Asymptotic solution to the Dirichlet problem for a two-dimensional Riccatti's type equation near a corner point

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Abstract. The Dirichlet problem for a quasilinear elliptic equation of the second order with quadratic nonlinearity in the first derivatives is considered in a plane domain with a corner point. An asymptotic solution which has a strong singularity at this point is constructed.

Keywords. quasilinear elliptic equations, asymptotic solutions, corner boundary points,

0. Introduction

Construction of asymptotic formulae for solutions to linear elliptic boundary value problems in domains with corner and conic points has been developed in numerous publications (see, for instance [1], [2] and the bibliography there). Less attention was paid to the asymptotics of solutions to nonlinear boundary value problems. In more detail properties of solutions to the p-Laplace equation were investigated (see [3] – [8]). In the case of weak singularities, when the problem can be linearized near the corner, boundary value problems for semilinear and more general quasilinear equations were considered in [9] – [11]. As for strong singularities, which correspond to boundary sources, the situation is quite different, since the principal terms of the asymptotics are determined by a nonlinear operator as a whole. This case was deal with [12], [13], where results relating some special equations of the type we investigate in the present paper were given without proofs. Description of asymptotic behavior of all solutions to the Neumann problem for the two-dimensional Riccatti equation near an angle point is given in [14]. Solutions to some nonlinear other equations having strong singularities were studied in [15] – [18].

The present paper concerns the equation

$$\mathbf{R}(\mathcal{U}(x)) = 0, \qquad x \in G, \tag{0.1}$$

where G is a domain in the plane $\mathbb{R}^2$ and $x = (x_1, x_2)$ is a point of $\mathbb{R}^2$. By $\mathbf{R}$ we denote an operator defined by

$$\mathbf{R}(\mathcal{U}(x))= \sum_{i,j=1}^{2} \mathbf{k}_{ij}(x,\mathcal{U}) \frac{\partial^2 \mathcal{U}}{\partial x_i \partial x_j} + \sum_{i,j=1}^{2} \mathbf{m}_{ij}(x,\mathcal{U}) \frac{\partial \mathcal{U}}{\partial x_i} \frac{\partial \mathcal{U}}{\partial x_j},$$

where $\{\mathbf{k}_{ij}\}_{i,j=1}^2, \{\mathbf{m}_{ij}\}_{i,j=1}^2$ are positive symmetric matrices. The part of the domain G coincides with an angle in a small disk $\mathcal{D}$ centered at the origin O. Our goal is to construct an asymptotic solution of the Dirichlet problem to (0.1) which has a strong singularity at O.

We assume that the coefficients $\mathbf{k}_{ij}(x,\mathcal{U})$ and $\mathbf{m}_{ij}(x,\mathcal{U})$ of the equation (0.1) approach some functions $\mathbf{k}^\bullet_{ij}(x)$ and $\mathbf{m}^\bullet_{ij}(x)$ exponentially as $\mathcal{U} \to +\infty$. Without loss of generality we admit that

$$\mathbf{k}^\bullet_{11}(0) = \mathbf{k}^\bullet_{22}(0) = 1, \quad \mathbf{k}^\bullet_{12}(0) = \mathbf{k}^\bullet_{21}(0) = 0.$$

At the boundary ∂G, the function $\mathcal{U}$ satisfies the Dirichlet boundary condition

$$\mathcal{U} = g, \tag{0.2}$$

where $g \in C^\infty(\partial G)$ and g vanishes on $\partial G \cap \mathcal{D}$.

Let (r, θ) denote polar coordinates with the origin O , $|\theta| < \pi$. We put $G \cap \mathcal{D} = \{(r, \theta) : 0 < r < r_0, |\theta| < \theta_0\}$.

We do not investigate the solvability of problem (0.1), (0.2) restricting ourselves to a construction of the asymptotic solution which is defined as follows.

Definition 1. *A function* $\mathbf{U}(x)$ *is called the asymptotic solution of problem* $(0.1), (0.2)$ *if for a certain* $\varepsilon > 0$

$$|\mathbf{R}(\mathbf{U}(x))| = O(D(x)^{\varepsilon - 2})$$

with $D(x) = \operatorname{dist}(x, \partial G)$, *and*

$$\mathbf{U}(r, \pm\theta_0) = O(r^\varepsilon). \tag{0.3}$$

We shall treat a simple model problem which can be solved explicitly and therefore gives a hint for the subsequent discussion of the general situation.

Example. Consider the equation

$$\frac{\partial^2 \mathcal{U}}{\partial x_1^2} + \frac{\partial^2 \mathcal{U}}{\partial x_2^2} + A\left(\left(\frac{\partial \mathcal{U}}{\partial x_1} \right)^2 + \left(\frac{\partial \mathcal{U}}{\partial x_2} \right)^2 \right) = 0 \tag{0.4}$$

in the domain G, where A is a constant, $A \neq 0$. Let $\mathcal{U}$ be subject to the boundary condition (0.2).

In order to construct a solution to problem (0.4), (0.2), one can linearize equation (0.4) by the substitution

$$\mathcal{U} = A^{-1} \log V. \tag{0.5}$$

The function V is positive in G and satisfies the Dirichlet problem

$$\begin{aligned} \Delta V = 0, \quad & x \in G, \\ V = e^{Ag}, \quad & x \in \partial G. \end{aligned} \tag{0.6}$$

It is a simple exercise to verify that any positive solution of problem (0.6) has the form

$$V(x) = \alpha \mathcal{H}(x) + \mathcal{V}(x),$$

where $\mathcal{V}$ satisfies (0.6) and belongs to $C(\overline{G})$, α is an arbitrary nonnegative constant and $\mathcal{H}$ is the solution of the homogeneous problem (0.6) which has the asymptotic form

$$\mathcal{H}(r,\theta) = r^{-\pi/\theta_0}\cos(\pi\theta/\theta_0) + O(r^{\pi/\theta_0}).$$

We have

$$\mathcal{V} = 1 + O(r^{\pi/\theta_0}) \tag{0.7}$$

as $r \to 0$. Hence the solution $\mathcal{U}$ of the original nonlinear problem, admits the asymptotic representation

$$\mathcal{U}(r,\theta) = A^{-1}\log\left(\alpha r^{-\pi/\theta_0}\cos(\pi\theta/\theta_0) + 1 + O(r^{\pi/\theta_0})\right). \tag{0.8}$$

Thus, each value $\alpha > 0$ generates a solution of problem (0.4), (0.2) which is unbounded in any neighbourhood of the origin O and is defined by (0.8).

