Notes on Hölder regularity of a boundary point with respect to an elliptic operator of second order

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A point O at the boundary of a domain $\Omega \subset \mathbb{R}^n$ is called Hölder regular with respect to a second order elliptic operator L if there exists $\alpha > 0$ such that the α-Hölder continuity at O of the Dirichlet data implies the α-Hölder continuity of a solution to the Dirichlet problem for the equation $Lu = 0$ in Ω. In this article, estimates of the continuity modulus of a solution are obtained which give directly a necessary and sufficient condition for the Hölder regularity of O, formulated in terms of the L-harmonic measure. Some conditions sufficient for the Hölder regularity of O are discussed. Bibliography: 12 titles

1 Continuity modulus of solutions and criterion of Hölder regularity of a point

Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, with compact closure and boundary $\partial\Omega$. We assume without loss of generality that the diameter of Ω is equal to 1 and we fix a non-isolated point $O \in \partial\Omega$. Let us say that a function u defined on Ω is α-Hölder continuous at O with $\alpha \in (0, \infty)$ if it has a limit $u(0)$ as $x \to 0$ and there exists $\alpha > 0$ such that

$$|u(x) - u(0)| \leq const. |x|^\alpha$$

for all $x \in \Omega$. Similarly, a function φ given on $\partial\Omega$ is called α-Hölder continuous at O if there is a limit $\varphi(0)$ of $\varphi(x)$ as $x \to 0$, $x \in \partial\Omega$, and

$$|\varphi(x) - \varphi(0)| \leq const. |x|^\alpha \tag{1.1}$$

Note that since we deal with only one point O at the boundary, the usual restriction $\alpha \leq 1$ is not needed.

By u_φ we mean a bounded solution to the Dirichlet problem

$$Lu = 0 \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega,$$

where φ is a bounded Borel function on $\partial\Omega$ and

$$(Lu)(x) = \operatorname{div}\big(\mathcal{A}(x)\,\operatorname{grad} u(x)\big)$$

is a uniformly elliptic operator with a measurable bounded coefficient matrix $\mathcal{A}$. Basic facts concerning solvability of this problem can be found in [1].

Let us assume that O is regular in the sense of Wiener, which means that the continuity of φ at O implies the continuity of u_φ at O. By [1], if $n > 2$, then the assumption of Wiener regularity is equivalent to the Wiener test

$$\int_0^1 \mathrm{cap}(B_\rho \backslash \Omega)\, \rho^{1-n}\, d\rho = \infty, \tag{1.2}$$

where $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$ and the abbreviation cap stands for the Wiener capacity. A similar criterion holds for $n = 2$.

We introduce the L-harmonic measure $\mathcal{H}_L(x, \mathcal{B})$, where $x \in \Omega$ and $\mathcal{B}$ is a Borel subset of $\partial\Omega$ (see e.g. Definition 1.2.6 in [1]).

Definition 1. The point O is called α-Hölder regular with respect to L if the α-Hölder continuity of φ at O implies the α-Hölder continuity of u_φ at O.

Definition 2. The point O is called Hölder regular with respect to L if there exists $\alpha > 0$ such that O is α-Hölder regular.

In what follows, by $\omega(t)$ and $\gamma(t)$ we denote increasing continuous functions defined on $[0, 1]$ such that $\omega(0) = \gamma(0) = 0$.

Proposition 1 (i) *If*

$$|\varphi(x) - \varphi(0)| \le \omega(|x|) \quad \text{for } x \in \partial\Omega \tag{1.3}$$

and if

$$\mathcal{H}_L(x, \partial\Omega \backslash B_r) \le \frac{\gamma(|x|)}{\gamma(r)} \tag{1.4}$$

for $x \in \Omega$ such that $|x| < r$, then

$$|u_\varphi(x) - \varphi(0)| \le \omega(|x|) + \gamma(|x|) \int_{|x|}^1 \frac{d\,\omega(t)}{\gamma(t)} \quad \text{for } x \in \Omega \tag{1.5}$$

and in particular for $\gamma(t) = \omega(t)$

$$|u_\varphi(x) - \varphi(0)| \le \omega(|x|) \log \frac{e\,\omega(1)}{\omega(|x|} \quad \text{for } x \in \Omega. \tag{1.6}$$

(ii) *If for any Dirichlet data φ subject to (1.3) the solution satisfies*

$$|u_\varphi(x) - \varphi(0)| \le \omega(|x|),$$

then

$$\mathcal{H}_L(x, \partial\Omega \backslash B_r) \le \frac{\omega(|x|)}{\omega(r)} \tag{1.7}$$

for $x \in \Omega$, $|x| < r$.

