymbol{\xi}, \boldsymbol{\eta})$ and $1 - P(\boldsymbol{\xi})$ restricted to the domain $B_R \setminus F$, where $B_R = \{\boldsymbol{\xi} : |\boldsymbol{\xi}| < R\}$ is the ball of a sufficiently large radius R. Taking into account that $P(\boldsymbol{\xi}) = 1$ and $g(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0$ when $\boldsymbol{\xi} \in \partial(F^c)$ we deduce

$$\int_{B_R \setminus F} \nabla_\xi g(\boldsymbol{\xi}, \boldsymbol{\eta}) \cdot \nabla_\xi P(\boldsymbol{\xi}) d\boldsymbol{\xi} = P(\boldsymbol{\eta}) - 1 - \int_{\partial B_R} (1 - P(\boldsymbol{\xi})) \frac{\partial}{\partial |\boldsymbol{\xi}|} g(\boldsymbol{\xi}, \boldsymbol{\eta}) ds_\xi, \tag{13}$$

and

$$\int_{B_R \setminus F} \nabla_\xi g(\boldsymbol{\xi}, \boldsymbol{\eta}) \cdot \nabla_\xi P(\boldsymbol{\xi}) d\boldsymbol{\xi} = \int_{\partial B_R} g(\boldsymbol{\xi}, \boldsymbol{\eta}) \frac{\partial}{\partial |\boldsymbol{\xi}|} P(\boldsymbol{\xi}) ds_\xi. \tag{14}$$

Hence,

$$1 - P(\boldsymbol{\eta}) = -\int_{\partial B_R} \Big(g(\boldsymbol{\xi}, \boldsymbol{\eta}) \frac{\partial}{\partial |\boldsymbol{\xi}|} P(\boldsymbol{\xi}) + (1 - P(\boldsymbol{\xi})) \frac{\partial}{\partial |\boldsymbol{\xi}|} g(\boldsymbol{\xi}, \boldsymbol{\eta}) \Big) ds_\xi. \tag{15}$$

It follows from (12) that

$$1 - P(\boldsymbol{\eta}) = -\lim_{R \to \infty} \int_{\partial B_R} \frac{\partial}{\partial |\boldsymbol{\xi}|} \frac{C(\boldsymbol{\eta})}{(n-2)|S^{n-1}||\boldsymbol{\xi}|^{n-2}} ds_\xi = C(\boldsymbol{\eta}).$$

Let $|\boldsymbol{\xi}| > 2$. Then for $\boldsymbol{\eta} \in \partial F^c$

$$|h(\boldsymbol{\xi}, \boldsymbol{\eta}) - (n-2)^{-1}|S^{n-1}|^{-1}|\boldsymbol{\xi}|^{2-n} P(\boldsymbol{\eta})| = (n-2)^{-1}|S^{n-2}|^{-1}\Big||\boldsymbol{\xi} - \boldsymbol{\eta}|^{2-n} - |\boldsymbol{\xi}|^{2-n}\Big|$$

$$\leq \text{Const } |\boldsymbol{\eta}||\boldsymbol{\xi}|^{1-n} \leq \text{Const } |\boldsymbol{\xi}|^{1-n}. \tag{16}$$

In the above estimate, we used the assumption (see Introduction) of the maximum distance between the origin and the points of ∂F^c being equal to 1. From

(16) and the maximum principle for functions harmonic in $\boldsymbol{\eta}$, we deduce

$$|h(\boldsymbol{\xi},\boldsymbol{\eta}) - \Big((n-2)|S^{n-1}|\Big)^{-1}|\boldsymbol{\xi}|^{2-n}P(\boldsymbol{\eta})| \le \text{Const } |\boldsymbol{\xi}|^{1-n}P(\boldsymbol{\eta}),$$

for all $\boldsymbol{\eta} \in F^c$ and $|\boldsymbol{\xi}| > 2$. □

Our main result concerning the uniform approximation of Green's function G_ε in the multi-dimensional case is given by

Theorem 1. *Green's function $G_\varepsilon(\mathbf{x},\mathbf{y})$ admits the representation*

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) + \varepsilon^{2-n} g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - ((n-2)|S^{n-1}||\mathbf{x}-\mathbf{y}|^{n-2})^{-1} \\ &+ H(0,\mathbf{y})P(\varepsilon^{-1}\mathbf{x}) + H(\mathbf{x},0)P(\varepsilon^{-1}\mathbf{y}) - H(0,0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y}) \\ &- \varepsilon^{n-2}\ \text{cap}(F)\ H(\mathbf{x},0)H(0,\mathbf{y}) + O\Big(\varepsilon^{n-1}(\min\{|\mathbf{x}|,|\mathbf{y}|\}+\varepsilon)^{2-n}\Big), \end{aligned} \tag{17}$$

uniformly with respect to $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$. Here, H and h are regular parts of Green's functions G and g, respectively (see (1), (2)*), and P is the capacitary potential of F.*

Before presenting a proof of this theorem, we give a *plausible formal argument* leading to (17).

Let G_ε be represented in the form

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = \Big((n-2)|S^{n-1}|\Big)^{-1}|\mathbf{x}-\mathbf{y}|^{2-n} - H_\varepsilon(\mathbf{x},\mathbf{y}) - h_\varepsilon(\mathbf{x},\mathbf{y}), \tag{18}$$

where H_ε and h_ε are solutions of the Dirichlet problems

$$\begin{aligned} \Delta_x H_\varepsilon(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon, \\ H_\varepsilon(\mathbf{x},\mathbf{y}) = \Big((n-2)|S^{n-1}|\Big)^{-1}|\mathbf{x}-\mathbf{y}|^{2-n}&, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon, \\ H_\varepsilon(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x} \in \partial F_\varepsilon^c,\ \mathbf{y} \in \Omega_\varepsilon. \end{aligned}$$

and

$$\begin{aligned} \Delta_x h_\varepsilon(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon, \\ h_\varepsilon(\mathbf{x},\mathbf{y}) = \Big((n-2)|S^{n-1}|\Big)^{-1}|\mathbf{x}-\mathbf{y}|^{2-n}&, \quad \mathbf{x} \in \partial F_\varepsilon^c,\ \mathbf{y} \in \Omega_\varepsilon, \\ h_\varepsilon(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon. \end{aligned} \tag{19}$$

By (18), it suffices to find asymptotic formulae for H_ε and h_ε.