Since a substitution of type (0.5), linearizing (0.1), is, in general, unknown and, most probably, does not exist, one can hardly hope to get an asymptotic formula for solutions of equation (0.1) in a form similar to (0.8). However, a way to construct an asymptotic solution of (0.1) can be anticipated by rewriting (0.8) in the form

$$\mathcal{U}(r,\theta) = \mathbf{v}(r,\theta) + \sum_{\pm}\mathbf{w}_\pm(r^{-\pi/\theta_0}(\theta_0 \mp \theta)) + O(r^{\pi/\theta_0}), \tag{0.9}$$

where the term

$$\mathbf{v}(r,\theta) = A^{-1}(\pi/\theta_0\log 1/r + \log\cos\pi\theta/\theta_0 + \log\alpha)$$

describes the solution $\mathcal{U}(r,\theta)$ in any angle $|\theta| < \theta_0 - \varepsilon$, $\varepsilon > 0$, and the two terms of boundary layer type

$$\mathbf{w}_\pm(r^{-\pi/\theta_0}(\theta_0 \mp \theta)) = A^{-1}\log\left(1 + \frac{r^{\pi/\theta_0}}{(\theta_0 \mp \theta)\alpha}\right)$$

characterize the asymptotic behavior of $\mathcal{U}(r,\theta)$ near the lines $\theta = \pm\theta_0$.

The above example shows the way to a treatment of the general equation (0.1). We shall construct the following asymptotic solution $\mathbf{U}$ of problem (0.1), (0.2) which contains (0.9) as a particular case

$$\mathbf{U}(r,\theta) = \Lambda\log 1/r + \mathbf{C} + \mathbf{Z}(\theta) + \sum_{\pm}\mathbf{w}_\pm\left((\theta_0 \mp \theta)r^{\Lambda c_\pm}\right). \tag{0.10}$$

Here $\mathbf{w}_\pm$ are polynomials in $\log r$. The functions $\mathbf{w}_\pm$ tend to zero as $(\theta_0 \pm \theta)r^{\Lambda c_\pm} \to +\infty$, i.e. the functions $\mathbf{w}_\pm$ describe the boundary layers near the rays $\theta = \mp\theta_0$. In (0.10), $\mathbf{C}$ is an arbitrary constant and Λ is a single positive number, for which there exists a solution $\mathbf{Z}$ of the problem

$$\begin{cases} \mathbf{Z}''(\theta) + \Lambda^2\mathbf{a}(\theta) + 2\Lambda\mathbf{b}(\theta)\mathbf{Z}'(\theta) + \mathbf{c}(\theta)(\mathbf{Z}'(\theta))^2 = 0, \\ \\ \mathbf{Z}(\pm\theta_0) = -\infty, \quad \mathbf{Z}(0) = 0, \end{cases}$$

where

$$\begin{aligned} \mathbf{a}(\theta) &= \mathbf{m}^\bullet_{11}(0)\cos^2\theta + 2\mathbf{m}^\bullet_{12}(0)\sin\theta\cos\theta + \mathbf{m}^\bullet_{22}(0)\sin^2\theta, \\ \mathbf{b}(\theta) &= (\mathbf{m}^\bullet_{11}(0) - \mathbf{m}^\bullet_{22}(0))\sin\theta\cos\theta - \mathbf{m}^\bullet_{12}(0)(\cos^2(\theta) - \sin^2\theta), \\ \mathbf{c}(\theta) &= \mathbf{m}^\bullet_{11}(0)\sin^2\theta - 2\mathbf{m}^\bullet_{12}(0)\sin\theta\cos\theta + \mathbf{m}^\bullet_{22}(0)\cos^2\theta. \end{aligned}$$

By $\mathbf{c}_\pm$ in (0.10) we denote the number $\mathbf{c}(\pm\theta_0)$.

It is convenient to map the domain $G \cap \mathcal{D}$ onto the semistrip $\Pi = (0, +\infty) \times (-l/2, l/2,)$ changing the variables $x \mapsto t = (t_1, t_2) = (y, z)$, where

$$y = \theta l/2\theta_0, \quad z = \log r^{-1} - \log r_0^{-1}.$$

We shall use the coordinates (y, z) throughout the text.

We obtain an asymptotic solution of the first boundary value problem in the semistrip Π. The asymptotic solution $\mathbf{U}$ results from the asymptotic solution in the semistrip after passing from the coordinates (y, z) to the coordinates (r, θ) by the formulae

$$r = r_0 e^{-z}, \quad \theta = 2\theta_0 y/l.$$

We consider a more general boundary value problem in Π. Suppose that

$$\mathcal{R}(u(t)) = 0, \qquad t \in \Pi, \tag{0.11}$$

where

$$\begin{aligned} \mathcal{R}(u(t)) =& \sum_{i,j=1}^{2} k_{ij}(t,u)\frac{\partial^2 u}{\partial t_i \partial t_j} + \sum_{j=1}^{2} l_j(t,u)\frac{\partial u}{\partial t_j} \\ & + \sum_{i,j=1}^{2} m_{ij}(t,u)\frac{\partial u}{\partial t_i}\frac{\partial u}{\partial t_j}, \end{aligned} \tag{0.12}$$

and $\{k_{ij}\}_{i,j=1}^2, \{m_{ij}\}_{i,j=1}^2$ are symmetric positive-definite matrices. We assume that $l_j(t,u)$ decay exponentially as $u \to +\infty$ and $k_{ij}(t,u), m_{ij}(t,u)$ approach some functions $k_{ij}^\bullet, m_{ij}^\bullet$ exponentially as $u \to +\infty$ (see (1.4), (1.5) below).

