Uniform asymptotic formulae for Green's kernels in regularly and singularly perturbed domains

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Abstract

Asymptotic formulae for Green's kernels $G_\varepsilon(\mathbf{x}, \mathbf{y})$ of various boundary value problems for the Laplace operator are obtained in regularly perturbed domains and certain domains with small singular perturbations of the boundary, as $\varepsilon \to 0$. The main new feature of these asymptotic formulae is their uniformity with respect to the independent variables $\mathbf{x}$ and $\mathbf{y}$.

Introduction. In 1907, Hadamard was awarded the Prix Vaillant by the Académie des Sciences de Paris for his work [1]. Among much else, this memoir includes investigation of Green's kernels of some boundary value problems, and, in particular, their dependence on a small regular perturbation of the domain. Hadamard's asymptotic formulae played a significant role at the dawn of functional analysis (see [2]). The list of subsequent applications includes extremal problems in the complex function theory [3], [4], a biharmonic maximum principle for hyperbolic surfaces [5], shape sensitivity and optimisation analysis [7], free boundary problems [8], Brownian motion on hypersurfaces [6], theory of reproducing kernels [9], [10], [11]. Analogues of Hadamard's formulae were obtained for general elliptic boundary value problems [13] and for the heat equation [14].

The issue of asymptotic formulae for Green's kernels in singularly perturbed domains attracted less attention. In this respect, we mention the papers [15], [16], where certain estimates for Green's functions in domains with small holes were obtained.

An important drawback of the estimates just mentioned, which is also inherent in the classical Hadamard formulae, is the non-uniformity with respect to the independent variables. The present article settles the question of uniformity left open in earlier work, both for regularly and singularly perturbed domains. First, we sharpen one of Hadamard's classical formulae in the case of a regular perturbation. Next, we consider several types of singularly perturbed domains including domains with a finite number of holes, thin rods and a truncated cone. For these geometries and different types of boundary conditions, we derive uniform asymptotic representations of Green's kernels for the Laplacian. The asymptotic formulae presented here for singularly perturbed problems involve Green's functions and other auxiliary solutions of certain model boundary value problems independent of ε.

1. Uniform Hadamard's type formula. Let Ω be a planar domain with compact closure $\overline{\Omega}$ and smooth boundary $\partial\Omega$. Also, let another domain $\Omega(\varepsilon)$, depending on a small positive parameter ε, lie inside Ω. By δ_z we denote a smooth positive function defined on $\partial\Omega$, and assume that $\varepsilon\delta_z$ is the distance between a point $\mathbf{z} \in \partial\Omega$ and $\partial\Omega_\varepsilon$. One of the results in Hadamard's paper [1] is the following formula, which relates Green's functions G and G_ε for the Dirichlet boundary value problem for the Laplacian in Ω and Ω_ε:

$$G_\varepsilon(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{y}) + \varepsilon \int\limits_{\partial\Omega} \frac{\partial G}{\partial \nu_z}(\mathbf{x},\mathbf{z}) \frac{\partial G}{\partial \nu_z}(\mathbf{z},\mathbf{y}) \delta_z ds_z = O(\varepsilon^2). \tag{1}$$

This asymptotic relation holds, in particular, when either $\mathbf{x}$ or $\mathbf{y}$ is placed at a positive distance from $\partial\Omega$, independent of ε. However, one can see from the simplest example of two concentric disks Ω and Ω_ε that (1) may fail when $\mathbf{x}$ and $\mathbf{y}$ approach the same point at $\partial\Omega_\varepsilon$. In the next theorem, we improve Hadamard's formula by obtaining a uniform asymptotic representation for $G_\varepsilon - G$.