Function H_ε. Obviously, $H_\varepsilon(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y})$ is harmonic in Ω_ε, and $H_\varepsilon(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) = 0$ for $\mathbf{x} \in \partial\Omega$. On the other hand, for $\mathbf{x} \in \partial F_\varepsilon^c$ the leading part of $H_\varepsilon(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y})$ is equal to the function $-H(0,\mathbf{y})$. This function can be extended onto F_ε^c, harmonically in $\mathbf{x}$, as $-H(0,\mathbf{y})P(\varepsilon^{-1}\mathbf{x})$, whose leading-order part is equal to $-\varepsilon^{n-2}\mathrm{cap}(F)\ H(\mathbf{x},0)H(0,\mathbf{y})$ for $\mathbf{x} \in \partial\Omega$. Hence,

$$H_\varepsilon(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) \sim -H(0,\mathbf{y})P(\varepsilon^{-1}\mathbf{x})$$
$$+\,\varepsilon^{n-2}\mathrm{cap}(F)\ H(\mathbf{x},0)H(0,\mathbf{y}) \ \text{ for all } \mathbf{x},\mathbf{y} \in \Omega_\varepsilon. \tag{20}$$

Function h_ε. By definitions (2) and (19) of h and h_ε,

$$h_\varepsilon(\mathbf{x},\mathbf{y}) - \varepsilon^{2-n}h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) = 0 \text{ for } \mathbf{x} \in \partial F_\varepsilon^c.$$

Furthermore, by Lemma 2

$$h_\varepsilon(\mathbf{x},\mathbf{y}) - \varepsilon^{2-n}h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})$$
$$\sim -\Big((n-2)|S^{n-1}|\Big)^{-1}|\mathbf{x}|^{2-n}P(\varepsilon^{-1}\mathbf{y}) \ \text{ for } \mathbf{x} \in \partial\Omega.$$

The harmonic function in $\mathbf{x} \in \Omega$, with the Dirichlet data

$$-\Big((n-2)|S^{n-1}|\Big)^{-1}|\mathbf{x}|^{2-n}P(\varepsilon^{-1}\mathbf{y})$$

on $\partial\Omega$, is $-H(\mathbf{x},0)P(\varepsilon^{-1}\mathbf{y})$, and it is asymptotically equal to $-H(0,0)P(\varepsilon^{-1}\mathbf{y})$ on ∂F_ε^c, which is not necessarily small. The harmonic in $\mathbf{x}$ extension of $H(0,0)P(\varepsilon^{-1}\mathbf{y})$ onto F_ε^c is given by $H(0,0)P(\varepsilon^{-1}\mathbf{y})P(\varepsilon^{-1}\mathbf{x})$. Since this function is small for $\mathbf{x} \in \partial\Omega$, one may assume the asymptotic representation

$$\begin{aligned} h_\varepsilon(\mathbf{x},\mathbf{y}) &- \varepsilon^{2-n}h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + H(\mathbf{x},0)P(\varepsilon^{-1}\mathbf{y}) \\ &\sim H(0,0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y}) \qquad \text{for all } \mathbf{x},\mathbf{y} \in \Omega_\varepsilon. \end{aligned} \tag{21}$$

Substituting (20) and (21) into (18), we deduce

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) \sim\ & \Big((n-2)|S^{n-1}|\Big)^{-1}|\mathbf{x}-\mathbf{y}|^{2-n} - H(\mathbf{x},\mathbf{y}) - \varepsilon^{2-n}h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\ & + H(0,\mathbf{y})P(\varepsilon^{-1}\mathbf{x}) + H(\mathbf{x},0)P(\varepsilon^{-1}\mathbf{y}) - H(0,0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y}) \\ & - \varepsilon^{n-2}\mathrm{cap}(F)\ H(\mathbf{x},0)H(0,\mathbf{y}), \end{aligned}$$

which is equivalent to

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) \sim\ & G(\mathbf{x},\mathbf{y}) + \varepsilon^{2-n}g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - ((n-2)|S^{n-1}|)^{-1}|\mathbf{x}-\mathbf{y}|^{2-n} \\ & + H(0,\mathbf{y})P(\varepsilon^{-1}\mathbf{x}) + H(\mathbf{x},0)P(\varepsilon^{-1}\mathbf{y}) - H(0,0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y}) \\ & - \varepsilon^{n-2}\mathrm{cap}(F)\ H(\mathbf{x},0)H(0,\mathbf{y}). \end{aligned}$$

Now, we give a rigorous proof of (17).

Proof of Theorem 1.

The remainder $r_\varepsilon(\mathbf{x},\mathbf{y})$ in (17) is a solution of the boundary value problem

$$\Delta_x r_\varepsilon(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon, \tag{22}$$

$$\begin{aligned} r_\varepsilon(\mathbf{x},\mathbf{y}) = & \; H(\mathbf{x},\mathbf{y}) - H(0,\mathbf{y}) \\ & -(H(\mathbf{x},0) - H(0,0))P(\varepsilon^{-1}\mathbf{y}) \\ & +\varepsilon^{n-2}\mathrm{cap}(F)\; H(\mathbf{x},0)H(0,\mathbf{y}), \quad \mathbf{x} \in \partial F_\varepsilon^c, \; \mathbf{y} \in \Omega_\varepsilon, \end{aligned} \tag{23}$$

$$\begin{aligned} r_\varepsilon(\mathbf{x},\mathbf{y}) = & \; \varepsilon^{2-n} h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - H(0,\mathbf{y})P(\varepsilon^{-1}\mathbf{x}) \\ & - H(\mathbf{x},0)P(\varepsilon^{-1}\mathbf{y}) + H(0,0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y}) \\ & +\varepsilon^{n-2}\mathrm{cap}(F)\; H(\mathbf{x},0)H(0,\mathbf{y}), \; \mathbf{x} \in \partial\Omega, \; \mathbf{y} \in \Omega_\varepsilon. \end{aligned} \tag{24}$$

The functions $H(\mathbf{x},0)$ and $H(0,\mathbf{y})$ are harmonic in Ω and are bounded by Const on $\partial\Omega$. Hence, they are bounded by Const for $\mathbf{x} \in \partial F_\varepsilon^c, \; \mathbf{y} \in \Omega_\varepsilon$ and for $\mathbf{x} \in \partial\Omega, \; \mathbf{y} \in \Omega_\varepsilon$, respectively. The terms $\varepsilon^{n-2}\mathrm{cap}(F)H(\mathbf{x},0)H(0,\mathbf{y})$ in the right-hand sides of (23) and (24) are bounded by Const ε^{n-2}.