Suppose that u vanishes on the lateral sides of the semistrip

$$u(\pm l/2, z) = 0. \tag{0.13}$$

Definition 2. *A function $U(y,z)$ is called the asymptotic solution of problem* (0.11), (0.13) *if there exists $\varepsilon > 0$, such that*

$$|\mathcal{R}(U(y,z))| = O(e^{-\varepsilon z}(l - 2|y|)^{\varepsilon - 2}),$$

for $2|y| < l$, and

$$U(\pm l/2, z) = O(e^{-\varepsilon z}).$$

We construct an asymptotic solution of problem (0.11), (0.13) in the form

$$\lambda z + Z(y) + \mathcal{C} + \sum_{\pm} \mathbf{w}_\pm((l/2 \mp y)e^{\lambda q_\pm z}).$$

Here $\mathcal{C}$ is an arbitrary constant, λ is an unique positive number, for which the following problem is solvable

$$\begin{cases} k_{22}^\bullet(y)Z''(y) + \lambda^2 m_{11}^\bullet(y) + 2\lambda m_{12}^\bullet(y)Z'(y) + m_{22}^\bullet(Z'(y))^2 = 0 \\ Z(\pm l/2) = -\infty, \quad Z(0) = 0, \end{cases} \tag{0.14}$$

and

$$q_\pm = m_{22}^\bullet(\pm l/2)/k_{22}^\bullet(\pm l/2).$$

We note that the solution (Z, λ) of problem (0.14) is defined in the case of constant coefficients $k_{22}^\bullet, m_{11}^\bullet, m_{12}^\bullet, m_{22}^\bullet$ by the formulae

$$Z(y) = \frac{k_{22}^\bullet}{m_{22}^\bullet} \log\cos\frac{\pi y}{l} \; - \; \frac{\pi m_{12}^\bullet k_{22}^\bullet y}{l m_{22}^\bullet (m_{11}^\bullet m_{22}^\bullet - {m_{12}^\bullet}^2)},$$

$$\lambda = \frac{\pi k_{22}^\bullet}{l(m_{11}^\bullet m_{22}^\bullet - {m_{12}^\bullet}^2)}.$$

1. Inner part of the asymptotic solution

Let us consider the boundary value problem

$$\mathcal{R}(u) = 0, \tag{1.1}$$

$$u(\pm l/2, z) = 0, \tag{1.2}$$

in the semistrip Π. We use the notation

$$\mathcal{R}(u) = \mathrm{Tr}(K\nabla_2 u) + (L, \nabla u) + (M\nabla u, \nabla u), \tag{1.3}$$

where

$$\nabla_2 = \begin{pmatrix} \dfrac{\partial^2}{\partial y^2} & \dfrac{\partial^2}{\partial y \partial z} \\ \\ \dfrac{\partial^2}{\partial y \partial z} & \dfrac{\partial^2}{\partial z^2} \end{pmatrix},$$

and K, M are symmetric matrices with elements k_{ij}, m_{ij} which are bounded functions of the variables e^{-z}, y, u defined on the set $\Sigma = (0, +\infty) \times [-l/2, l/2] \times (0, +\infty)$. By L we denote a vector functions of the same variables with bounded components l_j on Σ. We note that (1.3) is a different form of (0.12).

Here and elsewhere, the indices i, j take values 1, 2.

Suppose that

$$\begin{gathered} k_{ij}(0, y, u) - k^{\bullet}_{ij}(y) = O(e^{-\nu u}), \\ m_{ij}(0, y, u) - m^{\bullet}_{ij}(y) = O(e^{-\nu u}) \end{gathered} \tag{1.4}$$

for large positive values of u, where $k^{\bullet}_{ij}$ and $m^{\bullet}_{ij}$ are bounded functions on $[-l/2, l/2]$. Furthermore, we assume, that

$$l_j(0, y, u) = O(e^{-\nu u}). \tag{1.5}$$

Constructing an asymptotic solution of problem (1.1), (1.2), we require

$$|k_{ij}(e^z, y, \zeta) - k_{ij}(0, y, \zeta)| \leq \text{const } e^{-\delta z}, \tag{1.6}$$

$$|k_{ij}(0, y, \zeta) - k_{ij}(0, \pm l/2, \zeta)| \leq \text{const } (l/2 \mp y)^{\delta}, \tag{1.7}$$

$$|k_{ij}(0,y,\zeta_1) - k_{ij}(0,y,\zeta_2)| \le \mathrm{const}|\zeta_1 - \zeta_2|^\delta, \tag{1.8}$$

where $|\zeta_1 - \zeta_2| \le 1$ and $0 < \delta \le 1$. The inequalities (1.6) and (1.8) hold uniformly in y, ζ and (1.7) is supposed to hold uniformly in ζ.

Let the coefficients m_{ij}, l_j of $\mathcal{R}$ satisfy conditions (1.6), (1.7), (1.8) and let the functions $m^\bullet_{ij}$, $k^\bullet_{ij}$ be subject to (1.7).

By analogy with the model problem considered in Introduction, we look for an asymptotic solution of the equation (1.1), as $z \to +\infty$, in an arbitrary semistrip $\Pi_d = \{(y,z) : z \ge 0, l/2 - |y| > d\}$, $d > 0$, in the form

$$v(y,z) = \lambda z + \mathcal{C} + Z(y). \tag{1.9}$$

Here $\mathcal{C}$ is an arbitrary constant and the relations for the number λ and the function Z are to be found.

Substituting v into (1.3), we obtain the expression

$$k_{22}Z'' + \lambda^2 m_{11} + 2\lambda m_{12}Z' + m_{22}(Z')^2 + l_1\lambda + l_2 Z'. \tag{1.10}$$

By (1.9), (1.4) and (1.6) we note that in the semistrip Π_d:

$$\begin{aligned} k_{22}(e^{-z},y,v) &= k^\bullet_{22}(y) + O(e^{-\beta_0 z}),\\ m_{22}(e^{-z},y,v) &= m^\bullet_{ij}(y) + O(e^{-\beta_0 z}),\\ l_j(e^{-z},y,v) &= O(e^{-\beta_0 z}), \end{aligned} \tag{1.11}$$

where $\beta_0 = \min\{\nu\lambda, \delta\}$. Therefore, owing to (1.1) and (1.10), we define the pair (λ, Z) satisfying

$$k^\bullet_{22}Z'' + \lambda^2 m^\bullet_{11} + 2\lambda m^\bullet_{12}Z' + m^\bullet_{22}(Z')^2 = 0. \tag{1.12}$$

Since the constant $\mathcal{C}$ in (1.9) is arbitrary, we may put $Z(0) = 0$.