Theorem 1. *Let $\mathbf{x}$ and $\mathbf{y}$ be points of $\overline{\Omega}_\varepsilon$ situated in a sufficiently thin neighbourhood of $\partial\Omega$. The right-hand side in* (1) *can be replaced by*

$$\frac{1}{4\pi} \log \frac{|\mathbf{z}(\mathbf{x}) - \mathbf{z}(\mathbf{y})|^2 + (\rho_x - \varepsilon\delta_{z(x)} + \rho_y - \varepsilon\delta_{z(y)})^2}{|z(\mathbf{x}) - z(\mathbf{y})|^2 + (\rho_x + \rho_y)^2} \tag{2}$$

$$+\frac{1}{2\pi} \frac{\varepsilon(\delta_{z(x)} + \delta_{z(y)})(\rho_x + \rho_y)}{|z(\mathbf{x}) - z(\mathbf{y})|^2 + (\rho_x + \rho_y)^2} + O\Big(\frac{\varepsilon^2(\rho_x + \rho_y)}{|z(\mathbf{x}) - z(\mathbf{y})|^2 + (\rho_x + \rho_y)^2}\Big),$$

where $\mathbf{z}(\mathbf{x})$ *is the point of* $\partial\Omega$ *nearest to* $\mathbf{x}$*,* $\rho_x = |\mathbf{x} - \mathbf{z}(\mathbf{x})|$*, and the notations* $\mathbf{z}(\mathbf{y}), \rho_y$ *have the same meaning for the point* $\mathbf{y}$*.*

We note that the principal term in (2) plays the role of a boundary layer, and when $|\mathbf{z}(\mathbf{x}) - \mathbf{z}(\mathbf{y})| < \text{Const}\ \varepsilon$ and $\rho_x + \rho_y < \text{Const}\ \varepsilon$, it has the order $O(1)$, whereas the remainder term has the order $O(\varepsilon)$.

2. Dirichlet problem for a domain with a small inclusion. Let Ω and ω be bounded domains in $\mathbf{R}^n$. We assume that Ω and ω contain the origin O and introduce the domain $\omega_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in \omega\}$. Without loss of generality, it is assumed that the minimum distance between the origin and the points of $\partial\Omega$ as well as the maximum distance between the origin and the points of $\partial\omega$ are equal to 1. Let G_ε be Green's function of the Dirichlet problem for the Laplace operator in $\Omega_\varepsilon = \Omega \setminus \overline{\omega}_\varepsilon$. We use the notation $|S^{n-1}|$ for the $(n-1)$-dimensional measure of the unit sphere.

Theorem 2. *Let* $n > 2$*. By* G *and* G *we denote Green's functions of the Dirichelt problems in* Ω *and* $\mathbf{R}^n \setminus \overline{\omega}$*, respectively.*

Let H *be the regular part of* G*, that is* $H(\mathbf{x}, \mathbf{y}) = ((n-2)|S^{n-1}|)^{-1}|\mathbf{x}-\mathbf{y}|^{2-n} - G(\mathbf{x}, \mathbf{y})$*, and let* P *stand for the harmonic capacitary potential of* $\overline{\omega}$*. Then*

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) + \varepsilon^{2-n} G(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - ((n-2)|S^{n-1}|)^{-1}|\mathbf{x} - \mathbf{y}|^{2-n}$$

$$+ H(0, \mathbf{y})P(\varepsilon^{-1}\mathbf{x}) + H(\mathbf{x}, 0)P(\varepsilon^{-1}\mathbf{y}) - H(0, 0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y})$$

$$- \varepsilon^{n-2}\ \text{cap}\ \overline{\omega}\ H(\mathbf{x}, 0)H(0, \mathbf{y}) + O\Big(\frac{\varepsilon^{n-1}}{(\min\{|\mathbf{x}|, |\mathbf{y}|\})^{n-2}}\Big) \tag{3}$$

uniformly with respect to $\mathbf{x}$ *and* $\mathbf{y}$ *in* Ω_ε*. (Note that the remainder term in (9) is* $O(\varepsilon)$ *on* $\partial\omega_\varepsilon$ *and* $O(\varepsilon^{n-1})$ *on* $\partial\Omega$*.)*

The next theorem contains a result of the same nature for $n = 2$. As before, G and G are Green's functions for Ω and $\mathbf{R}^2 \setminus \overline{\omega}$, respectively, whereas H is the regular part of G.

