The Inhomogeneous Dirichlet Problem for the Stokes System in Lipschitz Domains with Unit Normals Close to VMO

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Dedicated to the memory of Solomon G. Mikhlin

Abstract

The goal of this work is to treat the inhomogeneous Dirichlet problem for the Stokes system in a Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$. Our main result is a well-posedness result formulated on the scales of Besov-Triebel-Lizorkin spaces, in the case in which the outward unit normal ν to Ω has small mean oscillation.

1 Introduction

The aim of this paper is to discuss the well-posedness of the inhomogeneous Dirichlet problem for the Stokes system of linearized hydrostatics in an arbitrary bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, when both the solution and the data belong to Besov spaces:

$$\begin{aligned} &\Delta\vec{u} - \nabla\pi = \vec{f} \in B^{p,q}_{s+\frac{1}{p}-2}(\Omega), \quad \operatorname{div}\vec{u} = g \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \\ &\vec{u} \in B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \operatorname{Tr}\vec{u} = \vec{h} \in B^{p,q}_{s}(\partial\Omega). \end{aligned} \tag{1.1}$$

As usual, $\vec{u}$ is the velocity field and π stands for the pressure function. Under the assumption that the outward unit normal ν of Ω has sufficiently small mean oscillation, relative to p, q, s and the Lipschitz constant of $\partial\Omega$, our main result states that the problem (1.1) is uniquely solvable, granted that the data satisfies some necessary compatibility conditions. We are also interested in the case in which the smoothness is measured on the Triebel-Lizorkin scale. More specifically, we have the following result (for notation, definitions and background material see § 2).

Theorem 1.1 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, of arbitrary topology, and denote by ν, σ, the outward unit normal and surface measure on $\partial\Omega$, respectively. Assume that $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, and consider the inhomogeneous Dirichlet problem for the Stokes system with (1.1), subject to the (necessary) compatibility condition*

$$\int_{\partial\mathcal{O}} \langle \nu, \vec{h} \rangle \, d\sigma = \int_{\mathcal{O}} g(X)\, dX, \quad \textit{for every component } \mathcal{O} \textit{ of } \Omega. \tag{1.2}$$

Then there exists $\delta > 0$ which depends only on the Lipschitz character of Ω and the exponent p, with the property that if

$$\{\nu\}_{\mathrm{Osc}(\partial\Omega)} := \lim_{\varepsilon\to 0} \left(\sup_{B_\varepsilon} ⨍_{B_\varepsilon\cap\partial\Omega} ⨍_{B_\varepsilon\cap\partial\Omega} \Big| \nu(X) - \nu(Y) \Big| \, d\sigma(X) d\sigma(Y) \right) < \delta, \tag{1.3}$$

*2000 *Math Subject Classification.* Primary: 35J25, 42B20, 46E35. Secondary 35J05, 45B05, 31B10.
Key words: Stokes system, Lipschitz domains, boundary problems, Besov-Triebel-Lizorkin spaces
The research of the authors was supported in part by the NSF

where the supremum is taken over the collection $\{B_\varepsilon\}$ of disks with centers on $\partial\Omega$ and of radius $\le \varepsilon$, then (1.1) is well-posed (with uniqueness modulo locally constant functions in Ω for the pressure). Hence, in the particular case when the Lipschitz domain Ω is such that $\nu \in \mathrm{VMO}\,(\partial\Omega)$, the problem (1.1) is well-posed whenever

$$\tfrac{n-1}{n} < p \le \infty, \quad 0 < q \le \infty \quad \textit{and} \quad (n-1)(\tfrac{1}{p} - 1)_+ < s < 1, \tag{1.4}$$

(where $(a)_+ := \max\{a, 0\}$). Consider three Furthermore, the solution has an integral representation formula in terms of hydrostatic layer potential operators and satisfies natural estimates. Concretely, there exists a finite, positive constant $C = C(\Omega, p, q, s, n)$ such that

$$\|\vec{u}\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} + \inf_c \|\pi - c\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)} \le C\|\vec{f}\|_{B^{p,q}_{s+\frac{1}{p}-2}(\Omega)} + C\|g\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)} + C\|\vec{h}\|_{B^{p,q}_s(\partial\Omega)}, \tag{1.5}$$

where the infimum is taken over all locally constant functions c in Ω.

Moreover, analogous well-posedness results hold on the Triebel-Lizorkin scale, i.e. for the problem

$$\begin{aligned} &\Delta\vec{u} - \nabla\pi = \vec{f} \in F^{p,q}_{s+\frac{1}{p}-2}(\Omega), \quad \operatorname{div}\vec{u} = g \in F^{p,q}_{s+\frac{1}{p}-1}(\Omega), \\ &\vec{u} \in F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in F^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \operatorname{Tr}\vec{u} = \vec{g} \in B^{p,p}_s(\partial\Omega), \end{aligned} \tag{1.6}$$

where the data is, once again, made subject to (1.2). This time, in addition to the previous conditions imposed on the indices p, q, it is also assumed that $p, q < \infty$.

Above, the Besov and Triebel-Lizorkin scales $A^{p,q}_s(\Omega)$, $A \in \{B, F\}$, are defined by restricting the (tempered) distributions from the corresponding spaces in $\mathbb{R}^n$ to the open set Ω. Also, $B^{p,q}_s(\partial\Omega)$ stands for the Besov class on the Lipschitz manifold $\partial\Omega$, obtained by transporting (via a partition of unity and pull-back) the standard scale $B^{p,q}_s(\mathbb{R}^{n-1})$. The reader should be advised that we make no notational distinction between these smoothness spaces of scalar-valued functions and their natural counterparts for vector-valued functions. It should be noted that conditions (1.4) describe the largest range of indices p, q, s for which the Besov spaces $B^{p,q}_s(\partial\Omega)$ can be meaningfully defined on the Lipschitz manifold $\partial\Omega$.

Regarding the nature of our main result, a few comments are in order. First, no topological restrictions have been imposed on the Lipschitz domain Ω (in particular, the boundary can be disconnected). This is significant since our approach is via boundary layer potentials, whose invertibility properties are directly affected by topological nature of the domain. We overcome this difficulty by devising a suitable combination of such layer potentials. Second, appropriate versions of the above results hold for unbounded Lipschitz domains with compact boundaries, for which (1.3) holds, and for Neumann-type boundary conditions. Third, one can show that actually $\{\nu\}_{\mathrm{Osc}(\partial\Omega)} \approx \operatorname{dist}(\nu, \mathrm{VMO}(\partial\Omega))$, where the distance is taken in $\mathrm{BMO}(\partial\Omega)$. Then condition (1.3) becomes equivalent to

$$\operatorname{dist}(\nu\,, \mathrm{VMO}\,(\partial\Omega)) < \delta \tag{1.7}$$

where the distance is measured in $\mathrm{BMO}\,(\partial\Omega)$. Fourth, codimension one surfaces in $\mathbb{R}^n$ whose unit normal has small BMO norm have been studied by S. Semmes in [50] from the perspective of Geometric Measure Theory, whereas, in the context of PDE's, the class of Lipschitz domains for which the smallness condition (1.3) holds has been first introduced by V. Maz'ya, M. Mitrea and T. Shaposhnikova in [41] (where the authors have established the well-posedness of the inhomogeneous Dirichlet problem for higher-order elliptic systems with bounded, complex-valued, coefficients). Elliptic PDE's (with L^p boundary data and non-tangential maximal function estimates for the solution) in the class of (two-sided) NTA domains (in the sense of Jerison and Kenig, [25]) with Ahlfors regular boundaries and satisfying the smallness condition (1.7) (with δ depending on p and the natural geometrical characteristics of the domain) have been recently treated by S. Hofmann, M. Mitrea and M. Taylor in [24].

The problem we address in this paper has a long history and the literature dealing with related issues is remarkably rich. When $\partial\Omega$ is sufficiently smooth, various cases (typically corresponding to Sobolev spaces with an integer amount of smoothness) have been dealt with by V.A. Solonnikov [54], L. Cattabriga [10], R. Temam [57], Y. Giga [23], W. Varnhorn [58], R. Dautray and J.-L. Lions [18], among others, when $\partial\Omega$ is at least of class C^2. This has been subsequently extended by C. Amrouche and V. Girault [6] to the case when $\partial\Omega \in C^{1,1}$ and, further, by G.P. Galdi, C.G. Simader and H. Sohr [21] when $\partial\Omega$ is Lipschitz, with a small Lipschitz constant. There is also a wealth of results related to Theorem 1.1 in the case when Ω is a polygonal domain in $\mathbb{R}^2$, or a polyhedral domain in $\mathbb{R}^3$. An extended account of this field of research can be found in V.A. Kozlov, V.G. Maz'ya and J. Rossmann's monograph [35], which also contains pertinent references to earlier work. Here we also wish to mention the recent work by V. Maz'ya and J. Rossmann [42].

In the case of a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, the Dirichlet and Regularity problem, with L^p nontangential maximal function estimates for the solution, have been solved when $|p-2|$ is small by E.B. Fabes, C.E. Kenig and G.C. Verchota in [20], and when $n = 3$ and $2 - \varepsilon < p < \infty$ by Z. Shen in [52]. Higher dimensional versions of the results in [52] have been worked out by J. Kilty in [31]. More recently, the homogeneous and inhomogeneous problem for the Stokes system equipped with Dirichlet, Neumann and transmission boundary conditions in arbitrary Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$, has been solved by M. Mitrea and M. Wright in [47]. To describe the portion of this work pertaining to the case of the inhomogeneous Dirichlet problem, for each $\varepsilon \in (0, 1]$ and $n \geq 2$ consider the two-dimensional region $\mathcal{R}_{n,\varepsilon}$ in the $(s, 1/p)$-plane, which depends on the dimension as follows:

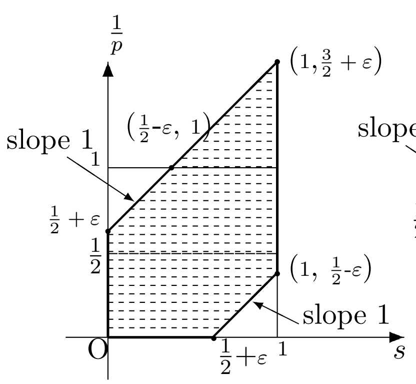

Figure 1:
$\mathcal{R}_{n,\varepsilon}$ for $n = 2$

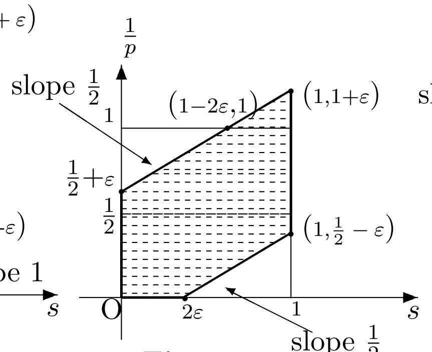

Figure 2:
$\mathcal{R}_{n,\varepsilon}$ for $n = 3$

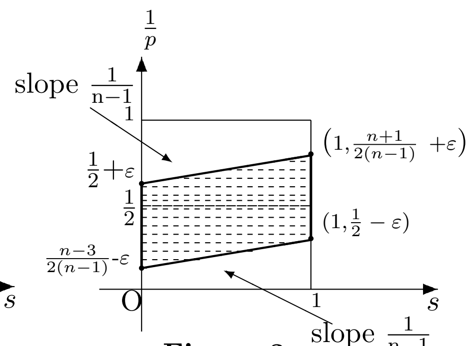

Figure 3:
$\mathcal{R}_{n,\varepsilon}$ for $n \geq 4$

It has been shown in [47] that, given a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, there exists $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ such that the problem (1.1) (subject to the compatibility conditions (1.2)) is well-posed whenever $0 < q \leq \infty$ and the pair $(s, 1/p)$ belongs to the region $\mathcal{R}_{n,\varepsilon}$, depicted above. Also, a similar result holds for (1.6) provided $p, q < \infty$. Related earlier results by R.M. Brown and Z. Shen for three-dimensional Lipschitz domains can be found in [9]. For Lipschitz and C^1 subdomains of Riemannian manifolds, see also the paper [19] by M. Dindoš and M. Mitrea, and the paper [46] by M. Mitrea and M. Taylor.

Examples of domains satisfying the hypotheses of Theorem 1.1 are:

(1) domains of class C^1 or, more generally, domains whose boundaries are locally the graphs of Lipschitz functions with gradients in VMO;

(2) domains whose boundaries are locally the graphs of Lipschitz functions with gradients sufficiently close −relatively to the exponent p− to VMO (in particular, of small BMO norm);

(3) Lipschitz domains with a sufficiently small Lipschitz constant (relatively to the exponent p);

(4) Lipschitz polyhedral domains with dihedral angles sufficiently close (depending on p) to π.

(5) polygonal domains with angles sufficiently close (relatively to the exponent p) to π.

The way in which one should interpret example (5) is as follows. Given an integrability exponent $p \in (\frac{n-1}{n}, \infty)$, there exists a small constant $c > 0$ which depends on p and the Lipschitz constant of the polygonal domain in question with the property that if its angles differ from π by at most c then the inhomogeneous Dirichlet problem (1.1), subject to the (necessary) compatibility conditions (1.2), is well-posed in that polygon (a remarkable feature is the fact that only the exponent p, and not the indices q and s, plays a role – compare with [41]). Similar interpretations apply to examples (2)-(4). In the case of example (1), the aforementioned problem is solvable for the full range of exponents p, q, s for which the intervening Besov and Triebel-Lizorkin spaces are meaningfully defined in the class of Lipschitz domains. In the case of example (5), our result is consistent with the predictions of the theory of BVP's in polygonal domains (where concrete calculations can be carried out, based on Mellin transform techniques).

We should also mention here that, if p is near 2, then no restriction on the size of the oscillation of the outward unit normal is necessary (that is, if $|p-2|$ is small and $0 < q \leq \infty$, $0 < s < 1$, then the above well-posedness result is valid in any Lipschitz domain). This follows from the work of E. Fabes, C. Kenig and G. Verchota, [20], according to which both the Dirichlet and the Regularity problem for the Stokes system in arbitrary Lipschitz domains are solvable (with nontangential maximal function L^p-estimates) when p is near 2 in any Lipschitz domain. Using a different approach (based on certain estimates obtained by G. Savaré in [49] and, more recently, by relying on certain stability interpolation results due to I. Ya. Šneiberg), M. Agranovich has extended in a series of papers, [2], [3], [4], [5], the scope of this type of result (i.e., when $|p-2|$ is small, and $0 < s < 1$) as to allow more general strongly elliptic systems with a Hermitian principal symbol, in arbitrary Lipschitz domains (Agranovich's results also touch on a number of other significant topics, such as resolvent estimates for spectral problems non-stationary problems, and transmission problems).

We also wish to note that one significant feature of our work is the fact that values of p below 1 are allowed. This is important since, in contrast with the scale of standard Sobolev spaces for which p is naturally restricted to $[1, \infty]$, the scales of Besov and Triebel-Lizorkin spaces continue to make sense for p below 1. For example, Triebel-Lizorkin spaces with $p \leq 1$, $q = 2$ and zero smoothness correspond to Hardy spaces. This is relevant since, for example, E. Stein and collaborators have treated in [11], [12], [13] the inhomogeneous Dirichlet problem for the Laplacian in smooth domains with data from Hardy spaces. In the process, they have conjectured that the smoothness condition on the domain can be relaxed considerably (depending on how much p is smaller than 1). Our main result contains, as a particular case, an answer to this conjecture (for the Stokes system) in the sense that the inhomogeneous Dirichlet problem for the Stokes system in a Lipschitz domain $\Omega \subset \mathbb{R}^n$ with data f from the Hardy space $H^p(\Omega) = F_0^{p,2}(\Omega)$ is well-posed whenever $\frac{n-1}{n} < p \leq 1$ provided the outward unit normal ν of Ω belongs to $\mathrm{VMO}(\partial\Omega)$.

In closing, it is worth mentioning that results of a somewhat similar nature have been proved in the case of the Laplace operator in [26], [44], [45], [59].

2 Background material

By a special Lipschitz domain Ω in $\mathbb{R}^n$ (in the sense of E. Stein [55]) we shall simply understand the over-graph region for a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. Also, call Ω a bounded Lipschitz domain in $\mathbb{R}^n$ if there exists a finite open covering $\{\mathcal{O}_j\}_{1 \leq j \leq N}$ of $\partial\Omega$ with the property that, for every $j \in \{1, ..., N\}$, $\mathcal{O}_j \cap \Omega$ coincides with the portion of $\mathcal{O}_j$ lying in the over-graph of a Lipschitz function $\varphi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ (where $\mathbb{R}^{n-1} \times \mathbb{R}$ is a new system of coordinates obtained from the original one via a rigid motion). We then define the *Lipschitz constant* of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ as

$$\inf \Big(\max\{ \|\nabla \varphi_j\|_{L^\infty(\mathbb{R}^{n-1})} : 1 \leq j \leq N \} \Big), \tag{2.1}$$

where the infimum is taken over all possible families $\{\varphi_j\}_{1 \leq j \leq N}$ as above. As is well-known, for a Lipschitz domain Ω, the surface measure $d\sigma$ is well-defined on $\partial\Omega$ and that there exists an outward pointing normal vector $\nu = (\nu_1, \cdots, \nu_n)$ at almost every point on $\partial\Omega$. For each $p \in (0, \infty]$, $L^p(\partial\Omega)$ will denote the Lebesgue scale of σ-measurable, p-th power integrable functions on $\partial\Omega$.

Assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain and consider the first-order tangential derivative operators $\partial_{\tau_{jk}}$ acting on a compactly supported function ψ of class C^1 in a neighborhood of $\partial\Omega$ by

$$\partial_{\tau_{jk}} \psi := \nu_j (\partial_k \psi)\Big|_{\partial\Omega} - \nu_k (\partial_j \psi)\Big|_{\partial\Omega}, \qquad j, k = 1, \ldots, n. \tag{2.2}$$

Repeated integrations by parts then show that, for every $j, k \in \{1, \ldots, n\}$,

$$\int_{\partial\Omega} \varphi \, (\partial_{\tau_{jk}} \psi) \, d\sigma = \int_{\partial\Omega} (\partial_{\tau_{kj}} \varphi) \, \psi \, d\sigma, \qquad \forall \, \varphi, \psi \in C^1_0(\mathbb{R}^n). \tag{2.3}$$

Assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain and that $1 < p, p' < \infty$ satisfy $\frac{1}{p} + \frac{1}{p'} = 1$. Inspired by (2.3), we then define the following L^p-based Sobolev space of order one on $\partial\Omega$:

$$\begin{aligned} L^p_1(\partial\Omega) \quad := \quad & \Big\{ f \in L^p(\partial\Omega) : \ \text{there exists a constant } c > 0 \text{ such that if } \psi \in C^1_0(\mathbb{R}^n) \\ & \quad \text{then } \Big| \int_{\partial\Omega} f \, (\partial_{\tau_{jk}} \psi) \, d\sigma \Big| \leq c \|\psi\|_{L^{p'}(\partial\Omega)} \ \text{ for } \ j, k = 1, \ldots, n \Big\}. \end{aligned} \tag{2.4}$$

Riesz's Theorem shows that if $f \in L^p_1(\partial\Omega)$ then for every $j, k \in \{1, \ldots, n\}$ there exists $g_{jk} \in L^p(\partial\Omega)$ such that

$$\int_{\partial\Omega} f \, (\partial_{\tau_{jk}} \psi) \, d\sigma = \int_{\partial\Omega} g_{jk} \, \psi \, d\sigma, \qquad \forall \, \psi \in C^1_0(\mathbb{R}^n). \tag{2.5}$$

In this situation, we agree to set $\partial_{\tau_{kj}} f := g_{jk}$. It follows that if Ω is a Lipschitz domain in $\mathbb{R}^n$ then the operators $\partial_{\tau_{kj}} : L^p_1(\partial\Omega) \to L^p(\partial\Omega)$, $1 \leq j, k \leq n$, are well-defined and bounded. Also, the following integration by parts formula holds:

$$\int\limits_{\partial\Omega} (\partial_{\tau_{jk}} f) \, g \, d\sigma = \int\limits_{\partial\Omega} f \, (\partial_{\tau_{kj}} g) \, d\sigma, \qquad 1 \leq j, k \leq n, \tag{2.6}$$

for every $f \in L^p_1(\partial\Omega)$ and $g \in L^{p'}_1(\partial\Omega)$ if $1 < p, p' < \infty$ satisfy $1/p + 1/p' = 1$. It can be easily shown that $L^p_1(\partial\Omega)$ becomes a Banach space when equipped with the natural norm

$$\|f\|_{L^p_1(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{L^p(\partial\Omega)}. \tag{2.7}$$

