Asymptotic analysis of the Navier-Stokes system in a plane domain with thin channels

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Abstract. The flow of viscous incompressible fluid in a domain Ω_ε depending on a small parameter ε is considered. The domain Ω_ε is the union of a domain Ω_0 with piecewise smooth boundary and thin channels with width of order ε. Every channel contains one angle point of the domain Ω_0 near the channels inlet.

We proof the existence of a solution $(\mathbf{v}_\varepsilon, p_\varepsilon)$ to the Navier-Stokes system such that in a neighbourhood of an angle point of the domain Ω_0 the pair $(\mathbf{v}_\varepsilon, p_\varepsilon)$ is equal, up to the term with finite kinetic energy, to the Jeffery-Hamel solution. In the channels the pair $(\mathbf{v}_\varepsilon, p_\varepsilon)$ asymptotically coincides with the Poiseuille solution. Asymptotic expressions for the kinetic energy and the Dirichlet integral of $(\mathbf{v}_\varepsilon, p_\varepsilon)$ is obtained.

Keywords. Navier-Stokes system, Jeffery-Hamel solution, Poiseuille solution, corner boundary points,

Introduction

We consider the flow of a viscous incompressible fluid in a domain Ω_ε depending on a small parameter ε. To describe Ω_ε we introduce a limit domain Ω_0 and

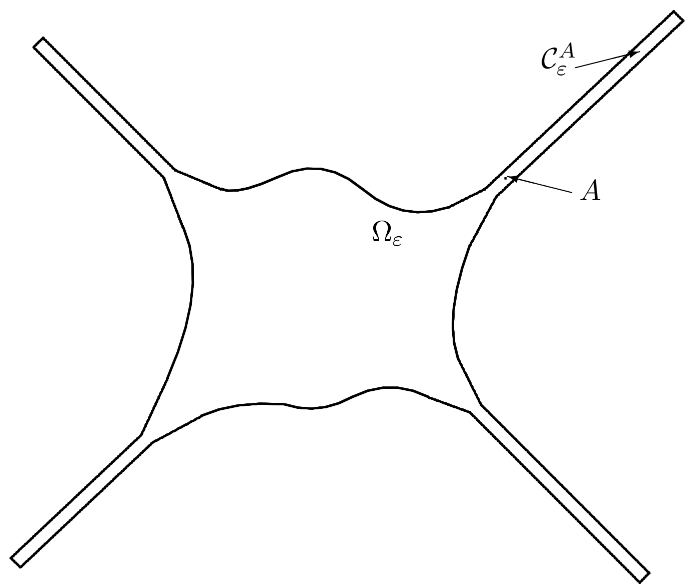

Fig.1. Domain Ω_ε.

thin channels. Let Ω_0 be a domain in $\mathbb{R}^2$ with compact closure and boundary $\partial\Omega_0$. We assume that $\partial\Omega_0$ is a union of smooth closed arcs and by $\{A\}$ we denote the finite set of all end points of these arcs. With every point $A \in \{A\}$ we associate a thin channel $\mathcal{C}_\varepsilon^A$ with A inside $\mathcal{C}_\varepsilon^A$ (see Fig.2, the formal description

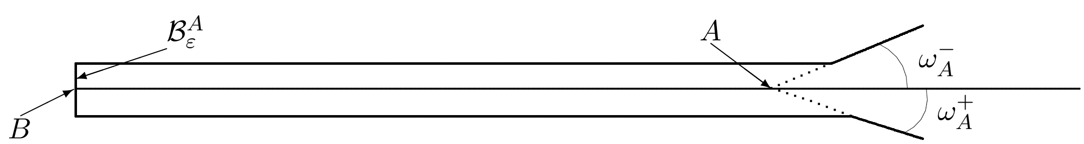

Fig.2. Channel $\mathcal{C}_\varepsilon^A$.

of $\mathcal{C}_\varepsilon^A$ will be given in Section 3).

Let (r, θ), $|\theta| < \pi$, be the polar coordinates with origin at A and the polar axis directed inside Ω_ε. Suppose that the domain Ω_0 is given by $-\omega_A^- < \theta < \omega_A^+$ in the disk with center A and diameter d_A. We assume that $0 < \omega_0^A < \omega_A^\pm < \pi/2$.

The domain Ω_ε is introduced by

$$\Omega_\varepsilon = \Omega_0 \cup \cup_{\{A\}} \mathcal{C}_\varepsilon^A.$$

We deal with the Navier-Stokes system

$$\langle \mathbf{v}_\varepsilon, \nabla \rangle \mathbf{v}_\varepsilon = -\rho^{-1} \operatorname{grad} p_\varepsilon + \nu \Delta \mathbf{v}_\varepsilon \quad \text{on} \quad \Omega_\varepsilon, \tag{0.1}$$

$$\operatorname{div} \mathbf{v}_\varepsilon = 0 \quad \text{on} \quad \Omega_\varepsilon. \tag{0.2}$$

Here $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^2$, ν is the viscosity, ρ is the density, $\mathbf{v}_\varepsilon$ is the velocity vector and p_ε is the pressure.

We assume that the vector-valued function $\mathbf{v}_\varepsilon$ satisfies the Dirichlet boundary condition at every interval $\mathcal{B}_\varepsilon^A$ (see Fig.2):

$$\mathbf{v}_\varepsilon = \varepsilon^{-1} \boldsymbol{\varphi}^A(\varepsilon^{-1}(x - B)), \quad x \in \mathcal{B}_\varepsilon^A, \tag{0.3}$$

where

$$\boldsymbol{\varphi}^A \in (C^{1,\alpha}(-b_A^-, b_A^+))^2$$

and $\boldsymbol{\varphi}^A$ is equal to zero at the end points of $\mathcal{B}_\varepsilon^A$. We suppose also that the velocity vector $\mathbf{v}_\varepsilon$ satisfies the homogeneous Dirichlet condition on the remaining part of the boundary $\partial\Omega_\varepsilon$:

$$\mathbf{v}_\varepsilon(x) = 0, \quad x \in \partial\Omega_\varepsilon \setminus \cup_{\{A\}} \mathcal{B}_\varepsilon^A. \tag{0.4}$$

Let the pressure p_ε be subject to the condition

$$\overline{p_\varepsilon} = 0, \tag{0.5}$$

where $\overline{f}$ is the mean value of the function f over the domain Ω_ε.

We introduce the notation

$$\Upsilon_A = -\int_{-b_A^-}^{b_A^+} \varphi_n^A(t)\, dt, \tag{0.6}$$

where (0.6) and henceforth a_n stands for the normal component of the vector $\mathbf{a}$. We assume that

$$\sum_{\{A\}} \Upsilon_A = 0. \tag{0.7}$$

We first construct an asymptotic solution $(\mathbf{V}_\varepsilon, P_\varepsilon)$ of problem (0.1)–(0.5) such that in Ω_0, outside the set $\{A\}$ there holds the asymptotic relation

$$(\mathbf{V}_\varepsilon(x), P_\varepsilon(x)) \sim (\mathbf{v}_0(x), p_0(x)), \quad \varepsilon \to 0, \tag{0.8}$$

where $(\mathbf{v}_0, p_0)$ is a solution of system (0.1), (0.2) in the domain Ω_0 with the flux

$$\int\limits_{\{x\in\Omega_0:|x-A|=\tau\}} \langle \mathbf{v}_0, \frac{x-A}{|x-A|}\rangle ds_x = \Upsilon_A \tag{0.9}$$

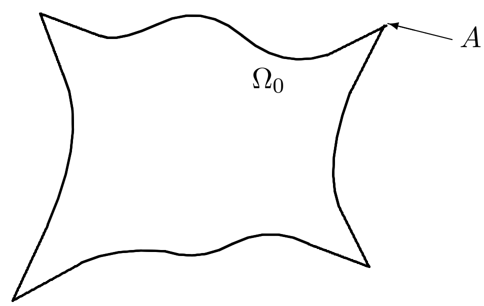

Fig.3. "Model" domain Ω_0.

given at every angle point A (τ being a sufficiently small positive number). Also let v_0 be subject to the boundary condition

$$\mathbf{v}_0(x) = 0, \quad x \in \partial\Omega_0.$$

In a neighbourhood of an angle point the pair $(\mathbf{v}_0, p_0)$ is equal, up to the term with finite Dirichlet integral, to the well-known exact solution of the Navier-Stokes system obtained by Jeffery(1915) and Hamel(1916) (see [1,2]). This solution $(\boldsymbol{\mathcal{H}}^A, \mathcal{Q}^A)$, which describes a plane viscous source (or sink) flow between straight walls has the following form in the polar coordinates (r, θ) with origin at A : /

$$\mathcal{H}_r^A(r,\theta) = r^{-1}\mathcal{V}^A(\theta),$$

$$\mathcal{H}_\theta^A(r,\theta) = 0, \tag{0.10}$$

$$\mathcal{Q}^A(r,\theta) = r^{-2}\mathcal{J}^A(\theta).$$

In a small neibourhood of the point $A \in \{A\}$ we look for $(\mathbf{V}_\varepsilon, P_\varepsilon)$ in the asymptotic form

$$(\mathbf{V}_\varepsilon(x), P_\varepsilon(x)) \sim \big(\varepsilon^{-1}\mathbf{v}^A(\varepsilon^{-1}(x-A)), \varepsilon^{-2}p^A(\varepsilon^{-1}(x-A))\big), \quad \varepsilon \to 0, \tag{0.11}$$

where $(\mathbf{v}^A, p^A)$ is a solution of the Navier-Stokes system considered in the model

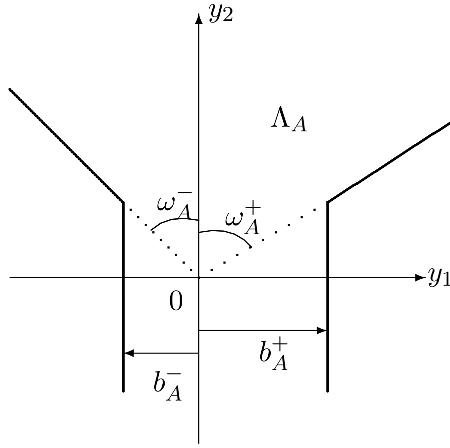

Fig.4. The "model" domain Λ_A.

domain Λ_A depicted in Fig.4. The velocity $\mathbf{v}^A$ satisfies the boundary condition

$$\mathbf{v}^A(y) = 0, \quad y \in \partial\Lambda_A \tag{0.12}$$

and the flux condition

$$\Upsilon_A = \int\limits_{y \in \Xi_1(\Lambda_A)} \langle \mathbf{v}^A, \frac{y}{|y|}\rangle \, ds_y, \tag{0.13}$$

which is equivalent to

$$\Upsilon_A = -\int\limits_{y \in \Xi_2(\Lambda_A)} v_2^A \, dy. \tag{0.14}$$

Here (y_1, y_2) are Cartesian coordinates with center A and with the axis Ay_2 directed along the axis of the channel (see Fig.4);

$$\Xi_1(\Lambda_A) = \{y \in \Lambda_A : y_2 > 0, \ |y| = T_1\},$$

$$\Xi_2(\Lambda_A) = \{y \in \Lambda_A : y_1 \in (-b_A^-, b_A^+), \ y_2 = -T_2\},$$

where T_1 and T_2 are sufficiently large positive numbers. By a_j, $j = 1, 2$, we denote the components of the vector $\mathbf{a}$. In particular, v_2 in (0.14) is the second component of $\mathbf{v}$.

The behavior of $(\mathbf{v}^A, p^A)$ as $|y| \to \infty$, $y_2 > 0$, is described, up to terms with finite Dirichlet integral, by the Jeffery-Hamel solution (0.10).

In the channel $\mathcal{C}_\varepsilon^A$ we have

$$(\mathbf{V}_\varepsilon(x), P_\varepsilon(x)) \\ \sim (\varepsilon^{-1}\mathbf{v}^C(\varepsilon^{-1}(x - C)), \varepsilon^{-2}p^C(\varepsilon^{-1}(x - C)) + \kappa^C\varepsilon^{-3}), \quad \varepsilon \to 0, \tag{0.15}$$

where C is the middle point of the axis of the channel, $(\mathbf{v}^C, p^C)$ is the Poiseuille solution to the Navier-Stokes system in an infinite strip, and κ^C is a constant.

In order to construct the asymptotic solution $(\mathbf{V}_\varepsilon, P_\varepsilon)$ near the end interval $\mathcal{B}_\varepsilon^A$ of the channel $\mathcal{C}_\varepsilon^A$ we introduce a solution $(\mathbf{v}^B, p^B)$ of the Navier-Stokes system (0.1), (0.2) in the semi-strip Π_B which does not depend on the parameter ε (see

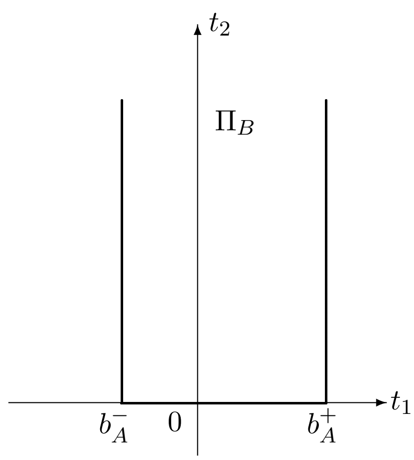

Fig.5. The "model" domain Π_B.

Fig.5). In a small neighbourhood of the end interval $\mathcal{B}_\varepsilon^A$ of the channel we have

$$(\mathbf{V}_\varepsilon(x), P_\varepsilon(x)) \\ \sim (\varepsilon^{-1}\mathbf{v}^B(\varepsilon^{-1}(x - B)), \varepsilon^{-2}p^B(\varepsilon^{-1}(x - B)) + \kappa^B\varepsilon^{-3}), \quad \varepsilon \to 0, \tag{0.16}$$

where $\kappa^B = const$. On the basement of Π_B the boundary condition

$$\mathbf{v}^B(t_1, 0) = \boldsymbol{\varphi}^A(t_1), \quad t_1 \in (-b_A^-, b_A^+) \tag{0.17}$$

is satisfied, where $\boldsymbol{\varphi}^A$ is the vector-valued function in the boundary condition (0.3) corresponding to the channel with the end interval $\mathcal{B}_\varepsilon^A$. On the lateral sides of Π_B the velocity vector $\mathbf{v}^B$ satisfies

$$\mathbf{v}^B(\pm b_A^\pm, t_2) = 0, \quad t_2 \in (0, +\infty) \tag{0.18}$$

and has the prescribed flux

$$\int\limits_{t \in \Xi(\Pi_B)} v_2^B \, dt = \Upsilon_A,$$

where

$$\Xi(\Pi_B) = \{t \in \Pi_B : t_1 \in (-b_A^-, b_A^+), t_2 = T\},$$

and $T > 0$.