By definition (1), $\nabla_x H(\mathbf{x},\mathbf{y})$ is bounded by Const uniformly with respect to $\mathbf{y} \in \Omega$ for every $\mathbf{x} \in B_{1/2}$. Hence, by (23) and the inequalities $0 < P(\mathbf{x}) \le 1$,

$$\begin{aligned} & |H(\mathbf{x},\mathbf{y}) - H(0,\mathbf{y}) - (H(\mathbf{x},0) - H(0,0))P(\varepsilon^{-1}\mathbf{y})| \\ & \le \text{Const } \varepsilon \sup_{\mathbf{z}\in B_\varepsilon} |\nabla_z H(\mathbf{z},\mathbf{y})| \le \text{Const } \varepsilon, \end{aligned}$$

for $\mathbf{x} \in \partial F_\varepsilon^c, \; \mathbf{y} \in \Omega_\varepsilon$. Thus, the following estimate holds when $\mathbf{x} \in \partial F_\varepsilon^c$ and $\mathbf{y} \in \Omega_\varepsilon$

$$|r_\varepsilon(\mathbf{x},\mathbf{y})| \le \text{Const } \varepsilon \sup_{\mathbf{z}\in B_\varepsilon} |\nabla_z H(\mathbf{z},\mathbf{y})| \le \text{Const } \varepsilon. \tag{25}$$

Next, we estimate $|r_\varepsilon(\mathbf{x},\mathbf{y})|$ for $\mathbf{x} \in \partial\Omega$ and $\mathbf{y} \in \Omega_\varepsilon$. By Lemma 1, the capacitary potential $P(\varepsilon^{-1}\mathbf{x})$ satisfies the inequalities

$$0 \le P(\varepsilon^{-1}\mathbf{x}) \le \text{Const } \frac{\varepsilon^{n-2}}{(|\mathbf{x}| + \varepsilon)^{n-2}}, \tag{26}$$

for $\mathbf{x} \in \Omega_\varepsilon$, and

$$\begin{aligned} & \Big| P(\varepsilon^{-1}\mathbf{x}) - \frac{\varepsilon^{n-2}\mathrm{cap}(F)}{(n-2)|S^{n-1}||\mathbf{x}|^{n-2}} \Big| \\ & \le \text{Const } \Big(\varepsilon/|\mathbf{x}|\Big)^{n-1} \le \text{Const } \varepsilon^{n-1}, \end{aligned} \tag{27}$$

for $\mathbf{x} \in \partial\Omega$. Now, (27) and the definition of $H(\mathbf{x}, \mathbf{y})$ imply

$$|\varepsilon^{n-2}\text{cap}(F)H(\mathbf{x}, 0)H(0, \mathbf{y}) - H(0, \mathbf{y})P(\varepsilon^{-1}\mathbf{x})| \leq \text{Const } \varepsilon^{n-1}. \tag{28}$$

Also, we have the estimate

$$\begin{aligned}
&|\varepsilon^{2-n}h(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - H(\mathbf{x}, 0)P(\varepsilon^{-1}\mathbf{y})| \\
&= \varepsilon^{2-n}\Big|h(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - \frac{P(\varepsilon^{-1}\mathbf{y})}{(n-2)|S^{n-1}||\mathbf{x}/\varepsilon|^{n-2}}\Big| \\
&\leq \text{Const } \varepsilon|\mathbf{x}|^{1-n}P(\varepsilon^{-1}\mathbf{y}) \\
&\leq \text{Const } \frac{\varepsilon^{n-1}}{(|\mathbf{y}| + \varepsilon)^{n-2}}, \ \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega_\varepsilon,
\end{aligned} \tag{29}$$

which follows from the definition (1) of $H(\mathbf{x}, \mathbf{y})$ and the estimates (8) and (26). Combining (26), (28) and (29) we obtain from (24) that the trace of the function $\mathbf{x} \to |r_\varepsilon(\mathbf{x}, \mathbf{y})|$ on $\partial\Omega$ does not exceed

$$\text{Const } \frac{\varepsilon^{n-1}}{(|\mathbf{y}| + \varepsilon)^{n-2}}.$$

for $\mathbf{y} \in \Omega_\varepsilon$. Using this and (25), we deduce by the maximum principle that

$$|r_\varepsilon(\mathbf{x}, \mathbf{y})| \leq \text{Const}\Big\{\varepsilon P\Big(\frac{\mathbf{x}}{\varepsilon}\Big) + \frac{\varepsilon^{n-1}}{(|\mathbf{y}| + \varepsilon)^{n-2}}\Big\},$$

for all $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$. Taking into account (26), we arrive at

$$|r_\varepsilon(\mathbf{x}, \mathbf{y})| \leq \text{Const } \frac{\varepsilon^{n-1}}{(\min\{|\mathbf{x}|, |\mathbf{y}|\} + \varepsilon)^{n-2}} \tag{30}$$

The proof is complete. □

3 Green's function for the Dirichlet problem in a planar domain with a small hole

In this section, we find an asymptotic approximation of G_ε in the two-dimensional case. We shall see that this approximation has new features in comparison with that in Theorem 1.

The notations $\Omega_\varepsilon, \Omega, F_\varepsilon, F$, introduced in Introduction, will be used here. As before, we assume that the minimum distance from the origin to $\partial\Omega$ and the maximum distance between the origin and the points of ∂F^c are equal to 1.