Theorem 3. *Let*

$$\zeta(\boldsymbol{\eta}) = \lim_{|\boldsymbol{\xi}| \to \infty} G(\boldsymbol{\xi}, \boldsymbol{\eta}) \quad and \quad \zeta_\infty = \lim_{|\boldsymbol{\eta}| \to \infty} \{\zeta(\boldsymbol{\eta}) - (2\pi)^{-1} \log |\boldsymbol{\eta}|\}.$$

Then the asymptotic representation, uniform with respect to $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$*, holds*

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) + G(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + (2\pi)^{-1} \log(\varepsilon^{-1}|\mathbf{x} - \mathbf{y}|)$$

$$+ \frac{\Big((2\pi)^{-1} \log \varepsilon + \zeta(\frac{\mathbf{x}}{\varepsilon}) - \zeta_\infty + H(\mathbf{x}, 0)\Big)\Big((2\pi)^{-1} \log \varepsilon + \zeta(\frac{\mathbf{y}}{\varepsilon}) - \zeta_\infty + H(0, \mathbf{y})\Big)}{(2\pi)^{-1} \log \varepsilon + H(0, 0) - \zeta_\infty}$$

$$- \zeta(\varepsilon^{-1}\mathbf{x}) - \zeta(\varepsilon^{-1}\mathbf{y}) + \zeta_\infty + O(\varepsilon). \tag{4}$$

The next assertion is a direct consequence of (4). It shows that asymptotic representation of $G_\varepsilon(\mathbf{x}, \mathbf{y})$ is simplified if $\mathbf{x}$ and $\mathbf{y}$ are subject to additional constraints. We use the same notations as in Theorem 3.

Corollary. The following assertions hold:

(a) Let $\mathbf{x}$ *and* $\mathbf{y}$ *be points of* $\Omega_\varepsilon \subset \mathbf{R}^n$ *such that* $\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon$. *Then for* $n = 2$,

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}) - \frac{G(\mathbf{x}, 0)G(0, \mathbf{y})}{(2\pi)^{-1}\log\varepsilon + H(0,0) - \zeta_\infty} = O(\frac{\varepsilon}{\min\{|\mathbf{x}|, |\mathbf{y}|\}}),$$

and for $n > 2$,

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}) + \varepsilon^{n-2}\mathrm{cap}\, \overline{\omega}\; G(\mathbf{x}, 0)G(0, \mathbf{y}) = O(\frac{\varepsilon^{n-1}}{(|\mathbf{x}||\mathbf{y}|)^{n-1}\min\{|\mathbf{x}|, |\mathbf{y}|\}}).$$

(b) If $\max\{|\mathbf{x}|, |\mathbf{y}|\} < 1/2$, *then for* $n = 2$,

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) - G(\frac{\mathbf{x}}{\varepsilon}, \frac{\mathbf{y}}{\varepsilon}) - \frac{\zeta(\varepsilon^{-1}\mathbf{x})\zeta(\varepsilon^{-1}\mathbf{y})}{(2\pi)^{-1}\log\varepsilon + H(0,0) - \zeta_\infty} = O(\max\{|\mathbf{x}|, |\mathbf{y}|\}),$$

and for $n > 2$,

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) - \varepsilon^{2-n}G(\frac{\mathbf{x}}{\varepsilon}, \frac{\mathbf{y}}{\varepsilon}) + H(0,0)(P(\varepsilon^{-1}\mathbf{x}) - 1)(P(\varepsilon^{-1}\mathbf{y}) - 1) = O(\max\{|\mathbf{x}|, |\mathbf{y}|\}).$$

One can say that these formulae are closer, in their spirit, to Hadamard's original formula (1) than Theorems 2 and 3 themselves. Analogous simplified formulae can be deduced easily from all other theorems of the present paper, but they are not included here.