We introduce a partition of unity $\{X_\varepsilon, \eta_\varepsilon^A \mu_\varepsilon^B, \xi_\varepsilon^B\}$ in Ω_ε, where η_ε^A and ξ_ε^B are cut-off functions supported by neighbourhoods of A and B respectively. By X_ε we denote cut-off function which vanishes in a neighbourhood of $\{A\}$. The cut-off function μ_ε^B is equal to 1 outside a neighbourhood of $\mathcal{B}_\varepsilon$.

We construct the asymptotic solution $(\mathbf{V}_\varepsilon, P_\varepsilon)$ of system (0.1)–(0.5) in the form

$$\begin{aligned} \mathbf{V}_\varepsilon(x) = \mathbf{v}_0(x) X_\varepsilon(x) + \varepsilon^{-1} \sum \Big\{ & \eta_\varepsilon^A(x) \mu_\varepsilon^B(x) \mathbf{v}^A(\varepsilon^{-1}(x - A)) \\ & + \xi_\varepsilon^B(x) \mathbf{v}^B\big(\varepsilon^{-1}(x - B)\big) \Big\}, \end{aligned} \tag{0.19}$$

$$\begin{aligned} P_\varepsilon(x) = p_0(x) X_\varepsilon(x) + \varepsilon^{-2} \sum \Big\{ & \eta_\varepsilon^A(x) \mu_\varepsilon^B(x) p^A(\varepsilon^{-1}(x - A)) \\ & + \xi_\varepsilon^B(x) p^B(\varepsilon^{-1}(x - B)) \Big\}, \end{aligned} \tag{0.20}$$

In (0.19), (0.20) and henceforth the summation is taken over all the channels i.e. over the set $\{A\}$.

We introduce the number

$$\mathcal{R} = \nu^{-1} \sum \|\boldsymbol{\varphi}^A\|_{(C^{1,\alpha}(-b_A^-, b_A^+))^2}$$

and suppose that $\mathcal{R}$ is sufficiently small:

$$\mathcal{R} \ll 1. \tag{0.21}$$

Our basic result is the existence theorem for a solution $(\mathbf{v}_\varepsilon, p_\varepsilon)$ of (0.1)–(0.5) such that

$$\begin{aligned} \mathbf{v}_\varepsilon(x) &= \mathbf{V}_\varepsilon(x) + \mathbf{w}_\varepsilon(x), \\ p_\varepsilon(x) &= P_\varepsilon(x) + q_\varepsilon(x), \end{aligned} \tag{0.22}$$

where

$$\|\mathbf{w}_\varepsilon\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2} + \|q_\varepsilon\|_{L_2(\Omega_\varepsilon)} \le c\varepsilon^\delta, \quad \delta > 0. \tag{0.23}$$

We also obtain asymptotic expressions for two integral characteristics of the solution $(\mathbf{v}_\varepsilon, p_\varepsilon)$. Let $L_A = |AB|$ and $b_A = \varepsilon^{-1}|\mathcal{B}^A_\varepsilon|$, i.e. L_A and εb_A are the length and the width of $\mathcal{C}^A_\varepsilon$ respectively. We show that the kinetic energy

$$\mathcal{E}(\mathbf{v}_\varepsilon) = \frac{\rho}{2} \int\limits_{\Omega_\varepsilon} \left|\mathbf{v}_\varepsilon(x)\right|^2 dx \tag{0.24}$$

admits the representation

$$\mathcal{E}(\mathbf{v}_\varepsilon) = \frac{3\rho}{5}\frac{1}{\varepsilon}\sum \Upsilon_A^2 L_A b_A^{-1} + \frac{\rho}{2}\log\frac{1}{\varepsilon}\sum \Upsilon_A^2 \int\limits_{\omega_A^-}^{\omega_A^+} \left(\mathcal{V}^A(\theta)\right)^2 d\theta + O(1). \tag{0.25}$$

For the Dirichlet integral

$$\mathcal{I}(\mathbf{v}_\varepsilon) = \int\limits_{\Omega_\varepsilon} \left(\nabla \mathbf{v}_\varepsilon(x)\right)^2 dx \tag{0.26}$$

we obtain the asymptotic formula

$$\mathcal{I}(\mathbf{v}_\varepsilon) = 12\varepsilon^{-3}\sum \Upsilon_A^2 L_A b_A^{-3} + O(\varepsilon^{-2}). \tag{0.27}$$

In *Section* 1 we consider the Dirichlet problem with prescribed fluxes at the points A for the Navier-Stokes system in the domain Ω_0. Auxiliary boundary value problems in model domains Λ_A and Π_B are considered in *Section* 2. The next *Section* 3 concerns the Stokes problem in Ω_ε. In *Section*

4 we derive the principal term $(\mathbf{V}_\varepsilon, P_\varepsilon)$ in the representation (0.22). The auxiliary *Section* 5 is a preparation to the proof of our principal result, an existence theorem for the solution $(\mathbf{v}_\varepsilon, p_\varepsilon)$ of problem (0.1) – (0.5) in the form (0.22). In the same section we study a boundary value problem for the remainder term $(\mathbf{w}_\varepsilon, q_\varepsilon)$. In *Section* 6 we prove the existence of $(\mathbf{v}_\varepsilon, p_\varepsilon)$. The last *Section* 7 contains a proof of asymptotic formulas (0.25), (0.27) for the kinetic energy and for the Dirichlet integral.

1 The flow in the limit domain Ω_0

Consider the system

$$\langle \mathbf{v}, \nabla \rangle \mathbf{v} = -\rho^{-1} \mathrm{grad}\, p + \nu \Delta \mathbf{v} \quad \text{on} \quad \Omega_0, \tag{1.1}$$

$$\mathrm{div}\, \mathbf{v} = 0 \quad \text{on} \quad \Omega_0 \tag{1.2}$$

with p and $\nabla \mathbf{v}$ square summable outside any neighborhood of $\{A\}$. Suppose that $\mathbf{v}$ satisfies

$$\mathbf{v} = 0 \quad \text{on} \quad \partial\Omega_0 \setminus \{A\}. \tag{1.3}$$

At every angle point $A \in \{A\}$ we prescribe the flux M_A,

$$\int\limits_{\{x \in \Omega_0 : |x-A| = \tau\}} \langle \mathbf{v}, \frac{x-A}{|x-A|} \rangle ds_x = M_A \tag{1.4}$$

and we suppose that

$$\sum M_A = 0.$$

Before proving the existence of the solution of (1.1)–(1.4) we note that the principal term of its asymptotics near the point A coincides with the Jeffery-Hamel solution $(\boldsymbol{\mathcal{H}}, \mathcal{Q})$ for the angle

$$\{(r, \theta) : -\omega_- < \theta < \omega_+ \ , \ 0 < r < +\infty\}$$

which is defined as follows. The vector-function $\boldsymbol{\mathcal{H}}$ satisfies the zero Dirichlet condition on the set

$$\{(r, \theta) : \theta = \omega_\pm \ , \ 0 < r < +\infty\}$$

and has the unit flux at A. The radial component $\mathcal{V}_r$ of the vector $\boldsymbol{\mathcal{V}} = r\boldsymbol{\mathcal{H}}$ satisfies

$$(\partial^2 \mathcal{V}_r/\partial\theta^2)(\theta, R) + 4(\mathcal{V}_r(\theta, R) - K) + R(\mathcal{V}_r(\theta, R))^2 = 0, \tag{1.5}$$

$$\int\limits_{-\omega_-}^{\omega_+} \mathcal{V}_r(\theta, R)\, d\theta = \sigma, \tag{1.6}$$

$$\mathcal{V}_r(\pm\omega_\pm, R) = 0, \tag{1.7}$$

where K is an unknown constant depending on $R > 0$, $\sigma = 1$ in the case of the source and $\sigma = -1$ in the case of the sink. The angle component $\mathcal{V}_\theta$ of $\boldsymbol{\mathcal{V}}$ is equal to zero and the function $\mathcal{J} = r^2\mathcal{Q}$ is found from

$$\mathcal{J} = 2\rho\nu(\mathcal{V}_r - K). \tag{1.8}$$

Properties of this solution, which is expressed in elliptic functions, have been investigated in detail in [3–6]. In particular, a complete information about its dependence on the Reynolds number has been obtained. A Jeffery-Hamel solution for the case of variable viscosity and density was considered in [7,8].

By using the Jeffery-Hamel solution obtained in [4], L.E. Fraenkel [9,10] and L.E.Fraenkel, P.M.Eagles [11] constructed an asymptotic series for the flow in channels with slightly curved walls. The stability of flow in an infinite channel of the same type was investigated in [12], [13]. In [14] P.M.Eagles showed that the Jeffery-Hamel solution appears as the first approximation of the boundary layer for the film flow over curved beds.

To study problem (1.1)–(1.4) we use weighted Hölder spaces $N_\tau^{j,\alpha}(\Omega_0)$ with $\alpha \in (0,1), \tau \in \mathbb{R}^1$ and $j = 0$ or 1 of functions on Ω_0 with finite norm

$$\|u\|_{N_\tau^{j,\alpha}(\Omega_0)} = \sup_{x,y\in\Omega_0} |x-y|^{-\alpha} |\nabla^j (r^\tau(x)u(x)) - \nabla^j (r^\tau(y)u(y))| + \sup_{x\in\Omega_0} r^{\tau-j-\alpha}(x)|u(x)|,$$

where $r(x) = \operatorname{dist}\{x, \{A\}\}$, $\nabla^j u = \nabla u$ if $j = 1$ and $\nabla^j u = u$ if $j = 0$. By $\overset{\circ}{N}{}_\tau^{0,\alpha}(\Omega_0)$ we denote the subset of $N_\tau^{0,\alpha}(\Omega_0)$ containing functions equal zero on $\partial\Omega_0 \setminus \{A\}$. Also, let $N_\tau^{-j,\alpha}(\Omega_0)$ be the space of distributions $\operatorname{div}\boldsymbol{\mathcal{W}} + r^{-1}\mathcal{W}_0$, where $\boldsymbol{\mathcal{W}} \in (N_\tau^{0,\alpha}(\Omega_0))^2$, $\mathcal{W}_0 \in N_\tau^{0,\alpha}(\Omega_0)$. The following auxiliary result on the Stokes system in the plane domain with angle points is known (see [15] , §5, where the three-dimensional case is considered).

Lemma 1.1 *The Stokes operator* $\mathcal{S}_0$ *defined by*

$$\mathcal{S}_0(\mathbf{V},P) = \big(-\Delta\mathbf{V} + (\nu\rho)^{-1}\mathrm{grad}\,P,\ \mathrm{div}\,\mathbf{V}\big)$$

performs the isomorphism

$$D^\alpha_\tau = (\mathring{N}^{1,\alpha}_\tau(\Omega_0))^2 \times N^{0,\alpha}_{\tau,\perp}(\Omega_0) \ \to R^\alpha_\tau = (N^{-1,\alpha}_\tau(\Omega_0))^2 \times N^{0,\alpha}_{\tau,\perp}(\Omega_0),$$

where $|\tau - 1 - \alpha| < 1$ *and* $N^{0,\alpha}_{\tau,\perp}(\Omega_0)$ *is the space of functions* $s \in N^{0,\alpha}_\tau(\Omega_0)$ *satisfying the condition*

$$\int\limits_{\Omega_0} s(x)\,dx = 0.$$

Now we are in a position to construct a solution $(\mathbf{v},p)$ of problem (1.1)–(1.4) in Ω_0. We formulate the principal result of this section. In its statement and in the sequel we put

$$\mathcal{M} = \sum |M_A|.$$

By $(\boldsymbol{\mathcal{V}}_A, \mathcal{J}_A)$ we denote the solution of problem (1.5)–(1.8), where $R = \nu^{-1}|M_A|$ and $\sigma = \mathrm{sign} M_A$ for the angle corresponding to A.

Let $\zeta \in C^\infty_0(\mathbb{D}_2(\mathbf{0}))$ and let $\zeta(x) = 1$ for $x \in \mathbb{D}_1(\mathbf{0})$ where $\mathbb{D}_d(a)$ is the disk of diameter d with center a.

We introduce the pair $(\mathbf{Y},\Theta)$ by

$$(\mathbf{Y},\Theta) = \sum |M_A|\zeta_A(\mathbf{H}^A, Q^A), \tag{1.9}$$

where $\zeta_A(x) = \zeta(2d_A^{-1}(x-A))$,

$$(\mathbf{H}^A, Q^A) = (r^{-1}\boldsymbol{\mathcal{V}}_A, r^{-2}\mathcal{J}_A + c^A), \tag{1.10}$$

and c^A is an arbitrary constant.

Theorem 1.1 *Let* $\nu^{-1}\mathcal{M} < C_0$*, where* C_0 *is a constant depending only on* Ω_0*. Then there exists a solution* $(\mathbf{v},p)$ *of problem* (1.1)–(1.4) *represented in the form*

$$(\mathbf{v},p) = (\mathbf{Y},\Theta) + (\mathbf{w},q), \tag{1.11}$$

where the pair $(\mathbf{w},q)$ *belongs to* $(\mathring{N}^{1,\alpha}_\tau(\Omega_0))^2 \times N^{0,\alpha}_{\tau,\perp}(\Omega_0)$ *and satisfies the estimate*

$$\|\mathbf{w}\|_{(\mathring{N}^{1,\alpha}_\tau(\Omega_0))^2} + \|q\|_{N^{0,\alpha}_{\tau,\perp}(\Omega_0)} \le c\mathcal{M} \tag{1.12}$$

with a constant c *independent of* $\mathcal{M}$.