Green's function $G(\mathbf{x}, \mathbf{y})$ for the unperturbed domain Ω has the form

$$G(\mathbf{x}, \mathbf{y}) = (2\pi)^{-1} \log |\mathbf{x} - \mathbf{y}|^{-1} - H(\mathbf{x}, \mathbf{y}), \tag{31}$$

where H is its regular part satisfying

$$\Delta_x H(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \Omega, \tag{32}$$

$$H(\mathbf{x}, \mathbf{y}) = (2\pi)^{-1} \log |\mathbf{x} - \mathbf{y}|^{-1}, \quad \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega. \tag{33}$$

The scaled coordinates $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$ will be used as in the multi-dimensional case. Similar to Section 2, $g(\boldsymbol{\xi}, \boldsymbol{\eta})$ and $h(\boldsymbol{\xi}, \boldsymbol{\eta})$ are Green's function and its regular part in F^c:

$$\Delta_\xi g(\boldsymbol{\xi}, \boldsymbol{\eta}) + \delta(\boldsymbol{\xi} - \boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi}, \boldsymbol{\eta} \in F^c, \tag{34}$$

$$g(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi} \in \partial F, \ \boldsymbol{\eta} \in F^c, \tag{35}$$

$$g(\boldsymbol{\xi}, \boldsymbol{\eta}) \ \text{ is bounded } \ \text{as } |\boldsymbol{\xi}| \to \infty \text{ and } \boldsymbol{\eta} \in F^c, \tag{36}$$

and

$$h(\boldsymbol{\xi}, \boldsymbol{\eta}) = (2\pi)^{-1} \log |\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1} - g(\boldsymbol{\xi}, \boldsymbol{\eta}). \tag{37}$$

We introduce a function ζ by

$$\zeta(\boldsymbol{\eta}) = \lim_{|\boldsymbol{\xi}| \to \infty} g(\boldsymbol{\xi}, \boldsymbol{\eta}), \tag{38}$$

and the constant

$$\zeta_\infty = \lim_{|\boldsymbol{\eta}| \to \infty} \{\zeta(\boldsymbol{\eta}) - (2\pi)^{-1} \log |\boldsymbol{\eta}|\}. \tag{39}$$

Lemma 3. *Let $|\boldsymbol{\xi}| > 2$. Then the regular part $h(\boldsymbol{\xi}, \boldsymbol{\eta})$ of Green's function g in F^c admits the asymptotic representation*

$$h(\boldsymbol{\xi}, \boldsymbol{\eta}) = -(2\pi)^{-1} \log |\boldsymbol{\xi}| - \zeta(\boldsymbol{\eta}) + O(|\boldsymbol{\xi}|^{-1}), \tag{40}$$

which is uniform with respect to $\boldsymbol{\eta} \in F^c$.

Proof: Following the inversion transformation, we use the variables:

$$\boldsymbol{\xi}' = |\boldsymbol{\xi}|^{-2}\boldsymbol{\xi}, \quad \boldsymbol{\eta}' = |\boldsymbol{\eta}|^{-2}\boldsymbol{\eta},$$

and the identity

$$|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1}|\boldsymbol{\xi}||\boldsymbol{\eta}| = |\boldsymbol{\xi}'-\boldsymbol{\eta}'|^{-1}.$$

Then, the boundary values of $h(\boldsymbol{\xi},\boldsymbol{\eta})$, as $\boldsymbol{\xi}\in\partial F^c, \boldsymbol{\eta}\in F^c$, can be expressed in the form

$$h(\boldsymbol{\xi},\boldsymbol{\eta}) = \mathfrak{H}(\boldsymbol{\xi}',\boldsymbol{\eta}') - (2\pi)^{-1}\log|\boldsymbol{\xi}||\boldsymbol{\eta}|, \tag{41}$$

where $\mathfrak{H}(\boldsymbol{\xi}',\boldsymbol{\eta}')$, $\boldsymbol{\xi}'\in\partial(F^c)'$, is the boundary value of the regular part of Green's function in the bounded transformed set $(F^c)'$. Namely, the function $\mathfrak{H}(\boldsymbol{\xi}',\boldsymbol{\eta}')$ is defined as a solution of the Dirichlet problem

$$\Delta_{\boldsymbol{\xi}'}\mathfrak{H}(\boldsymbol{\xi}',\boldsymbol{\eta}') = 0, \quad \boldsymbol{\xi}',\boldsymbol{\eta}'\in(F^c)', \tag{42}$$

$$\mathfrak{H}(\boldsymbol{\xi}',\boldsymbol{\eta}') = (2\pi)^{-1}\log|\boldsymbol{\xi}'-\boldsymbol{\eta}'|^{-1}, \quad \boldsymbol{\xi}'\in\partial(F^c)'. \tag{43}$$

It follows from (41) that the harmonic extension of $h(\boldsymbol{\xi},\boldsymbol{\eta})$ is

$$h(\boldsymbol{\xi},\boldsymbol{\eta}) = \mathfrak{H}(\boldsymbol{\xi}',\boldsymbol{\eta}') - (2\pi)^{-1}\log|\boldsymbol{\xi}||\boldsymbol{\eta}|, \quad \boldsymbol{\xi},\boldsymbol{\eta}\in F^c. \tag{44}$$

Since $\mathfrak{H}(\boldsymbol{\xi}',\boldsymbol{\eta}')$ is smooth in $(F^c)'\times(F^c)'$, we deduce

$$h(\boldsymbol{\xi},\boldsymbol{\eta}) = \mathfrak{H}(0,\boldsymbol{\eta}') - (2\pi)^{-1}\log|\boldsymbol{\xi}||\boldsymbol{\eta}| + O(|\boldsymbol{\xi}'|), \tag{45}$$

for $|\boldsymbol{\xi}'|<1/2$ and for all $\boldsymbol{\eta}'\in(F^c)'$. Also, by (44) and the definition of $h(\boldsymbol{\xi},\boldsymbol{\eta})$,

$$\mathfrak{H}(\boldsymbol{\xi}',\boldsymbol{\eta}') = -g(\boldsymbol{\xi},\boldsymbol{\eta}) + (2\pi)^{-1}\log|\boldsymbol{\xi}||\boldsymbol{\eta}| - (2\pi)^{-1}\log|\boldsymbol{\xi}-\boldsymbol{\eta}|. \tag{46}$$

Then, applying (38) and taking the limit in (46), as $|\boldsymbol{\xi}'|\to 0$, we arrive at

$$\mathfrak{H}(0,\boldsymbol{\eta}') = -\zeta(\boldsymbol{\eta}) + (2\pi)^{-1}\lim_{|\boldsymbol{\xi}|\to\infty}\log(|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1}|\boldsymbol{\xi}|) + (2\pi)^{-1}\log|\boldsymbol{\eta}|$$

$$= (2\pi)^{-1}\log|\boldsymbol{\eta}| - \zeta(\boldsymbol{\eta}).$$

Further substitution of $\mathfrak{H}(0,\boldsymbol{\eta}')$ into (45) leads to

$$h(\boldsymbol{\xi},\boldsymbol{\eta}) = -(2\pi)^{-1}\log|\boldsymbol{\xi}| - \zeta(\boldsymbol{\eta}) + O(|\boldsymbol{\xi}|^{-1}),$$

for $|\boldsymbol{\xi}|>2$ and for all $\boldsymbol{\eta}\in F^c$. The proof is complete $\Box$.

