Bounds for eigenfunctions of the Neumann p-Laplacian on noncompact Riemannian manifolds

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Abstract

Eigenvalue problems for the p-Laplace operator in domains with finite volume, on noncompact Riemannian manifolds, are considered. If the domain does not coincide with the whole manifold, Neumann boundary conditions are imposed. Sharp assumptions ensuring L^q or L^∞ bounds for eigenfunctions are offered either in terms of the isoperimetric function or of the isocapacitary function of the domain.

1 Introduction and main results

Let Ω be an open set in an n-dimensional Riemannian manifold $\mathbb{M}$, which will be assumed to be without boundary throughout. Assume that $n \geq 2$, and

$$\mathcal{H}^n(\Omega) < \infty, \tag{1.1}$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on $\mathbb{M}$, i.e. the volume measure on $\mathbb{M}$ induced by its Riemannian metric. In particular, if $\mathbb{M} = \mathbb{R}^n$ equipped with the Euclidean metric, then $\mathcal{H}^n$ agrees with the Lebesgue measure. The choice

$$\Omega = \mathbb{M}$$

is also admissible, provided that $\mathcal{H}^n(\mathbb{M}) < \infty$.

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We are concerned with eigenfunctions of the p-Laplace operator in Ω, subject to homogeneous Neumann boundary conditions on $\partial\Omega$, if $\Omega \neq \mathbb{M}$. Namely, we deal with solutions to the equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \gamma |u|^{p-2}u \quad \text{in } \Omega, \tag{1.2}$$

for some $\gamma \in \mathbb{R}$, satisfying the condition

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

if $\partial\Omega \neq \emptyset$. Here, $p > 1$, and $\mathbf{n}$ stands for the normal unit vector on $\partial\Omega$.

A unified definition of an eigenfunction u of the problems under consideration amounts to requiring that $u \in W^{1,p}(\Omega)$ and satisfies the equation

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, d\mathcal{H}^n = \gamma \int_\Omega |u|^{p-2}u\, \phi \, d\mathcal{H}^n \tag{1.4}$$

for some $\gamma \in \mathbb{R}$ and every test function $\phi \in W^{1,p}(\Omega)$, where the dot "$\cdot$" denotes the scalar product associated with the Riemannian structure on $\mathbb{M}$, and $|\nabla u|$ the norm of the gradient ∇u defined via this scalar product.

Classical variational methods ensure that the eigenvalue problems in question do admit non-trivial (i.e. non-constant) eigenfunctions under suitable assumptions on Ω – see e.g. [Le]. This is the case if Ω has a compact closure and a regular boundary – a Lipschitz domain, for instance. The same conclusion holds if $\Omega = \mathbb{M}$ and the latter is compact. The regularity of Ω also ensures that any eigenfunction of problem (1.4) does not merely belong to $L^p(\Omega)$, but is in fact globally essentialy bounded in Ω. On the other hand, membership of eigenfunctions in $L^\infty(\Omega)$, and even in $L^q(\Omega)$ for $q > p$, is not guaranteed if Ω is an arbitrary open set with $\mathcal{H}^n(\Omega) < \infty$.

The present paper is aimed at offering minimal assumptions on the geometry of Ω for any eigenfunction of problem (1.4) to belong to $L^\infty(\Omega)$, and to admit a corresponding bound of the form

$$\|u\|_{L^\infty(\Omega)} \leq c\|u\|_{L^p(\Omega)} \tag{1.5}$$

for some constant c depending on Ω, γ and p. Estimates in $L^q(\Omega)$, for every $q < \infty$, namely inequalities of the type

$$\|u\|_{L^q(\Omega)} \leq c\|u\|_{L^p(\Omega)} \tag{1.6}$$

for some constant c depending on Ω, γ, p and q, are also established under slightly weaker conditions on Ω.

The description of the geometry of Ω adopted in our results calls into play certain functions defined in terms of inequalities of geometric-functional nature for subsets of Ω. They are the isoperimetric function and the p-isocapacitary function of Ω.

The isoperimetric function λ_Ω is the largest non-decreasing function in $[0, \frac{\mathcal{H}^n(\Omega)}{2}]$ such that

$$\lambda_\Omega(\mathcal{H}^n(E)) \leq P(E;\Omega) \tag{1.7}$$

for every measurable set $E \subset \Omega$ with $\mathcal{H}^n(E) \leq \frac{\mathcal{H}^n(\Omega)}{2}$. Here, $P(E;\Omega)$ denotes the perimeter of E relative to Ω.

Let us emphasize that, in view of our applications, only the asymptotic behaviour of the isoperimetric function λ_Ω near 0 is relevant. If, for instance, Ω has a Lipschitz continuous boundary and $\overline{\Omega}$ is compact, or $\Omega = \mathbb{M}$ and the latter is compact, then

$$\lambda_\Omega(s) = O\big(s^{\frac{n-1}{n}}\big) \quad \text{near } 0. \tag{1.8}$$

Here, the relation "O near zero" between two functions means that they are bounded by each other, up to positive multiplicative constants, for small values of their argument.