Let the function Z satisfy the boundary condition

$$\lim_{y\to -l/2} Z(y) = \lim_{y\to l/2} Z(y) = -\infty. \tag{1.13}$$

Lemma 1. *There exists one and only one positive number λ, such that problem* (1.12), (1.13) *is uniquely solvable.*

Proof. Let $\Phi(y) = \Lambda^{-1}Z'(y)$. Then (1.12) can be rewritten as

$$k^\bullet_{22}\Phi' + \lambda(m^\bullet_{11} + 2m^\bullet_{12}\Phi + m^\bullet_{22}\Phi^2) = 0. \tag{1.14}$$

By (1.12) and the positivity of the quadratic form $m^\bullet_{11}\xi_1^2 + m^\bullet_{12}\xi_1\xi_2 + m^\bullet_{22}\xi_2^2$, relations (1.13) imply the boundary conditions

$$\Phi(-l/2) = +\infty, \quad \Phi(l/2) = -\infty.$$

By Φ_λ we denote a solution of (1.14) satisfying the initial condition $\Phi(-l/2) = +\infty$. The substitution $\Psi_\lambda = 1/\Phi_\lambda$ leads to the initial value problem

$$k^{\bullet}_{22}\Psi'_\lambda = \lambda(m^{\bullet}_{11}\Psi^2_\lambda + 2m^{\bullet}_{12}\Psi_\lambda + m^{\bullet}_{22}), \tag{1.15}$$

$$\Psi_\lambda(-l/2) = 0.$$

This problem is uniquely solvable in a small neighborhood of the point $y = -l/2$. Therefore, Φ_λ is uniquely defined near this point. Let $(0, P(\lambda))$ be the maximum interval where Φ_λ exists. Obviously, $\Phi_\lambda(P(\lambda)) = -\infty$. We show that Φ_λ is a strictly decreasing function of λ. Let $\lambda_1 < \lambda_2$. By (1.15), $k^{\bullet}_{22}(-l/2)\Psi'(-l/2) = \lambda m^{\bullet}_{22}(-l/2)$ and consequently the inequality $\Psi_{\lambda_1} < \Psi_{\lambda_2}$ holds near $y = -l/2$. Thus, $\Phi_{\lambda_1} > \Phi_{\lambda_2}$ in the same neighbourhood of $y = -l/2$. By y_0 we denote the first positive number such, that both Φ_{λ_1} and Φ_{λ_2} are defined and the inequality $\Phi_{\lambda_1} > \Phi_{\lambda_2}$ holds. Then it follows from (1.14), that $\Phi'_{\lambda_1}(y_0) > \Phi'_{\lambda_2}(y_0)$. Let the number $y_1, y_1 > y_0$, be so close to y_0 that $\Phi'_{\lambda_1}(y) > \Phi'_{\lambda_2}(y)$ for $y \in (y_0, y_1)$. Since

$$\Phi_{\lambda_j}(y_1) - \Phi_{\lambda_j}(y_0) = \int_{y_1}^{y_0} \Phi'_{\lambda_j}(y)dy,$$

and $\Phi_{\lambda_1}(y_0) = \Phi_{\lambda_2}(y_0)$, we arrive at

$$\Phi_{\lambda_1}(y_1) - \Phi_{\lambda_2}(y_1) = \int_{y_1}^{y_0} (\Phi'_{\lambda_2}(y) - \Phi'_{\lambda_1}(y))dy < 0.$$

The contradiction obtained shows that the inequality $\Phi_{\lambda_1} > \Phi_{\lambda_2}$ holds over the whole domain of both functions. This implies the inequality $P(\lambda_2) < P(\lambda_1)$. Thus P is a strictly decreasing function on $(0, +\infty)$. The smoothness of $P(\lambda)$ follows from the implicit function theorem and the relations $\Psi_\lambda(P(\lambda)) = 0, d\Psi_\lambda/dy > 0$. According to (1.14),

$$\lambda P(\lambda) = \int_{-\infty}^{+\infty} \frac{k^{\bullet}_{22}d\Phi}{m^{\bullet}_{11} + 2m^{\bullet}_{12}\Phi + m^{\bullet}_{22}\Phi^2}.$$

Let γ be a positive constant such that

$$\gamma^{-1}k^{\bullet}_{22}(y) \le \frac{m^{\bullet}_{11}(y)\xi_1^2 + m^{\bullet}_{12}(y)\xi_1\xi_2 + m^{\bullet}_{22}(y)\xi_2^2}{\xi_1^2 + \xi_2^2} \le \gamma k^{\bullet}_{22}(y)$$

for all $y \in (-l/2, l/2)$ and any real ξ_1, ξ_2. Then

$$\pi\gamma^{-1} \leq \lambda P(\lambda) \leq \pi\gamma.$$

Therefore, the positive semiaxis is the domain of the function P. The existence of a unique root of the equation $P(\lambda) = l$ follows from the monotonicity and continuity of P. ⊠

Lemma 2. *The solution Z of problem* (1.12), (1.13) *and its derivatives have the asymptotic representations*

$$Z(y) = q_\pm^{-1} \log(l/2 \mp y) + c_\pm + \mathcal{Z}_0^\pm, \tag{1.16}$$

$$(d^j Z/dy^j)(y) = q_\pm^{-1}(-1)^{j+1}(l/2 \mp y)^{-j} + \mathcal{Z}_j^\pm, \quad j = 1, 2, \tag{1.17}$$

in a small neighborhood of the points $y = \pm l/2$. Here, the functions $\mathcal{Z}_p^\pm, p = 0, 1, 2$ satisfy

$$|\mathcal{Z}_p^\pm(y)| \leq (l/2 \mp y)^{\delta - p}. \tag{1.18}$$

Proof. Near the points $y = \pm l/2$, the solution Ψ_λ of the initial value problem (1.15) admits the representation

$$\Psi_\lambda(y) = \lambda q_\pm(l/2 \mp y) + \Xi_\lambda(y), \tag{1.19}$$

where Ξ_λ is a function satisfying (1.18) with $p = -1$. After returning from Ψ to Z in (1.19), we get (1.17) for $j = 1$ and (1.16). Equality (1.17) follows from (1.14) for $j = 2$ and from (1.17) for $j = 1$. ⊠

2. Boundary layers near the lateral sides of the semistrip

By (1.13), function (1.9) equals $-\infty$ on the lateral sides of the semistrip Π. In order to obtain an asymptotic solution satisfying the boundary condition (1.2), we construct some boundary layers w_+ and w_-. The sign plus corresponds to the line $y = l/2$ and minus stands for the line $y = -l/2$.