From (2.2) and the fact that (cf. [43] for a proof)

$$C^\infty(\mathbb{R}^n)\Big|_{\partial\Omega} \hookrightarrow L^p_1(\partial\Omega) \quad \text{densely, whenever } 1 < p < \infty, \tag{2.8}$$

one can check that if Ω is a Lipschitz domain in $\mathbb{R}^n$, $1 < p < \infty$, and $f \in L^p_1(\partial\Omega)$, then

$$\partial_{\tau_{jk}} f = \nu_j (\nabla_{tan} f)_k - \nu_k (\nabla_{tan} f)_j, \qquad j, k = 1, ..., n, \tag{2.9}$$

σ-a.e. on $\partial\Omega$, where

$$\nabla_{tan} f := \Big(\sum_{k=1}^{n} \nu_k \partial_{\tau_{kj}} f\Big)_{1\leq j\leq n}, \qquad f \in L^p_1(\partial\Omega). \tag{2.10}$$

As a consequence of (2.9) and (2.10), we note that for each $p \in (1,\infty)$,

$$\|f\|_{L^p_1(\partial\Omega)} \approx \|f\|_{L^p(\partial\Omega)} + \|\nabla_{tan} f\|_{L^p(\partial\Omega)} \qquad \text{uniformly in } f \in L^p_1(\partial\Omega). \tag{2.11}$$

Moving on, let Ω be a Lipschitz domain in $\mathbb{R}^n$ and fix a sufficiently large constant $\kappa > 0$. We define the non-tangential maximal operator as the mapping which associates to a function $u : \Omega \to \mathbb{R}$ the function $M(u) : \partial\Omega \to [0,\infty]$ given by

$$M(u)(X) := \sup\{|u(Y)| : Y \in \Omega,\ |X-Y| < (1+\kappa)\,\mathrm{dist}\,(Y,\partial\Omega)\}, \qquad X \in \partial\Omega. \tag{2.12}$$

We also introduce the non-tangential restriction to the boundary of a function $u : \Omega \to \mathbb{R}$ as

$$u\Big|_{\partial\Omega}(X) := \lim_{\substack{\Omega\ni Y\to X\\ |X-Y|<(1+\kappa)\,\mathrm{dist}\,(Y,\partial\Omega)}} u(Y), \qquad X \in \partial\Omega, \tag{2.13}$$

whenever the limit exists.

Let us briefly digress for the purpose of recalling the VMO space on the boundary of a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$. For a measurable set $S \subseteq \partial\Omega$ we set

$$⨍_S f(Y)\,d\sigma(Y) := \frac{1}{|S|}\int_S f(Y)\,d\sigma(Y), \qquad \text{where } \ |S| := \sigma(S). \tag{2.14}$$

Call Δ_r a surface ball of radius $r > 0$ if there exists $X \in \partial\Omega$ such that $\Delta_r := \partial\Omega\cap B(X,r)$. Now, for some fixed $\eta \in (0, \mathrm{diam}\,(\partial\Omega))$, the John-Nirenberg space of functions of bounded mean oscillations on $\partial\Omega$ is defined as

$$f \in \mathrm{BMO}\,(\partial\Omega) \overset{def}{\Longleftrightarrow} f \in L^2(\partial\Omega) \quad \text{and} \quad \sup_{\substack{\Delta_r \text{ surface ball}\\ \text{with } r\leq\eta}} ⨍_{\Delta_r} |f - f_{\Delta_r}|\,d\sigma < \infty \tag{2.15}$$

where $f_{\Delta_r} := ⨍_{\Delta_r} f\,d\sigma$, and is equipped with the natural norm

$$\|f\|_{\mathrm{BMO}\,(\partial\Omega)} := \|f\|_{L^2(\partial\Omega)} + \left(\sup_{\substack{\Delta_r \text{ surface ball}\\ \text{with } r\leq\eta}} ⨍_{\Delta_r} |f - f_{\Delta_r}|\,d\sigma\right). \tag{2.16}$$

Sarason's space of functions of vanishing mean oscillation on $\partial\Omega$ is then defined by the demand

$$f \in \mathrm{VMO}\,(\partial\Omega) \overset{def}{\Longleftrightarrow} f \in \mathrm{BMO}\,(\partial\Omega) \ \text{ and } \ \lim_{R\to 0}\left(\sup_{\substack{\Delta_r \text{ surface ball}\\ \text{with } r\leq R}} ⨍_{\Delta_r} |f - f_{\Delta_r}|\,d\sigma\right) = 0. \tag{2.17}$$

Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$, denote by $\mathcal{D}'(\Omega)$ the space of distributions in Ω. Let $B^{p,q}_\alpha(\mathbb{R}^n)$ and $F^{p,q}_\alpha(\mathbb{R}^n)$ denote the standard Besov and Triebel-Lizorkin scales of spaces in $\mathbb{R}^n$. There is a wealth of material pertaining to this topic and the interested reader is referred to the monographs [8] by J. Bergh and J. Löfström, and [56] by H. Triebel. If $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$, we set

$$\begin{aligned} B^{p,q}_\alpha(\Omega) &:= \{u \in \mathcal{D}'(\Omega) : \exists\, v \in B^{p,q}_\alpha(\mathbb{R}^n) \text{ with } v|_\Omega = u\},\\ F^{p,q}_\alpha(\Omega) &:= \{u \in \mathcal{D}'(\Omega) : \exists\, v \in F^{p,q}_\alpha(\mathbb{R}^n) \text{ with } v|_\Omega = u\}. \end{aligned} \tag{2.18}$$

For $a \in \mathbb{R}$ set $(a)_+ := \max\{a, 0\}$. Consider three parameters p, q, s subject to the conditions in (1.4) and assume that $\Omega \subset \mathbb{R}^n$ is the over-graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. We then define $B_s^{p,q}(\partial\Omega)$ as the space of locally integrable functions f on $\partial\Omega$ for which the mapping $\mathbb{R}^{n-1} \ni x' \mapsto f(x', \varphi(x'))$ belongs to $B_s^{p,q}(\mathbb{R}^{n-1})$. We then define

$$\|f\|_{B_s^{p,q}(\partial\Omega)} := \|f(\cdot, \varphi(\cdot))\|_{B_s^{p,q}(\mathbb{R}^{n-1})}. \tag{2.19}$$

As is well-known, the case when $p = q = \infty$ corresponds to the usual (non-homogeneous) Hölder spaces $C^s(\partial\Omega)$.

As far as Besov spaces with a negative amount of smoothness are concerned, in the same context as above we set

$$f \in B_{s-1}^{p,q}(\partial\Omega) \Longleftrightarrow f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2} \in B_{s-1}^{p,q}(\mathbb{R}^{n-1}), \tag{2.20}$$

$$\|f\|_{B_{s-1}^{p,q}(\partial\Omega)} := \|f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2}\|_{B_{s-1}^{p,q}(\mathbb{R}^{n-1})}. \tag{2.21}$$

The above definitions then readily extend to the case of bounded Lipschitz domains in $\mathbb{R}^n$ via a standard partition of unity argument. The Besov scale on $\partial\Omega$ has been defined in such a way that a number of basic properties from the Euclidean setting carry over to spaces defined on $\partial\Omega$ in a rather direct fashion. We recall some of these properties below.

Proposition 2.1 *For $(n-1)/n < p \leq \infty$ and $(n-1)(1/p - 1)_+ < s < 1$,*

$$\|f\|_{B_s^{p,p}(\partial\Omega)} \approx \|f\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(X) - f(Y)|^p}{|X - Y|^{n-1+sp}}\, d\sigma(X) d\sigma(Y)\right)^{1/p}. \tag{2.22}$$

See [40] for a proof of the equivalence (2.22).

We continue by recording an interpolation result which is going to be very useful for us here. To state it, recall that $(\cdot,\cdot)_{\theta,q}$ and $[\cdot,\cdot]_\theta$ stand for the real and complex interpolation brackets.

Proposition 2.2 *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Then*

$$(L^p(\partial\Omega), L_1^p(\partial\Omega))_{\theta,q} = B_\theta^{p,q}(\partial\Omega), \tag{2.23}$$

if $1 < p, q < \infty$ and $0 < \theta < 1$.

The case $p = q$ of (2.23) appears on p. 200 of [26] but this restriction is inessential. The trace theorem below appears in [39], [40] (for related results, in more general domains but for more restrictive ranges of indices, the reader is also referred to [27]).

Theorem 2.3 *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that the indices p, s satisfy $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$. Then the following hold:*

(i) The restriction to the boundary extends to a linear, bounded operator

$$\mathrm{Tr} : B_{s+\frac{1}{p}}^{p,q}(\Omega) \longrightarrow B_s^{p,q}(\partial\Omega) \quad \textit{for} \quad 0 < q \leq \infty. \tag{2.24}$$

Moreover, for this range of indices, Tr *is onto and has a bounded right-inverse.*

(ii) Similar considerations hold for $\mathrm{Tr} : F_{s+\frac{1}{p}}^{p,q}(\Omega) \to B_s^{p,p}(\partial\Omega)$ *with the convention that $q = \infty$ if $p = \infty$. More specifically, this is a bounded operator which has a linear, bounded right-inverse.*

We continue to assume that $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, is an arbitrary Lipschitz domain. When equipped with the surface measure and the Euclidean distance, $\partial\Omega$ becomes a space of homogeneous type, in the sense of Coifman-Weiss [15]. Hence, if $\frac{n-1}{n} < p \leq 1$, we may consider the atomic Hardy space $h_{at}^p(\partial\Omega)$, as well as the Hardy-based Sobolev space of order one, $h_{at}^{1,p}(\partial\Omega)$. For further reference, it is then convenient to introduce

$$H^p(\partial\Omega) := \begin{cases} h_{at}^p(\partial\Omega) \ \text{if} \ \frac{n-1}{n} < p \leq 1, \\ L^p(\partial\Omega) \ \text{if} \ p > 1, \end{cases} \qquad H_1^p(\partial\Omega) := \begin{cases} h_{at}^{1,p}(\partial\Omega) \ \text{if} \ \frac{n-1}{n} < p \leq 1, \\ L_1^p(\partial\Omega) \ \text{if} \ p > 1. \end{cases} \tag{2.25}$$

3 The Mikhlin-Calderón-Zygmund theory of singular integral operators associated with the Stokes system

In this section we discuss the nature of the singular integral operators of layer potential type which are most relevant in the treatment of the Stokes system in Lipschitz domains.

