ON SHARP AGMON-MIRANDA MAXIMUM PRINCIPLES

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Dedicated to Shmuel Agmon with great respect

Abstract. In this survey we formulate our results on different forms of maximum principles for linear elliptic equations and systems. We start with necessary and sufficient conditions for validity of the classical maximum modulus principle for solutions of second order strongly elliptic systems. This principle holds under rather heavy restrictions on the coefficients of the systems, for instance, it fails for the Stokes and Lamé systems. Next, we turn to sharp constants in more general maximum principles due to S. Agmon and C. Miranda. We consider higher order elliptic equations, Stokes and Lamé systems in a half-space as well as the system of planar deformed state in a half-plane.

1. Introduction

Maximum principles are fundamental properties of partial differential operators, both linear and nonlinear. They have important applications to various facts in the theory of boundary value problems for these operators.

The present survey contains formulations of authors' results on the best constants in different forms of maximum principles for linear elliptic equations and systems.

Let Ω be a bounded domain in the Euclidean space $\mathbb{R}^n$. Consider the uniformly elliptic equation

$$\sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j} - a_0(x)u = 0\ , \qquad x\in\Omega, \tag{1.1}$$

with bounded coefficients, positive-definite matrix $((a_{jk}(x)))$, and with $a_0(x)\geq 0$. The following basic fact is called the classical maximum modulus principle.

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An arbitrary solution $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}(\overline{\Omega})$ of equation (1.1) satisfies

$$\max_{\overline{\Omega}} \ |u| \le \max_{\partial\Omega} \ |u| \ . \tag{1.2}$$

This principle was obtained for different classes of solutions and under various assumptions about the coefficients.

Henceforth by $|\cdot|$ we denote the Euclidean length of a vector. If, instead of equation (1.1), we consider a homogeneous second order elliptic system with vector-valued solutions $\boldsymbol{u} \in [\mathrm{C}^2(\Omega)]^m \cap [\mathrm{C}(\overline{\Omega})]^m$ subject to the inequality

$$\max_{\overline{\Omega}} \ |\boldsymbol{u}| \le \max_{\partial\Omega} \ |\boldsymbol{u}| \ , \tag{1.3}$$

then we say that the classical maximum modulus principle holds for the system in question.

In section 2 we state criteria for validity of the classical maximum modulus principle for second order strongly elliptic systems following our papers [16, 25]. Note that this principle holds for the systems under rather heavy conditions on the coefficients. In particular, Polya's paper [37] contains an example showing that the classical maximum modulus principle fails for the displacement vector satisfying the Lamé system. The same is true for the velocity vector subject to the Stokes system.

The not so sharp as the classical modulus principle but incomparably more general properties of the same nature, which hold for elliptic equations of arbitrary order in smooth domains, were discovered by S. Agmon and C. Miranda. Similar results for elliptic systems were obtained by S. Agmon, A. Douglis, L. Nirenberg.

For solutions of a homogeneous elliptic equation of order 2ℓ, Agmon-Miranda principle is the estimate

$$\max_{\overline{\Omega}} \ |\nabla_{\ell-1} u| \le c(\Omega) \max_{\partial\Omega} \ |\nabla_{\ell-1} u|. \tag{1.4}$$

A weaker variant of the Agmon-Miranda maximum principle runs as follows:

$$\max_{\overline{\Omega}} |\nabla_{\ell-1} u| \le K \max_{\partial\Omega} |\nabla_{\ell-1} u| + C\|u\|_{L^1(\Omega)}. \tag{1.5}$$

An estimate similar to (1.4) for the biharmonic equation with two variables was proved earlier by C. Miranda [31]. For strongly elliptic equations with real coefficients in the two-dimensional case, a result of type (1.4) was obtained by C. Miranda in [32] with the help of Agmon's result in [1]. In the general case of equation with complex coefficients with any number of independent variables, (1.4) was established in Agmon [2].

The inequality

$$\max_{\overline{\Omega}} |\mathbf{u}| \le k(\Omega) \max_{\partial\Omega} |\mathbf{u}|, \qquad \Omega \subset \mathbb{R}^n \tag{1.6}$$

was established by Fichera [9] for solutions of the Lamé system and is called the Fichera's maximum principle in elastostatics. The construction for $k(\Omega)$ proposed by Fichera, gives some information about dependence of $k(\Omega)$ on the geometry of Ω, and elastic constants λ and μ. More general estimates for solutions of elliptic systems were obtained in Agmon, Douglis and Nirenberg [4].

In section 3, following our paper [15], we describe a sharp constant K in inequality (1.5) for solutions to higher order elliptic equation with constant complex coefficients. Besides, we give an explicit formula for the sharp constant K in Miranda's inequality

$$\sup_{\overline{\mathbb{R}^n_+}} |\nabla u| \le K \sup_{\partial\mathbb{R}^n_+} |\nabla u|$$

for a biharmonic function u in the half-space $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$.

In section 4 we collect the results obtained in [25]. Here we state explicit formulas for the sharp constant $\mathcal{K}(\mathbb{R}^n_+)$ in inequality

$$|\boldsymbol{u}(x)| \le \mathcal{K}(\mathbb{R}^n_+) \sup\{|\boldsymbol{u}(x')| : x' \in \partial\mathbb{R}^n_+\}, \tag{1.7}$$

for bounded solutions of the Lamé and Stokes systems in a half-space. Besides, we give sharp majorant for two components of the stress tensor of the planar deformed state in a half-plane.

The above mentioned authors' results with detailed proofs have been collected in monograph [17].

2. Classical maximum modulus principle for solutions to second order strongly elliptic systems

The first articles aiming at the study and applications of the classical maximum modulus principle for solutions of elliptic second order systems concerned systems with scalar coefficients in the first and second derivatives of the unknown vector-valued function (see Bitsadze [6, 7], Pini [36], Szeptycki [49]). These systems are weakly coupled: a system of partial differential equations is called weakly coupled if there are no derivatives in the coupling terms.

Sufficient conditions for validity of the maximum modulus principle, its modifications and generalizations for non-weakly coupled systems were given by Hile and Protter [11], C. Miranda [33], Rus [40], Stys [48], Sabitov [42], Wasowski [51]. In particular, C. Miranda [33] considered elliptic second order systems with a scalar principal part and with arbitrary coefficients in derivatives of order less than two. He found an algebraic inequality sufficient for the classical maximum modulus principle (conditions in Remarks 2.1 and 2.3 with the strict inequality sign). A survey of maximum principles for elliptic equations and systems with a scalar principal part is given by Protter [39].