3. Dirichlet-Neumann problems in a planar domain with a small hole. The set Ω_ε is assumed to be the same as in Theorem 3, with the additional constraint that $\partial\omega$ is smooth. First, let G_ε denote the kernel of the inverse operator of the mixed boundary value problem in Ω_ε for the operator $-\Delta$, with the Dirichlet data on $\partial\Omega$ and the Neumann data on $\partial\omega_\varepsilon$. The notations G, G and H have the same meaning as in Theorem 3, and N is the Neumann function in $\mathbf{R}^2 \setminus \overline{\omega}$. Let $\boldsymbol{D}$ be a vector function, harmonic in $\mathbf{R}^2 \setminus \overline{\omega}$, vanishing at infinity and such that $\partial\boldsymbol{D}/\partial\nu = \boldsymbol{\nu}$ on $\partial\omega$. This vector function appears in the asymptotic representation

$$N(\boldsymbol{\xi}, \boldsymbol{\eta}) \sim (2\pi)^{-1}\log|\boldsymbol{\xi}|^{-1} + (\boldsymbol{D}(\boldsymbol{\eta}) - \boldsymbol{\eta}) \cdot \nabla((2\pi)^{-1}\log|\boldsymbol{\xi}|^{-1}), \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty.$$

Theorem 4. *The kernel $G_\varepsilon(\mathbf{x},\mathbf{y})$ of the inverse operator of the mixed boundary value problem in Ω_ε for the operator $-\Delta$, with the Dirichlet data on $\partial\Omega$ and the Neumann data on $\partial\omega_\varepsilon$, has the following asymptotic representation, which is uniform with respect to $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$,*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + (2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|)$$
$$+\varepsilon\boldsymbol{D}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y}) + \varepsilon\boldsymbol{D}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0)$$
$$-\varepsilon^2\boldsymbol{D}(\varepsilon^{-1}\mathbf{y})\cdot((\nabla_x\otimes\nabla_y)H(0,0))\boldsymbol{D}(\varepsilon^{-1}\mathbf{x}) + O(\varepsilon^2). \tag{5}$$

Next, consider the mixed boundary value problem in Ω_ε for the Laplace operator, with the Neumann data on the smooth boundary $\partial\Omega$ and the Dirichlet data on $\partial\omega_\varepsilon$, where $\omega_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x}\in\omega\subset\mathbf{R}^2\}$ with ω being an arbitray bounded domain. Let $N(\mathbf{x},\mathbf{y})$ be the Neumann function in Ω, i.e.

$$\Delta N(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y}\in\Omega,$$
$$\frac{\partial}{\partial\nu_x}\Big(N(\mathbf{x},\mathbf{y}) + (2\pi)^{-1}\log|\mathbf{x}|\Big) = 0, \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega,$$

and

$$\int\limits_{\partial\Omega} N(\mathbf{x},\mathbf{y})\frac{\partial}{\partial\nu_x}\log|\mathbf{x}|ds_x = 0,$$

with the last condition implying the symmetry of $N(\mathbf{x},\mathbf{y})$. The regular part of the Neumann function is defined by

$$R(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - N(\mathbf{x},\mathbf{y}).$$

Note that

$$R(0,\mathbf{y}) = -(2\pi)^{-2}\int\limits_{\partial\Omega}\log|\mathbf{x}|\frac{\partial}{\partial\nu}\log|\mathbf{x}|ds_x.$$

Let $\mathbf{D}$ be a vector function, harmonic in $\mathbf{R}^2\backslash\overline{\omega}$, bounded at infinity and subject to the Dirichlet condition $\mathbf{D}(\boldsymbol{\xi}) = \boldsymbol{\xi}, \quad \boldsymbol{\xi}\in\partial\omega$. This vector function appears in the asymptotic representation of Green's function in $\mathbf{R}^2\setminus\overline{\omega}$

$$G(\boldsymbol{\xi},\boldsymbol{\eta}) \sim G(\infty,\boldsymbol{\eta}) + (\mathbf{D}(\boldsymbol{\eta})-\boldsymbol{\eta})\cdot\nabla((2\pi)^{-1}\log|\boldsymbol{\xi}|^{-1}), \quad \text{as} \quad |\boldsymbol{\xi}|\to\infty.$$