3.1 Bilinear forms and conormal derivatives

For $\lambda \in \mathbb{R}$ fixed, let $a_{jk}^{\alpha\beta}(\lambda) := \delta_{jk}\delta_{\alpha\beta} + \lambda\,\delta_{j\beta}\delta_{k\alpha}$ for $1 \leq j,k,\alpha,\beta \leq n$, and, adopting the summation convention over repeated indices, consider the differential operator L_λ given by

$$(L_\lambda \vec{u})_\alpha := \partial_j(a_{jk}^{\alpha\beta}(\lambda)\partial_k u_\beta) = \Delta u_\alpha + \lambda\,\partial_\alpha(\operatorname{div}\vec{u}), \qquad 1 \leq \alpha \leq n. \tag{3.1}$$

Consider the linear first-order differential operator $Du := (\partial_k u_\beta)_{1\leq k,\beta\leq n}$ if $u = (u_\beta)_{1\leq\beta\leq n}$ along with the zero-order linear operator $Av := (a_{j,k}^{\alpha\beta}(\lambda)v_{k\beta})_{1\leq j,\alpha\leq n}$ if $v = (v_{k\beta})_{1\leq k,\alpha\leq n}$. Then we have $D^*v = -(\partial_k v_{k\beta})_{1\leq\beta\leq n}$ and, consequently,

$$L_\lambda u = -D^*ADu = \Big(\partial_j(a_{jk}^{\alpha\beta}(\lambda)\partial_k u_\beta)\Big)_{1\leq\alpha\leq n}. \tag{3.2}$$

One aspect which is directly affected by the choice of the parameter λ is the format of the conormal derivative for the Stokes system, which we define as

$$\partial_\nu^\lambda(\vec{u},\pi) := \Big(\nu_j a_{jk}^{\alpha\beta}(\lambda)\partial_k u_\beta - \nu_\alpha\pi\Big)_{1\leq\alpha\leq n} = \Big[(\nabla\vec{u})^\top + \lambda(\nabla\vec{u})\Big]\nu - \pi\nu \quad \text{on } \partial\Omega, \tag{3.3}$$

where $\nabla\vec{u} = (\partial_k u_j)_{1\leq j,k\leq n}$ denotes the Jacobian matrix of the vector-valued function $\vec{u}$, and $\top$ stands for transposition of matrices. As we shall see momentarily, the algebraic format of the conormal derivative affects the functional analytic properties of the double layer operator.

3.2 Hydrostatic layer potential operators

We continue to review background material by recalling the definitions and some basic properties of the layer potentials for the Stokes system in an arbitrary Lipschitz domain $\Omega \subset \mathbb{R}^n, n \geq 2$. Let ω_{n-1} denote the surface measure of S^{n-1}, the unit sphere in $\mathbb{R}^n$, and let $E(X) = (E_{jk}(X))_{1\leq j,k\leq n}$ be the Kelvin matrix of fundamental solutions for the Stokes system, where

$$E_{jk}(X) := -\frac{1}{2\omega_{n-1}}\left(\frac{1}{n-2}\frac{\delta_{jk}}{|X|^{n-2}} + \frac{x_j x_k}{|X|^n}\right), \quad X = (x_j)_{1\leq j\leq n} \in \mathbb{R}^n \setminus \{0\}, \quad n \geq 3, \tag{3.4}$$

with $\frac{1}{n-2}\frac{\delta_{jk}}{|X|^{n-2}}$ replaced by $\log|X|$ when $n = 2$. Let us also introduce a pressure vector given by

$$\vec{q}(X) = (q_j(X))_{1\leq j\leq n} := -\frac{1}{\omega_{n-1}}\frac{X}{|X|^n}, \quad X \in \mathbb{R}^n \setminus \{0\}. \tag{3.5}$$

Then, for $X \in \mathbb{R}^n \setminus \{0\}$, we have

$$\partial_k E_{jk}(X) = 0 \ \text{ for } \ 1 \leq j \leq n \ \text{ and } \ \partial_j E_{jk}(X) = 0 \ \text{ for } \ 1 \leq k \leq n, \tag{3.6}$$

$$\Delta E_{jk}(X) = \Delta E_{kj}(X) = \partial_k q_j(X) = \partial_j q_k(X) \quad \text{for} \quad 1 \leq j,k \leq n. \tag{3.7}$$

Now, fix $-1 < \lambda \leq 1$, and define the single and double layer potential operators $\mathcal{S}$ and $\mathcal{D}_\lambda$ by

$$\mathcal{S}\vec{f}(X) \ := \int_{\partial\Omega} E(X-Y)\,\vec{f}(Y)\,d\sigma(Y), \qquad X \in \Omega, \tag{3.8}$$

$$\mathcal{D}_\lambda\vec{f}(X) \ := \int_{\partial\Omega} [\partial_{\nu(Y)}^\lambda\{E,\vec{q}\}(Y-X)]^\top\,\vec{f}(Y)\,d\sigma(Y), \qquad X \in \Omega, \tag{3.9}$$

where $\partial^{\lambda}_{\nu(Y)}\{E,\vec{q}\}$ is defined to be the matrix obtained by applying ∂^{λ}_{ν}, in the variable Y, to each pair consisting of the j-th column in E and the j-th component of $\vec{q}$. More concretely,

$$\begin{aligned}(\partial^{\lambda}_{\nu(Y)}\{E,\vec{q}\}(Y-X))_{jk} \;&:=\; \nu_\alpha(Y)(\partial_\alpha E_{kj})(Y-X)\\ &\qquad +\lambda\nu_\alpha(Y)(\partial_k E_{\alpha j})(Y-X)-q_j(Y-X)\nu_k(Y).\end{aligned} \tag{3.10}$$

Let us also define corresponding potentials for the pressure by

$$\mathcal{Q}\vec{f}(X) \;:=\; \int_{\partial\Omega}\langle\vec{q}(X-Y),\vec{f}(Y)\rangle\,d\sigma(Y), \qquad X\in\Omega, \tag{3.11}$$

$$\mathcal{P}_\lambda\vec{f}(X) \;:=\; (1+\lambda)\int_{\partial\Omega}\nu_j(Y)\langle(\partial_j\vec{q})(Y-X),\vec{f}(Y)\rangle\,d\sigma(Y), \qquad X\in\Omega. \tag{3.12}$$

Then

$$\Delta\mathcal{S}\vec{f}-\nabla\mathcal{Q}\vec{f}=0 \quad\text{and}\quad \operatorname{div}\mathcal{S}\vec{f}=0 \quad\text{in}\quad \Omega, \tag{3.13}$$

and for each $\lambda\in\mathbb{R}$,

$$\Delta\mathcal{D}_\lambda\vec{f}-\nabla\mathcal{P}_\lambda\vec{f}=0 \quad\text{and}\quad \operatorname{div}\mathcal{D}_\lambda\vec{f}=0 \quad\text{in}\quad \Omega. \tag{3.14}$$

Let us also consider the fundamental solution for the Laplacian in $\mathbb{R}^n$,

$$E_\Delta(X):=-\frac{1}{(n-2)\omega_{n-1}|X|^{n-2}}, \qquad X\neq 0, \tag{3.15}$$

if $n\geq 3$ (with the usual modification if $n=2$), and the corresponding single and double harmonic layer potentials

$$\mathcal{S}_\Delta f(X):=\int_{\partial\Omega}E_\Delta(X-Y)f(Y)\,d\sigma(Y), \qquad X\in\Omega, \tag{3.16}$$

$$\mathcal{D}_\Delta f(X):=\int_{\partial\Omega}\partial_{\nu(Y)}E_\Delta(X-Y)f(Y)\,d\sigma(Y), \qquad X\in\Omega. \tag{3.17}$$

Then $\vec{q}=-\nabla E_\Delta$ in $\mathbb{R}^n\setminus\{0\}$ so

$$\mathcal{Q}\vec{f}=-\sum_{k=1}^{n}\partial_k(\mathcal{S}_\Delta f_k)=-\operatorname{div}\mathcal{S}_\Delta\vec{f} \quad\text{and}\quad \mathcal{P}_\lambda\vec{f}=(1+\lambda)\operatorname{div}\mathcal{D}_\Delta\vec{f}. \tag{3.18}$$

Let us now record a basic result from the theory of singular integral operators of Mikhlin-Calderón-Zygmund type on Lipschitz domains. In the present format, this result has been established in [44], following the work in [14] and [59].

Proposition 3.1 *Let $\Omega\subseteq\mathbb{R}^n$ be an arbitrary Lipschitz domain. There exists a positive integer $N=N(n)$ with the following significance. Let Ω be a Lipschitz domain in $\mathbb{R}^n$, fix some function*

$$k\in C^N(\mathbb{R}^n\setminus\{0\}) \quad\textit{with}\quad k(-X)=-k(X) \quad\textit{and}\quad k(\lambda X)=\lambda^{-(n-1)}k(X)\ \ \forall\,\lambda>0, \tag{3.19}$$

and define the singular integral operator

$$\mathcal{T}f(X):=\int_{\partial\Omega}k(X-Y)f(Y)\,d\sigma(Y), \qquad X\in\Omega. \tag{3.20}$$

Then for each $p \in (\frac{n-1}{n}, \infty)$ there exists a finite constant $C = C(p, n, \partial\Omega) > 0$ such that

$$\|M(\mathcal{T}f)\|_{L^p(\partial\Omega)} \leq C\|k|_{S^{n-1}}\|_{C^N}\|f\|_{H^p(\partial\Omega)}. \tag{3.21}$$

Furthermore, for each $p \in (1, \infty)$, $f \in L^p(\partial\Omega)$, the limit

$$Tf(X) := \text{p.v.} \int_{\partial\Omega} k(X-Y)f(Y)\,d\sigma(Y) := \lim_{\varepsilon\to 0^+} \int_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} k(X-Y)f(Y)\,d\sigma(Y) \tag{3.22}$$

exists for a.e. $X \in \partial\Omega$, and the jump-formula

$$\mathcal{T}f\Big|_{\partial\Omega}(X) = \tfrac{1}{2\sqrt{-1}}\mathcal{F}(k)(\nu(X))f(X) + Tf(X) \tag{3.23}$$

is valid at a.e. $X \in \partial\Omega$, where $\mathcal{F}$ denotes the Fourier transform.

Let us now specialize the jimp-formula (3.23) to the case of hydrostatic layer potentials.