An algebraic necessary and sufficient condition for validity of the maximum principle for the product $\alpha(x)|\boldsymbol{u}|$, where α is a certain function and $\boldsymbol{u}$ is a solution of the elliptic system with analytic coefficients, is due to Hong [12].

There is a number of results on the componentwise maximum principle for weakly coupled elliptic systems, in particular, on non-negativity of the components of a solution (see de Figueiredo and Mitidieri [10], Lenhart and Schaefer [20], López-Gómez and Molina-Meyer [22], Mitidieri and Sweers [34], Sirakov [45] and bibliography there). Various maximum principles for weakly coupled systems are discussed in the book by Protter and Weinberger [38]. Necessary and sufficient conditions for the componentwise and for the so-called "stochastic" extremum principles for solutions of elliptic systems of the second order are given by Kamynin and Khimchenko in [13] and [14], respectively.

In this section we describe criteria for validity of the classical maximum modulus principle for solutions of the strongly elliptic system

$$\sum_{j,k=1}^{n} \mathcal{A}_{jk}(x)\frac{\partial^2 \boldsymbol{u}}{\partial x_j \partial x_k} - \sum_{j=1}^{n} \mathcal{A}_j(x)\frac{\partial \boldsymbol{u}}{\partial x_j} - \mathcal{A}_0(x)\boldsymbol{u} = \boldsymbol{0} \tag{2.1}$$

with real or complex coefficients. Here $\mathcal{A}_{jk}, \mathcal{A}_j, \mathcal{A}_0$ are $(m\times m)$-matrix-valued functions and $\boldsymbol{u}$ is a m-component vector-valued function. Without loss of generality we assume that $\mathcal{A}_{jk} = \mathcal{A}_{kj}$.

2.1. **Model systems.** We begin with the simple case of the homogeneous operator with the constant coefficients.

2.1.1. *The case of real coefficients.* We introduce the operator

$$\mathfrak{A}_0(D_x) = \sum_{j,k=1}^{n} \mathcal{A}_{jk}\frac{\partial^2}{\partial x_j \partial x_k},$$

where $D_x = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ and $\mathcal{A}_{jk} = \mathcal{A}_{kj}$ are constant real $(m\times m)$-matrices. Assume that the operator $\mathfrak{A}_0$ is strongly elliptic, i.e. that for all $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_m) \in \mathbb{R}^m$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$, with

$\boldsymbol{\zeta}$, $\boldsymbol{\sigma} \neq \mathbf{0}$, we have the inequality

$$\left(\sum_{j,k=1}^{n} \mathcal{A}_{jk}\sigma_j\sigma_k \boldsymbol{\zeta},\ \boldsymbol{\zeta} \right) > 0.$$

Let Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$. Let $[\mathrm{C_b}(\overline{\Omega})]^m$ denote the space of bounded m-component vector-valued functions which are continuous in $\overline{\Omega}$. The norm on $[\mathrm{C_b}(\overline{\Omega})]^m$ is $\|\boldsymbol{u}\| = \sup\left\{|\boldsymbol{u}(x)| :\ x \in \overline{\Omega}\right\}$. The notation $[\mathrm{C_b}(\partial\Omega)]^m$ has a similar meaning. By $[\mathrm{C}^2(\Omega)]^m$ we denote the space of m-component vector-valued functions with continuous derivatives up to the second order in Ω. We omit the upper index m in notation of spaces in the case $m = 1$.

Let

$$\mathcal{K}(\Omega) = \sup \frac{\|\boldsymbol{u}\|_{[\mathrm{C_b}(\overline{\Omega})]^m}}{\|\boldsymbol{u}\|_{[\mathrm{C_b}(\partial\Omega)]^m}}, \tag{2.2}$$

where the supremum is taken over all vector-valued functions in the class $[\mathrm{C_b}(\overline{\Omega})]^m \cap [\mathrm{C}^2(\Omega)]^m$ satisfying the system $\mathfrak{A}_0(D_x)\boldsymbol{u} = \mathbf{0}$.

Clearly, $\mathcal{K}(\Omega)$ is the best constant in the inequality

$$|\boldsymbol{u}(x)| \leq \mathcal{K}(\Omega) \sup\{|\boldsymbol{u}(y)| : y \in \partial\Omega\}, \tag{2.3}$$

where $x \in \Omega$ and $\boldsymbol{u}$ is a solution of the system $\mathfrak{A}_0(D_x)\boldsymbol{u} = 0$ in the class $[\mathrm{C_b}(\overline{\Omega})]^m \cap [\mathrm{C}^2(\Omega)]^m$.

If $\mathcal{K}(\Omega) = 1$, then the classical maximum modulus principle holds for the system $\mathfrak{A}_0(D_x)\boldsymbol{u} = \mathbf{0}$.

According to Agmon, Douglis and Nirenberg [4], Lopatinskiĭ [21], Shapiro [44], Solonnikov [46] there exists a bounded solution of the Dirichlet problem

$$\mathfrak{A}_0(D_x)\boldsymbol{u} = \mathbf{0} \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{u} = \boldsymbol{f} \text{ on } \partial\mathbb{R}^n_+, \tag{2.4}$$

with $\boldsymbol{f} \in [\mathrm{C_b}(\partial\mathbb{R}^n_+)]^m$, such that $\boldsymbol{u}$ is continuous up to $\partial\mathbb{R}^n_+$, and can be represented in the form

$$\boldsymbol{u}(x) = \int_{\partial\mathbb{R}^n_+} \mathcal{M}\left(\frac{y-x}{|y-x|}\right) \frac{x_n}{|y-x|^n} \boldsymbol{f}(y')dy'. \tag{2.5}$$

Here $y = (y', 0),\ y' = (y_1, \dots, y_{n-1})$, and $\mathcal{M}$ is a continuous $(m \times m)$-matrix-valued function on the closure of the hemisphere $\mathbb{S}^{n-1}_- = \big\{x \in \mathbb{R}^n :\ |x| = 1, x_n < 0\big\}$ such that the integral

$$\int_{\mathbb{S}^{n-1}_-} \mathcal{M}(\sigma)d\sigma$$

is the identity matrix.

The uniqueness of the solution of the Dirichlet problem (2.4) in the class $[\mathrm{C_b}(\overline{\mathbb{R}^n_+})]^m \cap [\mathrm{C}^2(\mathbb{R}^n_+)]^m$ can be derived by means of standard arguments from (2.5) and by local estimates of the derivatives of solutions to elliptic systems (see Agmon, Douglis and Nirenberg [4], Solonnikov [46]).