We suppose that the boundary layers $w_\pm$ and their first and second order derivatives tend to zero as $z \to +\infty$ for any fixed $y > 0$. By U we denote the sum of w_+ , w_- and v. We substitute U into (1.1), (1.2). Making the principal terms (1.2) and the principal terms in (1.1) near the sides $y = \pm l/2$ of the semistrip Π equal to zero, we find the boundary layers w_+ and w_-.

In order to determine w_+ and w_-, we pass in (1.1), (1.2) to the coordinates (ξ, η_+) or (ξ, η_-) defined by

$$\xi = z \ , \ \eta_\pm = (l/2 \mp y) e^{q_\pm(\lambda z + c_\pm + \mathcal{C})}. \tag{2.1}$$

According to (1.16) and (2.1)

$$v(\xi, \eta_\pm) = q_\pm^{-1} \log \eta_\pm + o(1), \qquad \text{as} \eta_\pm \to 0. \tag{2.2}$$

By assumption, the boundary layers w_+ and w_- tend to zero far from the sides $y = l/2$ and $y = -l/2$. Therefore, in view of (1.2) and (2.2), we have the boundary condition:

$$\lim_{\eta_\pm \to 0} \{w_\pm(\xi, \eta_\pm) + q_\pm^{-1} \log \eta_\pm\} = 0. \tag{2.3}$$

Clearly,

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \xi} + q_\pm \lambda \eta_\pm \frac{\partial}{\partial \eta_\pm} \ , \quad \frac{\partial}{\partial y} = \mp e^{q_\pm(\lambda z + c_\pm + \mathcal{C})} \frac{\partial}{\partial \eta_\pm}. \tag{2.4}$$

As $\xi \to +\infty$, the coefficients k_{ij} of (1.3) have the representations

$$\begin{aligned} &k_{ij}(e^{-\xi}, \pm(l/2 - \eta_\pm e^{-q_\pm(\lambda\xi + c_\pm + \mathcal{C})}), U(\xi, \eta_\pm)) \\ &= k_{ij}(0, \pm l/2, w_\pm(\xi, \eta_\pm) + q_\pm^{-1} \log \eta_\pm) + o(1) \end{aligned} \tag{2.5}$$

near the sides $y = \pm l/2$ of the semistrip Π. Formula (2.5) follows from (2.2) and conditions (1.6), (1.7), (1.8).

Taking into account (2.4), (1.17), and (2.5), we rewrite $\mathcal{R}(u)$ in the coordinates (ξ, η_+) and (ξ, η_-). The principal term of $\mathcal{R}(u)$ is $O(e^{2\lambda q_\pm \xi})$ near the lines $y = \pm l/2$. Therefore, the equations for the boundary layers $w_\pm$ have the form

$$\mathbf{N}_\pm(w_\pm) = 0, \tag{2.6}$$

with

$$\begin{aligned} \mathbf{N}_\pm(w_\pm) = \frac{d^2 w_\pm}{d\eta_\pm^2} + Q_\pm \bigg(\left(\frac{dw_\pm}{d\eta_\pm} \right)^2 + \frac{2}{q_\pm \eta_\pm} \frac{dw_\pm}{d\eta_\pm} \bigg) \\ + \frac{1}{q_\pm \eta_\pm^2} \left(\frac{Q_\pm}{q_\pm} - 1 \right), \end{aligned} \tag{2.7}$$

where

$$Q_\pm(W_\pm(\eta_\pm)) = m_{22}(0, \pm l/2, W_\pm(\eta_\pm))/k_{22}(0, \pm l/2, W_\pm(\eta_\pm))$$

and

$$W_\pm(\eta_\pm) = w_\pm(\eta_\pm) + q_\pm^{-1} \log \eta_\pm. \tag{2.8}$$

Thus, the boundary layers $w_\pm$ near the sides $y = \pm l/2$ are solutions of problem (2.6), (2.3).

By (2.6), (2.3), function (2.8) satisfies the equation

$$\frac{d^2 W_\pm}{d\eta_\pm^2}(\eta_\pm) + Q_\pm(W_\pm(\eta_\pm)) \left(\frac{dW_\pm}{d\eta_\pm}(\eta_\pm) \right)^2 = 0 \tag{2.9}$$

and the boundary condition

$$\lim_{\eta_\pm \to 0} W_\pm(\eta_\pm) = 0. \tag{2.10}$$

From (2.9), we obtain

$$\frac{dW_\pm}{d\eta_\pm}(\eta_\pm) = \exp\left(-\int_0^{W_\pm(\eta_\pm)} Q_\pm(\rho)d\rho + \text{const} \right). \tag{2.11}$$

Thus, by (2.11), (2.10), the solution $W_\pm$ of (2.9) satisfies

$$\int_0^{W_\pm(\eta_\pm)} \exp\left(\int_0^{\zeta} Q_\pm(\rho)d\rho \right) d\zeta = \varkappa \eta_\pm, \tag{2.12}$$

where $\varkappa$ is a constant to be determined. It follows from (2.8) and the condition $\lim\limits_{\eta_\pm \to +\infty} w_\pm(\eta_\pm) = 0$ that

$$\lim_{\eta_\pm \to +\infty} (W_\pm(\eta_\pm) - q_\pm^{-1} \log \eta_\pm) = 0. \tag{2.13}$$

Let $\eta_\pm \to +\infty$ in (2.11). By (2.12) and the equality $\lim\limits_{\eta_\pm \to +\infty} Q_\pm = q_\pm$, we arrive at

$$\lim_{\eta_\pm \to +\infty} \{ e^{(q_\pm W_\pm(\eta_\pm) + h_\pm)} - 1 - \varkappa q_\pm \eta_\pm \} = 0,$$

where

$$h_\pm = \int\limits_0^{+\infty} (Q_\pm(\eta) - q_\pm) d\eta.$$

Combining the last formula with (2.13), we find $\varkappa = q_\pm^{-1} e^{h_\pm}$.