Proof. The pair $(\mathbf{w}, q)$ satisfies the equation

$$\mathcal{S}_0(\mathbf{w}, q) + \nu^{-1}\mathcal{T}_0(\mathbf{w}, q) = (\mathbf{\Phi}, \psi), \tag{1.13}$$

where

$$\mathcal{T}_0(\mathbf{w}, q) = \big(\langle \mathbf{w}, \nabla\rangle\mathbf{w} + \langle \mathbf{Y}, \nabla\rangle\mathbf{w} + \langle \mathbf{w}, \nabla\rangle\mathbf{Y}, 0\big),$$

$$\mathbf{\Phi} = \sum |M_A| \Big\{ \mathbf{H}^A \Delta\zeta_A(x) + 2\langle \nabla\zeta_A, \nabla\rangle\mathbf{H}^A - \nu^{-1}\big(\rho^{-1}Q^A\nabla\zeta_A$$

$$+\eta_A(\mathbf{H}^A\langle \mathbf{H}^A, \nabla\zeta_A\rangle + |M_A|(\zeta_A - 1)\langle \mathbf{H}^A, \nabla\rangle\mathbf{H}^A)\big)\Big\},$$

$$\psi = -\sum |M_A|\langle \mathbf{H}^A, \nabla\zeta_A\rangle.$$

For any $\mathbf{S}$ and $\mathbf{T}$ one has

$$\begin{aligned}\langle \mathbf{S}, \nabla\rangle\mathbf{T} + \langle \mathbf{T}, \nabla\rangle\mathbf{S} &= -\mathbf{S}\mathrm{div}\mathbf{T} - \mathbf{T}\mathrm{div}\mathbf{S} \\ &+\big(\mathrm{div}(S_1\mathbf{T} + T_1\mathbf{S}), \mathrm{div}(S_2\mathbf{T} + T_2\mathbf{S})\big).\end{aligned} \tag{1.14}$$

We put here $\mathbf{S} = \mathbf{w}$, $\mathbf{T} = \mathbf{Y}$ and $\mathbf{S} = \mathbf{w}$, $\mathbf{T} = \mathbf{w}$. Taking into account the resulting relations and equations

$$\mathrm{div}(\mathbf{Y} + \mathbf{w}) = 0, \quad \mathrm{div}\mathbf{Y} = \sum |M_A|\langle \mathbf{H}^A, \nabla\zeta_A\rangle,$$

we write (1.13) in the form

$$\mathcal{S}_0(\mathbf{w}, q) + \mathcal{N}_0(\mathbf{w}, q) = (\mathbf{\Psi}, \psi).$$

Here

$$\mathbf{\Psi} = \mathbf{\Phi} - \nu^{-1}\sum |M_A|\eta_A\mathbf{H}^A\langle \mathbf{H}^A, \nabla\zeta_A\rangle,$$

and $\mathcal{N}_0 : D^\alpha_\tau \to R^\alpha_\tau$ is the operator defined by

$$\mathcal{N}_0(\mathbf{w}, q) = \Big(\mathrm{div}\big(\mathbf{N}^{(1)}(\mathbf{Y}; (\mathbf{w}, q))\big), \mathrm{div}\big(\mathbf{N}^{(2)}(\mathbf{Y}; (\mathbf{w}, q))\big)\Big),$$

where

$$N_i^{(j)}((\mathbf{w}, q)) = \nu^{-1}(Y_i^A w_j + Y_j^A w_i + w_i w_j).$$

By using (1.14) and definition (1.9) of $\mathbf{H}^A$ we represent $(\boldsymbol{\Psi}, \psi)$ in the form

$$(\boldsymbol{\Psi}, \psi) = \big(\operatorname{div} \mathbf{X}^{(1)}(x), \operatorname{div} \mathbf{X}^{(2)}(x), -\operatorname{div} \Theta(x)\big)\Big|_{x \in \mathfrak{Z}},$$

where $\mathfrak{Z} = \cup_{\{A\}} \operatorname{supp} \nabla \zeta_A$ and $\mathbf{X}^{(k)}$, $k = 1, 2$, are given by

$$\mathbf{X}^{(k)} = \sum |M_A| \big(\nabla \zeta_A H_k^A - \nu^{-1}\big(\rho^{-1} \zeta_A \mathcal{Q}^A \mathbf{e}^{(k)} - \zeta_A^2 H_k^A \mathbf{H}^A\big)$$

with

$$\mathbf{e}^{(1)} = (1, 0), \quad \mathbf{e}^{(2)} = (0, 1).$$

In accordance with the inequalities

$$\|\mathbf{X}^{(k)}\|_{N_\tau^{0,\alpha}(\mathfrak{Z})} \leq c\mathcal{M}, \quad \|\operatorname{div} \Theta\|_{N_\tau^{0,\alpha}(\mathfrak{Z})} \leq c\mathcal{M}$$

the estimates hold

$$\|\boldsymbol{\Psi}\|_{N_\tau^{-1,\alpha}(\Omega_0)} + \|\psi\|_{N_\tau^{0,\alpha}(\Omega_0)} \leq c\mathcal{M}.$$

Let $\mathbb{B}_\delta$ be a ball in the space D_τ^α of sufficiently small radius δ centered at $\mathcal{S}_0^{-1}\big((\boldsymbol{\Psi}, \psi)\big)$. If $(\mathbf{w}^{(j)}, q^{(j)}) \in \mathbb{B}_\delta$, $j = 1, 2$, for sufficiently small $\nu^{-1}|M_A|$ and δ, we obtain from the standard inequality

$$\|r^{-1}\mathbf{u}\|_{\overset{\circ}{N}{}_\tau^{0,\alpha}(\Omega_0)} \leq c\|\mathbf{u}\|_{N_\tau^{1,\alpha}(\Omega_0)}$$

that

$$\|N_i^{(j)}((\mathbf{w}^{(1)}, q^{(1)})) - N_i^{(j)}((\mathbf{w}^{(2)}, q^{(2)}))\|_{(N_\tau^{0,\alpha}(\Omega_0))^2}$$

$$\leq m\|\mathbf{w}^{(1)} - \mathbf{w}^{(2)}\|_{(\overset{\circ}{N}{}_\tau^{1,\alpha}(\Omega_0))^2}$$

for $m < 1$, and

$$\|N_i^{(j)}((\mathbf{w}^{(j)}, q^{(j)}))\|_{(N_\tau^{0,\alpha}(\Omega_0))^2} \leq c\|\mathbf{w}^{(j)}\|_{(\overset{\circ}{N}{}_\tau^{1,\alpha}(\Omega_0))^2}.$$

Hence, the operator

$$\mathcal{S}_0^{-1}(\mathcal{N}_0) : D_\tau^\alpha \to D_\tau^\alpha$$

is a contraction mapping. Therefore, there exists one and only one solution $(\mathbf{w}, q) \in \mathbb{B}_\delta$ of equation (1.13) subject to (1.12). ∎

Remark 1.1 The Jeffery-Hamel solution $(\mathbf{H}^A, Q^A)$ is defined up to an arbitrary constant c^A (see (1.9)). Let $(\mathbf{H}_1^A, Q_1^A)$ and $(\mathbf{H}_2^A, Q_2^A)$ be the pairs defined by (1.9) with different constants c_1^A and c_2^A. To the pairs $(\mathbf{H}_1^A, Q_1^A)$, $(\mathbf{H}_2^A, Q_2^A)$ there correspond the solutions $(\mathbf{v}_1, p_1), (\mathbf{v}_2, p_2)$ given by (1.11) with the remainders $(\mathbf{w}_1, q_1)$ and $(\mathbf{w}_2, q_2)$ respectively. The pairs $(\mathbf{w}_1, q_1)$ and $(\mathbf{w}_2, q_2)$ can by found by (1.13) with the right-hand sides $(\mathbf{\Phi}_1, 0)$ and $(\mathbf{\Phi}_2, 0)$, subject to

$$(\mathbf{\Phi}_2, 0) = (\mathbf{\Phi}_1, 0) + ((c_1^A - c_2^A)\nabla\zeta_A, 0).$$

Hence and by (1.13)

$$(\mathbf{w}_2, q_2) = (\mathbf{w}_1, q_1) + (\mathbf{0}, (c_1^A - c_2^A)\zeta_A). \tag{1.15}$$

Combining (1.9), (1.15) and (1.11) we have

$$(\mathbf{v}_2, p_2) = (\mathbf{v}_1, p_1).$$

Therefore the pressure does not depend on the choice of the constant c^A in (1.9) and we set $c^A = 0$ in the sequel.

Remark 1.2 Let the domain Ω_0 be prescribed by

$$\lambda_-(r) - \omega/2 < \theta < \lambda_+(r) + \omega/2$$

near the point A , where $\lambda_\pm$ are smooth functions, $\lambda_\pm(0) = 0$. The difference between the present situation and Theorem 1.1 is that the function $r^{-1}\boldsymbol{\mathcal{V}}(\theta)$ does not satisfy the zero Dirichlet condition near A and therefore the principal term in the asymptotics of the solution $(\mathbf{v}, p)$ becomes more complicated.

One can show that the velocity vector and the pressure are represented in the form

$$r^{-1}\boldsymbol{\mathcal{V}}(\theta, R) + \boldsymbol{\mathcal{V}}^*(\theta, R), \quad r^{-2}\mathcal{J}(\theta, R) + r^{-1}\mathcal{J}^*(\theta, R)$$

modulo terms with finite energy. Here $\boldsymbol{\mathcal{V}}^*$ and $\mathcal{J}^*$ are analytic in R at $R = 0$ and

$$\mathcal{V}_\theta^*(\theta, 0) = Z(\omega)\sum_\pm \pm\gamma_\pm(\omega(\theta \pm \omega/2)\sin(\theta \mp \omega/2) \\ - \sin\omega(\theta \mp \omega/2)\sin(\theta \pm \omega/2)),$$

$$\mathcal{V}_r^*(\theta, 0) = -(d\mathcal{V}_\theta^*/d\theta)(\theta, 0),$$

$$\mathcal{J}^*(\theta, 0) = Z(\omega)\sum_\pm \gamma_\pm(\omega\sin(\theta \pm \omega/2) - \sin\omega\sin(\theta \mp \omega/2)),$$

where

$$Z(\omega) = \sin\omega/((\sin\omega - \omega\cos\omega)(\sin^2\omega - \omega^2))$$

and $\gamma_\pm$ is the curvature of the arc $\theta = \pm\omega/2 + \lambda_\pm$ at the point A, i.e. $\gamma_\pm = 2(d\lambda_\pm/dr)(0)$.

In principle, our main result could be generalized to the case of curved angle considered here. However, we shall not dwell upon this extension for the sake of simplicity of presentation.

2 Navier-Stokes system in the model domains

2.1. Navier-Stokes system in an infinite channel. Let (z_1, z_2) be a Cartesian system and let Σ_A be the strip

$$\Sigma_A = \{(z_1, z_2) : -b_A^- < z_1 < b_A^+,\ z_2 \in \mathbb{R}^1\}.$$

By $(\boldsymbol{\mathcal{U}}_M^A, \mathcal{P}_M^A)$ we denote a solution of the Navier-Stokes system satisfying the zero Dirihlet condition on the boundary $\partial\Sigma_A$ and such that

$$M = \int\limits_{\substack{z_1\in(-b_A^-, b_A^+),\\ z_2=T}} \boldsymbol{\mathcal{U}}(z)\, dz_1.$$

This solution has the form

$$(\boldsymbol{\mathcal{U}}_M^A, \mathcal{P}_M^A) = M(\boldsymbol{\mathcal{U}}_A, \mathcal{P}_A) + (\mathbf{0}, \kappa), \tag{2.1}$$

where κ is an arbitrary constant and $(\boldsymbol{\mathcal{U}}_A, \mathcal{P}_A)$ is explicitly given by

$$\begin{gathered} \boldsymbol{\mathcal{U}}_A(z) = -6b_A^{-3}\big(0, (z_1 - b_A^+)(z_1 + b_A^-)\big), \\ \mathcal{P}_A(z) = -12\rho\nu b_A^{-3} z_2 \end{gathered} \tag{2.2}$$

(we remind that $b_A = b_A^+ + b_A^-$).

2.2. Navier-Stokes system in Λ_A. We introduce a smooth partition of unity $\{\zeta_+^A, \zeta_-^A, \zeta_0^A\}$ on the domain Λ_A (see Fig.4), where $\zeta_0^A(y) = \zeta(b_A^{-1}y)$, $\zeta_-^A(y) = 0$ for positive y_2 and $\zeta_+^A(y) = 0$ for $y_2 > b_A$.

Let w be a function on Λ_A and let

$$[w]_\alpha = \sup_{y,z\in\Lambda_A} \frac{|w(y) - w(z)|}{|y - z|^\alpha}.$$

By $r(y)$ we denote the distance between y and the nearest angle point on $\partial\Lambda_A$.

We say that a function u on Λ_A belongs to the space $K^{l,\alpha}_{\delta,\tau,\beta}(\Lambda_A), l = 0, 1,$ and $\alpha \in (0,1), \delta, \tau, \beta \in \mathbb{R}^1$, if it has the finite norm

$$\|u\|_{K^{l,\alpha}_{\delta,\tau,\beta}(\Lambda_A)} = \lfloor r^{l+\delta+\alpha+1}\nabla^l(\zeta^A_+ u)\rfloor_\alpha + \lfloor r^{l-\tau+\alpha}\nabla^l(\zeta^A_0 u)\rfloor_\alpha$$

$$+\lfloor e^{\beta r}\nabla^l(\zeta^A_- u)\rfloor_\alpha + \|r^{1+\delta}\zeta^A_+ u\|_{L_\infty(\Lambda_A)}$$

$$+\|r^{-\tau}\zeta^A_0 u\|_{L_\infty(\Lambda_A)} + \|e^{\beta r}\zeta^A_- u\|_{L_\infty(\Lambda_A)}.$$

The space of distributions $\operatorname{div}\mathbf{h} + r^{-1}h_0$, where

$$\mathbf{h} \in (K^{0,\alpha}_{\delta,\tau,\beta}(\Lambda_A))^2, \quad h_0 \in K^{0,\alpha}_{\delta,\tau,\beta}(\Lambda_A),$$

will be denoted by $K^{-1,\alpha}_{\delta+1,\tau-1,\beta}(\Lambda_A)$.