The assertion below contains a representation of the sharp constant $\mathcal{K}(\mathbb{R}^n_+)$ in the pointwise estimate (2.3) for the case $\Omega = \mathbb{R}^n_+$.

Theorem 2.1. *The formula*

$$\mathcal{K}(\mathbb{R}^n_+) = \sup_{|\boldsymbol{z}|=1} \int_{\mathbb{S}^{n-1}_-} |\mathcal{M}^*(\sigma)\boldsymbol{z}| d\sigma \tag{2.6}$$

is valid, where asterisk denotes passage to the transposed matrix and $\boldsymbol{z} \in \mathbb{R}^m$.

The next statement gives a criterion for validity of the classical modulus principle for the system

$$\sum_{j,k=1}^{n} \mathcal{A}_{jk} \frac{\partial^2 \boldsymbol{u}}{\partial x_j \partial x_k} = \boldsymbol{0} \tag{2.7}$$

in $\mathbb{R}^n_+$ with the real constant coefficients.

Theorem 2.2. *The equality* $\mathcal{K}(\mathbb{R}^n_+) = 1$ *is satisfied if and only if*

$$\mathfrak{A}_0(D_x) = \mathcal{A} \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k}, \tag{2.8}$$

where $\mathcal{A}$ *and* $((a_{jk}))$ *are positive-definite constant matrices of orders* m *and* n*, respectively.*

The theorem below, together with Theorem 2.2 form a necessary condition for the validity of the classical maximum modulus principle for system (2.7).

Theorem 2.3. *Let* Ω *be a domain in* $\mathbb{R}^n$ *with compact closure and* C^1*-boundary. Then*

$$\mathcal{K}(\Omega) \geq \sup\{\mathcal{K}(\mathbb{R}^n_+(\boldsymbol{\nu})) : \boldsymbol{\nu} \in \mathbb{S}^{n-1}\},$$

where $\mathbb{R}^n_+(\boldsymbol{\nu})$ *is the half-space with inward normal* $\boldsymbol{\nu}$ *and* $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

Now, we formulate a necessary and sufficient condition ensuring the classical modulus principle for system (2.7) in a bounded domain with a smooth boundary.

Theorem 2.4. *Let Ω be a domain in $\mathbb{R}^n$ with compact closure and* C^1*-boundary. The equality $\mathcal{K}(\Omega) = 1$ holds if and only if the operator $\mathfrak{A}_0(D_x)$ is defined by* (2.8).

2.1.2. *The case of complex coefficients.* We introduce the operator

$$\mathfrak{C}_0(D_x) = \sum_{j,k=1}^{n} \mathcal{C}_{jk} \frac{\partial^2}{\partial x_j \partial x_k},$$

where $\mathcal{C}_{jk} = \mathcal{C}_{kj}$ are constant complex $(m \times m)$-matrices. Assume that the operator $\mathfrak{C}_0$ is strongly elliptic, that is

$$\Re \left(\sum_{j,k=1}^{n} \mathcal{C}_{jk} \sigma_j \sigma_k \boldsymbol{\zeta},\ \boldsymbol{\zeta} \right) > 0$$

for all $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$, with $\boldsymbol{\zeta},\ \boldsymbol{\sigma} \neq \mathbf{0}$. Here and henceforth $\mathbb{C}^m$ is a complex linear m-dimensional space with the elements $\boldsymbol{a} + i\boldsymbol{b}$, where $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$. The inner product in $\mathbb{C}^m$ is $(\boldsymbol{c}, \boldsymbol{d}) = c_1\overline{d}_1 + \cdots + c_m\overline{d}_m,\ \boldsymbol{c} = (c_1, \dots, c_m),\ \boldsymbol{d} = (d_1, \dots, d_m)$. The length of the vector $\boldsymbol{d}$ in $\mathbb{C}^m$ is $|\boldsymbol{d}| = (\boldsymbol{d}, \boldsymbol{d})^{1/2}$.

Let $\mathcal{R}_{jk}$ and $\mathcal{H}_{jk}$ be constant real $(m \times m)$-matrices such that $\mathcal{C}_{jk} = \mathcal{R}_{jk} + i\mathcal{H}_{jk}$, $i = \sqrt{-1}$. We define the operators

$$\mathfrak{R}_0(D_x) = \sum_{j,k=1}^{n} \mathcal{R}_{jk} \frac{\partial^2}{\partial x_j \partial x_k}, \quad \mathfrak{H}_0(D_x) = \sum_{j,k=1}^{n} \mathcal{H}_{jk} \frac{\partial^2}{\partial x_j \partial x_k}.$$

Separating the real and imaginary parts of the system $\mathfrak{C}_0(D_x)\boldsymbol{u} = 0$, where $\boldsymbol{u} = \boldsymbol{v} + i\boldsymbol{w}$, we get a system with real coefficients

$$\mathfrak{R}_0(D_x)\boldsymbol{v} - \mathfrak{H}_0(D_x)\boldsymbol{w} = \mathbf{0}, \quad \mathfrak{H}_0(D_x)\boldsymbol{v} + \mathfrak{R}_0(D_x)\boldsymbol{w} = \mathbf{0},$$

which, like the original system, is strongly elliptic.

We introduce the matrix operator

$$\mathfrak{K}_0(D_x) = \begin{pmatrix} \mathfrak{R}_0(D_x) & -\mathfrak{H}_0(D_x) \\ \mathfrak{H}_0(D_x) & \mathfrak{R}_0(D_x) \end{pmatrix}.$$

Let $[\mathbf{C_b}(\Omega)]^m$ be the space of m-component complex vector-valued functions $\boldsymbol{u} = \boldsymbol{v} + i\boldsymbol{w}$ which are bounded and continuous on $\Omega \subset \mathbb{R}^n$. The norm on $[\mathbf{C_b}(\Omega)]^m$ is $\|\boldsymbol{u}\| = \sup \big\{ (|\boldsymbol{v}(x)|^2 + |\boldsymbol{w}(x)|^2)^{1/2} \ :\ x \in \Omega \big\}$. The notation $[\mathbf{C_b}(\partial\Omega)]^m$ has a similar meaning. By $[\mathbf{C}^2(\Omega)]^m$ we denote the space of m-component complex vector-valued functions with continuous derivatives up to the second order in Ω.

By analogy with the definition (2.2) of $\mathcal{K}(\Omega)$, let

$$\mathcal{K}'(\Omega) = \sup \frac{\|\boldsymbol{u}\|_{[\mathbf{C_b}(\overline{\Omega})]^m}}{\|\boldsymbol{u}\|_{[\mathbf{C_b}(\partial\Omega)]^m}},$$

where the supremum is extended over all vector-valued functions in the class $[\mathbf{C}^2(\Omega)]^m \cap [\mathbf{C_b}(\overline{\Omega})]^m$ subject to the system $\mathfrak{C}_0(D_x)\boldsymbol{u} = 0$ in Ω.