Hence, the boundary layers $w_\pm$ can be found from (2.8), where $W_\pm$ are defined by (2.12) with $\varkappa = q_\pm^{-1} e^{h_\pm}$.

Remark 1. *In the case of constant coefficients m_{22} and k_{22}, the function $Q_\pm$ in* (2.7) *is equal to the constant*

$$Q_\pm = q_\pm = m_{22}^\bullet / k_{22}^\bullet.$$

Therefore, the solution $W_\pm$ of problem $(2.9), (2.10), (2.13)$ *has the form* $W_\pm(\eta_\pm) = q_\pm^{-1} \log(\eta_\pm + 1)$. *Hence*

$$w_\pm = q_\pm^{-1} \log(1 + 1/\eta_\pm) \tag{2.14}$$

(*compare with Example in Introduction*) .

Now, we estimate the boundary layers at infinity.

Lemma 3. (i) *In the case $\nu \neq q_\pm$, the solutions $w_\pm$ of problem* (2.6), (2.3) *vanishing at $+\infty$ admit the estimate*

$$w_\pm(\eta_\pm) = O(\eta_\pm^{-\chi_\pm}), \tag{2.15}$$

where $\chi_\pm = \min\{1, \nu/q_\pm\}$.

If $\nu = q_\pm$, then

$$w_\pm(\eta_\pm) = O(\eta_\pm^{-1} \log \eta_\pm). \tag{2.16}$$

(ii) *The derivatives of the boundary layers $w_\pm$ satisfy*

$$\begin{aligned} d^j w_\pm / d\eta_\pm^j(\eta_\pm) &= O(\eta_\pm^{-\chi_\pm - j}), \quad \nu \neq q_\pm, \\ d^j w_\pm / d\eta_\pm^j(\eta_\pm) &= O(\eta_\pm^{-1-j} \log \eta_\pm), \quad \nu = q_\pm. \end{aligned} \tag{2.17}$$

Proof. (i). By (1.4)

$$Q_\pm(\zeta) = q_\pm + O(e^{-\nu\zeta}), \tag{2.18}$$

which, along with (2.12), implies

$$\frac{1}{q_\pm} e^{q_\pm W_\pm(\eta_\pm)} + \alpha_\pm(W_\pm(\eta_\pm)) = \frac{e^{h_\pm}}{q_\pm}\eta_\pm,$$

where $\alpha(w) = O(e^{(q_\pm - \nu)W_\pm})$. By the Banach principle we obtain

$$W_\pm(\eta_\pm) = \frac{1}{q_\pm}\log\eta_\pm + O(\eta_\pm^{-\chi_\pm})$$

which completes the proof.

(ii). From (2.11), (2.18) and (2.15), (2.16), we obtain (2.17) with $j = 1$. According to (2.9) and (2.17) with $j = 1$, the function $d^2 w_\pm/d\eta_\pm^2$ admits estimate (2.17) with $j = 2$. ⊠

3. Estimates of $\mathcal{R}(u_0)$

We define asymptotic solution U of problem (1.1), (1.2) in the form

$$\begin{aligned} U = \lambda z + Z(y) + \mathcal{C} + w_+((l/2 - y)e^{q_+(\lambda z + c_+ + \mathcal{C})}) \\ + w_-((l/2 + y)e^{q_-(\lambda z + c_- + \mathcal{C})}). \end{aligned}$$

Here, the pair λ, Z is determined by (1.12), (1.13), and the boundary layers $w_\pm$ satisfy (2.7). We prove that the discrepancy generated by U in equations (1.1), (1.2) vanishes exponentially as $z \to +\infty$.

Let K_0 and M_0 be the matrices with the elements $k_{ij}(0, y, u)$, $m_{ij}(0, y, u)$ and let L_0 be the vector with the components $l_j(0, y, u)$. Also let

$$\mathcal{R}_0(u) = \mathrm{Tr}(K_0 \nabla_2 u) + (L_0, \nabla u) + (M_0 \nabla u, \nabla u). \tag{3.1}$$

The next assertion is the main result of this section.

Theorem 1. *For $(y, z) \in \Pi$, the functions $\mathcal{R}(U)$ and $U(\pm l/2, z)$ satisfy*

$$|\mathcal{R}(U)(y, z)|\, dy = O(e^{-\varepsilon z}(l - 2|y|)^{-2+\varepsilon}), \tag{3.2}$$

and

$$U(\pm l/2, z) = O(z^{-\varepsilon z}) \tag{3.3}$$

with a small positive ε.

Proof. By (1.16) and (2.3), one has

$$u(\pm l/2, z) = w_\mp(le^{-q_\pm(\lambda z + c_\pm + \mathcal{C})}). \tag{3.4}$$

The functions $w_\mp$ are represented in the form (2.15) as $y \to \pm l/2$. Hence, (3.4) implies

$$U(\pm l/2, z) = O(z^s e^{-\lambda q_\mp \chi_\mp z}),$$

with $\chi_\mp = \min\{1, \nu/q_\mp\}$ and $s = 1$ when $\nu = q_\mp$ and $s = 0$ in other cases. Thus, we obtain (3.3).