Proposition 3.2 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an arbitrary Lipschitz domain and assume that $1 < p < \infty$. Then for each $\lambda \in \mathbb{R}$, $\vec{f} \in L^p(\partial\Omega)$, and a.e. $X \in \partial\Omega$,*

$$\begin{aligned}
\mathcal{Q}\vec{f}\Big|_{\partial\Omega}(X) &= \tfrac{1}{2}\langle\nu(X), \vec{f}(X)\rangle + \text{p.v.}\int_{\partial\Omega}\langle\vec{q}(X-Y), \vec{f}(Y)\rangle\,d\sigma(Y), &(3.24)\\
\mathcal{D}_\lambda\vec{f}\Big|_{\partial\Omega}(X) &= \Big(\tfrac{1}{2}I + K_\lambda\Big)\vec{f}(X), &(3.25)
\end{aligned}$$

where I denotes the identity operator, the traces are taken in the sense of (2.13), and

$$K_\lambda\vec{f}(X) := \text{p.v.}\int_{\partial\Omega}[\partial^\lambda_{\nu(Y)}\{E, \vec{q}\}(Y-X)]^\top\vec{f}(Y)\,d\sigma(Y), \qquad X \in \partial\Omega. \tag{3.26}$$

Proof. Recall that if m is an integer and P_j is a harmonic, homogeneous polynomial of degree $j \geq 0$ in $\mathbb{R}^n$ then, as is well-known (cf., e.g., p. 73 in [55]),

$$\mathcal{F}(Q_j)(X) = \frac{P_j(X)}{|X|^{j+n-m}} \tag{3.27}$$

where, with Γ denoting the standard Gamma function,

$$Q_j(X) := (-1)^j\gamma_{j,m}\frac{P_j(X)}{|X|^{j+m}} \quad \text{and} \quad \gamma_{j,m} := (-1)^{j/2}\pi^{\frac{n}{2}-m}\frac{\Gamma(\frac{j}{2}+\frac{m}{2})}{\Gamma(\frac{j}{2}+\frac{n}{2}-\frac{m}{2})}, \tag{3.28}$$

provided either $0 < m < n$, or $m \in \{0, n\}$ and $j \geq 1$. Based on this and (3.23), a straightforward calculation gives the following trace formulas (with the boundary restriction considered in the sense of (2.13))

$$\partial_j\Big(\mathcal{S}_{\alpha\beta}\,g\Big)\Big|_{\partial\Omega}(X) = -\tfrac{1}{2}\nu_j(X)\Big(\delta_{\alpha\beta} - \nu_\alpha(X)\nu_\beta(X)\Big)g(X) + \partial_j S_{\alpha\beta}\,g(X) \tag{3.29}$$

valid at a.e. $X \in \partial\Omega$, for every $g \in L^p(\partial\Omega)$, $1 < p < \infty$, where for each $\alpha, \beta, j \in \{1, ..., n\}$, we have used the abbreviations

$$\mathcal{S}_{\alpha\beta}\,g(X) := \int_{\partial\Omega} E_{\alpha\beta}(X-Y)g(Y)\,d\sigma(Y), \qquad X \in \Omega, \tag{3.30}$$

$$\partial_j S_{\alpha\beta}\,g(X) := \text{p.v.}\int_{\partial\Omega}(\partial_j E_{\alpha\beta})(X-Y)g(Y)\,d\sigma(Y), \qquad X \in \partial\Omega. \tag{3.31}$$

Since, from (3.10) we have that

$$(\mathcal{D}_\lambda \vec{f})_j = -\partial_\alpha \mathcal{S}_{kj}(\nu_\alpha f_k) - \lambda \partial_k \mathcal{S}_{\alpha j}(\nu_\alpha f_k) - \partial_j \mathcal{S}_\Delta(\nu_k f_k) \tag{3.32}$$

for $j \in \{1, ..., n\}$, on account of (3.29) and the fact that

$$\partial_j \mathcal{S}_\Delta g\Big|_{\partial\Omega}(X) \;=\; -\tfrac{1}{2}\nu_j(X)\, g(X) + \text{p.v.}\int_{\partial\Omega} (\partial_j E_\Delta)(X-Y) g(Y)\, d\sigma(Y), \tag{3.33}$$

for a.e. $X \in \partial\Omega$, we obtain (with the boundary restriction taken as in (2.13))

$$\begin{aligned}(\mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega} \;=\; & \tfrac{1}{2}\nu_\alpha\Big(\delta_{kj} - \nu_k\nu_j\Big)\nu_\alpha f_k - (\partial_\alpha S_{kj})(\nu_\alpha f_k) \\ & + \tfrac{1}{2}\lambda\nu_k\Big(\delta_{\alpha j} - \nu_\alpha\nu_j\Big)\nu_\alpha f_k - \lambda(\partial_k S_{\alpha j})(\nu_\alpha f_k) \\ & + \tfrac{1}{2}\nu_j\nu_k f_k - (\partial_j S_\Delta)(\nu_k f_k),\end{aligned} \tag{3.34}$$

where $\partial_j S_\Delta$ is the principal-value singular integral operator with kernel $(\partial_j E_\Delta)(X - Y)$. Since

$$\tfrac{1}{2}\nu_\alpha\Big(\delta_{kj} - \nu_k\nu_j\Big)\nu_\alpha f_k + \tfrac{1}{2}\lambda\nu_k\Big(\delta_{\alpha j} - \nu_\alpha\nu_j\Big)\nu_\alpha f_k + \tfrac{1}{2}\nu_j\nu_k f_k = \tfrac{1}{2} f_j \tag{3.35}$$

and

$$-(\partial_\alpha S_{kj})(\nu_\alpha f_k) - \lambda(\partial_k S_{\alpha j})(\nu_\alpha f_k) - (\partial_j S_\Delta)(\nu_k f_k) = \Big(K_\lambda \vec{f}\Big)_j, \tag{3.36}$$

formula (3.25) follows. Formula (3.24) is proved in a similar fashion. □

Corollary 3.3 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain, and fix $\lambda \in \mathbb{R}$. Define*

$$S\vec{f} := \mathcal{S}\vec{f}\Big|_{\partial\Omega}. \tag{3.37}$$

Then the operators

$$K_\lambda : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \qquad S : L^p(\partial\Omega) \longrightarrow L^p_1(\partial\Omega) \tag{3.38}$$

are well-defined, linear, and bounded whenever $1 < p < \infty$.

Proof. The L^p-boundedness of K_λ is a consequence of Proposition 3.2 and Proposition 3.1. That S maps $L^p(\partial\Omega)$ boundedly into $L^p_1(\partial\Omega)$ follows upon noticing that $\partial_{\tau_{jk}} S$ is a singular integral operator of the type treated in Proposition 3.1. □

Next, we wish to discuss the action of these operators on Sobolev-Hardy spaces. We first note that, from (3.9)-(3.10), for each $\lambda \in \mathbb{R}$, $j \in \{1, ..., n\}$, and $\vec{f} \in L^p(\partial\Omega)$, $1 < p < \infty$,

$$\begin{aligned}\Big(\mathcal{D}_\lambda \vec{f}\Big)_j(X) \;=\; & \int_{\partial\Omega}\Big(\nu_\alpha(Y)(\partial_\alpha E_{jk})(Y - X) + \lambda\, \nu_\alpha(Y)(\partial_j E_{\alpha k})(Y - X) \\ & -\nu_j(Y) q_k(Y - X)\Big) f_k(Y)\, d\sigma(Y), \qquad X \in \Omega.\end{aligned} \tag{3.39}$$

Then for each $\vec{f} \in H^p_1(\partial\Omega)$, $\frac{n-1}{n} < p < \infty$, $r, j \in \{1, ..., n\}$, and $X \in \Omega$, we may write (based on (3.6)-(3.7) and integrations by parts –cf. (2.6))

$$\partial_r(\mathcal{D}_\lambda \vec{f})_j = -\partial_\alpha \mathcal{S}_{jk}(\partial_{\tau_{\alpha r}} f_k) - \lambda \partial_j \mathcal{S}_{\alpha k}(\partial_{\tau_{\alpha r}} f_k) - \partial_k \mathcal{S}_\Delta(\partial_{\tau_{jr}} f_k) \quad \text{in } \Omega. \tag{3.40}$$

The same type of reasoning applies to (3.12). Specifically, we have for each $X \in \Omega$,

$$\mathcal{P}_\lambda \vec{f}(X) = (1+\lambda) \int\limits_{\partial\Omega} (\partial_r E_\Delta)(Y-X)(\partial_{\tau_{rk}} f_k)(Y)\, d\sigma(Y) = (1+\lambda)\partial_r \mathcal{S}_\Delta(\partial_{\tau_{rk}} f_k)(X), \tag{3.41}$$

whenever $\vec{f} \in H_1^p(\partial\Omega)$, $\frac{n-1}{n} < p < \infty$. With these identities in mind, we can prove the following results.

Proposition 3.4 *Fix $\lambda \in \mathbb{R}$. Then for Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, and $\frac{n-1}{n} < p < \infty$, there exists a finite constant $C = C(\partial\Omega, p) > 0$ such that*

$$\|M(\nabla \mathcal{D}_\lambda \vec{f})\|_{L^p(\partial\Omega)} + \|M(\mathcal{P}_\lambda \vec{f})\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{H_1^p(\partial\Omega)}, \quad \forall \vec{f} \in H_1^p(\partial\Omega). \tag{3.42}$$

Proof. This is a direct consequence of Proposition 3.1, (3.40), (3.41) and the fact that for each $j,k \in \{1,...,n\}$, the operator $\partial_{\tau_{jk}} : H_1^p(\partial\Omega) \to H^p(\partial\Omega)$ is bounded if $\frac{n-1}{n} < p < \infty$. □

Proposition 3.5 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a Lipschitz domain. Then for each $\lambda \in \mathbb{R}$,*

$$K_\lambda : H_1^p(\partial\Omega) \longrightarrow H_1^p(\partial\Omega) \tag{3.43}$$

is a well-defined, bounded operator for every $p \in (\frac{n-1}{n}, \infty)$.