Theorem 5. *The function $G_\varepsilon(\mathbf{x},\mathbf{y})$ has the asymptotic behaviour, uniform with respect to $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$:*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + N(\mathbf{x},\mathbf{y}) - (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} + R(0,0)$$
$$+\varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y R(\mathbf{x},0) + \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x R(0,\mathbf{y})$$
$$-\varepsilon^2\mathbf{D}(\varepsilon^{-1}\mathbf{y})\cdot\Big(((\nabla_x\otimes\nabla_y)R)(0,0)\Big)\mathbf{D}(\varepsilon^{-1}\mathbf{x}) + O(\varepsilon^2). \tag{6}$$

4. The Dirichlet-Neumann problem in a thin rod. Let C be the infinite cylinder $\{(\mathbf{x}', x_n) : \ \mathbf{x}' \in \omega, \ x_n \in \mathbf{R}\}$, where ω is a bounded domain in $\mathbf{R}^{n-1}$ with smooth boundary. Also let $C^{\pm}$ denote Lipschitz subdomains of C separated from $\pm\infty$ by surfaces $\gamma^{\pm}$, respectively. We introduce a positive number a, the vector $\mathbf{a} = (\mathbf{O}', a)$, where $\mathbf{O}'$ is the origin of $\mathbf{R}^{n-1}$, and the small parameter $\varepsilon > 0$,.

By G_ε we denote the fundamental solution for $-\Delta$ in the thin domain $C_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}(\mathbf{x} - \mathbf{a}) \in C^+, \ \varepsilon^{-1}(\mathbf{x} + \mathbf{a}) \in C^-\}$ subject to zero Neumann condition on the cylindrical part of C and zero Dirichlet condition on the remaining part of ∂C_ε. Similarly, G^+ and G^- stand for the fundamental solutions for $-\Delta$ in the domains $C^{\pm}$, and satisfy zero Dirichlet condition on $\gamma^{\pm}$, zero Neumann condition on $\partial C^{\pm} \setminus \gamma^{\pm}$, and are bounded as $x_n \to \mp\infty$. Let $\zeta^{\pm}$ be harmonic functions in $C^{\pm}$ subject to the same boundary conditions as $G^{\pm}$ and asymptotically equivalent to $\mp|\omega|^{-1}x_n + \zeta^{\pm}_\infty$ as $x_n \to \mp\infty$, where $\zeta^{\pm}_\infty$ are constants. We note that $\zeta^{\pm}(\mathbf{y}) = \lim_{\mathbf{x}\to\infty} G^{\pm}(\mathbf{x}, \mathbf{y})$. Finally, by $G_\infty(\mathbf{x}, \mathbf{y})$ we denote Green's function of the Neumann problem in C such that

$$G_\infty(\mathbf{x}, \mathbf{y}) = -(2|\omega|)^{-1}|x_n - y_n| + O(\exp(-\alpha|x_n - y_n|)) \quad \text{as} \quad |x_n| \to \infty,$$

where α is a positive constant, and $|\omega|$ is the $(n-1)$-dimensional measure of ω.

Theorem 6. *The following asymptotic formula for $G_\varepsilon(\mathbf{x}, \mathbf{y})$, uniform with respect to $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$, holds*

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) = \varepsilon^{2-n}\Big\{G^+(\varepsilon^{-1}(\mathbf{x}-\mathbf{a}), \varepsilon^{-1}(\mathbf{y}-\mathbf{a})) + G^-(\varepsilon^{-1}(\mathbf{x}+\mathbf{a}), \varepsilon^{-1}(\mathbf{y}+\mathbf{a})) - G^\infty(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y})$$
$$-\varepsilon\frac{(\frac{x_n}{\varepsilon|\omega|} - \frac{1}{2}(\zeta^-_\infty - \zeta^+_\infty) + \zeta^+(\frac{\mathbf{x}-\mathbf{a}}{\varepsilon}) - \zeta^-(\frac{\mathbf{x}+\mathbf{a}}{\varepsilon}))(\frac{y_n}{\varepsilon|\omega|} - \frac{1}{2}(\zeta^-_\infty - \zeta^+_\infty) + \zeta^+(\frac{\mathbf{y}-\mathbf{a}}{\varepsilon}) - \zeta^-(\frac{\mathbf{y}+\mathbf{a}}{\varepsilon}))}{2|\omega|^{-1}a + \varepsilon(\zeta^+_\infty + \zeta^-_\infty)}$$
$$+\frac{1}{4}\Big((\varepsilon|\omega|)^{-1}2a + \zeta^-_\infty + \zeta^+_\infty - 2\sum_{\pm}(\zeta^{\pm}(\varepsilon^{-1}(\mathbf{x}\mp\mathbf{a})) + \zeta^{\pm}(\varepsilon^{-1}(\mathbf{y}\mp\mathbf{a})))\Big)$$
$$+ O(\exp(-\beta/\varepsilon))\Big\}, \tag{7}$$