It is clear that the constant $\mathcal{K}'(\Omega)$ for the system $\mathfrak{C}_0(D_x)\boldsymbol{u} = 0$ with complex coefficients coincides with the constant $\mathcal{K}(\Omega)$ for the system $\mathfrak{K}_0(D_x)\big\{\boldsymbol{v}, \boldsymbol{w}\big\} = 0$ with real coefficients if we replace m by $2m$, $\mathfrak{A}_0(D_x)$ by $\mathfrak{K}_0(D_x)$, and $\boldsymbol{u}$ by $\big\{\boldsymbol{v}, \boldsymbol{w}\big\}$ in definition (2.2). Therefore, all assertions about $\mathcal{K}'(\Omega)$ are direct consequences of the analogous assertions about $\mathcal{K}(\Omega)$. Using this fact, for solutions of the system

$$\sum_{j,k=1}^{n} \mathcal{C}_{jk} \frac{\partial^2 \boldsymbol{u}}{\partial x_j \partial x_k} = \mathbf{0} \tag{2.9}$$

with constant complex coefficients, we obtain the following four theorems.

Theorem 2.5. *The formula*

$$\mathcal{K}'(\mathbb{R}^n_+) = \sup_{|\boldsymbol{z}|=1} \int_{\mathbb{S}^{n-1}_-} |\mathcal{U}^*(\sigma)\boldsymbol{z}| d\sigma$$

is valid, where $\mathcal{U}$ is the $(2m \times 2m)$-matrix-valued function on $\mathbb{S}^{n-1}_-$ appearing in the integral representation for a solution of the Dirichlet problem in $\mathbb{R}^n_+$ for the system $\mathfrak{K}_0(D_x)\big\{\boldsymbol{v}, \boldsymbol{w}\big\} = 0$ (analogous to representation (2.5)) and $\boldsymbol{z} \in \mathbb{R}^{2m}$.

Theorem 2.6. *The equality $\mathcal{K}'(\mathbb{R}^n_+) = 1$ holds if and only if*

$$\mathfrak{C}_0(D_x) = \mathcal{C} \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k}, \tag{2.10}$$

where $\mathcal{C}$ is a constant complex-valued $(m\times m)$-matrix such that $\Re(\mathcal{C}\boldsymbol{\zeta}, \boldsymbol{\zeta}) > 0$ for all $\boldsymbol{\zeta} \in \mathbb{C}^m$, $\boldsymbol{\zeta} \neq \mathbf{0}$, and $((a_{jk}))$ is a real positive-definite $(n \times n)$-matrix.

Theorem 2.7. *Let Ω be a domain in $\mathbb{R}^n$ with compact closure and C^1-boundary. Then*

$$\mathcal{K}'(\Omega) \geq \sup\{\mathcal{K}'(\mathbb{R}^n_+(\boldsymbol{\nu})) : \boldsymbol{\nu} \in \mathbb{S}^{n-1}\},$$

where $\mathbb{R}^n_+(\boldsymbol{\nu})$ is a half-space with inward normal $\boldsymbol{\nu}$.

A necessary and sufficient condition for validity of the classical modulus principle for system (2.9) in a bounded domain runs as follows.

Theorem 2.8. *Let Ω be a domain in $\mathbb{R}^n$ with compact closure and C^1-boundary. The equality $\mathcal{K}'(\Omega) = 1$ holds if and only if the operator $\mathfrak{C}_0(D_x)$ has the form (2.10).*

2.2. **Systems with lower order terms and variable coefficients.** Now, we turn to linear elliptic system (2.1) of the general form.

2.2.1. *The case of real coefficients.* Let Ω be a domain in $\mathbb{R}^n$ with compact closure $\overline{\Omega}$ and with boundary $\partial\Omega$ of the class $\mathrm{C}^{2,\alpha}, 0 < \alpha \leq 1$. The space of $(m \times m)$-matrix-valued functions whose elements have continuous derivatives up to order k and satisfy the Hölder condition with exponent $\alpha, 0 < \alpha \leq 1$, on $\overline{\Omega}$ is denoted by $[\mathrm{C}^{k,\alpha}(\overline{\Omega})]^{m\times m}$.

We introduce the operator

$$\mathfrak{A}(x, D_x) = \sum_{j,k=1}^{n} \mathcal{A}_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} - \sum_{j=1}^{n} \mathcal{A}_j(x) \frac{\partial}{\partial x_j} - \mathcal{A}_0(x),$$

where $\mathcal{A}_{jk} = \mathcal{A}_{kj}, \mathcal{A}_j, \mathcal{A}_0$ are real $(m \times m)$-matrix-valued functions in the spaces

$$[\mathrm{C}^{2,\alpha}(\overline{\Omega})]^{m\times m}, \qquad [\mathrm{C}^{1,\alpha}(\overline{\Omega})]^{m\times m}, \qquad [\mathrm{C}^{\alpha}(\overline{\Omega})]^{m\times m},$$

respectively. If the coefficients of the operator $\mathfrak{A}(x, D_x)$ do not depend on x we use the notation $\mathfrak{A}(D_x)$. Let the principal homogeneous part of the operator $\mathfrak{A}(x, D_x)$ be denoted by $\mathfrak{A}_0(x, D_x)$.

We assume that $\mathfrak{A}(x, D_x)$ is strongly elliptic in $\overline{\Omega}$, which means that for all $x \in \overline{\Omega}$, $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_m) \in \mathbb{R}^m$, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$, with $\boldsymbol{\zeta}$, $\boldsymbol{\sigma} \neq \mathbf{0}$, the inequality

$$\left(\sum_{j,k=1}^{n} \mathcal{A}_{jk}(x) \sigma_j \sigma_k \boldsymbol{\zeta},\ \boldsymbol{\zeta} \right) > 0 \tag{2.11}$$

is satisfied.

The next assertion gives necessary and sufficient conditions for validity of the classical maximum modulus principle for system (2.1) in any subdomain ω of a bounded domain Ω with smooth boundary.