Proof. (i) It suffices to estimate $(u_\varphi(x) - \varphi(0))_+$. Since

$$u_\varphi(x) = \int_{\partial\Omega} \varphi(y)\, \mathcal{H}_L(x, dy), \tag{1.8}$$

it follows that

$$(u_\varphi(x)-\varphi(0))_+ \le \omega(|x|) + \int_{\partial\Omega} (\omega(|y|)-\omega(|x|))_+ \mathcal{H}_L(x,dy).$$

Therefore,

$$(u_\varphi(x)-\varphi(0))_+ \le \omega(|x|) + \int_{|x|}^1 \mathcal{H}_L(x,\partial\Omega\backslash B_t)\, d\omega(t). \tag{1.9}$$

Now, (1.5) follows from (1.4).

(ii) We choose $\varphi(x)=\omega(|x|)$. By (1.8),

$$\omega(|x|) \ge \int_{\partial\Omega\backslash B_r} \omega(|y|)\mathcal{H}_L(x,dy) \ge \omega(r)\mathcal{H}_L(x,\partial\Omega\backslash B_r) \quad \text{for } |x|<r$$

and (1.7) follows. ∎

A necessary and sufficient condition for the Hölder regularity of O is contained in the following assertion.

Corollary 1 *The point $O\in\partial\Omega$ is Hölder regular with respect to L if and only for some positive constants λ and C*

$$\mathcal{H}_L(x,\partial\Omega\backslash B_r) \le c\Big(\frac{|x|}{r}\Big)^\lambda \tag{1.10}$$

for all $r>0$ and $x\in\Omega\cap B_r$.

Note that condition (1.10) does not guarantee the λ-regularity of O. In fact, if (1.10) holds with $L=\Delta$ and we suppose that

$$|\varphi(x)-\varphi(0)| \le c\,|x|^\lambda, \tag{1.11}$$

then (1.6) implies

$$|u_\varphi(x)-\varphi(0)| \le c|x|^\lambda \log\frac{e}{|x|} \tag{1.12}$$

which is weaker than the λ-Hölder continuity of u_φ.

We shall see that this logarithmic worsening of the boundary λ-Hölder condition is sharp. Consider the plane sector

$$\Omega = \{x=(\rho,\theta): 0<\rho<1,\ |\theta|<\Theta/2\}$$

and the Dirichlet problem in Ω for the Laplace operator. It is standard that $\mathcal{H}_\Delta(x,\partial\Omega\backslash B_r)$ is asymptotically equivalent to $c\rho^{\pi/\Theta}\sin(\pi\theta/\Theta)$ for small $\rho=|x|$. Hence condition (1.11) holds with $\lambda=\pi\Theta^{-1}$. Now we notice that the boundary data φ defined by

$$\varphi(\rho,\pm\Theta/2) = \pm\frac{\Theta}{2}\rho^{\pi/\Theta} \quad \text{for } \rho<1,$$

$$\varphi(1,\theta) = -\theta\sin\Big(\frac{\pi}{\Theta}\theta\Big) \qquad \text{for } |\theta|<\Theta/2,$$

satisfy (1.11) with $\lambda = \pi/\Theta$ but the harmonic extension of φ

$$u_\varphi(x) = \Big(\log\rho\,\cos\Big(\frac{\pi}{\Theta}\theta\Big) - \theta\,\sin\Big(\frac{\pi}{\Theta}\theta\Big)\Big)\rho^{\pi/\Theta}$$

satisfies (1.12) and it is not π/Θ-Hölder continuous.

We give a sufficient condition for the λ-Hölder regularity.

Proposition 2 *Let $\lambda > 0$ and let $\mathcal{H}_L$ satisfy*

$$\mathcal{H}_L(x, \partial\Omega\backslash B_r) \le c\Big(\frac{|x|}{r}\Big)^\lambda \sigma\Big(\frac{t}{|x|}\Big) \tag{1.13}$$

for all $r > 0$ and $x \in \Omega \cap B_r$, where σ is a continuous and decreasing function on $[1, \infty)$ subject to the Dini condition

$$\int_1^\infty \sigma(\tau)\frac{d\tau}{\tau} < \infty.$$

Then the λ-Hölder continuity of φ at O implies the λ-Hölder continuity of u_φ at O.

Proof. The result follows by substituting (1.13) in (1.9), where $\alpha = \lambda$. ∎

2 Sufficient conditions for Hölder regularity

Remark 1. In contrast with the Wiener criterion (1.2) it is not known whether the α-Hölder regularity of a boundary point is independent of the operator L.

Remark 2. The following upper estimate for the harmonic measure $\mathcal{H}_\Delta$, valid for any domain Ω, is formulated in [2] and proved in [3], Theorem 9. For arbitrary $a > 1$ and $b < 1$ there exists a constant $c = c(a, b)$ such that

$$\mathcal{H}_\Delta(x, \partial\Omega\backslash B_r) \le c\,\frac{\mathcal{Z}(a\,|x|)}{\mathcal{Z}(b\,r)} \tag{2.1}$$

where $a|x| \le br$ and $\mathcal{Z}$ is a bounded solution of the ordinary differential equation

$$\mathcal{Z}''(t) + \frac{n-1}{t}\mathcal{Z}'(t) - \frac{\mu(t)}{t^2}\mathcal{Z}(t) = 0 \tag{2.2}$$