Now, we prove (3.2). By Lemma 2 and Lemma 3, we have

$$\frac{\partial^{i+j} U}{\partial y^i \partial z^j} = O\left(\frac{(\min\{1, \eta_+\} \min\{1, \eta_-\})^\delta}{(l/2 - |y|)^2}\right) \tag{3.5}$$

which along with (1.6) implies

$$\mathcal{R}(U) - \mathcal{R}_0(U) = O\left(e^{-\delta z} \frac{(\min\{1, \eta_+\} \min\{1, \eta_-\})^\delta}{(l/2 - |y|)^2}\right). \tag{3.6}$$

According to (1.9) and (1.12), the equality

$$\mathrm{Tr}(K_0 \nabla_2 v) + (M_0 \nabla v, \nabla v) = 0$$

holds. Hence

$$\begin{aligned}\mathcal{R}_0(U) = \mathcal{R}_0(U) - \mathrm{Tr}(K_0 \nabla_2 v) - (M_0 \nabla v, \nabla v) \\ - \sum_{\pm} k_{22}(0, \pm l/2, W_\pm) \mathbf{N}_\pm(w_\pm).\end{aligned}$$

The right-hand side of the last equality can be represented as the sum of nine terms

$$\sum_{q=1}^{3} \sum_{p=1}^{3} S_q^{(p)} \tag{3.7}$$

which will be described and estimated in what follows.

In order to treat $S_q^{(p)}$, we introduce the sets

$$\Gamma_\gamma^0 = \{(y, z) : \eta_+(y, z) \geq a e^{\gamma z}, \eta_-(y, z) \geq a e^{\gamma z}\},$$

$$\Gamma_\gamma^\pm = \{(y, z) : \eta_\pm(y, z) \leq a e^{\gamma z}\},$$

where a is a sufficiently large positive number and $\gamma \in (0, \lambda \min\{q_+, q_-\})$.

Let us consider the functions

$$\begin{aligned}S_1^{(1)} &= \mathrm{Tr}((K_0(y, U) - K_0(y, W_+) - K_0(y, W_-) + K^\bullet(y)) \nabla_2 U), \\ S_1^{(2)} &= (L_0(y, U) - L_0(y, W_+) - L_0(y, W_-), \nabla U), \\ S_1^{(3)} &= ((M_0(y, U) - M_0(y, W_+) - M_0(y, W_-) + M^\bullet(y)) \nabla U, \nabla U),\end{aligned}$$

where $K^\bullet$ and $M^\bullet$ are the matrices with the elements $k^\bullet_{ij}(0,y,u)$, $m^\bullet_{ij}(0,y,u)$. We begin with the term $S_1^{(1)}$ on the set $\Gamma^\pm_\gamma$. Using (1.4) and the inequality

$$\eta_\mp(y,z) > le^{q_\mp \lambda z} \qquad \text{for} \quad (y,z) \in \Gamma^\pm_\gamma \tag{3.8}$$

we obtain the estimate

$$|k_{ij}(0,y,W_\mp) - k^\bullet_{ij}(0,y)| \le ce^{-\lambda\nu z}, \qquad \text{for} \quad (y,z) \in \Gamma^\pm_\gamma. \tag{3.9}$$

Inequality (1.8) implies

$$|k_{ij}(0,y,U) - k_{ij}(0,y,W_\pm)| \le c(|w_\mp|^\delta + |\mathcal{Z}^\pm_0|^\delta). \tag{3.10}$$

Hence, by Lemmas 2 and 3, as well as the estimate

$$l/2 \mp y \le ce^{-(q_\mp\lambda-\gamma)z}, \qquad \text{for} \quad (y,z) \in \Gamma^\pm_\gamma \tag{3.11}$$

we have

$$|k_{ij}(0,y,U) - k_{ij}(0,y,W_\pm)| \le cz^{\delta s}e^{-\varepsilon_\pm z}, \tag{3.12}$$

where

$$\varepsilon_\pm = \min\{(q_\pm\lambda - \gamma)\delta, \lambda\nu, \lambda q_\pm\},$$

$s = 1$ for $\nu = q_\pm$, and $s = 0$ in other cases. According to definition of the asymptotic solution U, and Lemmas 2 and 3, there hold the inequalities

$$\begin{gathered} |U(y,z)| \ge \gamma(\min\{q_+, q_-\})^{-1} z, \\ |W_\pm(\eta_\pm(y,z))| \ge \gamma(\min\{q_+, q_-\})^{-1} z \end{gathered} \tag{3.13}$$

for $(y,z) \in \Gamma^0_\gamma$. Thus, by (1.4)

$$|k_{ij}(0,y,U) - k_{ij}(0,y,W_+) - k_{ij}(0,y,W_-) + k^\bullet(0,y)| = O(e^{-\varepsilon_\bullet z}), \tag{3.14}$$

where

$$\varepsilon_\bullet = \frac{\gamma\nu}{\min\{q_+, q_-\}}.$$

The value

$$\varepsilon = \min\{\varepsilon_\bullet, \varepsilon_+, \varepsilon_-\}$$

which depends on γ, attains its minimum

$$\varepsilon_1 := \nu\lambda\delta \min\{\frac{q_-}{q_+\delta+\nu}, \frac{q_+}{q_-\delta+\nu}\}$$

if $\gamma = \gamma_1$ with

$$\gamma_1 = \frac{q_- q_+ \lambda\delta}{\delta \max\{q_+, q_-\} + \nu}.$$

Note that γ_1 satisfies the inequality

$$\gamma_1 < \lambda \min\{q_+, q_-\}.$$

Thus, (3.5), (3.9), (3.12) and (3.14) imply the estimate

$$\left|S_1^{(1)}\right| = O\left(e^{-\varepsilon_1 z} z^{\delta s} \frac{(\min\{1,\eta_+\}\min\{1,\eta_-\})^\delta}{(l/2-|y|)^2}\right). \tag{3.15}$$

The functions $S_1^{(2)}$ and $S_1^{(3)}$ are estimated in the same way.