Theorem 2.9. *The classical maximum modulus principle*

$$\|\boldsymbol{u}\|_{[\mathrm{C}(\overline{\omega})]^m} \leq \|\boldsymbol{u}|_{\partial\omega}\|_{[\mathrm{C}(\partial\omega)]^m}, \tag{2.12}$$

holds for solutions of the system $\mathfrak{A}(x, D_x)\boldsymbol{u} = \mathbf{0}$ *in an arbitrary domain* $\omega \subset \Omega$ *with boundary from the class* $\mathrm{C}^{2,\alpha}$ *if and only if:*

(i) *for all* $x \in \overline{\Omega}$ *the equalities hold*

$$\mathcal{A}_{jk}(x) = \mathcal{A}(x) a_{jk}(x), \ \ 1 \leq j,\ k \leq n,$$

where $\mathcal{A}$ *and* $((a_{jk}))$ *are real positive-definite matrices in* $\overline{\Omega}$ *of orders* m *and* n*, respectively;*

(ii) *for all* $x \in \Omega$ *and any* $\boldsymbol{\xi}_j,\ \boldsymbol{\zeta} \in \mathbb{R}^m,\ j = 1, \ldots, n,$ *with* $(\boldsymbol{\xi}_j, \boldsymbol{\zeta}) = 0$ *the inequality*

$$\sum_{j,k=1}^{n} a_{jk}(x)(\boldsymbol{\xi}_j, \boldsymbol{\xi}_k) + \sum_{j=1}^{n} (\mathcal{A}^{-1}(x)\mathcal{A}_j(x)\boldsymbol{\xi}_j, \boldsymbol{\zeta}) + (\mathcal{A}^{-1}(x)\mathcal{A}_0(x)\boldsymbol{\zeta}, \boldsymbol{\zeta}) \geq 0$$

is valid.

The next assertion is a consequence of Theorem 2.9.

Corollary 2.1. *The classical maximum modulus principle*

$$\|\boldsymbol{u}\|_{[\mathrm{C}(\overline{\omega})]^m} \leq \|\boldsymbol{u}|_{\partial\omega}\|_{[\mathrm{C}(\partial\omega)]^m}$$

holds for solutions of the system

$$\sum_{j,k=1}^{n} \mathcal{A}_{jk}(x) \frac{\partial^2 \boldsymbol{u}}{\partial x_j \partial x_k} - \sum_{j=1}^{n} \mathcal{A}_j(x) \frac{\partial \boldsymbol{u}}{\partial x_j} = \mathbf{0}$$

in an arbitrary domain $\omega \subset \Omega$ *with boundary from the class* $\mathrm{C}^{2,\alpha}$ *if and only if:*

$$\mathcal{A}_{jk}(x) = \mathcal{A}(x) a_{jk}(x), \quad \mathcal{A}_j(x) = \mathcal{A}(x) a_j(x), \quad 1 \leq j,\ k \leq n.$$

Here $\mathcal{A}$ *and* $((a_{jk}))$ *are positive-definite matrix-valued functions in* $\overline{\Omega}$ *of orders* m *and* n*, respectively, and* a_j *are scalar functions with the same smoothness as* $\mathcal{A}_j$*.*

Remark 2.1. Condition (ii) of Theorem 2.9 can be replaced by the following:

for all $x \in \Omega$ *and for any* $\boldsymbol{\zeta} \in \mathbb{R}^m \backslash \{\mathbf{0}\}$ *the inequality holds:*

$$|\boldsymbol{\zeta}|^{-2} \sum_{j,k=1}^{n} b_{jk}(x)(\mathcal{A}^{-1}(x)\mathcal{A}_j(x)\boldsymbol{\zeta}, \boldsymbol{\zeta})(\mathcal{A}^{-1}(x)\mathcal{A}_k(x)\boldsymbol{\zeta}, \boldsymbol{\zeta})$$

$$- \sum_{j,k=1}^{n} b_{jk}(x)(\mathcal{A}_j^*(x)(\mathcal{A}^*(x))^{-1}\boldsymbol{\zeta},\ \mathcal{A}_k^*(x)(\mathcal{A}^*(x))^{-1}\boldsymbol{\zeta})$$

$$+4(\mathcal{A}^{-1}(x)\mathcal{A}_0(x)\boldsymbol{\zeta}, \boldsymbol{\zeta}) \geq 0.$$

Here $((b_{ij}))$ *is the inverse matrix of* $((a_{ij}))$ *and* $*$ *means passage to the transposed matrix.*

Remark 2.2. In [16] we showed by an example that the possibility to represent the principal part of the system $\mathfrak{A}(x, D_x)\boldsymbol{u} = \mathbf{0}$ in the form

$$\mathcal{A}(x) \sum_{j,k=1}^{n} a_{jk}(x) \frac{\partial^2 \boldsymbol{u}}{\partial x_j \partial x_k}$$

everywhere in $\overline{\Omega}$ is not necessary for validity of the classical maximum modulus principle

$$\|\boldsymbol{u}\|_{[\mathrm{C}(\overline{\Omega})]^m} \le \|\boldsymbol{u}|_{\partial\Omega}\|_{[\mathrm{C}(\partial\Omega)]^m},$$

where $\boldsymbol{u}$ is a solution of the system $\mathfrak{A}(x, D_x)\boldsymbol{u} = \mathbf{0}$ in Ω which belongs to $[\mathrm{C}^2(\Omega)]^m \cap [\mathrm{C}(\overline{\Omega})]^m$.

2.2.2. *The case of complex coefficients.* In this section we extend basic results of subsection 2.2.1 to system (2.1) with complex coefficients with solutions $\boldsymbol{u} = \boldsymbol{v} + i\boldsymbol{w}$, where $\boldsymbol{v}$ and $\boldsymbol{w}$ are m-component vector-valued functions with real-valued components. Here, similarly to subsection 2.2.1, we assume that Ω is a domain in $\mathbb{R}^n$ with compact closure $\overline{\Omega}$ and with boundary $\partial\Omega$ in the class $\mathrm{C}^{2,\alpha}, 0 < \alpha \le 1$.

For the spaces of matrix-valued functions with complex components we retain the same notation as in the case of real components but use bold letters.

We introduce the operator

$$\mathfrak{C}(x, D_x) = \sum_{j,k=1}^{n} \mathcal{C}_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} - \sum_{j=1}^{n} \mathcal{C}_j(x)\frac{\partial}{\partial x_j} - \mathcal{C}_0(x),$$

where $\mathcal{C}_{jk} = \mathcal{C}_{kj}, \mathcal{C}_j, \mathcal{C}_0$ are complex $(m \times m)$-matrix-valued functions in the spaces

$$[\mathbf{C}^{2,\alpha}(\overline{\Omega})]^{m\times m}, \qquad [\mathbf{C}^{1,\alpha}(\overline{\Omega})]^{m\times m}, \qquad [\mathbf{C}^{\alpha}(\overline{\Omega})]^{m\times m},$$

respectively. Suppose that the operator $\mathfrak{C}(x, D_x)$ is strongly elliptic in $\overline{\Omega}$, that is for all $x \in \overline{\Omega}$, $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$, with $\boldsymbol{\zeta}$, $\boldsymbol{\sigma} \ne \mathbf{0}$, the inequality

$$\Re\left(\sum_{j,k=1}^{n} \mathcal{C}_{jk}(x)\sigma_j\sigma_k\boldsymbol{\zeta},\ \boldsymbol{\zeta} \right) > 0$$

holds.