where β is a positive constant independent of ε.

5. The Dirichlet problem in a truncated cone. Let K be an infinite cone $\{\mathbf{x} : |\mathbf{x}| > 0, |\mathbf{x}|^{-1}\mathbf{x} \in \omega\}$, where ω is a subdomain of the unit sphere S^{n-1} such that $S^{n-1} \setminus \omega$ has a positive $(n-1)$-dimensional harmonic capacity. Also let K_0 and K_∞ denote subdomains of K separated from the vertex of K and from ∞ by surfaces γ and Γ, respectively.

By G_ε and G_{cone} we denote Green's function of the Dirichlet problem for $-\Delta$ in the domains $K_\varepsilon = \{\mathbf{x} \in K_0 : \varepsilon^{-1}\mathbf{x} \in K_\infty\}$ and K, respectively. Similarly,

G_0 and G_∞ stand for Green's functions of the Dirichlet problem for $-\Delta$ in the domains K_0 an K_∞ satisfying zero Dirichlet boundary conditions on $\partial K_0 \setminus \{O\}$ and ∂K_∞, with the asymptotic representations

$$Z_0(\mathbf{x}) = |\mathbf{x}|^{2-n-\lambda}\Psi(|\mathbf{x}|^{-1}\mathbf{x})(1+o(1)) \ \text{ as } |\mathbf{x}| \to 0,$$

and

$$Z_\infty(\mathbf{x}) = |\mathbf{x}|^{\lambda}\Psi(|\mathbf{x}|^{-1}\mathbf{x})(1+o(1)) \ \text{ as } |\mathbf{x}| \to \infty,$$

where λ is a positive number such that $\lambda(\lambda+n-2)$ is the first eigenvalue of the Dirichlet spectral problem in ω for the Beltrami operator on S^{n-1}, and Ψ is the corresponding eigenfunction. Similarly, we introduce $\lambda_2 > 0$ such that $\lambda_2(\lambda_2+n-2)$ is the second eigenvalue of the same spectral problem.

Theorem 7. *Let $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$. Green's function $G_\varepsilon(\mathbf{x},\mathbf{y})$ has the asymptotic behaviour*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G_0(\mathbf{x},\mathbf{y}) + \varepsilon^{2-n}G_\infty(\boldsymbol{\xi},\boldsymbol{\eta}) - G_{cone}(\mathbf{x},\mathbf{y})$$
$$+\frac{\varepsilon^\lambda}{2\lambda+n-2}\Big\{\Big(|\boldsymbol{\xi}|^\lambda\Psi\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) - Z_\infty(\boldsymbol{\xi})\Big)\Big(|\mathbf{y}|^{2-n-\lambda}\Psi\Big(\frac{\mathbf{y}}{|\mathbf{y}|}\Big) - Z_0(\mathbf{y})\Big)$$
$$+\Big(|\boldsymbol{\eta}|^\lambda\Psi\Big(\frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}\Big) - Z_\infty(\boldsymbol{\eta})\Big)\Big(|\mathbf{x}|^{2-n-\lambda}\Psi\Big(\frac{\mathbf{x}}{|\mathbf{x}|}\Big) - Z_0(\mathbf{x})\Big)\Big\} + O(\varepsilon^{\min(2\lambda,\lambda_2)}). \quad (8)$$

This representation is uniform with respect to $\mathbf{x}$ *and* $\mathbf{y}$ *in the truncated cone* K_ε.