3.1 Asymptotic approximation of the equilibrium potential

The *equilibrium potential* $P_\varepsilon(\mathbf{x})$ is introduced as a solution of the following Dirichlet problem in Ω_ε

$$\Delta P_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{47}$$
$$P_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega, \tag{48}$$
$$P_\varepsilon(\mathbf{x}) = 1, \quad \mathbf{x} \in \partial F_\varepsilon^c. \tag{49}$$

Lemma 4. *The asymptotic approximation of $P_\varepsilon(\mathbf{x})$ is given by the formula*

$$P_\varepsilon(\mathbf{x}) = \frac{-G(\mathbf{x},0) + \zeta(\frac{\mathbf{x}}{\varepsilon}) - \frac{1}{2\pi}\log\frac{|\mathbf{x}|}{\varepsilon} - \zeta_\infty}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty} + p_\varepsilon(\mathbf{x}), \tag{50}$$

where ζ_∞ is defined by (39), and p_ε is the remainder term such that

$$|p_\varepsilon(\mathbf{x})| \le Const\, \varepsilon(\log\varepsilon)^{-1}$$

uniformly with respect to $\mathbf{x} \in \Omega_\varepsilon$.

Proof. Direct substitution of (50) into (47)–(49) yields the Dirichlet problem for the remainder term p_ε

$$\Delta p_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{51}$$
$$p_\varepsilon(\mathbf{x}) = -\frac{\zeta(\varepsilon^{-1}\mathbf{x}) - \frac{1}{2\pi}\log(\varepsilon^{-1}|\mathbf{x}|) - \zeta_\infty}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty}, \quad \mathbf{x} \in \partial\Omega, \tag{52}$$
$$p_\varepsilon(\mathbf{x}) = 1 - \frac{H(\mathbf{x},0) + \frac{1}{2\pi}\log\varepsilon - \zeta_\infty}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty}, \quad \mathbf{x} \in \partial F_\varepsilon^c. \tag{53}$$

Using (39) and the expansion of $\zeta(\boldsymbol{\xi})$ in spherical harmonics, we deduce

$$\zeta(\varepsilon^{-1}\mathbf{x}) - (2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}|) - \zeta_\infty = O(\varepsilon),$$

as $|\mathbf{x}| \in \partial\Omega$, and hence the right-hand side in (52) is $O(\varepsilon(\log\varepsilon)^{-1})$. Since $H(\mathbf{x},0)$ is smooth in Ω, we have

$$H(\mathbf{x},0) - H(0,0) = O(\varepsilon),$$

as $\mathbf{x} \in \partial F_\varepsilon^c$, and therefore the right-hand side in (53) is also $O(\varepsilon(\log\varepsilon)^{-1})$. Applying the maximum principle, we arrive at the result of Lemma. □

Remark. For the case when Ω is a Jordan domain and F is the closure of a Jordan domain, we can adopt the notions of [4]: the inner conformal radius r_F of F, with respect to $\mathbf{O}$, and the outer conformal radius R_Ω of Ω, with respect to $\mathbf{O}$, are defined as

$$r_F = \exp(-2\pi\zeta_\infty), \;\; R_\Omega = \exp(-2\pi H(0,0)),$$

respectively. In this case, the equilibrium potential $P_\varepsilon(\mathbf{x})$ can be represented in the form

$$P_\varepsilon(\mathbf{x}) = \frac{-G(\mathbf{x},0) + \zeta(\frac{\mathbf{x}}{\varepsilon}) - \frac{1}{2\pi}\log\frac{|\mathbf{x}|}{\varepsilon r_F}}{\frac{1}{2\pi}\log\frac{\varepsilon r_F}{R_\Omega}} + p_\varepsilon(\mathbf{x}).$$

3.2 Uniform asymptotic approximation

Theorem 2. *Green's function G_ε for the operator $-\Delta$ in $\Omega_\varepsilon \subset \mathbb{R}^2$ admits the representation*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + g(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + (2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|)$$
$$+\frac{\Big((2\pi)^{-1}\log\varepsilon + \zeta(\frac{\mathbf{x}}{\varepsilon}) - \zeta_\infty + H(\mathbf{x},0)\Big)\Big((2\pi)^{-1}\log\varepsilon + \zeta(\frac{\mathbf{y}}{\varepsilon}) - \zeta_\infty + H(0,\mathbf{y})\Big)}{(2\pi)^{-1}\log\varepsilon + H(0,0) - \zeta_\infty}$$
$$-\zeta(\varepsilon^{-1}\mathbf{x}) - \zeta(\varepsilon^{-1}\mathbf{y}) + \zeta_\infty + O(\varepsilon), \tag{54}$$

which is uniform with respect to $(\mathbf{x},\mathbf{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$.

Proof. Let

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - H_\varepsilon(\mathbf{x},\mathbf{y}) - h_\varepsilon(\mathbf{x},\mathbf{y}), \tag{55}$$

where H_ε and h_ε are defined as solutions of the Dirichlet problems

$$\Delta_x H_\varepsilon(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x},\ \mathbf{y} \in \Omega_\varepsilon, \tag{56}$$
$$H_\varepsilon(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1}, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon, \tag{57}$$
$$H_\varepsilon(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x} \in \partial F_\varepsilon,\ \mathbf{y} \in \Omega_\varepsilon, \tag{58}$$

and

$$\Delta_x h_\varepsilon(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x},\ \mathbf{y} \in \Omega_\varepsilon, \tag{59}$$
$$h_\varepsilon(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon, \tag{60}$$
$$h_\varepsilon(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1}, \quad \mathbf{x} \in \partial F_\varepsilon,\ \mathbf{y} \in \Omega_\varepsilon. \tag{61}$$