Proof. Note that for each $j \in \{1,...,n\}$ we have

$$\begin{aligned}\partial_{\tau_{rs}}(K_\lambda \vec{f})_j(X) &= \partial_{\tau_{rs}}(\tfrac{1}{2}\vec{f} + K_\lambda \vec{f})_j(X) - \tfrac{1}{2}\partial_{\tau_{rs}} f_j(X) \\ &= \nu_r(\partial_s \mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega}(X) - \nu_s(\partial_r \mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega}(X) - \tfrac{1}{2}\partial_{\tau_{rs}} f_j(X),\end{aligned} \tag{3.44}$$

at almost every $X \in \partial\Omega$. Now, if $\partial_j S_\Delta$ stands for the principal-value integral operator on $\partial\Omega$ with kernel $(\partial_j E_\Delta)(X-Y)$, then at almost every point on $\partial\Omega$, we have from (3.40) and (3.29)

$$\begin{aligned}\partial_s(\mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega} &= \tfrac{1}{2}\nu_\alpha(\delta_{jk} - \nu_j\nu_k)\partial_{\tau_{\alpha s}} f_k - \partial_\alpha S_{jk}(\partial_{\tau_{\alpha s}} f_k) \\ &\quad + \lambda\, \tfrac{1}{2}\nu_j(\delta_{\alpha k} - \nu_\alpha\nu_k)\partial_{\tau_{\alpha s}} f_k - \lambda\, \partial_j S_{\alpha k}(\partial_{\tau_{\alpha s}} f_k) \\ &\quad - \tfrac{1}{2}\nu_k \partial_{\tau_{sj}} f_k + \partial_k S_\Delta(\partial_{\tau_{sj}} f_k),\end{aligned} \tag{3.45}$$

with a similar formula for $\partial_r(\mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega}$. Note that

$$\begin{aligned}\nu_\alpha(\delta_{jk} - \nu_j\nu_k)\partial_{\tau_{\alpha s}} f_k &= \nu_\alpha(\delta_{jk} - \nu_j\nu_k)(\nu_\alpha(\nabla_{tan} f_k)_s - \nu_s(\nabla_{tan} f_k)_\alpha) \\ &= (\nabla_{tan} f_j)_s - \nu_j\nu_k(\nabla_{tan} f_k)_s,\end{aligned} \tag{3.46}$$

and similarly,

$$\nu_j(\delta_{\alpha k} - \nu_\alpha\nu_k)\partial_{\tau_{\alpha s}} f_k = -\nu_j\nu_s(\nabla_{tan} f_k)_k, \tag{3.47}$$

$$\nu_k \partial_{\tau_{sj}} f_k = \nu_k\nu_s(\nabla_{tan} f_k)_j - \nu_k\nu_j(\nabla_{tan} f_k)_s. \tag{3.48}$$

Thus, the jump-terms in $\nu_r\partial_s(\mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega} - \nu_s\partial_r(\mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega}$ amount to $\frac{1}{2}J_1 + \frac{\lambda}{2}J_2 - \frac{1}{2}J_3$ where

$$\begin{aligned}J_1 &:= \nu_r(\nabla_{tan} f_j)_s - \nu_s(\nabla_{tan} f_j)_r - \nu_r\nu_j\nu_k(\nabla_{tan} f_k)_s + \nu_s\nu_j\nu_k(\nabla_{tan} f_k)_r \\ &= \partial_{\tau_{rs}} f_j - \nu_j\nu_k \partial_{\tau_{rs}} f_k,\end{aligned} \tag{3.49}$$

$$J_2 \quad := \quad -\nu_s\nu_j\nu_r(\nabla_{tan}f_k)_k + \nu_r\nu_j\nu_s(\nabla_{tan}f_k)_k = 0, \tag{3.50}$$

and

$$\begin{aligned} J_3 \quad &:= \quad \nu_r\nu_k\nu_s(\nabla_{tan}f_k)_j - \nu_r\nu_k\nu_j(\nabla_{tan}f_k)_s - \nu_s\nu_k\nu_r(\nabla_{tan}f_k)_j + \nu_s\nu_k\nu_j(\nabla_{tan}f_k)_r \\ &= \quad -\nu_j\nu_k\partial_{\tau_{rs}}f_k. \end{aligned} \tag{3.51}$$

Thus, $\frac{1}{2}J_1 + \frac{\lambda}{2}J_2 - \frac{1}{2}J_3 = \frac{1}{2}\partial_{\tau_{rs}}f_j$, which cancels the last term in (3.44). In summary, all the jump-terms cancel out, and we arrive at the identity

$$\begin{aligned} \partial_{\tau_{rs}}(K_\lambda\vec{f})_j \quad = \quad & \nu_s\partial_\alpha S_{jk}(\partial_{\tau_{\alpha r}}f_k) + \lambda\,\nu_s\partial_j S_{\alpha k}(\partial_{\tau_{\alpha r}}f_k) - \nu_s\partial_k S_\Delta(\partial_{\tau_{rj}}f_k) \\ & -\nu_r\partial_\alpha S_{jk}(\partial_{\tau_{\alpha s}}f_k) - \lambda\,\nu_r\partial_j S_{\alpha k}(\partial_{\tau_{\alpha s}}f_k) + \nu_r\partial_k S_\Delta(\partial_{\tau_{sj}}f_k), \end{aligned} \tag{3.52}$$

valid at almost every boundary point. Since we have that $\partial_{\tau_{\alpha\beta}}f_k \in H^p(\partial\Omega)$, the desired conclusion follows easily from this identity and the mapping properties of the operators involved. $\square$

4 Singular integral operators on Besov-Triebel-Lizorkin spaces

In this section we extend the scope of our previous results in order to deduce some useful mapping properties for the singular integral operators associated with the Stokes system on the scales of Besov and Triebel-Lizorkin spaces.

4.1 Spaces of null-solutions of elliptic operators

Let $L = \sum_{|\gamma|=m} a_\gamma\partial^\gamma$ be a constant coefficient, elliptic differential operator of order $m \in 2\,\mathbb{N}$ in $\mathbb{R}^n$. For a fixed, bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, denote by $\operatorname{Ker} L$ the space of functions satisfying $Lu = 0$ in Ω. Then, for $0 < p \leq \infty$ and $\alpha \in \mathbb{R}$, introduce $\mathbb{H}^p_\alpha(\Omega; L)$ the space of functions $u \in \operatorname{Ker} L$ subject to the condition

$$\|u\|_{\mathbb{H}^p_\alpha(\Omega;L)} := \|\delta^{\langle\alpha\rangle-\alpha}|\nabla^{\langle\alpha\rangle}u|\|_{L^p(\Omega)} + \sum_{j=0}^{\langle\alpha\rangle-1}\|\nabla^j u\|_{L^p(\Omega)} < +\infty. \tag{4.1}$$

Above, ∇^j stands for vector of all mixed-order partial derivatives of order j and $\langle\alpha\rangle$ is the smallest nonnegative integer greater than or equal to α. The following theorem has been established in [40] and [28]. It extends results from [26], where the authors have dealt with the case in which $1 < p, q < \infty$, $s > 0$, $L = \Delta$, and [1] where the case $1 < p, q < \infty$, $s > 0$, $L = \Delta^2$ is treated.

Theorem 4.1 *Assume that L is a homogeneous, constant coefficient, elliptic differential operator and that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain. Then*

$$\mathbb{H}^p_\alpha(\Omega; L) = F^{p,q}_\alpha(\Omega) \cap \operatorname{Ker} L \tag{4.2}$$

for every $\alpha \in \mathbb{R}$, $0 < p < \infty$, and $0 < q < \infty$. In particular, for each fixed $\alpha \in \mathbb{R}$ and $0 < p < \infty$, the space $F^{p,q}_\alpha(\Omega) \cap \operatorname{Ker} L$ is independent of $q \in (0, \infty)$.

Furthermore, corresponding to $p = \infty$, for each $k \in \mathbb{N}_0$ and $s \in (0, 1)$ one has

$$\mathbb{H}^\infty_{k+s}(\Omega; L) = B^{\infty,\infty}_{k+s}(\Omega) \cap \operatorname{Ker} L. \tag{4.3}$$

4.2 Operator estimates on Besov-Triebel-Lizorkin scales

Here we record some results describing mapping properties on Besov spaces of integral operators. The first such result is modeled upon the harmonic and hydrostatic double layer potential operators.

Proposition 4.2 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and consider an integral operator*

$$Tf(X) := \int_{\partial\Omega} K(X,Y)f(Y)d\sigma(Y), \qquad X \in \Omega, \tag{4.4}$$

with the property that $T1$ is a constant function in Ω and

$$|\nabla_X^k K(X,Y)| \leq C|X-Y|^{-(n+k-1)}, \quad k = 1, ..., N, \tag{4.5}$$

for some positive integer N. Then, with $\delta := \mathrm{dist}\,(\cdot, \partial\Omega)$,

$$\|\delta^{k-\frac{1}{p}-s}|\nabla^k Tf|\|_{L^p(\Omega)} + \sum_{j=0}^{k-1} \|\nabla^j Tf\|_{L^p(\Omega)} \leq C\|f\|_{B_s^{p,p}(\partial\Omega)}, \tag{4.6}$$

granted that $k \in \{1, ..., N\}$, $\frac{n-1}{n} < p \leq \infty$, and $(n-1)(\frac{1}{p}-1)_+ < s < 1$.

For a proof of Proposition 4.2 see [40]. The next result gives an analogue of Theorem 4.2 for single layer-like integral operators. Once again, see [40] for a proof.

Proposition 4.3 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and consider the integral operator*

$$Rf(X) := \int_{\partial\Omega} K(X,Y)f(Y)d\sigma(Y), \qquad X \in \Omega, \tag{4.7}$$

whose kernel satisfies the conditions

$$|\nabla_X^k \nabla_Y^j K(X,Y)| \leq C|X-Y|^{-(n-2+k+j)}, \quad j = 0, 1, \tag{4.8}$$

for $k = 1, 2, ..., N$, where N is some positive integer. Then

$$\|\delta^{k-\frac{1}{p}-s}|\nabla^k Rf|\|_{L^p(\Omega)} + \sum_{j=0}^{k-1} \|\nabla^j Rf\|_{L^p(\Omega)} \leq C\|f\|_{B_{s-1}^{p,p}(\partial\Omega)}, \quad k = 1, 2, ..., N, \tag{4.9}$$

granted that $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$.

We are now ready to discuss the mapping properties for the hydrostatic layer potentials on Besov and Triebel-Lizorkin spaces in Lipschitz domains.

Theorem 4.4 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and assume that $\lambda \in \mathbb{R}$, $\frac{n-1}{n} < p \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, and $0 < q \leq \infty$. Then*

$$\mathcal{D}_\lambda : B_s^{p,q}(\partial\Omega) \longrightarrow B_{s+\frac{1}{p}}^{p,q}(\Omega), \qquad \mathcal{D}_\lambda : B_s^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}}^{p,q}(\Omega), \tag{4.10}$$

$$\mathcal{P}_\lambda : B_s^{p,q}(\partial\Omega) \longrightarrow B_{s+\frac{1}{p}-1}^{p,q}(\Omega), \qquad \mathcal{P}_\lambda : B_s^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}-1}^{p,q}(\Omega), \tag{4.11}$$

$$\mathcal{Q} : B_{s-1}^{p,q}(\partial\Omega) \longrightarrow B_{s+\frac{1}{p}-1}^{p,q}(\Omega), \qquad \mathcal{Q} : B_s^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}-1}^{p,q}(\Omega), \tag{4.12}$$

$$\mathcal{S} : B_{s-1}^{p,q}(\partial\Omega) \longrightarrow B_{s+\frac{1}{p}}^{p,q}(\Omega), \qquad \mathcal{S} : B_{s-1}^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}}^{p,q}(\Omega), \tag{4.13}$$

are well-defined, bounded operators (with the additional demand that $p \neq \infty$ in the case of Triebel-Lizorkin spaces).