6. Dirichlet problem in a domain containing several inclusions. It is straightforward to generalise the results of sections dealing with a domain containing a small inclusion/void to the case of a body containing a finite number of inclusions. As an example, we formulate a generalisation of Theorem 2. Let $\Omega \subset \mathbf{R}^n, n > 2$, be a bounded domain, and let $\mathbf{O}^{(1)}, \mathbf{O}^{(2)}, \dots, \mathbf{O}^{(N)}$ be interior points in Ω. Small sets $\omega_\varepsilon^{(j)}, j = 1,\dots,N$, are defined by $\omega_\varepsilon^{(j)} = \{\mathbf{x} : \varepsilon^{-1}(\mathbf{x}-\mathbf{O}^{(j)}) \in \omega^{(j)} \subset \mathbf{R}^n\}$, where $\omega^{(j)},\ j = 1,\dots,N$, are bounded domains in $\mathbf{R}^n$, and they contain the origin $\mathbf{O}$. Similar to Section 2, it is assumed that the minimum distance between $\mathbf{O}^{(j)}, j = 1,\dots,N$, and the points of $\partial\Omega$ is equal to 1. Also, it is supposed that the distance between $\mathbf{O}$ and the points of $\partial\omega^{(j)}, j = 1,\dots,N$, does not exceed 1. By G_ε we denote Green's function for the Laplacian in the domain $\Omega_\varepsilon = \Omega \setminus \cup_j \overline{\omega}_\varepsilon^{(j)}$.

Theorem 8. *Let G and $G^{(j)}$ stand for Green's functions of the Dirichlet problems in Ω and $\mathbf{R}^n \setminus \overline{\omega}^{(j)}$, respectively. Also let H be the regular part of G, and $P^{(j)}$ denote the harmonic capacitary potentials of the sets $\overline{\omega}^{(j)}$. Then*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + \varepsilon^{2-n}\sum_{j=1}^{N} G\Big(\frac{\mathbf{x}-\mathbf{O}^{(j)}}{\varepsilon}, \frac{\mathbf{y}-\mathbf{O}^{(j)}}{\varepsilon}\Big) - \frac{N}{(n-2)|S^{n-1}||\mathbf{x}-\mathbf{y}|^{n-2}}$$

$$+\sum_{j=1}^{N}\Big\{H(\mathbf{O}^{(j)},\mathbf{y})P^{(j)}(\frac{\mathbf{x}-\mathbf{O}^{(j)}}{\varepsilon})+H(\mathbf{x},\mathbf{O}^{(j)})P^{(j)}(\frac{\mathbf{y}-\mathbf{O}^{(j)}}{\varepsilon}) \qquad (9)$$

$$-H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j)}(\frac{\mathbf{x}-\mathbf{O}^{(j)}}{\varepsilon})P^{(j)}(\frac{\mathbf{y}-\mathbf{O}^{(j)}}{\varepsilon})-\varepsilon^{n-2}\,\mathrm{cap}\,\overline{\omega}^{(j)}\ H(\mathbf{x},\mathbf{O}^{(j)})H(\mathbf{O}^{(j)},\mathbf{y})\Big\}$$

$$+\sum_{j=1}^{N}\sum_{1\le j\le N, i\ne j}G(\mathbf{O}^{(j)},\mathbf{O}^{(i)})P^{(j)}(\frac{\mathbf{x}-\mathbf{O}^{(j)}}{\varepsilon})P^{(i)}(\frac{\mathbf{y}-\mathbf{O}^{(i)}}{\varepsilon})$$

$$+O\Big(\sum_{j=1}^{N}\frac{\varepsilon^{n-1}}{(\min\{|\mathbf{x}-\mathbf{O}^{(j)}|,|\mathbf{y}-\mathbf{O}^{(j)}|\})^{n-2}}\Big)$$

uniformly with respect to $\mathbf{x}$ *and* $\mathbf{y}$ *in* Ω_ε. *(Note that the remainder term in (9) is* $O(\varepsilon)$ *on* $\partial(\cup_j\omega_\varepsilon^{(j)})$ *and* $O(\varepsilon^{n-1})$ *on* $\partial\Omega$. *)*

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