The term $S_2^{(1)}$ is defined by the formula

$$\begin{aligned} S_2^{(1)} &= \mathrm{Tr}((K_0(y, W_+) + K_0(y, W_-) - K^\bullet(y))\nabla_2 U) \\ &-(k_{22}(y, W_+) + k_{22}(y, W_-) - k^\bullet(y))\frac{\partial^2 U}{\partial y^2}. \end{aligned}$$

Let us take $\gamma = \gamma_2$, where

$$\gamma_2 = \frac{q_- q_+ \lambda}{\max\{q_+, q_-\} + \nu}.$$

According to (3.5), (3.11) we have

$$\left|S_2^{(1)}\right| = O\left(e^{-\varepsilon_2 z} z^{s} \frac{(\min\{1,\eta_+\}\min\{1,\eta_-\})^\delta}{(l/2-|y|)^2}\right), \tag{3.16}$$

$$\varepsilon_2 = \nu\lambda \min\{\frac{q_-}{q_+ + \nu}, \frac{q_+}{q_- + \nu}\}$$

on the sets $\Gamma_\pm$. In the case $(y,z) \in \Gamma_0$ formula (3.16) follows from the equalities $\partial^2 U/\partial x^2 = \partial^2 U/\partial x \partial y = 0$ and Lemma 3.

Similar arguments lead to estimate (3.16) for the functions

$$S_2^{(2)} = (L_0(y, W_+) + L_0(y, W_-), \nabla U),$$

$$S_2^{(3)} = ((M_0(y, W_+) + M_0(y, W_-) - M^\bullet(y))\nabla U, \nabla U) - (m_{22}(y, W_+) \\ + m_{22}(y, W_-) - m_{22}^\bullet(y))\frac{\partial U}{\partial y}\frac{\partial U}{\partial y} - (m_{22}(y, W_+) + m_{22}(y, W_-) \\ - m_{22}^\bullet(y))\frac{\partial v}{\partial x}\frac{\partial v}{\partial x} - 2(m_{22}(y, W_+) + m_{22}(y, W_-) - m_{22}^\bullet(y))\frac{\partial U}{\partial y}\frac{\partial v}{\partial x},$$

$$S_3^{(2)} = (m_{11}(y, W_+) + m_{11}(y, W_-) - 2m_{11}^\bullet(y))\lambda^2.$$

Let us consider

$$S_3^{(1)} = \left(\sum_{\pm} k_{22}(0, y, W_\pm) - k_{22}^\bullet(0, y)\right)\frac{\partial^2 U}{\partial y^2} \\ - k_{22}^\bullet(0, y)\frac{\partial^2 Z}{\partial y^2} - \sum_{\pm}\Big\{k_{22}(0, \pm l/2, W_\pm)\frac{\partial^2 w_\pm}{\partial y^2} \\ - (k_{22}(0, \pm l/2, W_\pm) - k_{22}^\bullet(0, \pm l/2))\frac{1}{q_\pm(l/2 \mp y)^2}\Big\}.$$

In order to prove (3.15) in $\Gamma_\gamma^\pm$ with $\gamma = \gamma_1$, we divide $S_3^{(1)}$ into three parts. The estimate (3.15) for

$$\Big\{k_{22}(0, y, W_\pm) - k_{22}(0, \pm l/2, W_\pm)\Big\}\frac{\partial^2 w_\pm}{\partial y^2} \\ + \Big\{k_{22}(0, y, W_\pm)\frac{\partial^2 Z}{\partial y^2} + k_{22}(0, \pm l/2, W_\pm)\frac{1}{q_\pm(l/2 \mp y)^2}\Big\} \\ + \Big\{k_{22}^\bullet(0, y)\frac{\partial^2 Z}{\partial y^2} + k_{22}^\bullet(0, \pm l/2)\frac{1}{q_\pm(l/2 \mp y)^2}\Big\}$$

follows from (3.11), (1.7) and (1.18). By (3.9), (3.5), we obtain the estimate (3.15) for

$$(k_{22}(0, y, W_\mp) - k_{22}^\bullet(0, y))\frac{\partial^2 U}{\partial y^2} \\ - (k_{22}(0, \mp l/2, W_\mp) + k_{22}^\bullet(0, \mp l/2))\frac{1}{q_\mp(l/2 \pm y)^2}.$$

By (3.8) and Lemma 3, the function

$$k_{22}(0, y, W_\pm)\frac{\partial^2}{\partial y^2}w_\mp(\eta_\mp)$$

satisfies (3.15) in $\Gamma_\gamma^\pm$.

Let us consider $S_3^{(1)}$ in Γ_γ^0, $\gamma=\gamma_1$. The inequality (3.9) implies (3.15) for

$$\left(\sum_\pm k_{22}(0,y,W_\pm)-2k_{22}^\bullet(0,y)\right)\frac{\partial^2 Z}{\partial y^2}$$
$$+\sum_\pm\left(k_{22}(0,\pm l/2,W_\pm)-k_{22}^\bullet(0,\pm l/2)\right)\frac{1}{q_\pm(l/2\mp y)^2}.$$

The other parts of $S_3^{(1)}$ are estimated in Γ_γ^0 by Lemma 3. The last term in (3.7)

$$S_3^{(3)}=\left(\sum_\pm m_{22}(0,y,W_\pm)-m_{22}^\bullet(0,y)\right)\left(\frac{\partial U}{\partial y}\right)^2-m_{22}^\bullet(0,y)\left(\frac{\partial Z}{\partial y}\right)^2$$
$$+2\lambda\left(\sum_\pm m_{12}(0,y,W_\pm)-m_{12}^\bullet(0,y)\right)\frac{\partial U}{\partial y}-2\lambda m_{12}^\bullet(0,y)\frac{\partial Z}{\partial y}$$
$$-\sum_\pm\left\{m_{22}(0,\pm l/2,W_\pm)\left(\frac{\partial w_\pm}{\partial y}\right)^2\right.$$
$$\left.-m_{22}(0,\pm l/2,W_\pm)\frac{1}{q_\pm(l/2\mp y)^2}\left(2\frac{\partial w_\pm}{\partial y}-\frac{1}{q_\pm(l/2\mp y)^2}\right)\right\}$$

is majorized in the same way as $S_3^{(1)}$.

By (3.15), (3.16) for $S_q^{(p)}$ $(p,q=1,2,3)$, we arrive at the estimate

$$|\mathcal{R}(U(y,z))|=O(z^{\delta s}e^{-\varepsilon_1 z}(l-2|y|)^{-2+\delta}). \tag{3.17}$$

The proof is complete.

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