on the interval $[0, 1]$. By μ we denote any function not exceeding the first eigenvalue $\Lambda(t)$ of the Dirichlet problem for the Laplace-Beltrami operator on the radial projection of $\Omega \cap \partial B_t$ to ∂B_1. If for instance

$$\Lambda(t) \ge \lambda(\lambda + n - 2), \qquad \lambda = const > 0, \tag{2.3}$$

for any small t, we may put $\mu(t) = \lambda(\lambda + n - 2)$ in (2.2). Then (2.1) with $\mathcal{Z}(t) = t^\lambda$ gives the inequality

$$\mathcal{H}_\Delta(x, \partial\Omega\backslash B_r) \le c\Big(\frac{|x|}{r}\Big)^\lambda.$$

By Proposition 1, we see that the geometrical condition (2.3) implies the α-regularity of O with respect to the Laplace operator for any $\alpha < \lambda$.

Note that using (2.1) we arrive at the inequality

$$|u_\varphi(x) - \varphi(0)| \le \omega(|x|) + c\,\mathcal{Z}(a\,|x|) \int_{|x|}^1 \frac{d\,\omega(t)}{\mathcal{Z}(b\,t)},$$

where u_φ is harmonic in Ω and ω is the continuity modulus of φ at 0 (compare with (1.5)).

Remark 3. Let $n > 2$. By [4], [5], the estimate

$$\mathcal{H}_L(x, \partial\Omega\backslash B_r) \le c_0 \exp\Big(-c \int_{|x|}^1 \mathrm{cap}(B_t\backslash\Omega) \frac{dt}{t^{n-1}}\Big) \tag{2.4}$$

holds with positive constants c_0 and c. Clearly, it implies the following condition sufficient for the Hölder regularity of O:

$$\frac{1}{|\log r|} \int_r^1 \mathrm{cap}(B_t\backslash\Omega) \frac{dt}{t^{n-1}} \ge const > 0 \tag{2.5}$$

for small r. Although this condition is sharp in a certain sense (see [6], [7]), it is not necessary in general for the Hölder regularity (cf. [8], [9]). In [9] the estimate (2.4) is improved and the following better sufficient condition for the Hölder regularity of a point with respect to Δ is obtained:

$$\frac{1}{|\log r|}\Big(\int \mathrm{cap}(B_t\backslash\Omega)\frac{dt}{t^{n-1}} + \int \frac{dt}{\delta_{\mathrm{cap}}(t)}\Big) > 0. \tag{2.6}$$

Here $\delta_{\mathrm{cap}}(r)$ is the interior capacitary radius of $\Omega \cap B_r$, i.e.

$$\delta_{\mathrm{cap}}(r) = \inf\{\delta > 0 : \mathrm{cap}(B_\delta(x)\backslash\Omega) \ge \varkappa\,\delta^{n-2} \ \text{ for all } x \in \partial B_r\}$$

with $B_\delta(x) = \{y : |y - x| < \delta\}$ and a sufficiently small constant $\varkappa$ (e.g. $\varkappa < 4^{-2n}$). The first integration in (2.6) is over all $t \in [r, 1]$ such that

$$\mathrm{cap}(B_t\backslash\Omega) \ge \varkappa\,(2t)^{n-2}$$

and the second integration is over the rest of the interval $[r, 1]$.

The following condition sufficient for the Hölder regularity with respect to the general operator L is found in [10]:

$$\liminf_{r\to 0} \frac{1}{|\log r|}\Big(N(r) + \int_r^1 \frac{\mathrm{cap}(B_\rho\backslash\Omega)}{\rho^{n-2}}\frac{d\rho}{\rho}\Big) > 0 \tag{2.7}$$

and

$$\limsup_{r\to 0} \frac{N(r)}{|\log r|} < \infty.$$

Here by $N(r)$ we denote the maximal number of pairwise disjoint intervals

$$(\rho - \delta_{\mathrm{cap}}(\rho),\ \rho + \delta_{\mathrm{cap}}(\rho))$$

which are contained in $(r, 1)$.

Needless to say, the conditions (2.5) and (2.7) ensuring the Hölder regularity of a boundary point do not depend on the operator L. However, the counterexamples constructed in [10] show that these conditions are not necessary for the Hölder regularity and therefore they do not imply independence of Hölder regularity of the operator L.

Remark 4. For the time being, we spoke about the Hölder regularity of *one* boundary point. A characterization of the simultaneous Hölder regularity of *all* points of $\partial\Omega$ with respect to the Laplace operator is given in [11], where it is shown that the global Hölder regularity is equivalent to the positivity of capacitary density of every point of $\partial\Omega$:

$$\liminf_{r\to 0} r^{2-n}\mathrm{cap}(B_r(x)\backslash\Omega) > const. > 0.$$

It is also demonstrated in [11] that there is no bounded domain that preserves the Hölder exponent 1 of the Dirichlet data. (The 1-Hölder regularity of one point is obviously possible).

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