Proof. From Proposition 4.2 and Proposition 4.1 it follows that

$$\mathcal{D}_\lambda : B^{p,p}_s(\partial\Omega) \longrightarrow \mathbb{H}^p_{s+\frac{1}{p}}(\Omega;\Delta^2) = F^{p,q}_{s+\frac{1}{p}}(\Omega) \cap \mathrm{Ker}\,\Delta^2 \tag{4.14}$$

is well-defined and bounded whenever $0 < p, q \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, provided $q = \infty$ if $p = \infty$. This and real interpolation then give that the operators (4.10) are bounded (in the second case, we also use monotonicity of the Triebel-Lizorkin scale in the second index to cover the case $q = \infty$). The operators (4.11)-(4.13) are handled similarly. □

Proposition 4.5 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. If p, q, s are as in (1.4) and $\lambda \in \mathbb{R}$, then the operators*

$$K_\lambda : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_s(\partial\Omega), \qquad S : B^{p,q}_{s-1}(\partial\Omega) \longrightarrow B^{p,q}_s(\partial\Omega), \tag{4.15}$$

are well-defined, linear, and bounded.

Proof. Since $\mathrm{Tr} \circ \mathcal{D}_\lambda = \frac{1}{2}I + K_\lambda$ and $\mathrm{Tr} \circ \mathcal{S} = S$ the claim about the operators in (4.15) follows from Proposition 4.4 and Theorem 2.3. □

5 The proof of Theorem 1.1

We debut with a few preliminaries. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we set (with χ_E denoting the characteristic function of E):

$$\mathbb{R}_{\partial\Omega} := \Big\{ \sum_j c_j \chi_{\Sigma_j} : c_j \in \mathbb{R} \text{ and } \Sigma_j \text{ connected component of } \partial\Omega \Big\}, \tag{5.1}$$

$$\mathbb{R}_{\Omega_+} := \Big\{ \sum_j c_j \chi_{\mathcal{O}_j} : c_j \in \mathbb{R} \text{ and } \mathcal{O}_j \text{ connected component of } \Omega \Big\}, \tag{5.2}$$

and set

$$\nu\mathbb{R}_{\partial\Omega} := \{\psi\nu : \psi \in \mathbb{R}_{\partial\Omega}\}, \quad \mathbb{R}_{\partial\Omega_+} := (\mathbb{R}_{\Omega_+})\Big|_{\partial\Omega}, \quad \nu\mathbb{R}_{\partial\Omega_+} := \{\psi\nu : \psi \in \mathbb{R}_{\partial\Omega_+}\}. \tag{5.3}$$

Next, let Ψ be the $n(n+1)/2$-dimensional linear space of $\mathbb{R}^n$-valued functions $\psi = (\psi_j)_{1\leq j\leq n}$ defined in $\mathbb{R}^n$ and satisfying $\partial_j\psi_k + \partial_k\psi_j = 0$ for $1 \leq j, k \leq n$, and note that

$$\Psi = \Big\{ \psi(X) = AX + \vec{a} : A,\ n\times n \text{ antisymmetric matrix, and } \vec{a} \in \mathbb{R}^n \Big\}. \tag{5.4}$$

Finally, set

$$\Psi(\partial\Omega_+) := \Big\{ \sum_j \psi_j \chi_{\partial\mathcal{O}_j} : \psi_j \in \Psi,\ \mathcal{O}_j \text{ bounded component of } \Omega \Big\}. \tag{5.5}$$

To proceed, we shall now introduce some versions of the boundary Besov spaces which are well-suited for the formulation and treatment of boundary value problems for the Stokes system in Lipschitz domains. Concretely, if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and $(n-1)/n < p \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, $0 < q \leq \infty$, we set:

$$B^{p,q}_{s,\nu_+}(\partial\Omega) := \Big\{ \vec{f} \in B^{p,q}_s(\partial\Omega) : \int_{\partial\Omega} \langle \psi, \vec{f}\rangle\, d\sigma = 0,\ \forall\, \psi \in \nu\,\mathbb{R}_{\partial\Omega_+} \Big\}, \tag{5.6}$$

$$B^{p,q}_{s,\nu}(\partial\Omega) := \Big\{ \vec{f} \in B^{p,q}_s(\partial\Omega) : \int_{\partial\Omega} \langle \psi, \vec{f}\rangle\, d\sigma = 0,\ \forall\, \psi \in \nu\,\mathbb{R}_{\partial\Omega} \Big\}. \tag{5.7}$$

The key analytical step in the proof of Theorem 1.1 is establishing the fact that, under the hypotheses stipulated in the statement of this theorem, there exists $\lambda \in (-1,1]$ such that

$$\tfrac{1}{2}I + K_\lambda : B^{p,q}_{s,\nu_+}(\partial\Omega)/\Psi^\lambda(\partial\Omega_+) \longrightarrow B^{p,q}_{s,\nu_+}(\partial\Omega)/\Psi^\lambda(\partial\Omega_+) \quad \text{is invertible.} \tag{5.8}$$

Once this has been justified more elementary and well-understood considerations (cf. [47] for details) yield that

$$S : B^{p,q}_{s-1}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega} \longrightarrow B^{p,q}_{s,\nu}(\partial\Omega) \ \text{ is invertible if } \ n \geq 3. \tag{5.9}$$

We proceed to complete the proof of Theorem 1.1 before returning to the claim (5.8). To this end, consider the integral operators

$$\Pi\vec{u}(X) := \int_{\mathbb{R}^n} E(X-Y)\vec{u}(Y)\,dY, \qquad \Theta\vec{u}(X) := \int_{\mathbb{R}^n} \langle q(X-Y), \vec{u}(Y)\rangle\,dY, \qquad X \in \mathbb{R}^n. \tag{5.10}$$

Then these are smoothing operators of order two and one, respectively, both on the Besov and Triebel-Lizorkin scale. Furthermore,

$$\Delta\Pi - \nabla\Theta = I, \qquad \operatorname{div}\Pi = 0, \qquad \Delta\Pi_\Delta = I, \tag{5.11}$$

where I stands for the identity operator. Thanks to the extension results in [48], any distribution can be extended from Ω to the entire Euclidean space with preservation of smoothness on the Besov and Triebel-Lizorkin scales. Below, we shall use such extensions tacitly, whenever convenient. Let $\vec{v}$ be such that $\vec{v} \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega)$ and $\operatorname{div}\vec{v} = g$ in Ω. For example, we may take $\vec{v} := \nabla\Pi_\Delta g$ where $\Pi_\Delta : B^{p,q}_{s+\frac{1}{p}-1}(\Omega) \to B^{p,q}_{s+\frac{1}{p}+1}(\Omega)$ is the harmonic Newtonian potential in Ω (i.e., the operator of convolution with E_Δ from (3.15)). Next, consider $\vec{w}$, ρ for which

$$(\vec{w},\rho) \in B^{p,q}_{s+\frac{1}{p}}(\Omega) \oplus B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \Delta\vec{w} - \nabla\rho = \vec{f} - \Delta\vec{v} \ \text{ and } \ \operatorname{div}\vec{w} = 0 \ \text{ in } \Omega. \tag{5.12}$$

For this, we may take $\vec{w} := \Pi(\vec{f} - \Delta\vec{v})$ and $\rho := \Theta(\vec{f} - \Delta\vec{v})$, where Π, Θ are as in (5.10). We now claim that

$$\operatorname{Tr}\vec{v} + \operatorname{Tr}\vec{w} - \vec{h} \in B^{p,q}_{s,\nu_+}(\partial\Omega). \tag{5.13}$$

To see this, we first observe that $\operatorname{Tr}\vec{v} + \operatorname{Tr}\vec{w} - \vec{h} \in B^{p,q}_{s}(\partial\Omega)$. To check the orthogonality condition on $\nu\mathbb{R}_{\partial\Omega_+}$, by virtue of (5.3) it suffices to note that for every $\psi \in \mathbb{R}_{\Omega_+}$ we have

$$\int_{\partial\Omega} \langle (\operatorname{Tr}\vec{v} + \operatorname{Tr}\vec{w})\,,\,\nu\rangle\,\psi\,d\sigma = \int_\Omega \psi\operatorname{div}(\vec{v}+\vec{w})\,dX = \int_\Omega g\,\psi\,dX = \int_{\partial\Omega}\langle\nu,\vec{h}\rangle\psi\,d\sigma, \tag{5.14}$$

by (1.2). This proves the claim made in (5.13). Next, we make the claim that if $n \geq 3$, then

$$T : B^{p,q}_{s,\nu_+}(\partial\Omega) \oplus B^{p,q}_{s-1}(\partial\Omega) \to B^{p,q}_{s,\nu_+}(\partial\Omega), \quad T(\vec{g}_1,\vec{g}_2) := (\tfrac{1}{2}I + K_\lambda)\vec{g}_1 + S\vec{g}_2 \quad \text{is onto.} \tag{5.15}$$

To justify this claim, observe that

$$\Psi^\lambda(\partial\Omega_+) \hookrightarrow B^{p,q}_{s,\nu}(\partial\Omega). \tag{5.16}$$

Consider next an arbitrary $\vec{f} \in B^{p,q}_{s,\nu_+}(\partial\Omega)$. Then (5.8) gives that there exists $\vec{g}_1 \in B^{p,q}_{s,\nu_+}(\partial\Omega)$ such that $\vec{\psi} := \vec{f} - (\frac{1}{2}I + K_\lambda)\vec{g}_1 \in \Psi^\lambda(\partial\Omega_+)$. This, (5.16), and (5.9) then guarantee the existence of some $\vec{g}_2 \in B^{p,q}_{s-1}(\partial\Omega)$ with the property that $S\vec{g}_2 = \vec{\psi}$. Consequently, $T(\vec{g}_1,\vec{g}_2) = \vec{f}$, proving the claim. Having established (5.13) and (5.15), we can now produce a solution for (1.1) in the form

$$\vec{u} := \vec{v} + \vec{w} + \mathcal{D}_\lambda\vec{g}_1 + \mathcal{S}\vec{g}_2, \qquad \pi := \rho + \mathcal{P}_\lambda\vec{g}_1 + \mathcal{Q}\vec{g}_2, \tag{5.17}$$

where

$$(\vec{g}_1,\vec{g}_2)\in B^{p,q}_{s,\nu_+}(\partial\Omega)\oplus B^{p,q}_{s-1}(\partial\Omega)\ \text{ is such that }\ T(\vec{g}_1,\vec{g}_2)=\vec{h}-\mathrm{Tr}\,\vec{v}-\mathrm{Tr}\,\vec{w}. \tag{5.18}$$

Furthermore, it is implicit in the above construction that (1.5) holds. The case $n=2$ is handled analogously. Finally, uniqueness can be established in a more straightforward fashion, using the the existence part. For the Triebel-Lizorkin scale a very similar approach works as well.