The function H_ε is represented in the form

$$H_\varepsilon(\mathbf{x},\mathbf{y}) = C(\mathbf{y},\log\varepsilon)G(\mathbf{x},0) + H(\mathbf{x},\mathbf{y}) + R_\varepsilon(\mathbf{x},\mathbf{y},\log\varepsilon), \tag{62}$$

where $C(\mathbf{y}, \log \varepsilon)$ is to be determined, G and H are defined by (31)–(33), and the third term R_ε satisfies the boundary value problem

$$\Delta_x R_\varepsilon(\mathbf{x}, \mathbf{y}, \log \varepsilon) = 0, \quad \mathbf{x},\ \mathbf{y} \in \Omega_\varepsilon, \tag{63}$$

$$R_\varepsilon(\mathbf{x}, \mathbf{y}, \log \varepsilon) = 0, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon, \tag{64}$$

$$R_\varepsilon(\mathbf{x}, \mathbf{y}, \log \varepsilon) = -CG(\mathbf{x}, 0) - H(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \partial F_\varepsilon,\ \mathbf{y} \in \Omega_\varepsilon, \tag{65}$$

and it is approximated by a function $R(\varepsilon^{-1}\mathbf{x}, \mathbf{y}, \log \varepsilon)$ defined in scaled coordinates in such a way that

$$\Delta_\xi R(\boldsymbol{\xi}, \mathbf{y}, \log \varepsilon) = 0, \quad \boldsymbol{\xi} \in F^c, \tag{66}$$

$$\begin{aligned} R(\boldsymbol{\xi}, \mathbf{y}, \log \varepsilon) = {} & C(2\pi)^{-1}(\log |\boldsymbol{\xi}| + \log \varepsilon) \\ & + CH(0, 0) - H(0, \mathbf{y}), \quad \boldsymbol{\xi} \in \partial F^c, \end{aligned} \tag{67}$$

$$R(\boldsymbol{\xi}, \mathbf{y}, \log \varepsilon) \to 0 \ \text{ as } |\boldsymbol{\xi}| \to \infty, \tag{68}$$

where $\mathbf{y} \in \Omega_\varepsilon$. The solution of the above problem has the form

$$\begin{aligned} R(\boldsymbol{\xi}, \mathbf{y}, \log \varepsilon) = {} & -C\{(2\pi)^{-1} \log |\boldsymbol{\xi}|^{-1} + \zeta(\boldsymbol{\xi})\} \\ & + C\{(2\pi)^{-1} \log \varepsilon + H(0, 0)\} - H(0, \mathbf{y}), \end{aligned} \tag{69}$$

with ζ defined by (38).

The condition (68) is satisfied provided

$$C(\mathbf{y}, \log \varepsilon) = \frac{H(0, \mathbf{y})}{H(0, 0) + \frac{1}{2\pi} \log \varepsilon - \zeta_\infty}. \tag{70}$$

Combining (69), (70), and (62), we deduce

$$H_\varepsilon(\mathbf{x}, \mathbf{y}) = -H(0, \mathbf{y}) P_\varepsilon(\mathbf{x}) + H(\mathbf{x}, \mathbf{y}) + \tilde{H}_\varepsilon(\mathbf{x}, \mathbf{y}), \tag{71}$$

where $\tilde{H}_\varepsilon$ is the remainder term, such that

$$\Delta_x \tilde{H}_\varepsilon(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \Omega_\varepsilon, \tag{72}$$

$$\tilde{H}_\varepsilon(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon, \tag{73}$$

$$\tilde{H}_\varepsilon(\mathbf{x}, \mathbf{y}) = H(0, \mathbf{y}) - H(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \partial F_\varepsilon,\ \mathbf{y} \in \Omega_\varepsilon, \tag{74}$$

where the modulus of the right-hand side in (74) is estimated by Const ε, uniformly with respect to $\mathbf{x} \in \partial F_\varepsilon^c$ and $\mathbf{y} \in \Omega_\varepsilon$. The maximum principle leads to the estimate $|\tilde{H}(\mathbf{x}, \mathbf{y})| \leq$ Const ε, which is uniform for $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$.

The approximation of h_ε (see (59)–(61)) also involves the equilibrium potential P_ε from Section 3.1. The harmonic function h_ε satisfies the homogeneous Dirichlet condition on $\partial\Omega$, and the boundary condition on ∂F_ε^c is rewritten as

$$h_\varepsilon(\mathbf{x},\mathbf{y}) = -(2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|) - (2\pi)^{-1}\log\varepsilon, \quad \mathbf{x}\in\partial F_\varepsilon^c, \mathbf{y}\in\Omega_\varepsilon.$$

Hence $h_\varepsilon(\mathbf{x},\mathbf{y})$ is sought in the form

$$h_\varepsilon(\mathbf{x},\mathbf{y}) = h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - (2\pi)^{-1}\log\varepsilon + \tilde{h}_\varepsilon^{(1)}(\mathbf{x},\mathbf{y}), \tag{75}$$

where the harmonic function $\tilde{h}_\varepsilon^{(1)}$ vanishes when $\mathbf{x}\in\partial F_\varepsilon^c,\ \mathbf{y}\in\Omega_\varepsilon$, and

$$\tilde{h}_\varepsilon^{(1)}(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log\varepsilon - h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}), \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon. \tag{76}$$

Representing the right-hand side in (76) according to Lemma 3, we obtain

$$\tilde{h}_\varepsilon^{(1)}(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}| + \zeta(\varepsilon^{-1}\mathbf{y}) + O(\varepsilon),$$

uniformly for $\mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon$. Using the capacitary potential P_ε and the definition (1) of $H(\mathbf{x},\mathbf{y})$, we write $\tilde{h}_\varepsilon^{(1)}$ as

$$\tilde{h}_\varepsilon^{(1)}(\mathbf{x},\mathbf{y}) = -H(\mathbf{x},0) + \zeta(\varepsilon^{-1}\mathbf{y})(1-P_\varepsilon(\mathbf{x})) + \tilde{h}_\varepsilon^{(2)}(\mathbf{x},\mathbf{y}), \tag{77}$$

where $\tilde{h}_\varepsilon^{(2)}$ is a harmonic function, which is $O(\varepsilon)$ for all $\mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon$, and satisfies

$$\tilde{h}_\varepsilon^{(2)}(\mathbf{x},\mathbf{y}) = H(\mathbf{x},0) = H(0,0) + O(\varepsilon),$$

for all $\mathbf{x}\in\partial F_\varepsilon^c, \mathbf{y}\in\Omega_\varepsilon$. Hence,

$$\tilde{h}_\varepsilon^{(2)}(\mathbf{x},\mathbf{y}) = H(0,0)P_\varepsilon(\mathbf{x}) + O(\varepsilon), \tag{78}$$

uniformly with respect to $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$.