Let $\mathcal{R}_{jk}, \mathcal{H}_{jk}, \mathcal{R}_j, \mathcal{H}_j, \mathcal{R}_0, \mathcal{H}_0$ be real $(m\times m)$-matrix-valued functions such that

$$\mathcal{C}_{jk} = \mathcal{R}_{jk} + i\mathcal{H}_{jk},\ \ \mathcal{C}_j = \mathcal{R}_j + i\mathcal{H}_j,\ \ \mathcal{C}_0 = \mathcal{R}_0 + i\mathcal{H}_0.$$

We use the notation

$$\mathfrak{R}(x, D_x) = \sum_{j,k=1}^{n} \mathcal{R}_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} - \sum_{j=1}^{n} \mathcal{R}_j(x)\frac{\partial}{\partial x_j} - \mathcal{R}_0(x),$$

$$\mathfrak{H}(x, D_x) = \sum_{j,k=1}^{n} \mathcal{H}_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} - \sum_{j=1}^{n} \mathcal{H}_j(x)\frac{\partial}{\partial x_j} - \mathcal{H}_0(x).$$

Separating the real and imaginary parts of the system $\mathfrak{C}(x, \partial/\partial x)\boldsymbol{u} = \boldsymbol{0}$, where $\boldsymbol{u} = \boldsymbol{v} + i\boldsymbol{w}$, we get the following system with real coefficients,

$$\mathfrak{R}(x, D_x)\boldsymbol{v} - \mathfrak{H}(x, D_x)\boldsymbol{w} = \boldsymbol{0}, \quad \mathfrak{H}(x, D_x)\boldsymbol{v} + \mathfrak{R}(x, D_x)\boldsymbol{w} = \boldsymbol{0},$$

which, like the original system, is strongly elliptic.

All the assertions below in this subsection are corollaries of the corresponding results in subsection 2.2.1. The next statement is analogous to Theorem 2.9.

Theorem 2.10. *The classical maximum modulus principle*

$$\|\boldsymbol{u}\|_{[\mathbf{C}(\overline{\omega})]^m} \le \|\boldsymbol{u}|_{\partial\omega}\|_{[\mathbf{C}(\partial\omega)]^m}$$

is valid for solutions of the system $\mathfrak{C}(x, D_x)\boldsymbol{u} = \boldsymbol{0}$ *in an arbitrary domain* $\omega \subset \Omega$ *with boundary from the class* $\mathrm{C}^{2,\alpha}$ *if and only if:*

(i) *for all* $x \in \overline{\Omega}$ *the equalities*

$$\mathcal{C}_{jk}(x) = \mathcal{C}(x) a_{jk}(x), \;\; 1 \le j, k \le n,$$

hold, where $\mathcal{C}$ *is a complex* $(m \times m)$*-matrix-valued function such that* $\Re(\mathcal{C}(x)\boldsymbol{\zeta}, \boldsymbol{\zeta}) > 0$ *for all* $x \in \overline{\Omega}, \boldsymbol{\zeta} \in \mathbb{C}^m \backslash \{\boldsymbol{0}\}, ((a_{jk}))$ *is a real positive-definite* $(n \times n)$*-matrix-valued function in* $x \in \overline{\Omega}$ *of order n;*

(ii) *for all* $x \in \Omega$ *and any* $\boldsymbol{\xi}_j, \boldsymbol{\zeta} \in \mathbb{C}^m, j = 1, \dots, n,$ *such that* $\Re(\boldsymbol{\xi}_j, \boldsymbol{\zeta}) = 0$ *the inequality*

$$\Re\left\{\sum_{j,k=1}^{n} a_{jk}(x)(\boldsymbol{\xi}_j, \boldsymbol{\xi}_k) + \sum_{j=1}^{n} (\mathcal{C}^{-1}(x)\mathcal{C}_j(x)\boldsymbol{\xi}_j, \boldsymbol{\zeta}) + (\mathcal{C}^{-1}(x)\mathcal{C}_0(x)\boldsymbol{\zeta}, \boldsymbol{\zeta})\right\} \ge 0$$

is valid.

The following assertion is a consequence of Theorem 2.10.

Corollary 2.2. *The classical maximum modulus principle*

$$\|\boldsymbol{u}\|_{[\mathbf{C}(\overline{\omega})]^m} \le \|\boldsymbol{u}|_{\partial\omega}\|_{[\mathbf{C}(\partial\omega)]^m}$$

holds for solutions of the system

$$\sum_{j,k=1}^{n} \mathcal{C}_{jk}(x)\frac{\partial^2 \boldsymbol{u}}{\partial x_j \partial x_k} - \sum_{j=1}^{n} \mathcal{C}_j(x)\frac{\partial \boldsymbol{u}}{\partial x_j} = \boldsymbol{0}$$

in an arbitrary domain $\omega \subset \Omega$ *with boundary from the class* $\mathrm{C}^{2,\alpha}$ *if and only if*

$$\mathcal{C}_{jk}(x) = \mathcal{C}(x) a_{jk}(x), \quad \mathcal{C}_j(x) = \mathcal{C}(x) a_j(x), \;\; 1 \le j, \; k \le n.$$

Here $\mathcal{C}(x)$ *and* $((a_{jk}))$ *are the matrix-valued functions defined in Theorem* 2.10 *and* a_j *are real scalar functions with the same smoothness as* $\mathcal{C}_j$*.*

Remark 2.3. As in Remark 2.1, condition (ii) in Theorem 2.10 can be replaced by the following one:

for all $x \in \Omega$ and for any $\boldsymbol{\zeta} \in \mathbb{C}^m \backslash \{\mathbf{0}\}$ the inequality

$$|\boldsymbol{\zeta}|^{-2} \sum_{j,k=1}^{n} b_{jk}(x) \Re(\mathcal{C}^{-1}(x)\mathcal{C}_j(x)\boldsymbol{\zeta}, \boldsymbol{\zeta})(\mathcal{C}^{-1}(x)\mathcal{C}_k(x)\boldsymbol{\zeta}, \boldsymbol{\zeta})$$

$$- \sum_{j,k=1}^{n} b_{jk}(x)(\mathcal{C}_j^*(x)(\mathcal{C}^*(x))^{-1}\boldsymbol{\zeta},\ \mathcal{C}_k^*(x)(\mathcal{C}^*(x))^{-1}\boldsymbol{\zeta})$$

$$+4\Re(\mathcal{C}^{-1}(x)\mathcal{C}_0(x)\boldsymbol{\zeta}, \boldsymbol{\zeta}) \geq 0$$

is valid, where $((b_{jk}))$ is the inverse matrix of $((a_{jk}))$ and $\mathcal{C}_j^*(x)$ is the adjoint matrix of $\mathcal{C}_j(x)$.