Let us consider the Dirichlet problem for the Stokes system

$$\begin{gathered} \nu\Delta\mathbf{V} - \rho^{-1}\operatorname{grad} P = \mathbf{F} \quad \text{on} \quad \Lambda_A, \\ \operatorname{div}\mathbf{V} = f \quad \text{on} \quad \Lambda_A, \\ \mathbf{V}\Big|_{\partial\Lambda_A} = 0. \end{gathered} \tag{2.3}$$

We suppose that the velocity $\mathbf{V}$ has the prescribed flux :

$$M = \int\limits_{y\in\Xi_1(\Lambda_A)} \langle\mathbf{V}, \frac{y}{|y|}\rangle\, ds_y, \tag{2.4}$$

which is equivalent to

$$M = -\int\limits_{y\in\Xi_2(\Lambda_A)} V_2\, dy. \tag{2.5}$$

Let

$$\boldsymbol{\mathcal{H}}_0(\tau,\theta) = \frac{\mathfrak{V}}{\tau}, \quad \mathcal{Q}_0(\tau,\theta) = \frac{\mathfrak{I}(\theta)}{\tau^2},$$

where $\tau = |y|$ and $(\mathfrak{V}, \mathfrak{I})$ is a solution of (1.5)–(1.8) for $R = 0$ and $\sigma = 1$. The following result is essentially known (see, for example,[15]).

Lemma 2.1 i) *For $\mathbf{F}=0, f=0$ and $M=1$ there exists one and only one solution $(\mathbf{V}_0, P_0)$ of problem (2.3)–(2.5) which can be represented in the form*

$$(\mathbf{V}_0, P_0) = \zeta_+^A\big(\boldsymbol{\mathcal{H}}_0, \mathcal{Q}_0\big) + \zeta_-^A\big(\boldsymbol{\mathcal{U}}_1^A, \mathcal{P}_1^A\big) + \zeta_-^A\big(\mathbf{0}, \mathbb{C}_{\mathbf{0}}\big) + \big(\mathbf{W}_0, Q_0\big),$$

where $(\mathbf{W}_0, Q_0) \in (K^{1,\alpha}_{\delta,\tau,\beta}(\Lambda_A))^2 \times K^{0,\alpha}_{\delta+1,\tau-1,\beta}(\Lambda_A)$ and

$$\mathbb{C}_0 = 2\int\limits_{\Lambda_A} \big\{\mathbf{V}_0\big(\mathcal{Q}_0\nabla\zeta_+^A + \mathcal{P}_1^A\nabla\zeta_-^A - \rho\big(\boldsymbol{\mathcal{H}}_0\Delta\zeta_+^A + \boldsymbol{\mathcal{U}}_1^A\Delta\zeta_-^A$$
$$+2(\langle\nabla\zeta_+^A, \nabla\rangle\boldsymbol{\mathcal{H}}_0 + \langle\nabla\zeta_-^A, \nabla\rangle\boldsymbol{\mathcal{U}}_1^A))\big)$$
$$-\big(\zeta_+^A\mathcal{Q}_0 + \zeta_-^A\mathcal{P}_1^A + q_0\big)\big(\langle\boldsymbol{\mathcal{H}}_0, \nabla\rangle\zeta_+^A + \big(\langle\boldsymbol{\mathcal{U}}_1^A, \nabla\rangle\zeta_-^A\big)\big\}\, dx.$$

ii) *Let*

$$\int\limits_{\Lambda_A} f(x)\, dx = 0.$$

For $(\mathbf{F}, f) \in (K^{-1,\alpha}_{\delta+2,\tau-2,\beta}(\Lambda_A))^2 \times K^{0,\alpha}_{\delta+1,\tau-1,\beta}(\Lambda_A)$ there exists one and only one solution $(\mathbf{V}, P)$ of problem (2.3)–(2.5) represented as

$$\big(\mathbf{V}, P\big) = M\zeta_+^A\big(\boldsymbol{\mathcal{H}}_0, \mathcal{Q}_0\big) + M\zeta_-^A\big(\boldsymbol{\mathcal{U}}_1^A, \mathcal{P}_1^A\big) + \zeta_-^A\big(\mathbf{0}, \mathbb{C}\big) + \big(\mathbf{W}, Q\big)\big).$$

Here

$$\mathbb{C} = \int\limits_{\Lambda_A} \{\rho\langle\mathbf{F}, \mathbf{V}_0\rangle + fP_0\}\; dx + M\mathbb{C}_0$$

and the pair $(\mathbf{W}, Q) \in (K^{1,\alpha}_{\delta,\tau,\beta}(\Lambda_A))^2 \times K^{0,\alpha}_{\delta+1,\tau-1,\beta}(\Lambda_A)$ satisfies

$$\|\mathbf{W}\|_{(K^{1,\alpha}_{\delta,\tau,\beta}(\Lambda_A))^2} + \|Q\|_{K^{0,\alpha}_{\delta+1,\tau-1,\beta}(\Lambda_A)}$$
$$\le c\nu^{-1}\big(\|\mathbf{F}\|_{(K^{-1,\alpha}_{\delta+2,\tau-2,\beta}(\Lambda_A))^2} + \|f\|_{K^{0,\alpha}_{\delta+1,\tau-1,\beta}(\Lambda_A)}\big),$$

where the constant c depends only on ρ and Λ_A.

Consider the Dirichlet problem

$$\nu\Delta\mathbf{v} - \rho^{-1}\mathrm{grad}\, p = \langle\mathbf{v}, \nabla\rangle\mathbf{v} \quad \text{on} \;\; \Lambda_A,$$
$$\mathrm{div}\,\mathbf{v} = 0 \quad \text{on} \;\; \Lambda_A, \tag{2.6}$$
$$\mathbf{v}\Big|_{\partial\Lambda_A} = 0.$$

Suppose that the velocity $\mathbf{v}$ satisfies (2.4) with a given M.

Let

$$\boldsymbol{\mathcal{H}}_M^A(y) = |y|^{-1}\boldsymbol{\mathcal{V}}_M^A(\theta), \quad \mathcal{Q}_M^A(y) = |y|^{-2}\mathcal{J}_M^A(\theta),$$

where $(\boldsymbol{\mathcal{V}}_M^A, \mathcal{J}_M^A)$ is the solution of (1.5)–(1.8) with $\omega_\pm = \omega_\pm^A$, $\sigma = \mathrm{sign} M$ and $R = \nu^{-1}|M|$.

By Lemma 2.1 and the contraction mapping principle we arrive at the following assertion

Lemma 2.2 *For sufficiently small positive values* $\alpha, \tau, \delta, \beta, \nu^{-1}|M|$ *there exists a unique solution* $(\mathbf{v}, p)$ *of problem* (2.6), (2.4), (2.5) *represented in the form*

$$(\mathbf{v}, p) = (\mathfrak{W}_M, \mathfrak{P}_M) + (\mathbf{w}, q) + \zeta_-^A(\mathbf{0}, \mathbb{C}),$$

where

$$\begin{aligned} \mathfrak{W}_M(y) &= |M|\zeta_+^A(y)\boldsymbol{\mathcal{H}}_M^A(y) + M\zeta_-^A(y)\boldsymbol{\mathcal{U}}_M^A(y), \\ \mathfrak{P}_M(y) &= |M|\zeta_+^A(y)\mathcal{Q}_M^A(y) + M\zeta_-^A(y)\mathcal{P}_M^A(y) \end{aligned} \tag{2.7}$$

and $(\mathbf{w}, q, \mathbb{C}) \in (K^{1,\alpha}_{\delta,\tau,\beta}(\Lambda_A))^2 \times K^{0,\alpha}_{\delta+1,\tau-1,\beta}(\Lambda_A) \times \mathbb{R}^1$. *Moreover,*

$$\|\mathbf{w}\|_{(K^{1,\alpha}_{\delta,\tau,\beta}(\Lambda_A))^2} + \|q\|_{K^{0,\alpha}_{\delta+1,\tau-1,\beta}(\Lambda_A)} + |\mathbb{C}| \le c|M|, \tag{2.8}$$

where c *is a constant independent of* M.

2.3. The case of the semistrip. Let Π_B be the semistrip $\{(t_1, t_2) : -b_A^- < t_1 < b_A^+, t_2 > 0\}$. We shall use the space $C^{l,\alpha}(\Pi_B)$, $l = 0, 1$, $\alpha \in (0, 1)$ of functions on Π_B with finite norm

$$\|u\|_{C^{l,\alpha}(\Pi_B)} = \sup_{t,s \in \Pi_B} |t-s|^{-\alpha}|\nabla^l u(t) - \nabla^l u(s)| + \sup_{t \in \Pi_B} |u(t)|.$$

By definition, $u \in C^{l,\alpha}_\delta(\Pi_B)$ if $\exp(\delta t_2)u \in C^{l,\alpha}(\Pi_B)$.

Consider the boundary value problem

$$\begin{aligned} \nu\Delta\mathbf{V} - \rho^{-1}\mathrm{grad}\, P = 0 \quad &\text{on } \Pi_B, \\ \mathrm{div}\,\mathbf{V} = 0 \quad &\text{on } \Pi_B, \\ \mathbf{V}(t_1, 0) = \mathbf{g}(t_1), \quad &t_1 \in [-b_A^-, b_A^+], \\ \mathbf{V}(\pm b_A^\pm, t_2) = 0, \quad &t_2 \ge 0, \end{aligned} \tag{2.9}$$

where $\mathbf{g} \in (C^{1,\alpha}(b_A^-, b_A^+))^2$ and $\mathbf{g}(\pm b_A^{\pm}) = 0$. Suppose that

$$\int\limits_{t\in\Xi(\Pi_B)} V_2(t)\,dt = M \tag{2.10}$$

with

$$M = -\int\limits_{b_A^-}^{b_A^+} g_2(t)\,dt.$$

The following result is well-known (see [17], [18]).

Lemma 2.3 *There exists one and only one solution of problem* (2.9), (2.10) *represented in the form*

$$(\mathbf{V}, P) = M(\boldsymbol{\mathcal{U}}_M^A, \mathcal{P}_M^A) + (\mathbf{W}, Q),$$

where $(\mathbf{W}, Q) \in (C_\delta^{1,\alpha}(\Pi_B))^2 \times C_\delta^{0,\alpha}(\Pi_B)$ *and the estimate*

$$\|\mathbf{W}\|_{(C_\delta^{1,\alpha}(\Pi_B))^2} + \|Q\|_{C_\delta^{0,\alpha}(\Pi_B)} \leq c\|\mathbf{g}\|_{(C^{1,\alpha}(-b_A^-, b_A^+))^2}$$

holds with a constant c depending only on ρ and the domain Π_B.

By this Lemma and contraction mapping principle we obtain the following solvability result for the Navier-Stokes system

$$\begin{gathered}
\nu\Delta\mathbf{v} - \rho^{-1}\mathrm{grad}\, p = \langle \mathbf{v}, \nabla\rangle\mathbf{v} \quad \text{on} \;\; \Pi_B, \\
\mathrm{div}\,\mathbf{v} = 0 \quad \text{on} \;\; \Pi_B, \\
\mathbf{v}(t_1, 0) = \mathbf{g}(t_1), \quad t_1 \in [-b_A^-, b_A^+], \\
\mathbf{v}(\pm b_A^{\pm}, t_2) = 0, \quad t_2 \geq 0.
\end{gathered} \tag{2.11}$$

Lemma 2.4 *If $\nu^{-1}M$ is sufficiently small, there exists a single solution* $(\mathbf{v}, p)$ *of problem* (2.9), (2.10) *represented in the form*

$$\mathbf{v}(t) = \boldsymbol{\mathcal{U}}_M^A(t) + \mathbf{w}(t),$$

$$p(t) = \mathcal{P}_M^A(t) + q(t),$$

where $(\mathbf{w}, q) \in (C_\delta^{1,\alpha}(\Pi_B))^2 \times C_\delta^{0,\alpha}(\Pi_B)$*, and the estimate*

$$\|\mathbf{w}\|_{(C_\delta^{1,\alpha}(\Pi_B))^2} + \|q\|_{C_\delta^{0,\alpha}(\Pi_B)} \leq c\|\mathbf{g}\|_{(C^{1,\alpha}(-b_A^-, b_A^+))^2} \tag{2.12}$$

is valid.

3 Stokes system in Ω_ε

Let Ω_ε be the domain depicted in Fig.1. In order to determine $\mathcal{C}_\varepsilon^A$ we introduce a local system of Cartesian coordinates (y_1^A, y_2^A) with origin A and with the axis Ay_2^A directed into Ω_0. The thin channel $\mathcal{C}_\varepsilon^A$ will be defined as

$$\mathcal{C}_\varepsilon^A = \{(y_1^A, y_2^A) : -\varepsilon b_A^- < y_1^A < \varepsilon b_A^+, -L_A^- < y_2^A < L_A^+\}.$$

The values $b_A^\pm, L_A^\pm$ are subject to the inequalities

$$b_A^\pm > b_0^A > 0, \quad L_A^\pm > L_0 > 0,$$

where b_0, L_0 are constants independent of ε. The interval $\mathcal{B}_\varepsilon^A = \{(y_1^A, y_2^A) : -\varepsilon b_A^- < y_1^A < \varepsilon b_A^+, y_2 = -L_A^-\}$ will be called the end of the channel $\mathcal{C}_\varepsilon^A$. This interval $\mathcal{B}_\varepsilon^A$ is orthogonal to the walls and placed at a finite distance $L_A = L_A^-$ from A. By $B \in \mathcal{B}_\varepsilon^A$ we denote the point with coordinates $(y_1^A, y_2^A) = (0, -L_A)$.