Combining (75), (77) and (78), we deduce

$$\begin{aligned} h_\varepsilon(\mathbf{x},\mathbf{y}) = {} & h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - (2\pi)^{-1}\log\varepsilon - H(\mathbf{x},0) \\ & +\zeta(\varepsilon^{-1}\mathbf{y})(1-P_\varepsilon(\mathbf{x})) + H(0,0)P_\varepsilon(\mathbf{x}) + O(\varepsilon), \end{aligned} \tag{79}$$

uniformly with respect to $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$.

Furthermore, it follows from (55), (71) and (79) that Green's function G_ε admits the representation

$$
\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) = {} & (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - H(\mathbf{x},\mathbf{y}) - h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\
& +(2\pi)^{-1}\log\varepsilon - \zeta(\boldsymbol{\eta}) + H(\mathbf{x},0) \\
& -P_\varepsilon(\mathbf{x})(H(0,0) - H(0,\mathbf{y}) - \zeta(\varepsilon^{-1}\mathbf{y})) + O(\varepsilon),
\end{aligned} \tag{80}
$$

which is uniform with respect to $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$.

By Lemma 4, (80) takes the form

$$
\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) = {} & (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - H(\mathbf{x},\mathbf{y}) - h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\
& + \frac{(H(0,0) - H(\mathbf{x},0) - \zeta(\varepsilon^{-1}\mathbf{x}))(H(0,0) - H(0,\mathbf{y}) - \zeta(\varepsilon^{-1}\mathbf{y}))}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty} \\
& + (2\pi)^{-1}\log\varepsilon + H(\mathbf{x},0) + H(0,\mathbf{y}) - H(0,0) + O(\varepsilon).
\end{aligned} \tag{81}
$$

Also with the use of Lemma 4, for all $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$, the above formula can be written as

$$
\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) = {} & (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - H(\mathbf{x},\mathbf{y}) - h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\
& +((2\pi)^{-1}\log\varepsilon + H(0,0) - \zeta_\infty)(1-P_\varepsilon(\mathbf{x}))(1-P_\varepsilon(\mathbf{y})) \\
& +(2\pi)^{-1}\log\varepsilon + H(\mathbf{x},0) + H(0,\mathbf{y}) - H(0,0) + O(\varepsilon) \\
= {} & (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - H(\mathbf{x},\mathbf{y}) - h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\
& +((2\pi)^{-1}\log\varepsilon + H(0,0) - \zeta_\infty)P_\varepsilon(\mathbf{x})P_\varepsilon(\mathbf{y}) \\
& -\zeta(\varepsilon^{-1}\mathbf{x}) - \zeta(\varepsilon^{-1}\mathbf{y}) + \zeta_\infty + O(\varepsilon),
\end{aligned} \tag{82}
$$

which is equivalent to (54). The proof is complete. □

4 Corollaries

The asymptotic formulae of sections 2 and 3 can be simplified under constraints on positions of the points $\mathbf{x},\mathbf{y}$ within Ω_ε.

Corollary 1.

(a) Let $\mathbf{x}$ and $\mathbf{y}$ be points of $\Omega_\varepsilon \subset \mathbb{R}^n, n > 2$, such that

$$
\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon. \tag{83}
$$

Then

$$
G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) \; -\varepsilon^{n-2}\mathrm{cap}(F) \; G(\mathbf{x},0)G(0,\mathbf{y})
$$

$$+O\Big(\frac{\varepsilon^{n-1}}{(|\mathbf{x}||\mathbf{y}|)^{n-2}\min\{|\mathbf{x}|,|\mathbf{y}|\}}\Big). \tag{84}$$

(b) If $\max\{|\mathbf{x}|,|\mathbf{y}|\}<1/2$*, then*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = \varepsilon^{2-n}g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})$$

$$-H(0,0)(P(\varepsilon^{-1}\mathbf{x})-1)(P(\varepsilon^{-1}\mathbf{y})-1)+O(\max\{|\mathbf{x}|,|\mathbf{y}|\}). \tag{85}$$

Both (84) *and* (85) *are uniform with respect to* ε *and* $(\mathbf{x},\mathbf{y})\in\Omega_\varepsilon\times\Omega_\varepsilon$.

Proof.

(a) The formula (17) is equivalent to

$$\begin{aligned}G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) - \varepsilon^{2-n}h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})\\ &+H(0,\mathbf{y})P(\varepsilon^{-1}\mathbf{x})+H(\mathbf{x},0)P(\varepsilon^{-1}\mathbf{y})-H(0,0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y})\\ &-\varepsilon^{n-2}\mathrm{cap}(F)\ H(\mathbf{x},0)H(0,\mathbf{y})+O\Big(\frac{\varepsilon^{n-1}}{(\min\{|\mathbf{x}|,|\mathbf{y}|\}^{n-2})}\Big).\end{aligned} \tag{86}$$

By Lemmas 1 and 2

$$P(\varepsilon^{-1}\mathbf{x}) = \frac{\varepsilon^{n-2}\ \mathrm{cap}(F)}{(n-2)|S^{n-1}||\mathbf{x}|^{n-2}}+O\Big(\frac{\varepsilon^{n-1}}{|\mathbf{x}|^{n-1}}\Big). \tag{87}$$

and

$$\varepsilon^{2-n}h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) = \frac{P(\varepsilon^{-1}\mathbf{y})}{(n-2)|S^{n-1}||\mathbf{x}|^{n-2}}+O\Big(\frac{\varepsilon^{n-1}}{|\mathbf{x}|^{n-1}|\mathbf{y}|^{n-2}}\Big) \tag{88}$$

$$=\frac{\varepsilon^{n-2}\mathrm{cap}(F)}{((n-2)|S^{n-1}|)^2|\mathbf{x}|^{n-2}|\mathbf{y}|^{n-2}}+O\Big(\frac{\varepsilon^{n-1}}{(|\mathbf{x}||\mathbf{y}|)^{n-2}\min\{|\mathbf{x}|,|\mathbf{y}|\}}\Big).$$