Thus, the proof of the theorem is complete at this point, modulo the claim (5.8). Note that we only need to know the invertibility of $\frac{1}{2}I+K_\lambda$ for just one value of $\lambda\in(-1,1]$. Due to space limitations, we shall indicate how (5.8) can be proved for $\lambda=1$ when $\frac{n-1}{n}<p\leq\infty$, $(n-1)(\frac{1}{p}-1)_+<s<1$ and $0<q\leq\infty$, in the case in which $\nu\in\mathrm{VMO}\,(\partial\Omega)$, and then comment on the necessary alterations in the perturbation case when (1.7) holds. In this vein, the following theorem, itself a particular case of a more general result from from [24], is most useful. To state it, denote by $\mathcal{L}(\mathcal{X})$ the Banach space of bounded linear operators from the Banach space $\mathcal{X}$ into itself, and by $\mathrm{Comp}\,(\mathcal{X})$ the closed two-sided ideal consisting of compact mappings of $\mathcal{X}$ into itself.

Theorem 5.1 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with unit normal vector ν and boundary surface measure σ. Then for every $\varepsilon>0$ the following holds. Given a function $k\in C^\infty(\mathbb{R}^n\setminus\{0\})$ even and homogeneous of degree $-n$, set*

$$Tf(X):=\lim_{\varepsilon\to 0}\int_{|X-Y|>\varepsilon,Y\in\partial\Omega}\langle X-Y,\nu(Y)\rangle k(X-Y)f(Y)\,d\sigma(Y),\qquad X\in\partial\Omega. \tag{5.19}$$

Then there exist an integer $N=N(n)$, along with a small number $\delta>0$ which depends only on ε, n, p, $\|k|_{S^n}\|_{C^N}$, and the Lipschitz character of Ω (more specifically, the geometrical characteristics of Ω regarded as a non-tangentially accessible domain, in the sense of Jerison and Kenig [25]), with the property that

$$\mathrm{dist}\,(\nu\,,\mathrm{VMO}\,(\partial\Omega))<\delta\Longrightarrow\mathrm{dist}\,(T\,,\,\mathrm{Comp}\,(L^p(\partial\Omega))<\varepsilon, \tag{5.20}$$

where the distance in the right-hand side is measured in $\mathcal{L}(L^p(\partial\Omega,d\sigma))$.

As a corollary, granted the initial geometrical assumptions on Ω and assuming that T is as above, then for every $p\in(1,\infty)$ the following implication is valid:

$$\nu\in\mathrm{VMO}\,(\partial\Omega)\Longrightarrow T:L^p(\partial\Omega)\longrightarrow L^p(\partial\Omega)\ \ \textit{is a compact operator.} \tag{5.21}$$

The proof in [24] of this result is rather long and involved. It relies on a splitting of $\partial\Omega$ (into two pieces: one of which is close to a Lipschitz surface with small constant, and one which has small surface measure), which is a sharper version of Semmes' decomposition theorem (stated as Proposition 5.1 on p. 212 of [51]; cf. also Theorem 4.1 on p. 398 of [30]), and other harmonic analysis tools, such as "good-λ" inequalities.

Let us now discuss the prospect of using Theorem 5.1 for the principal value hydrostatic double layer, i.e., for

$$K_\lambda\vec{f}(X):=\lim_{\varepsilon\to 0^+}\int_{\substack{Y\in\partial\Omega\\|X-Y|>\varepsilon}}[\partial^\lambda_{\nu(Y)}\{E,\vec{q}\}(Y-X)]^\top\vec{f}(Y)\,d\sigma(Y),\qquad X\in\partial\Omega. \tag{5.22}$$

The integral kernel of the operator (5.22) is a $n\times n$ matrix whose (j,k)-entry is

$$\begin{aligned}-(1-\lambda)\frac{\delta_{jk}}{\omega_{n-1}}\frac{\langle X-Y,\nu(Y)\rangle}{|X-Y|^n}-(1+\lambda)\frac{n}{\omega_{n-1}}\frac{\langle X-Y,\nu(Y)\rangle(x_j-y_j)(x_k-y_k)}{|X-Y|^{n+2}}\\ -(1-\lambda)\frac{1}{\omega_{n-1}}\frac{(x_j-y_j)\nu_k(Y)-(x_k-y_k)\nu_j(Y)}{|X-Y|^n}.\end{aligned} \tag{5.23}$$

For $\lambda = 1$, in which case the operator (5.22) is known as the slip hydrostatic double layer (cf., e.g., [37]), the last term in (5.23) vanishes. Thus, for this particular choice of the parameter λ, the operator (5.22) becomes of the type (5.19). Hence,

$$\nu \in \mathrm{VMO}\,(\partial\Omega) \Longrightarrow K_1 : L^p(\partial\Omega) \to L^p(\partial\Omega) \text{ is a compact operator, } \forall\, p \in (1,\infty). \tag{5.24}$$

Extending this compactness property to the scale of boundary Besov spaces is done using Proposition 2.2 and the following remarkable one-sided compactness property for the real method of interpolation for (compatible) Banach couples proved by M. Cwikel in [16]:

Theorem 5.2 *Assume that X_j, Y_j, $j = 0, 1$, are two compatible Banach couples and suppose that the linear operator $T : X_j \to Y_j$ is bounded for $j = 0$ and compact for $j = 1$. Then the operator $T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$ is compact for all $\theta \in (0, 1)$ and $q \in [1,\infty]$.*

Granted (5.24), this shows that K_1 is compact on $B^{p,q}_s(\partial\Omega)$ for most of the portion of the Besov scale consisting of Banach spaces, i.e., when $1 < p, q < \infty$ and $0 < s < 1$. There remains to treat the case of quasi-Banach Besov spaces. Incidentally, let us note that the corresponding result in Theorem 5.2 for the complex method of interpolation remains open. However, in [16] M. Cwikel has shown that the property of being compact can be extrapolated on complex interpolation scales of Banach spaces:

Theorem 5.3 *Assume that X_j, Y_j, $j = 0, 1$, are two compatible Banach couples and suppose that $T : X_j \to Y_j$, $j = 0, 1$, is a bounded, linear operator with the property that there exists $\theta^* \in (0, 1)$ such that $T : [X_0, X_1]_{\theta^*} \to [Y_0, Y_1]_{\theta^*}$ is compact. Then $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is compact for all values of θ in $(0, 1)$.*

It is unclear whether a similar result holds for arbitrary compatible quasi-Banach couples. Nonetheless, in [28] the authors have shown that such an extrapolation result holds for the entire scale of Besov spaces. More specifically, we have:

Theorem 5.4 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a Lipschitz domain and assume that R is an open, convex subset of*

$$\Big\{(s, 1/p, 1/q) : \tfrac{n-1}{n} < p < \infty,\ 0 < q \leq \infty,\ (n-1)(\tfrac{1}{p} - 1)_+ < s < 1\Big\}. \tag{5.25}$$

Also, assume that T is a linear operator such that

$$T : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_s(\partial\Omega), \tag{5.26}$$

is bounded whenever $(s, 1/p, 1/q) \in R$. If there exists $(s^*, 1/p^*, 1/q^*) \in R$ such that T maps $B^{p^*,q^*}_{s^*}(\partial\Omega)$ compactly into itself then the operator (5.26) is in fact compact for all $(s, 1/p, 1/q) \in R$.

In summary, the above analysis shows that if the bounded Lipschitz domain Ω is such that $\nu \in \mathrm{VMO}\,(\partial\Omega)$ then K_1 is compact on $B^{p,q}_s(\partial\Omega)$ whenever $\frac{n-1}{n} < p < \infty$, $0 < q \leq \infty$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$. In particular, in this case,

$$\tfrac{1}{2}I + K_1 : B^{p,q}_{s,\nu_+}(\partial\Omega)/\Psi^1(\partial\Omega_+) \to B^{p,q}_{s,\nu_+}(\partial\Omega)/\Psi^1(\partial\Omega_+) \text{ is Fredholm with index zero,} \tag{5.27}$$

from which (5.8) with $\lambda = 1$ now follows from routine arguments. Finally, when in place of $\nu \in \mathrm{VMO}\,(\partial\Omega)$ we only have (1.7), using (5.20) it can be be shows that (5.27) continues to hold, so the same endgame in the proof of Theorem 1.1 works.

We conclude with some comments pertaining to the nature of condition (1.7) in the context of Theorem 1.1. Consider the two-dimensional setting, when Ω is a bounded curvilinear polygon (i.e., a piece-wise smooth domain). From the Mellin analysis of the structure of the spectra of singular integral operators, it is well-known that the presence of any boundary angle $\theta \neq \pi$ prevents K_1 from being compact on $L^p(\partial\Omega)$, for any $p \in (1, \infty)$. This failure of K_1 to be compact can be quantified in a more precise fashion. Concretely, consider the case when Ω is a curvilinear polygon with precisely one angular point located at the origin $0 \in \mathbb{R}^2$. Furthermore, assume that, in a neighborhood of 0, $\partial\Omega$ agrees with a sector of aperture $\theta \in (0, \pi)$ with vertex at 0. In particular, the outward unit normal ν to Ω is smooth on $\partial\Omega \setminus \{0\}$ and is piecewise constant near 0, where it assumes two values, say, ν_+ and ν_-. As a result,

$$\{\nu\}_{\mathrm{Osc}(\partial\Omega)} \approx \|\nu_+ - \nu_-\| \approx \sqrt{1 + \cos\theta}, \tag{5.28}$$

which shows that there exists a family of domains $\Omega = \Omega_\theta$ as above for which

$$\mathrm{dist}\,(\nu, \mathrm{VMO}(\partial\Omega_\theta)) \longrightarrow 0, \qquad \text{as} \quad \theta \to \pi. \tag{5.29}$$

Based on this analysis, we may conclude that for each $\delta > 0$ there exists a bounded Lipschitz domain Ω (whose Lipschitz character is controlled by a universal constant) with the property that $\mathrm{dist}\,(\nu, \mathrm{VMO}(\partial\Omega)) < \delta$ and yet for each $p \in (1, \infty)$ the operator K_1 fails to be compact on $L^p(\partial\Omega)$.

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