We introduce the norm in the Sobolev space $H^1(\Omega_\varepsilon)$:

$$\|u\|_{H^1(\Omega_\varepsilon)} = \Big(\int\limits_{\Omega_\varepsilon} |\nabla u|^2 \, dx + \int\limits_{\Omega_\varepsilon} r_\varepsilon^{-2} |u|^2 \, dx \Big)^{1/2},$$

where

$$r_\varepsilon(x) = \begin{cases} r & \text{when } x \in \Omega_0 \cap (\mathbb{D}_d(x-A) \setminus \mathbb{D}_{\varepsilon a}(x-A)) \\ \varepsilon & \text{when } x \in (\Omega_\varepsilon \cap \mathbb{D}_{\varepsilon a}(x-A)) \cup \mathcal{C}_\varepsilon^A \\ 1 & \text{when } x \in \Omega_0 \setminus \cup_{\{A\}} \mathbb{D}_d(x-A) \end{cases}$$

and

$$d = \min_{\{A\}} d_A, \quad a = 2\max_{\{A\}} \{b_0^A / \cos\omega_0^A\}.$$

By $\overset{\circ}{H}{}^1(\Omega_\varepsilon)$ we denote the completion of $C_0^\infty(\Omega_\varepsilon)$ with respect to this norm and we set

$$\|\varphi\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^*} = \sup\{\varphi(u) : \|u\|_{\overset{\circ}{H}{}^1(\Omega_\varepsilon)} = 1\}.$$

Before studying the structure of the solutions to the Navier-Stokes problem (0.1)–(0.5) consider an auxiliary linear Stokes system in Ω_ε.

Lemma 3.1 *Let*

$$\mathcal{S} : (\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2 \times L_2(\Omega_\varepsilon) \to ((\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2)^* \times L_2(\Omega_\varepsilon) \tag{3.1}$$

be the operator, which transforms $(\mathbf{U}_\varepsilon, \pi_\varepsilon)$ *to* $(-\Delta \mathbf{U}_\varepsilon + \rho^{-1}\nu^{-1}\nabla\pi_\varepsilon, \mathrm{div}\mathbf{U}_\varepsilon)$.

Suppose that $(\mathbf{F}_\varepsilon, f_\varepsilon) \in ((\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2)^* \times L_2(\Omega_\varepsilon)$ *and that* f_ε *is subject to*

$$\overline{f_\varepsilon} = 0. \tag{3.2}$$

Then there exists a single solution $(\mathbf{U}_\varepsilon, \pi_\varepsilon) \in (\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2 \times L_2(\Omega_\varepsilon)$ *of the problem*

$$\mathcal{S}(\mathbf{U}_\varepsilon, \pi_\varepsilon) = (\mathbf{F}_\varepsilon, f_\varepsilon), \quad \overline{\pi_\varepsilon} = 0, \tag{3.3}$$

and the estimate holds

$$\|\pi_\varepsilon\|_{L_2(\Omega_\varepsilon)} + \|\mathbf{U}_\varepsilon\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2} \le \mathrm{c}(\|\mathbf{F}_\varepsilon\|_{((\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2)^*} + \|f_\varepsilon\|_{L_2(\Omega_\varepsilon)}), \tag{3.4}$$

where c *does not depend on* ε.

Proof. The unique solvability of (3.3) is well-known [20]. We only need to check estimate (3.4). By using an argument from [19] we shall construct a vector function $\mathbf{Z}_\varepsilon \in \overset{\circ}{H}{}^1(\Omega_\varepsilon)$ satisfying the equation

$$\mathrm{div}\, \mathbf{Z}_\varepsilon = f_\varepsilon \tag{3.5}$$

and the inequality

$$\|\mathbf{Z}_\varepsilon\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2} \le c\|f_\varepsilon\|_{L_2(\Omega_\varepsilon)}, \tag{3.6}$$

where c does not depend on ε and f_ε. We consider Ω_0 as a sum of domains $\Omega^{(l)}$ star-shaped with respect to a ball, $l = 1, \dots, L$. The channels $\mathcal{C}_\varepsilon^{(j)}$ are represented as unions of the squares $\mathcal{T}_\varepsilon^{(k)}, k = 1, 2, \dots, K$, with the side length ε. So we have

$$\Omega_\varepsilon = \cup_{l=1}^{L} \Omega^{(l)} \cup \cup_{k=1}^{K} \mathcal{T}_\varepsilon^{(k)}.$$

By (3.2) f_ε can be written as

$$f_\varepsilon(x) = \sum_{l=1}^{L} F^{(l)}(x) + \sum_{k=1}^{K} f_\varepsilon^{(k)}(x),$$

where $\mathrm{supp} F^{(l)} \subset \Omega^{(l)}$, $\mathrm{supp} f_\varepsilon^{(k)} \subset \mathcal{T}_\varepsilon^{(k)}$ and

$$\int\limits_{\Omega^{(l)}} F^{(l)}(x)dx = 0, \qquad \int\limits_{\mathcal{T}^{(k)}} f_\varepsilon^{(k)}(x)dx = 0 \tag{3.7}$$

(see [19]). According to (3.7) there exist vector-functions $\mathbf{Z}^{(l)} \in (\overset{\circ}{H}{}^1(\Omega^{(l)}))^2$, $\mathbf{z}_\varepsilon^{(k)} \in (\overset{\circ}{H}{}^1(\mathcal{T}_\varepsilon^{(k)}))^2$ satisfying the equations

$$\operatorname{div} \mathbf{Z}^{(l)} = F^{(l)}, \quad \operatorname{div} \mathbf{z}_\varepsilon^{(k)} = f_\varepsilon^{(k)},$$

and the inequalities

$$\|\mathbf{Z}^{(l)}\|_{(\overset{\circ}{H}{}^1(\Omega^{(l)}))^2} \leq c\|F^{(l)}\|_{L_2(\Omega^{(l)})}, \quad \|\,|\nabla \mathbf{z}_\varepsilon^{(k)}|\,\|_{L_2(\mathcal{T}_\varepsilon^{(k)})} \leq c\|f_\varepsilon^{(k)}\|_{L_2(\mathcal{T}_\varepsilon^{(k)})}$$

([19],Lemma 1). We extend $\mathbf{Z}^{(l)}, \mathbf{z}_\varepsilon^{(k)}$ by zero to Ω_ε . Then, the vector function

$$\mathbf{Z}_\varepsilon = \sum_{l=1}^{L} \mathbf{Z}^{(l)} + \sum_{k=1}^{K} \mathbf{z}_\varepsilon^{(k)}$$

satisfies both (3.5) and (3.6).

Let $(\mathbf{U}_\varepsilon, \pi_\varepsilon) \in (\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2 \times L_2(\Omega_\varepsilon)$ be a solution of (3.3). Then $(\mathbf{\Gamma}_\varepsilon, \pi_\varepsilon) = (\mathbf{U}_\varepsilon + \mathbf{Z}_\varepsilon, \pi_\varepsilon)$ is a solution of

$$\mathcal{S}(\mathbf{\Gamma}_\varepsilon, \pi_\varepsilon) = (\mathbf{F}_\varepsilon + \Delta \mathbf{Z}_\varepsilon, 0).$$

By the standard energy estimate

$$\|\mathbf{\Gamma}_\varepsilon\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2} \leq c\|\mathbf{F}_\varepsilon + \Delta \mathbf{Z}_\varepsilon\|_{((\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2)^*}$$

and by (3.5), it follows

$$\|\mathbf{U}_\varepsilon\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2} \leq c(\|\mathbf{F}_\varepsilon\|_{((\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2)^*} + \|f_\varepsilon\|_{L_2(\Omega_\varepsilon)}). \tag{3.8}$$

In order to estimate the pressure π_ε, we introduce a function $\mathbf{I}_\varepsilon \in (\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2$ satisfying

$$\operatorname{div} \mathbf{I}_\varepsilon = \pi_\varepsilon, \tag{3.9}$$

$$\|\mathbf{I}_\varepsilon\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2} \leq c\|\pi_\varepsilon\|_{L_2(\Omega_\varepsilon)}. \tag{3.10}$$

By (3.9), we have

$$\|\pi_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} = -\int_{\Omega_\varepsilon} \langle \nabla \pi_\varepsilon, \mathbf{I}_\varepsilon \rangle dx \leq c\|\nabla \pi_\varepsilon\|_{((\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2)^*} \|\mathbf{I}_\varepsilon\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2}.$$

Hence and from (3.10) we obtain

$$\|\pi_\varepsilon\|_{L_2(\Omega_\varepsilon)} \leq c\|\nabla \pi_\varepsilon\|_{((\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2)^*}.$$

Now (3.4) follows from (3.3) and (3.8). ∎

4 The flow in Ω_ε. Calculation of the principal term $(\mathbf{V}_\varepsilon, P_\varepsilon)$

As already mentioned in Introduction, the principal term $(\mathbf{V}_\varepsilon, P_\varepsilon)$ of representation (0.22) for the solution $(\mathbf{v}_\varepsilon, p_\varepsilon)$ to problem (0.1)–(0.5) is defined by (0.19), (0.20). We give now more details for calculation of the term in (0.19), (0.20) and study their asymptotic behavior.

We define X_ε, η_ε^A, μ_ε^B and ξ_ε^B by formulas

$$\eta_\varepsilon^A(x) = \begin{cases} \zeta(\varepsilon^{-1/2}(x-A)) & \text{for} \quad x \in \Omega_\varepsilon \setminus \mathcal{C}_\varepsilon^A \\ 1 & \text{for} \quad x \in \mathcal{C}_\varepsilon^A, \end{cases}$$

$$\chi_\varepsilon^A(x) = 1 - \zeta(\varepsilon^{-1/2}(x-A)), \quad \xi_\varepsilon^B(x) = \zeta(\varepsilon^{-1/2}(x-B)),$$

$$\mu_\varepsilon^B(x) = 1 - \zeta(\varepsilon^{-1/2}(x-B)), \quad X_\varepsilon(x) = \prod \chi_\varepsilon^A(x),$$

where $\mathcal{C}_\varepsilon^A$ is the channel, which starts at the point A and the product is taken over all points of the set $\{A\}$. By definition of the cut-off functions we have

$$\mu_\varepsilon^B(x) + \xi_\varepsilon^B(x) = 1, \quad \eta_\varepsilon^A(x) = 1, \quad \chi_\varepsilon^A(x) = 0 \quad \text{for } x \in \mathcal{C}_\varepsilon^A$$

and

$$X_\varepsilon(x) + \sum_{A \in \{A\}} \eta_\varepsilon^A(x) = 1, \quad \mu_\varepsilon^B(x) = 1, \quad \xi_\varepsilon^B(x) = 0, \quad \text{for } x \in \Omega_0.$$

Hence, the collection of cut-off functions $\{X_\varepsilon, \eta_\varepsilon^A \mu_\varepsilon^B, \xi_\varepsilon^B\}$ forms a partition of unity on Ω_ε.

The pair $(\mathbf{v}_0, p_0)$ is determined from problem (1.1)–(1.4), with the prescribed fluxes

$$M_A = \Upsilon_A$$

at the points $A \in \{A\}$. According to Theorem 1.1, one has

$$(\mathbf{v}_0, p_0) = (\mathbf{Y}_0, \Theta_0) + (\mathbf{w}_0, q_0 + K_\varepsilon), \tag{4.1}$$

where $(\mathbf{w}_0, q_0) \in (\mathring{N}_\tau^{1,\alpha}(\Omega_0))^2 \times N_{\tau,\perp}^{0,\alpha}(\Omega_0)$, the pair $(\mathbf{Y}_0, \Theta_0)$ is defined by (1.9) with $M_A = \Upsilon_A$ and K_ε is a constant.

The term $(\mathbf{v}^A, p^A)$ is a solution of problem (2.9), (2.10), there $M = \Upsilon_A$ in the domain Λ_A(cf. Fig.4). By Lemma 2.2 $(\mathbf{v}^A, p^A)$ can be represented as

$$(\mathbf{v}^A, p^A) = (\mathfrak{W}^A, \mathfrak{P}^A) + (\mathbf{w}^A, q^A + k_\varepsilon^A) + \zeta_-(\mathbf{0}, \mathbb{C}_\mathbf{0}^\mathbf{A}), \tag{4.2}$$

where $(\mathbf{w}^A, q^A), \mathbb{C}_0^A$ satisfy (2.8) with $M = \Upsilon_A$, k_ε^A is a constant and $(\mathfrak{W}^A, \mathfrak{P}^A) = (\mathfrak{W}_M^A, \mathfrak{P}_M^A)$, where $M = \Upsilon_A$.

The pair $(\mathbf{v}^B, p^B)$ is sought from problem (2.11) in the domain Π_B with $\mathbf{g} = \boldsymbol{\varphi}^A$. According to Lemma 2.4 the solution $(\mathbf{v}^B, p^B)$ has the form

$$(\mathbf{v}^B, p^B) = \Upsilon_A(\boldsymbol{\mathcal{U}}^A, \mathcal{P}^A) + (\mathbf{w}^B, q^B + k_\varepsilon^B), \tag{4.3}$$

where $(\mathbf{w}^B, q^B)$ is subject to (2.12) with $\mathbf{g} = \boldsymbol{\varphi}^A$, $(\boldsymbol{\mathcal{U}}^A, \mathcal{P}^A) = (\boldsymbol{\mathcal{U}}_M^A, \mathcal{P}_M^A)$ with $M = \Upsilon_A$ and k_ε^B is a constant.

In order to obtain representation (0.22) of the solution $\mathbf{v}_\varepsilon, p_\varepsilon$ of problem (0.1)–(0.5) satisfying estimate (0.23) we find the constants $K_\varepsilon, k_\varepsilon^A, k_\varepsilon^B$ from the condition

$$\overline{P}_\varepsilon = O(\varepsilon^D), \tag{4.4}$$

where D is a positive number. By (4.1)–(4.3) one has

$$\int\limits_{\Omega_\varepsilon} P_\varepsilon(x)\,dx = \sum \{I_1^A + I_2^A + I_3^A + I^B\} + I_0 + J, \tag{4.5}$$

where

$$I_1^A = \int\limits_{\Omega_\varepsilon} \zeta_A(x)\zeta_+^A(\varepsilon^{-1}(x-A))\mathcal{Q}^A(x)\,dx,\ I_0 = \int\limits_{\Omega_\varepsilon} q_0(x)X_\varepsilon(x)\,dx,$$

$$I_2^A = \frac{1}{\varepsilon^2}\int\limits_{\mathcal{C}_\varepsilon^A} \zeta_-^A(\varepsilon^{-1}(x-A))(\mathcal{P}^A(x) + \mathbb{C}^A)\,dx,$$

$$I_3^A = \frac{1}{\varepsilon^2}\int\limits_{\mathcal{C}_\varepsilon^A} \eta_\varepsilon^A(x)\mu_\varepsilon^B(x)q^A(x)\,dx,\ I^B = \frac{1}{\varepsilon^2}\int\limits_{\mathcal{C}_\varepsilon^A} \xi_\varepsilon^B(x)q^B(x)\,dx,$$

$$J = \int\limits_{\Omega_\varepsilon} \{K_\varepsilon X_\varepsilon(x) + \sum(\eta_\varepsilon^A(x)\mu_\varepsilon^B(x)k_\varepsilon^A + \xi_\varepsilon^A(x)k_\varepsilon^B)\}\,dx$$

and $\mathcal{Q}^A = \mathcal{Q}_M^A$ with $M = \Upsilon_A$. We shall calculate the integral I_1^A and I_2^A.