Direct substitution of (88) and (87) into (86) leads to

$$\begin{aligned}G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) - \frac{\varepsilon^{n-2}\mathrm{cap}(F)}{(n-2)^2|S^{n-1}|^2|\mathbf{x}|^{n-2}|\mathbf{y}|^{n-2}}\\ &\quad+\varepsilon^{n-2}\mathrm{cap}(F)\Big(\frac{H(0,\mathbf{y})}{(n-2)|S^{n-1}||\mathbf{x}|^{n-2}}+\frac{H(\mathbf{x},0)}{(n-2)|S^{n-1}||\mathbf{y}|^{n-2}}\\ &\quad-H(\mathbf{x},0)H(0,\mathbf{y})\Big)+O\Big(\frac{\varepsilon^{n-1}}{(|\mathbf{x}||\mathbf{y}|)^{n-2}\min\{|\mathbf{x}|,|\mathbf{y}|\}}\Big)\\ &= G(\mathbf{x},\mathbf{y}) - \varepsilon^{n-2}\mathrm{cap}(F)\Big\{\Big((n-2)^{-1}|S^{n-1}|^{-1}|\mathbf{x}|^{2-n}-H(\mathbf{x},0)\Big)\\ &\quad\times\Big((n-2)^{-1}|S^{n-1}|^{-1}|\mathbf{y}|^{2-n}-H(0,\mathbf{y})\Big)\\ &\quad+O\Big(\frac{\varepsilon^{n-1}}{(|\mathbf{x}||\mathbf{y}|)^{n-2}\min\{|\mathbf{x}|,|\mathbf{y}|\}}\Big),\end{aligned}$$

which is equivalent to (84).

(b) Since $H(\mathbf{x},\mathbf{y})$ is smooth in the vicinity of $(\mathbf{O},\mathbf{O})$ formula (17) can be presented in the form

$$\begin{aligned}G_\varepsilon(\mathbf{x},\mathbf{y}) = {}& \varepsilon^{2-n} g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - H(0,0)\\ &+(H(0,0)+O(|\mathbf{y}|))P(\varepsilon^{-1}\mathbf{x}) + (H(0,0)+O(|\mathbf{x}|))P(\varepsilon^{-1}\mathbf{y})\\ &-H(0,0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y}) + O(\max\{|\mathbf{x}|,|\mathbf{y}|\}),\end{aligned}$$

which is equivalent to (85). The proof is complete. □

We give an analogue of Corollary 1 for the planar case.

Corollary 2. *(a) Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be points of* $\Omega_\varepsilon \subset \mathbb{R}^2$ *subject to (83). Then*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + \frac{G(\mathbf{x},0)G(0,\mathbf{y})}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty} + O\Big(\frac{\varepsilon}{\min\{|\mathbf{x}|,|\mathbf{y}|\}}\Big), \tag{89}$$

(b) If $\max\{|\mathbf{x}|,|\mathbf{y}|\} < 1/2$*, then*

$$\begin{aligned}G_\varepsilon(\mathbf{x},\mathbf{y}) = {}& g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})\\ &+\frac{\zeta(\varepsilon^{-1}\mathbf{x})\zeta(\varepsilon^{-1}\mathbf{y})}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty} + O(\max\{|\mathbf{x}|,|\mathbf{y}|\}),\end{aligned} \tag{90}$$

Both (89) *and* (90) *are uniform with respect to* ε *and* $(\mathbf{x},\mathbf{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$*.*

Proof. *(a)* Formula (54) can be written as

$$\begin{aligned}G_\varepsilon(\mathbf{x},\mathbf{y}) = {}& (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - H(\mathbf{x},\mathbf{y}) - h(\boldsymbol{\xi},\boldsymbol{\eta})\\ &+\frac{(G(\mathbf{x},0) - \zeta(\boldsymbol{\xi}) + \frac{1}{2\pi}\log|\boldsymbol{\xi}| + \zeta_\infty)(G(0,\mathbf{y}) - \zeta(\boldsymbol{\eta}) + \frac{1}{2\pi}\log|\boldsymbol{\eta}| + \zeta_\infty)}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty}\\ &- \zeta(\boldsymbol{\xi}) - \zeta(\boldsymbol{\eta}) + \zeta_\infty + O(\varepsilon).\end{aligned} \tag{91}$$

It follows from Lemma 3 and definition (38) that

$$h(\boldsymbol{\xi},\boldsymbol{\eta}) = -(2\pi)^{-1}\log|\boldsymbol{\xi}| - \zeta(\boldsymbol{\eta}) + O(\varepsilon/|\mathbf{x}|), \tag{92}$$

and

$$\zeta(\boldsymbol{\xi}) = (2\pi)^{-1}\log|\boldsymbol{\xi}| + \zeta_\infty + O(\varepsilon/|\mathbf{x}|). \tag{93}$$

Direct substitution of (92) and (93) into (91) yields

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - H(\mathbf{x},\mathbf{y}) + \frac{(-G(\mathbf{x},0)+O(\varepsilon/|\mathbf{x}|))(-G(0,\mathbf{y})+O(\varepsilon/|\mathbf{y}|))}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty} + O(\varepsilon), \quad (94)$$

and hence we arrive at (89).

(b) When $\max\{|\mathbf{x}|,|\mathbf{y}|\} < 1/2$, (54) is presented in the form:

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - H(\mathbf{x},\mathbf{y}) + \frac{(H(0,0)-H(\mathbf{x},0)-\zeta(\varepsilon^{-1}\mathbf{x}))(H(0,0)-H(0,\mathbf{y})-\zeta(\varepsilon^{-1}\mathbf{y}))}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty} + H(\mathbf{x},0) + H(0,\mathbf{y}) - H(0,0) + O(\varepsilon)$$

(compare with (81)). Since $H(\mathbf{x},\mathbf{y})$ is smooth in a vicinity of $(\mathbf{O},\mathbf{O})$, we obtain

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + \frac{(-\zeta(\varepsilon^{-1}\mathbf{x})+O(|\mathbf{x}|))(-\zeta(\varepsilon^{-1}\mathbf{y})+O(|\mathbf{y}|))}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty} + O(\max\{|\mathbf{x}|,|\mathbf{y}|\})$$

$$= g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + \frac{\zeta(\varepsilon^{-1}\mathbf{x})\zeta(\varepsilon^{-1}\mathbf{y}) + O(|\mathbf{y}|\log(|\mathbf{x}|/\varepsilon)) + O(|\mathbf{x}|\log(|\mathbf{y}|/\varepsilon))}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty} + O(\max\{|\mathbf{x}|,|\mathbf{y}|\}),$$

which implies (90). □

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