For the scalar uniformly elliptic equation with complex coefficients of the general form

$$\sum_{j,k=1}^{n} c_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{j=1}^{n} c_j(x) \frac{\partial u}{\partial x_j} - c_0(x)u = 0, \tag{2.13}$$

Theorem 2.10 and Remark 2.3 imply

Corollary 2.3. *The classical maximum modulus principle*

$$\|u\|_{\mathbf{C}(\overline{\omega})} \leq \|u|_{\partial\omega}\|_{\mathbf{C}(\partial\omega)}$$

is valid for solutions of equation (2.13) *in an arbitrary subdomain ω of Ω with boundary $\partial\omega$ of the class $\mathrm{C}^{2,\alpha}$ if and only if for all $x \in \overline{\Omega}$:*

(i) *$c_{jk}(x) = c(x)a_{jk}(x)$, $1 \leq j, k \leq n$, where $\Re c(x) > 0$ and $((a_{jk}))$ is a real positive-definite $(n \times n)$-matrix-valued function;*

(ii) *the inequality*

$$4\Re\left(\frac{c_0(x)}{c(x)}\right) \geq \sum_{j,k=1}^{n} b_{jk}(x) \Im\left(\frac{c_j(x)}{c(x)}\right) \Im\left(\frac{c_k(x)}{c(x)}\right)$$

holds, where $((b_{jk}))$ is the $(n \times n)$-matrix inverse of $((a_{jk}))$.

3. Sharp Agmon-Miranda estimates for the gradients of solutions to higher order elliptic equations

Everywhere in this section, by smoothness we mean the membership in C^∞. Suppose that Ω is a domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and a compact closure $\overline{\Omega}$. We consider the elliptic operator

$$P(D_x) = \sum_{|\beta| \leq 2\ell} a_\beta D_x^\beta$$

with constant complex coefficients, where $D_x^\beta = \partial^{|\beta|}/\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}$, and $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index of order $|\beta| = \beta_1 + \dots + \beta_n$. By $P_0(\xi)$ we denote the principal homogeneous part of the polynomial $P(\xi)$. For $n = 2$ we assume also that all ξ_2-roots of the polynomial $P_0(\xi)$ for all $\xi_1 \in \mathbb{R}\backslash\{0\}$.

Let $\mathbb{R}^n_+(\boldsymbol{\nu}) = \big\{x \in \mathbb{R}^n : (x, \boldsymbol{\nu}) > 0\big\}$, where $\boldsymbol{\nu}$ is a unit vector and let $K(\boldsymbol{\nu})$ be the best constant in the Agmon-Miranda inequality

$$\sup_{\overline{\mathbb{R}^n_+(\boldsymbol{\nu})}} |\nabla_{\ell-1} u| \le K(\boldsymbol{\nu}) \sup_{\partial\mathbb{R}^n_+(\boldsymbol{\nu})} |\nabla_{\ell-1} u|. \tag{3.1}$$

Here

$$|\nabla_{\ell-1} u| = \left(\sum_{|\beta|=\ell-1} \frac{(\ell-1)!}{\beta!} |D_x^\beta u|^2 \right)^{1/2},$$

and u is an arbitrary solution of the equation $P_0(D_x)u = 0$, smooth in $\overline{\mathbb{R}^n_+(\boldsymbol{\nu})}$ and such that $u(x) = O(|x|^{\ell-1})$ for large $|x|$.

The following assertion gives a best constant in a weak form (1.5) of the Agmon-Miranda inequality.

Theorem 3.1. *For any solution of the equation $P(D_x)u = 0$, smooth on $\overline{\Omega}$, the inequality*

$$\max_{\overline{\Omega}} |\nabla_{\ell-1} u| \le \left(\sup_{|\boldsymbol{\nu}|=1} K(\boldsymbol{\nu}) \; + \; \varepsilon \right) \max_{\partial\Omega} |\nabla_{\ell-1} u| + c(\varepsilon) \|u\|_{L^1(\Omega)} \tag{3.2}$$

is valid, where ε is any positive number and $c(\varepsilon)$ is a positive constant independent of u.

In the next theorem we give we the sharp constant K in the C. Miranda inequality

$$\sup_{\overline{\mathbb{R}^n_+}} |\nabla u| \le K \sup_{\partial\mathbb{R}^n_+} |\nabla u|, \tag{3.3}$$

where u is a solution of the biharmonic equation in $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ from $\mathrm{C}^\infty(\overline{\mathbb{R}^n_+})$ and $u(x) = O(|x|)$ for large $|x|$.

Theorem 3.2. *The sharp constant K in inequality* (3.3) *is given by*

$$K = \frac{2\Gamma(\frac{n}{2})}{\sqrt{\pi}\; \Gamma(\frac{n-1}{2})} \int_0^{\pi/2} \left[4 + n(n-4)\cos^2\vartheta\right]^{1/2} \sin^{n-2}\vartheta d\vartheta.$$

In particular, $K = 4/\pi$ for $n = 2, K = 1/2 + 2\pi\sqrt{3}/9$ for $n = 3$ and $K = 2$ for $n = 4$.

The last assertion was proved in [25] for the case $n = 2$ and in [15] for any n.

4. Sharp Agmon-Miranda estimates for solutions of the Lamé, Stokes and planar deformed state systems

Polya's example [37] demonstrated that the best factor in the inequality between the modulus of the elastic displacement inside the three-dimensional ball and its maximum value on the boundary of that ball exceeds 1. A similar inequality (1.6) with coefficient depending on the domain holds for domains with smooth boundary, and this inequality for solutions of the Lamé system is called Fichera's maximum principle (see Fichera [9]). This principle is a particular case of the maximum principles for general elliptic systems (see Agmon, Douglis and Nirenberg [4], Cannarsa [8], Schulze [43], Solonnikov [46], Zhou [53]).