We have

$$\begin{aligned} I_1^A &= \int\limits_{\varepsilon b_A}^{a} \int\limits_{-\omega_A^-}^{\omega_A^+} \zeta_A(x)\zeta_+^A(\varepsilon^{-1}(x-A))\mathcal{J}^A(\theta)r^{-1}\,d\theta dr \\ &= \int\limits_{a/2}^{a} \int\limits_{-\omega_A^-}^{\omega_A^+} \zeta_A(x)\mathcal{J}^A(\theta)r^{-1}\,d\theta dr + \int\limits_{2\varepsilon b_A}^{a/2} \int\limits_{-\omega_A^-}^{\omega_A^+} \mathcal{J}^A(\theta)r^{-1}\,d\theta dr \\ &+ \int\limits_{b_A}^{2b_A} \int\limits_{-\omega_A^-}^{\omega_A^+} \rho^{-1}\zeta_+^A(y(\rho,\theta))\,d\theta d\rho = \log 1/\varepsilon \int\limits_{-\omega_A^-}^{\omega_A^+} \mathcal{J}^A(\theta)\,d\theta + c_1^A, \end{aligned} \tag{4.6}$$

where $\mathcal{J}^A = \mathcal{J}_M^A$ with $M = \Upsilon_A$ and

$$\begin{aligned} c_1^A = &\int\limits_{a/2}^{a} \int\limits_{-\omega_A^-}^{\omega_A^+} \log r \frac{\partial \eta_A}{\partial r}(r,\theta)\mathcal{J}^A(\theta)\,d\theta dr \\ &+ \int\limits_{b_A}^{2b_A} \int\limits_{-\omega_A^-}^{\omega_A^+} \log \rho \frac{\partial \zeta_+^A}{\partial \rho}(\rho,\theta)\mathcal{J}^A(\theta)\,d\theta d\rho. \end{aligned}$$

By (2.2) with $b = b_A$, the integral I_2^A is

$$I_2^A = \varepsilon^{-2} 6\rho\nu L_A^2 b_A^{-2} + \varepsilon^{-1}\mathbb{C}^A L_A b_A - \varepsilon^{-2}\mathbb{C}^A \int\limits_{\mathcal{C}_\varepsilon^A} \xi_\varepsilon^B(x)\,dx + c_2^A, \tag{4.7}$$

where L_A is the distance between A and B and

$$c_2^A = \int\limits_{-2b_A}^{0} \int\limits_{-b_A^-}^{b_A^+} (1-\zeta_A^-(y))(\mathcal{P}^A(y) + \mathbb{C}^A)\,dy_1 dy_2.$$

We pass to the estimates of I_3^A, I^B and I_0. We begin with the equality

$$\begin{aligned} I_3^A - \int\limits_{\Lambda_A} q^A(y)\,dy = \frac{1}{\varepsilon^2} \int\limits_{-\omega_A^-}^{\omega_A^+} \int\limits_{\varepsilon^{-1/2}}^{\infty} (1-\eta_\varepsilon^A(x))q^A(\varepsilon^{-1}(x-A))r\,drd\theta \\ + \frac{1}{\varepsilon^2} \int\limits_{-\infty}^{\varepsilon^{-1/2}} \int\limits_{-\varepsilon b_A^-}^{\varepsilon b_A^+} (1-\mu_\varepsilon^B(x))q^A(\varepsilon^{-1}(x-A))\,dx_1dx_2. \end{aligned} \tag{4.8}$$

Since $q^A \in K^{0,\alpha}_{\delta+1,\tau-1,\beta}(\Lambda_A)$, we have

$$\begin{aligned} \left| q^A(\varepsilon^{-1}(x-A)) \right| \le c\varepsilon^{\delta+2} r^{-\delta-2} \quad \text{for} \quad x \in \operatorname{supp}(1-\eta_\varepsilon^A), \\ \left| q^A(\varepsilon^{-1}(x-A)) \right| \le ce^{-\beta/\varepsilon} \qquad \text{for} \quad x \in \operatorname{supp}(1-\mu_\varepsilon^B). \end{aligned} \tag{4.9}$$

Hence by (4.8), (4.9) we obtain

$$I_3^A = \int\limits_{\Lambda_A} q^A(y)\,dy + O(\varepsilon^{\delta/2}). \tag{4.10}$$

Similarly, using the equality

$$I^B - \int\limits_{\Pi_B} q^B(t)\,dt = \frac{1}{\varepsilon^2} \int\limits_{\varepsilon^{-1/2}}^{\infty} \int\limits_{-\varepsilon b_A^-}^{\varepsilon b_A^+} (1-\xi_\varepsilon^B(x))q^B(\varepsilon^{-1}(x-B))\,dx_1dx_2$$

and the inclusion $q^B \in C^{0,\alpha}_\delta(\Pi_B)$, we find

$$I^B = \int\limits_{\Pi_B} q^B(t)\,dt + O(\varepsilon^{\delta/2}). \tag{4.11}$$

Since $q_0 \in N^{0,\alpha}_{\tau,\perp}$, $|\tau-1-\alpha|<1$, it follows that the equality

$$\int\limits_{\Omega_\varepsilon} X_\varepsilon(x)q_0(x)\,dx = \int\limits_{\Omega_0} q_0(x)\,dx + \sum \int\limits_{-\omega_A^-}^{\omega_A^+} \int\limits_{0}^{2\varepsilon^{-1/2}} (1-\chi_\varepsilon^A(x))q_0(x)r\,drd\theta$$

implies

$$I_0 = O(\varepsilon). \tag{4.12}$$

Thus, by (4.6), (4.7), (4.10)–(4.12) we arrive at the formula

$$\begin{aligned}\int\limits_{\Omega_\varepsilon} P_\varepsilon(x)\,dx = J + \sum \{\varepsilon^{-2} 6\rho\nu L_A^2 b_A^{-2} - \varepsilon^{-2}\mathbb{C}^A \int\limits_{\mathcal{C}_\varepsilon^A} \xi_\varepsilon^B(x)\,dx \\ +\varepsilon^{-1}\mathbb{C}^A L_A b_A + \log 1/\varepsilon \int\limits_{-\omega_A^-}^{\omega_A^+} \mathcal{J}^A(\theta)\,d\theta \\ +c_1^A + c_2^A + \int\limits_{\Lambda_A} q^A(y)\,dy + \int\limits_{\Pi_B} q^B(t)\,dt\} + O(\varepsilon).\end{aligned} \tag{4.13}$$

In order to equate p_0 to $\varepsilon^{-2}p^A$ as well as $\varepsilon^{-2}p^A$ to $\varepsilon^{-2}p^B$ in the domains $\mathrm{supp}\nabla\eta_\varepsilon^A$ and $\mathrm{supp}\nabla\mu_\varepsilon^B$ respectively, we put

$$k_\varepsilon^A = \varepsilon^2 K_\varepsilon, \quad k_\varepsilon^B = k_\varepsilon^A + \mathbb{C}^A. \tag{4.14}$$

Let us calculate the integral J. Taking into consideration (4.14) we have

$$J = K_\varepsilon|\Omega_\varepsilon| + \varepsilon^{-2}\sum \mathbb{C}^A \int\limits_{\mathcal{C}_\varepsilon^A} \xi_\varepsilon^B(x)\,dx. \tag{4.15}$$

By direct calculation we obtain

$$|\Omega_\varepsilon| = |\Omega_0| + \varepsilon \sum b_A L_A + \varepsilon^2 \frac{1}{2} \sum ((b_A^+)^2 \mathrm{ctg}\omega_A^+ + (b_A^-)^2 \mathrm{ctg}\omega_A^-). \tag{4.16}$$

Let us substitute (4.15), (4.16) into (4.13). Condition (4.4) implies

$$\begin{aligned}K_\varepsilon\{|\Omega_0| + \varepsilon \sum b_A L_A + \varepsilon^2 \frac{1}{2} \sum ((b_A^+)^2 \mathrm{ctg}\omega_A^+ + (b_A^-)^2 \mathrm{ctg}\omega_A^-))\} \\ = \varepsilon^{-2} 6\rho\nu \sum L_A^2 b_A^{-2} + \varepsilon^{-1} \sum \mathbb{C}^A L_A b_A + \log 1/\varepsilon \sum \int\limits_{-\omega_A^-}^{\omega_A^+} \mathcal{J}^A(\theta)\,d\theta \\ + \sum\{c_1^A + c_2^A + \int\limits_{\Lambda_A} q^A(y)\,dy + \int\limits_{\Pi_B} q^B(t)\,dt\}.\end{aligned} \tag{4.17}$$

Hence, we look for K_ε in the form

$$K_\varepsilon = K^{(2)}\varepsilon^{-2} + K^{(1)}\varepsilon^{-1} + K^{(\log)}\log 1/\varepsilon + K^{(0)}. \tag{4.18}$$

After substituting (4.18) into (4.17) we have

$$K^{(2)} = -|\Omega_0|^{-1} 6\rho\nu \sum (L_A/b_A)^2,$$

$$K^{(1)} = -|\Omega_0|^{-1} \sum b_A L_A(\mathbb{C}^A + K^{(2)}),$$

$$K^{(\log)} = -|\Omega_0|^{-1} \sum \int\limits_{-\omega_A^-}^{\omega_A^+} \mathcal{J}^A(\theta)\, d\theta, \tag{4.19}$$

$$K^{(0)} = -|\Omega_0|^{-1} \sum \{c_1^A + c_2^A + \int\limits_{\Lambda_A} q^A(y)\, dy + \int\limits_{\Pi_B} q^B(t)\, dt$$

$$+K^{(1)} b_A L_A + K^{(2)}/2((b_A^+)^2 \mathrm{ctg}\omega_A^+ + (b_A^-)^2 \mathrm{ctg}\omega_A^-)\}.$$

Thus, the constants $K_\varepsilon, k_\varepsilon^A, k_\varepsilon^B$ are defined by (4.19), (4.14).

5 The boundary value problem for the remainder $(\mathbf{w}_\varepsilon, p_\varepsilon)$

In the previous section we were concerned with the principal term $(\mathbf{V}_\varepsilon, P_\varepsilon)$ in the asymptotic representation (0.22) for the solution $(\mathbf{v}_\varepsilon, p_\varepsilon)$ of problem (0.1)–(0.5). To justify representation (0.22), consider the problem for the remainder $(\mathbf{w}_\varepsilon, q_\varepsilon)$. Let

$$\mathcal{T} : (\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2 \times L_2(\Omega_\varepsilon) \to ((H^1(\Omega_\varepsilon))^2)^* \times L_2(\Omega_\varepsilon)$$

be the operator defined by

$$\mathcal{T}(\mathbf{w}, q) = \big(\langle \mathbf{w}, \nabla\rangle \mathbf{w} + \langle \mathbf{V}_\varepsilon, \nabla\rangle \mathbf{w} + \langle \mathbf{w}, \nabla\rangle \mathbf{V}_\varepsilon, 0\big).$$

The pair $(\mathbf{w}_\varepsilon, q_\varepsilon)$ satisfies the equation

$$\mathcal{S}(\mathbf{w}_\varepsilon, q_\varepsilon) + \nu^{-1}\mathcal{T}(\mathbf{w}_\varepsilon, q_\varepsilon) = (\mathbf{F}_\varepsilon, h_\varepsilon), \tag{5.1}$$

where $\mathcal{S}$ is the operator of the Stokes system in Ω_ε (cf. Section 3) and

$$\mathbf{F}_\varepsilon = -\mathcal{S}(\mathbf{V}_\varepsilon, P_\varepsilon) - \nu^{-1}\langle \mathbf{V}_\varepsilon, \nabla\rangle \mathbf{V}_\varepsilon, \quad h_\varepsilon = -\mathrm{div}\mathbf{V}_\varepsilon.$$

Using (1.14) with $\mathbf{S}=\mathbf{w}_\varepsilon$, $\mathbf{T}=\mathbf{V}_\varepsilon$ and $\mathbf{S}=\mathbf{T}=\mathbf{w}_\varepsilon$ as well as the equality $\mathrm{div}(\mathbf{V}_\varepsilon+\mathbf{w}_\varepsilon)=0$, we write (5.1) in the form

$$\mathcal{S}(\mathbf{w}_\varepsilon,q_\varepsilon)+\mathcal{N}(\mathbf{w}_\varepsilon,q_\varepsilon)=(\mathbf{G}_\varepsilon,h_\varepsilon). \tag{5.2}$$

Here

$$\mathcal{N}=\big(\mathrm{div}\boldsymbol{\mathcal{N}}^{(1)},\mathrm{div}\boldsymbol{\mathcal{N}}^{(2)},0\big),\quad \mathbf{G}_\varepsilon=\nu^{-1}\big(\mathrm{div}\boldsymbol{\mathcal{G}}_\varepsilon^{(1)},\mathrm{div}\boldsymbol{\mathcal{G}}_\varepsilon^{(2)}\big), \tag{5.3}$$

with

$$\boldsymbol{\mathcal{N}}^{(k)}=\nu^{-1}(w_{\varepsilon k}\mathbf{w}_\varepsilon+V_{\varepsilon k}\mathbf{w}_\varepsilon+w_{\varepsilon k}\mathbf{V}_\varepsilon),$$

$$\boldsymbol{\mathcal{G}}_\varepsilon^{(k)}=\nabla V_{\varepsilon k}-\rho^{-1}p_\varepsilon\mathbf{e}^{(k)}-V_{\varepsilon k}\mathbf{V}_\varepsilon,$$

where $k=1,2$.