There are works on the Agmon-Miranda type maximum principle for elliptic systems in domains with singularities at the boundary (Maz'ya and Plamenevskiĭ [26], Albinus [5], Maz'ya and Rossmann [27, 28, 30] and the bibliography there).

Estimates for the maximum modulus of velocity vector subject to the nonlinear Navier-Stokes system were obtained by Solonnikov [47] for smooth domains, by Maz'ya and Rossmann [29] for polyhedral domains, by Russo [41] for Lipschitz domains. Agmon-Miranda maximum principle as well as existence and uniqueness of solutions to Stokes system and elastostatics were treated by Maremonti and Russo [23, 24], and Tartaglione [50]. A survey of maximum principles for the elasticity theory is given by Wheeler [52].

4.1. **The Lamé and Stokes systems.** In the half-space $\mathbb{R}^n_+$, $n \geq 2$, let us consider the Lamé system

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\text{grad div } \boldsymbol{u} = \boldsymbol{0}, \tag{4.1}$$

and the Stokes system

$$\nu\Delta\boldsymbol{u} - \text{grad } p = 0, \quad \text{div } \boldsymbol{u} = \boldsymbol{0}, \tag{4.2}$$

with the Dirichlet boundary condition

$$\boldsymbol{u}\big|_{x_n=0} = \boldsymbol{f}, \tag{4.3}$$

where λ and μ are the Lamé constants, ν is the kinematic coefficient of viscosity, $\boldsymbol{f} \in [\mathrm{C}_b(\partial\mathbb{R}^n_+)]^n$, $\boldsymbol{u} = (u_1, \dots, u_n)$ is the displacement vector of an elastic medium or the velocity vector of a fluid, and p is the pressure in the fluid.

For the solution $\boldsymbol{u} \in [\mathrm{C}^2(\mathbb{R}^n_+)]^n \cap [\mathrm{C_b}(\overline{\mathbb{R}^n_+})]^n$ of problems (4.1), (4.3) and (4.2), (4.3) we have the representation (see Kupradze, Gegelia, Basheleishvili and Burchuladze[18], Ladyzhenskaya [19])

$$\boldsymbol{u}(x) = \int_{\partial\mathbb{R}^n_+} U_\kappa\left(\frac{y-x}{|y-x|}\right)\frac{x_n}{|y-x|^n}\boldsymbol{f}(y')dy',$$

where $x \in \mathbb{R}^n_+$, $y = (y', 0)$, $y' = (y_1, \dots, y_{n-1})$. Here $\kappa = 1$ for the Stokes system, $\kappa = (\lambda+\mu)(\lambda+3\mu)^{-1}$ for the Lamé system, and U_κ is the $(n \times n)$-matrix-valued function on $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ with the entries

$$\frac{2}{\omega_n}\Big[(1-\kappa)\delta_{ij} + n\kappa\frac{(y_i-x_i)(y_j-x_j)}{|y-x|^2}\Big],$$

ω_n being the area of the sphere $\mathbb{S}^{n-1}$.

Now, we give explicit formulas for sharp constant in inequality (1.7) for bounded solutions of Lamé and Stokes systems in a half-space.

Theorem 4.1. *The sharp constant $\mathcal{K}(\mathbb{R}^n_+)$ for the Lamé and the Stokes systems in*

$$|\boldsymbol{u}(x)| \le \mathcal{K}(\mathbb{R}^n_+)\sup\{|\boldsymbol{u}(x')| : x' \in \partial\mathbb{R}^n_+\}$$

has the form

$$\mathcal{K}(\mathbb{R}^n_+) = \frac{2\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}\int_0^{\pi/2}\left[(1-\kappa)^2 + n\kappa(n\kappa-2\kappa+2)\cos^2\vartheta\right]^{1/2}\sin^{n-2}\vartheta\, d\vartheta$$

and the inequality $\mathcal{K}(\mathbb{R}^n_+) > 1$ holds for $\kappa \ne 0$.

In the case $\kappa = 1$, i.e., for an n-dimensional Stokes system,

$$\mathcal{K}(\mathbb{R}^n_+) = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2})}.$$

Below we give consequences of Theorem 4.1 for the cases $n = 2$ and $n = 3$, respectively.

Corollary 4.1. *The equality*

$$\begin{aligned}\mathcal{K}(\mathbb{R}^2_+) &= \frac{2}{\pi}(1+\kappa)E\left(\frac{2\sqrt{\kappa}}{1+\kappa}\right)\\ &= 1 + \frac{1}{2^2}\kappa^2 + \frac{1}{2^2 4^2}\kappa^4 + \dots + \left[\frac{(2m-3)!!}{2^m m!}\right]^2\kappa^{2m} + \dots\end{aligned}$$

is valid, where E is the complete elliptic integral of the second kind. In particular, $\mathcal{K}(\mathbb{R}^2_+) = 4/\pi$ for $\kappa = 1$.

Corollary 4.2. *The equality*

$$\mathcal{K}(\mathbb{R}^3_+) = \frac{1}{2}\left[1 + 2\kappa + \frac{(1-\kappa)^2}{\sqrt{3\kappa(\kappa+2)}} \log \frac{1 + 2\kappa + \sqrt{3\kappa(\kappa+2)}}{1-\kappa}\right]$$

is valid. In particular, $\mathcal{K}(\mathbb{R}^3_+) = 3/2$ *for* $\kappa = 1$.

4.2. **Planar deformed state.** Let σ_{11}, σ_{12} and σ_{22} be the components of the stress tensor in the half-plane $\mathbb{R}^2_+$. Consider the system of equations in $\mathbb{R}^2_+$ for the stresses in a planar deformed state (see, for example, Muskhelishvili [35]):

$$\partial\sigma_{11}/\partial x_1 + \partial\sigma_{12}/\partial x_2 = 0,$$
$$\partial\sigma_{12}/\partial x_1 + \partial\sigma_{22}/\partial x_2 = 0,$$
$$\Delta(\sigma_{11} + \sigma_{22}) = 0,$$

with the boundary conditions

$$\sigma_{12}(x_1, 0) = p_1(x_1), \quad \sigma_{22}(x_1, 0) = p_2(x_1),$$

where p_1 and p_2 are continuous and bounded functions on $\partial\mathbb{R}^2_+$.

Theorem 4.3. *The sharp constant in the inequality*

$$||(\sigma_{12}^2 + \sigma_{22}^2)^{1/2}||_{C(\overline{\mathbb{R}^2_+})} \leq \mathcal{K}||(\sigma_{12}^2 + \sigma_{22}^2)^{1/2}||_{C(\partial\mathbb{R}^2_+)}$$

is equal to $4/\pi$.

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