To estimate the right-hand side $(\mathbf{G}_\varepsilon,h_\varepsilon)$ of (5.2) we represent Ω_ε in the form

$$\Omega_\varepsilon=\cup_{\{A\}}\big(\Gamma_\varepsilon^A\cup\mathbb{G}_\varepsilon^A\big)\cup\cup_{\{B\}}\big(\Gamma_\varepsilon^B\cup\mathbb{G}_\varepsilon^B\big),$$

where

$$\Gamma_\varepsilon^A=\{x\in\Omega_\varepsilon: x\in\Omega_0\cap\big(\mathbb{D}_{2\varepsilon^{-1/2}}(x-A)\setminus\mathbb{D}_{\varepsilon^{-1/2}}(x-A)\big)\},$$

$$\Gamma_\varepsilon^B=\{x\in\Omega_\varepsilon: x\in\mathcal{C}_\varepsilon^A\cap\mathbb{D}_{2\varepsilon^{-1/2}}(x-B)\},$$

$$\mathbb{G}_\varepsilon^0=\Omega_0\setminus\cup_{\{A\}}\mathbb{D}_{2\varepsilon^{-1/2}}(x-A),\quad \mathbb{G}_\varepsilon^B=\Omega_\varepsilon\cap\cup_{\{B\}}\mathbb{D}_{\varepsilon^{-1/2}}(x-B),$$

$$\mathbb{G}_\varepsilon^A=\cup_{\{A\}}\big(\Omega_0\cap\mathbb{D}_{\varepsilon^{-1/2}}(x-A)\big)\cup\big(\cup_{\{A\}}\mathcal{C}_\varepsilon^A\setminus\cup_{\{B\}}\mathbb{D}_{2\varepsilon^{-1/2}}(x-B)\big)$$

and $\{B\}$ is the union the points B with coordinates $(y_1^A,y_2^A)=(0,-L_A)$ which is extended over all channels $\mathcal{C}^A$.

According to (0.19), (0.20) we have

$$(\mathbf{V}_\varepsilon,P_\varepsilon)\equiv\begin{cases}(\mathbf{v}_0,p_0) & \text{on}\quad x\in\mathbb{G}_\varepsilon^0\\ (\varepsilon^{-1}\mathbf{v}^A,\varepsilon^{-2}p^A) & \text{on}\quad x\in\mathbb{G}_\varepsilon^A\\ (\varepsilon^{-1}\mathbf{v}^B,\varepsilon^{-2}p^B) & \text{on}\quad x\in\mathbb{G}_\varepsilon^B.\end{cases}$$

Hence, by definition of $(\mathbf{v}_0,p_0),(\mathbf{v}^A,p^A)$ and $(\mathbf{v}^B,p^B)$ we obtain

$$(\mathbf{G}_\varepsilon,h_\varepsilon)=0\quad\text{on}\quad\mathbb{G}_\varepsilon^0\cup\mathbb{G}_\varepsilon^A\cup\mathbb{G}_\varepsilon^B. \tag{5.4}$$

To simplify the notation, in Section 5 we omit the indices A,B for χ_ε^A, η_ε^A, μ_ε^B, ξ_ε^B.

Lemma 5.1 *The inequality*

$$\|\boldsymbol{\mathcal{G}}_\varepsilon^{(1)}\|_{L_2(\mathbb{G}_\varepsilon)} + \|\boldsymbol{\mathcal{G}}_\varepsilon^{(2)}\|_{L_2(\mathbb{G}_\varepsilon)} + \|h_\varepsilon\|_{L_2(\mathbb{G}_\varepsilon)} \le c\varepsilon^D \tag{5.5}$$

is valid with $D > 0$ and with a constant c independent of ε.

Proof. By (5.4)

$$\operatorname{supp}\{(\mathbf{G}_\varepsilon, h_\varepsilon)\} = \cup_{\{A\}}\Gamma_\varepsilon^A \cup \cup_{\{B\}}\Gamma_\varepsilon^B.$$

For $x \in \Gamma_\varepsilon^A$ one has

$$\chi_\varepsilon(x) + \eta_\varepsilon(x) = 1, \quad \operatorname{div} \boldsymbol{\mathcal{H}}^A = 0,$$

$$\operatorname{div}\big(\nabla\mathcal{H}_k^A - \varepsilon^{-1}\{\rho^{-1}\mathcal{Q}^A\mathbf{e}^{(k)} + \mathcal{H}_k^A\boldsymbol{\mathcal{H}}^A\}\big) = 0,$$

where $\boldsymbol{\mathcal{H}}^A = \boldsymbol{\mathcal{H}}_M^A$ with $M = \Upsilon_A$. Consequently,

$$\boldsymbol{\mathcal{G}}_\varepsilon^{(k)} = g_{k,1}^A + g_{k,2}^A + g_{k,3}^A, \quad h_\varepsilon = -\varepsilon^{-1}\operatorname{div}(\eta_\varepsilon\mathbf{w}^A + \chi_\varepsilon\mathbf{w}^0),$$

where

$$g_{k,1}^A = \varepsilon^{-1}\big\{\nabla(\eta_\varepsilon w_k^A) - \varepsilon^{-1}\{\rho^{-1}\mathbf{e}^{(k)}\eta_\varepsilon q^A$$
$$+\mathcal{H}_k^A\eta_\varepsilon\mathbf{w}^A + \eta_\varepsilon w_k^A\boldsymbol{\mathcal{H}}^A + \eta_\varepsilon^2 w_k^A\mathbf{w}^A\}\big\},$$

$$g_{k,2}^A = \varepsilon^{-1}\big\{\nabla(\chi_\varepsilon w_{0k}) - \varepsilon^{-1}\{\rho^{-1}\mathbf{e}^{(k)}\chi_\varepsilon q_0$$
$$+\mathcal{H}_k^A\chi_\varepsilon\mathbf{w}_0 + \chi_\varepsilon w_{0k}\boldsymbol{\mathcal{H}}^A + \chi_\varepsilon^2 w_{0k}\mathbf{w}_0\}\big\},$$

$$g_{k,3}^A = 2\varepsilon^{-2}\chi_\varepsilon\eta_\varepsilon\mathbf{w}_0\mathbf{w}^A.$$

Estimate (2.8) implies

$$\begin{gathered}\varepsilon^{-1}|\nabla^j\mathbf{w}^A(\varepsilon^{-1}(x-A))| \le c\varepsilon^{\delta+j}r^{-\delta-j-1}, \quad j = 0,1,\\ \varepsilon^{-2}|q^A(\varepsilon^{-1}(x-A))| \le c\varepsilon^{\delta+1}r^{-\delta-2}\end{gathered} \tag{5.6}$$

for $x \in \Gamma_\varepsilon^A$. Hence

$$\|g_{k,1}^A\|_{L_2(\Gamma_\varepsilon^A)} \le c\varepsilon^{\delta/2},\ k = 1,2, \quad \varepsilon^{-1}\|\operatorname{div}(\eta_\varepsilon\mathbf{w}^A)\|_{L_2(\Gamma_\varepsilon^A)} \le c\varepsilon^{\delta/2}. \tag{5.7}$$

Since $(\mathbf{w}_0, q_0) \in (\overset{\circ}{N}{}^{1,\alpha}_{\tau}(\Omega_0))^2 \times N^{0,\alpha}_{\tau,\perp}(\Omega_0)$, we have

$$|\nabla^j \mathbf{w}_0(x)| \le c r^{\delta-j},\ j=0,1, \quad |q_0(x)| \le c r^{\delta-1} \tag{5.8}$$

for $x \in \Gamma^A_\varepsilon$. By (5.8)

$$\|g^A_{k,2}\|_{L_2(\Gamma^A_\varepsilon)} \le c\varepsilon^{\delta/2},\ k=1,2, \quad \varepsilon^{-1}\|\mathrm{div}\,(\chi_\varepsilon \mathbf{w}_0)\|_{L_2(\Gamma^A_\varepsilon)} \le c\varepsilon^{\delta/2}. \tag{5.9}$$

The estimate

$$\|g^A_{k,3}\|_{L_2(\Gamma^A_\varepsilon)} \le c\varepsilon^{\delta/2} \tag{5.10}$$

for $x \in \Gamma^A_\varepsilon$ follows from (5.6), (5.8). Unifying (5.7), (5.9), (5.10) we have

$$\|\boldsymbol{\mathcal{G}}^{(k)}_\varepsilon\|_{L_2(\Gamma^A_\varepsilon)} \le c\varepsilon^{\delta/2}, k=1,2, \quad \|h_\varepsilon\|_{L_2(\Gamma^A_\varepsilon)} \le c\varepsilon^{\delta/2}. \tag{5.11}$$

For $x \in \Gamma^B_\varepsilon$ using the equalities

$$\mu_\varepsilon(x) + \xi_\varepsilon(x) = 1, \quad \mathrm{div}\,\boldsymbol{\mathcal{U}}^A = 0,$$

$$\mathrm{div}\big(\nabla \mathcal{U}^A_k - \varepsilon^{-1}\{\rho^{-1}\mathcal{P}^A \mathbf{e}^{(k)} + \mathcal{U}^A_k \boldsymbol{\mathcal{U}}^A\}\big) = 0$$

we find

$$\boldsymbol{\mathcal{G}}^{(k)}_\varepsilon = g^B_{k,1} + g^B_{k,2} + g^B_{k,3}, \quad h_\varepsilon = -\varepsilon^{-1}\mathrm{div}\,(\mu_\varepsilon \mathbf{w}^A + \xi_\varepsilon \mathbf{w}^B),$$

where

$$g^B_{k,1} = \varepsilon^{-1}\big\{\nabla(\xi_\varepsilon w^B_k) - \varepsilon^{-1}\{\rho^{-1}\mathbf{e}^{(k)}\xi_\varepsilon q^B$$

$$+\mathcal{U}^A_k \xi_\varepsilon \mathbf{w}^B + \xi_\varepsilon w^B_k \boldsymbol{\mathcal{U}}^A + \xi^2_\varepsilon w^B_k \mathbf{w}^B\}\big\},$$

$$g^B_{k,2} = \varepsilon^{-1}\big\{\nabla(\mu_\varepsilon w^A_k) - \varepsilon^{-1}\{\rho^{-1}\mathbf{e}^{(k)}\mu_\varepsilon q^A$$

$$+\mathcal{U}^A_k \mu_\varepsilon \mathbf{w}^A + \mu_\varepsilon w^A_k \boldsymbol{\mathcal{U}}^A + \mu^2_\varepsilon w^A_k \mathbf{w}^A\}\big\},$$

$$g^B_{k,3} = 2\varepsilon^{-2}\mu_\varepsilon \xi_\varepsilon w^A_k w^B_k.$$

By (2.8) for $x \in \Gamma^B_\varepsilon$ we obtain

$$|\nabla^j \mathbf{w}^A(\varepsilon^{-1}(x-A))| \le c e^{-d/\varepsilon},\ j=0,1, \quad |q^A(\varepsilon^{-1}(x-A))| \le c e^{-d/\varepsilon} \tag{5.12}$$

with $d>0$. The similar estimate

$$|\nabla^j \mathbf{w}^B(\varepsilon^{-1}(x-B))| \le c e^{-d/\varepsilon}, \quad j=0,1,$$
$$|q^B(\varepsilon^{-1}(x-B))| \le c e^{-d/\varepsilon}, \quad d>0 \tag{5.13}$$

for $x\in\Gamma_\varepsilon^B$ follows from (2.12). Using (5.12) for $g^B_{k,2}$, (5.13) for $g^B_{k,1}$ and both estimates for h_ε, $g^B_{k,3}$ we arrive to the inequalities

$$\|g^B_{k,m}\|_{L_2(\Gamma_\varepsilon^B)} \le c e^{-d/\varepsilon},\ m=1,2,3, \quad \|h_\varepsilon\|_{L_2(\Gamma_\varepsilon^B)} \le c e^{-d/\varepsilon}. \tag{5.14}$$

Unifying (5.11) and (5.14) we complete the proof. ∎

Thus, by Lemma 5.1 and representation (5.3) for the function $\mathbf{G}_\varepsilon$ the right-hand side of (5.2) admits the estimate

$$\|(\mathbf{G}_\varepsilon, h_\varepsilon)\|_{((\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2)^*\times L_2(\Omega_\varepsilon)} \le c\varepsilon^D. \tag{5.15}$$

6 The existence theorem

In Section 4 we obtained the constants $K_\varepsilon, k^A_\varepsilon, k^B_\varepsilon$ and the pairs $(\mathbf{v}_0, p_0)$, $(\mathbf{v}^A, p^A)$, $(\mathbf{v}^B, p^B)$ which enter formulas (0.19), (0.20) for the principal term $(\mathbf{V}_\varepsilon, P_\varepsilon)$ of representation (0.22). In Section 5 we considered the problem for the remainder term $(\mathbf{w}_\varepsilon, p_\varepsilon)$. Now we are in a position to prove the main result of the paper.

Theorem 6.1 *There exists a solution* $(\mathbf{v}_\varepsilon, p_\varepsilon)$ *of problem* (0.1)–(0.5) *represented in the form* (0.22)*, where* $(\mathbf{w}_\varepsilon, q_\varepsilon)\in(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2\times L_2(\Omega_\varepsilon)$ *is subject to* (0.23)*.*

Proof. Let

$$l_\varepsilon = q_\varepsilon + \overline{P_\varepsilon}. \tag{6.1}$$

The pair $(\mathbf{w}_\varepsilon, l_\varepsilon)$ satisfies equation (5.2) with

$$\mathcal{N} : (\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2\times L_2(\Omega_\varepsilon) \to ((H^1(\Omega_\varepsilon))^2)^*\times L_2(\Omega_\varepsilon)$$

being the operator acting by formula (5.3). Let $\mathfrak{B}_\kappa$ be the ball in $(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2\times L_2(\Omega_\varepsilon)$ with center at $\mathcal{S}^{-1}(\mathbf{G}_\varepsilon, h_\varepsilon)$ and with a small radius κ and let $(\mathbf{U}^{(j)}, T^{(j)}) \in \mathfrak{B}_\kappa,\ j=1,2$. We shall show that if the right-hand side of the boundary condition (0.3) satisfies (0.21), then, for a sufficiently small κ, the operator

$$\mathcal{S}^{-1}(\mathcal{N}) : (\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2\times L_2(\Omega_\varepsilon) \to (\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2\times L_2(\Omega_\varepsilon)$$

is a contraction operator in $\mathfrak{B}_\kappa$, i.e. the inequality

$$\|\mathcal{N}(\mathbf{U}^{(1)},T^{(1)})-\mathcal{N}(\mathbf{U}^{(2)},T^{(2)})\|_{((\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2)^*\times L_2(\Omega_\varepsilon)}$$
$$\leq k\|(\mathbf{U}^{(1)},T^{(1)})-(\mathbf{U}^{(2)},T^{(2)})\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2\times L_2(\Omega_\varepsilon)} \tag{6.2}$$

holds with a constant $m<1$ and

$$\|\mathcal{N}(\mathbf{U}^{(j)},T^{(j)})\|_{((\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2)^*\times L_2(\Omega_\varepsilon)}\leq\kappa. \tag{6.3}$$

By (5.3) in order to prove (6.2) it is sufficient to check inequalities

$$\nu^{-1}\|V_{\varepsilon k}U_i^{(j)}\|_{L_2(\Omega_\varepsilon)}\leq C_{\mathcal{R}}\|U_i^{(j)}\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2}, \tag{6.4}$$

$$\nu^{-1}\|U_i^{(j)}U_m^{(k)}\|_{L_2(\Omega_\varepsilon)}\leq C_\kappa\|U_i^{(j)}\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2} \tag{6.5}$$

with $i,j,k,m=1,2$ and constants $C_{\mathcal{R}},\ C_\kappa$ satisfying the conditions

$$C_{\mathcal{R}}\to 0 \text{ as } \mathcal{R}\to 0,\quad C_\kappa\to 0 \text{ as } \kappa\to 0.$$

We begin with (6.4). By (0.19), (4.1)–(4.3)

$$\|V_\varepsilon U\|_{L_2(\Omega_\varepsilon)}\leq c\big(\|w_0UX\|_{L_2(\Omega_\varepsilon)}$$
$$+\sum\big\{\|\zeta_A\eta_\varepsilon^A\mathcal{H}^AU\|_{L_2(\Omega_\varepsilon)}+\varepsilon^{-1}\big\{\|\eta_\varepsilon^A\mu_\varepsilon^Bw^AU\|_{L_2(\Omega_\varepsilon)} \tag{6.6}$$
$$+\|\zeta_-^A\mathcal{U}^AU\|_{L_2(\Omega_\varepsilon)}+\|\xi_\varepsilon^Bw^BU\|_{L_2(\Omega_\varepsilon)}\big)\big\}\big\}$$

(To simplify the notation, in (6.6) and henceforth we have omitted the indices j,k for $U_k^{(j)}$ as well as the index k for the components $V_{\varepsilon k},\mathcal{U}_k,\mathcal{H}_k$ of the vectors $\mathbf{V}_\varepsilon,\boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{H}}$.) Using the estimates

$$\|u\|_{L_2(\mathcal{C}_\varepsilon^A)}\leq\varepsilon C\|\nabla u\|_{L_2(\mathcal{C}_\varepsilon^A)},\quad\|r^{-1}u\|_{L_2(\Omega_0)}\leq C\|\nabla u\|_{L_2(\Omega_0)} \tag{6.7}$$

for $u\in\overset{\circ}{H}{}^1(\Omega_\varepsilon)$, we find

$$\varepsilon^{-2}\|\zeta_-^A\mathcal{U}^AU\|^2_{L_2(\Omega_\varepsilon)}+\|\zeta_A\eta_\varepsilon^A\mathcal{H}^AU\|^2_{L_2(\Omega_\varepsilon)}\leq C|\Upsilon_A|\|\nabla U\|_{L_2(\Omega_\varepsilon)}. \tag{6.8}$$

Here and below we denote constants independent of $\varepsilon,\nu,\boldsymbol{\varphi}$ by C.

According to (2.12) with $\mathbf{g}=\boldsymbol{\varphi}^A$ and the Sobolev inequality

$$\|u\|_{L_4(\mathcal{C}_\varepsilon^A)}\leq\varepsilon^{1/2}C\|\nabla u\|_{L_2(\mathcal{C}_\varepsilon^A)}$$

the last term in (6.6) is estimated as follows

$$\|\xi_\varepsilon^B w^B U\|_{L_2(\Omega_\varepsilon)} \leq C\|w^B\|_{L_4(\mathcal{C}_\varepsilon^A)}\|U\|_{L_4(\mathcal{C}_\varepsilon^A)}$$

$$\leq C\varepsilon\|w^B\|_{L_4(\Pi_B)}\|\nabla U\|_{L_4(\mathcal{C}_\varepsilon^A)} \leq C\varepsilon\|w^B\|_{\overset{\circ}{H}{}^1(\Pi_B)}\|U\|_{\overset{\circ}{H}{}^1(\Omega_\varepsilon)} \tag{6.9}$$

$$\leq C\varepsilon\|\boldsymbol{\varphi}^A\|_{(C^{1,\alpha}(-b_A^-,b_A^+))^2}\|U\|_{\overset{\circ}{H}{}^1(\Omega_\varepsilon)}.$$

We represent the function $\eta_\varepsilon^A\mu_\varepsilon^B w^A U$ in the form

$$\eta_\varepsilon^A\mu_\varepsilon^B w^A U = (1-\zeta_+^A)\mu_\varepsilon^B w^A U + \zeta_+^A\eta_\varepsilon^A w^A U.$$

Using (2.8) with $M = \Upsilon_A$ and a chain of inequalities similar to (6.9) we obtain

$$\|(1-\zeta_+^A)\mu_\varepsilon^B w^A U\|_{L_2(\Omega_\varepsilon)} \leq C\varepsilon\|\boldsymbol{\varphi}^A\|_{(C^{1,\alpha}(-b_A^-,b_A^+))^2}\|U\|_{\overset{\circ}{H}{}^1(\Omega_\varepsilon)}. \tag{6.10}$$

By (1.12) with

$$\mathcal{M} = \sum|\Upsilon_A|$$

and the Sobolev inequality

$$\|u\|_{L_4(\Omega_\varepsilon)} \leq C\|u\|_{\overset{\circ}{H}{}^1(\Omega_\varepsilon)} \tag{6.11}$$

we have

$$\|Xw_0U\|_{L_2(\Omega_\varepsilon)} \leq C\|w_0\|_{L_4(\Omega_0)}\|U\|_{L_4(\Omega_\varepsilon)} \leq C\|U\|_{\overset{\circ}{H}{}^1(\Omega_\varepsilon)}\sum|\Upsilon_A|. \tag{6.12}$$

Let us introduce the set

$$\mathbb{S}_\varepsilon^A = \{x\in\Omega_\varepsilon\,:\,x\in\Omega_0\cap\big(\mathbb{D}_{2\varepsilon^{1/2}}(x-A)\setminus\mathbb{D}_{b_A}(x-A)\big).$$

By (2.8) with $M = \Upsilon_A$ the estimate

$$\mathbf{w}^A(\varepsilon^{-1}(x-A)) \leq C|\Upsilon_A|(x/\varepsilon)^{-1-\delta}, \quad x\in\mathbb{S}_\varepsilon^A$$

holds. Hence

$$\|w^A\|_{L_4(\mathbb{S}_\varepsilon^A)} \leq C|\Upsilon_A|(x/\varepsilon)^{3/4+\delta/2}. \tag{6.13}$$

Using (6.13) and the inequality

$$\|u\|_{L_4(\mathbb{S}_\varepsilon^A)} \leq \varepsilon^{1/2}C\|\nabla u\|_{L_2(\mathbb{S}_\varepsilon^A)}$$

we arrive at

$$\|\zeta_+^A \eta_\varepsilon^A w^A U\|_{L_2(\Omega_\varepsilon)} \le C\varepsilon |\Upsilon_A| \, \|U\|_{\overset{\circ}{H}{}^1(\Omega_\varepsilon)}. \tag{6.14}$$

Since $|\Upsilon_A| \le C\nu\mathcal{R}$ and $\sum |\Upsilon_A| \le C\nu\mathcal{R}$, by combining (6.6) with (6.8)–(6.10), (6.12)–(6.14) we obtain (6.4) with $C_{\mathcal{R}} = \mathcal{R}C$.

The estimate (6.3) with a sufficiently small κ and the estimate (6.5) with $C_\kappa = \kappa C$ follow from (6.11).

Thus, $\mathcal{N}$ is a contraction operator in $\mathfrak{B}_\kappa$ and therefore, according to the Banach principle, there exists a unique solution $(\mathbf{w}_\varepsilon, l_\varepsilon) \in \mathfrak{B}_\kappa$ of equation (5.2). Putting $\kappa = \varepsilon^D$ and taking into account (6.1), (4.4), (5.15) we complete the proof. ∎

7 Asymptotic representations for the kinetic energy and Dirichlet integral

The asymptotic behavior of the kinetic energy $\mathcal{E}(\mathbf{v}_\varepsilon)$ is described in the following assertion.

Theorem 7.1 *Kinetic energy $\mathcal{E}(\mathbf{v}_\varepsilon)$ of the fluid in the domain Ω_ε has the asymptotic representation* (0.25)*, where $\boldsymbol{\mathcal{V}}^A = \boldsymbol{\mathcal{V}}_M^A$ with $M = \Upsilon_A$.*

Proof. We write the velocity $\mathbf{v}_\varepsilon$ in the form

$$\mathbf{v}_\varepsilon = \mathbf{u}_\varepsilon + \mathbf{W}_\varepsilon + \mathbf{w}_\varepsilon, \tag{7.1}$$

where

$$\mathbf{u}_\varepsilon(x) = \varepsilon^{-1} \sum \Big\{ \zeta_A(x-A)\zeta_+^A(\varepsilon^{-1}(x-A))\mathcal{H}(\varepsilon^{-1}(x-A))$$

$$+\zeta_-^A(\varepsilon^{-1}(x-A))\boldsymbol{\mathcal{U}}(\varepsilon^{-1}(x-A)) \Big\},$$

$$\mathbf{W}_\varepsilon(x) = X_\varepsilon(x)\mathbf{w}_\varepsilon(x)$$

$$+\varepsilon^{-1} \sum \Big\{ \eta_\varepsilon^A(x)\mu_\varepsilon^B(x)\mathbf{w}^A(\varepsilon^{-1}(x-A)) + \xi_\varepsilon^B(x)\mathbf{w}^B(\varepsilon^{-1}(x-B)) \Big\}.$$

We remind that the summation is taken over all the channels. By (7.1) we have

$$\mathcal{E}(\mathbf{v}_\varepsilon) = \frac{\rho}{2}\big(\|\mathbf{u}_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} + \|\mathbf{W}_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} + \|\mathbf{w}_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} + J_1 + J_2\big), \tag{7.2}$$

where

$$J_1 = 2\int\limits_{\Omega_\varepsilon} \mathbf{u}_\varepsilon(\mathbf{w}_\varepsilon + \mathbf{W}_\varepsilon)\,dx, \quad J_2 = 2\int\limits_{\Omega_\varepsilon} \mathbf{w}_\varepsilon \mathbf{W}_\varepsilon\,dx.$$

Straightforward calculation gives

$$\int\limits_{\Omega_\varepsilon} \mathbf{u}_\varepsilon^2\,dx = \frac{6}{5}\frac{1}{\varepsilon}\sum \Upsilon_A^2 L_A b_A^{-1} + \log\frac{1}{\varepsilon}\sum \Upsilon_A^2 \int\limits_{\omega_A^-}^{\omega_A^+} \big(\mathcal{V}^A(\theta)\big)^2\,d\theta + O(1). \quad (7.3)$$

Now we estimate other terms in the right-hand side of (7.2). Since

$$\|\mathbf{W}_\varepsilon\|_{(\overset{\circ}{H}{}^1(\Omega_\varepsilon))^2} \le c\big(\|\mathbf{w}_0\|_{(\overset{\circ}{H}{}^1(\Omega_0))^2}$$

$$+\sum\{\|\mathbf{w}^A\|_{(\overset{\circ}{H}{}^1(\Lambda_A))^2} + \|\mathbf{w}^B\|_{(\overset{\circ}{H}{}^1(\Pi_B))^2}\}\big),$$

then (1.12) with $\mathcal{M} = \sum|\Upsilon_A|$, (2.8) with $M = \Upsilon_A$ and (2.12) with $\mathbf{g} = \boldsymbol{\varphi}^A$ imply

$$\|\mathbf{W}_\varepsilon\|_{\overset{\circ}{H}{}^1(\Omega_\varepsilon)} \le C. \quad (7.4)$$

By (0.23) we have

$$\|\mathbf{w}_\varepsilon\|_{\overset{\circ}{H}{}^1(\Omega_\varepsilon)} \le c\varepsilon^\delta. \quad (7.5)$$

The estimate

$$|J_1| \le c \quad (7.6)$$

follows from (7.4) and (7.5). According to (7.4), (7.5) and (6.7)

$$|J_2| \le c. \quad (7.7)$$

Unifying (7.2)–(7.7) we arrive at (0.25). ∎

Now we calculate the principal term of the asymptotic representation of the Dirichlet integral $\mathcal{I}(\mathbf{v}_\varepsilon)$ of problem (0.1)–(0.5).

Theorem 7.2 *Dirichlet integral* (0.26) *of problem* (0.1)–(0.5) *admits representation* (0.27).

Proof. We make use of expression (7.1) for the velocity vector $\mathbf{v}_\varepsilon$. A straightforward calculation gives

$$\|\nabla \mathbf{u}_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} = 12\varepsilon^{-3} \sum \Upsilon_A^2 L_A b_A^{-3} + O(\varepsilon^{-2}). \tag{7.8}$$

It follows by (7.4), (7.5) that

$$\mathcal{I}(\mathbf{w}_\varepsilon + \mathbf{W}_\varepsilon) \leq c. \tag{7.9}$$

The inequality

$$|\mathcal{I}(\mathbf{v}_\varepsilon) - \mathcal{I}(\mathbf{u}_\varepsilon)| \leq c\mathcal{I}(\mathbf{w}_\varepsilon + \mathbf{W}_\varepsilon)^{1/2}\big(\mathcal{I}(\mathbf{w}_\varepsilon + \mathbf{W}_\varepsilon)^{1/2} + \mathcal{I}(\mathbf{u}_\varepsilon)^{1/2}\big)$$

combined with (7.8), (7.9) completes the proof. ∎

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