The behaviour near 0 is also the only piece of information about the p-isocapacitary function which is needed for our purposes. This function is denoted by $\nu_{\Omega,p}$, and is defined in analogy with λ_Ω, save that the perimeter of a set $E \subset \Omega$ is replaced by its condenser capacity $\mathrm{cap}_p(E,G)$ relative to any set G such that $E \subset G \subset \Omega$. Thus, $\nu_{\Omega,p}$ is the largest non-decreasing function in $[0, \frac{\mathcal{H}^n(\Omega)}{2}]$ which renders the inequality

$$\nu_{\Omega,p}(\mathcal{H}^n(E)) \leq \mathrm{cap}_p(E,G) \tag{1.9}$$

true for every measurable sets $E \subset G \subset \Omega$ such that $\mathcal{H}^n(G) \leq \frac{\mathcal{H}^n(\Omega)}{2}$.
If Ω is sufficiently regular, as specified in connection with equation (1.8) for instance, then

$$\nu_{\Omega,p}(s) = \begin{cases} O\big(s^{\frac{n-p}{n}}\big) & \text{if } 1 \leq p < n \\ O\big(\big(\log \frac{1}{s}\big)^{1-n}\big) & \text{if } p = n \\ O(1) & \text{if } p > n \end{cases} \qquad \text{near } 0. \tag{1.10}$$

More details can be found in the next section. Let us just mention here that the regularity of a domain Ω affects the decay of the functions λ_Ω and $\nu_{\Omega,p}$ near 0. A more irregular geometry of the domain Ω is reflected in a faster decay to 0 when their argument approaches 0. In particular, neither $\lambda_\Omega(s)$ nor $\nu_{\Omega,p}(s)$ can decay more slowly to 0 as $s \to 0$ than the functions appearing on the right-hand sides of equations (1.8) and (1.10), respectively, whatever Ω is.

The isoperimetric function and the p-isocapacitary function were introduced in the papers [Ma1, Ma2] to characterize the domains Ω in $\mathbb{R}^n$ supporting a Sobolev type inequality for weakly differentiable functions whose gradient belongs to $L^1(\Omega)$ and to $L^p(\Omega)$, respectively. Their use in various question beyond the theory of Sobolev spaces, including the theory of partial differential equations, the spectral theory of differential operators and Riemannian geometry, has become apparent over the years. Besides the early contributions [Ma3, Ma4, Ma5, Ma6, MaNe] and the monograph [Ma7], a sample of the developments on these topics is provided by the papers [ACMM, Ba, BeCa, BuZa, CMS, ChFe, Ci1, Ci2, CiMa1, CiMa2, CGL, CoMa, Ga, GrPa, HaKi, KiMa, Kl, Ku, LaMa, LiPa, MoJo, Mi, NaPa, Pa, Pi, Ps, Ri, StZu].

In particular, the special choice $p = 2$ in (1.4) reproduces the linear eigenvalue problem for the Neumann Laplacian. The analysis of spectral problems for this classical operator has been the center of numerous investigations, especially in the case when Ω agrees with a compact manifold. The vast bibliography on this topic includes the monographs [Cha, BGM] and the papers [Bou, BD, Che, CGY, DS, Do1, Ga, Gr2, HSS, JMS, Na, SS, SZ, Ya]. Results for the Laplace operator in the noncompact case, in the same vein as those established here, can be found in [CiMa4]. The papers [CiMa3, CiMa2] deal with related topics. Our approach in the nonlinear framework at hand has a start reminiscent of that of [CiMa4]. However, different techiniques have to be exploited in fundamental steps of the proofs of estimates (1.5) and (1.6). For instance, certain customary fixed point theorems, which are well suited for the linear case, do not fit the nonlinear setting for general $p \neq 2$.

All criteria that will be proposed are invariant under replacement of $\nu_{\Omega,p}$ or λ_Ω with equivalent functions near 0, in the sense of the relation $\approx$ defined as follows. Given two functions functions $f, g : (0,\infty) \to [0,\infty)$, the notation

$$f \approx g \qquad \text{near } 0 \tag{1.11}$$

means that $c_1 g(c_1 s) \leq f(s) \leq c_2 g(c_2 s)$ if $0 < s \leq s_0$, for suitable positive constants c_1, c_2 and s_0.

The conditions in terms of the isoperimetric functon λ_Ω ensuring bounds in $L^q(\Omega)$ or $L^\infty(\Omega)$ for eigenfunctions are presented in our first result. Interestingly enough, the condition for L^q-estimates is independent of p and q. The dependence on these exponents only enters the constant involved in the estimates. By contrast, the dependence on p is crucial in the condition for L^∞-estimates.

Theorem 1.1. [Bounds for eigenfunctions via λ_Ω] *Assume that $n \geq 2$ and $p > 1$. Let Ω be an open subset of an n-dimensional Riemannian manifold $\mathbb{M}$, such that $\mathcal{H}^n(\Omega) < \infty$. Let u be any eigenfunction of problem* (1.4) *associated with any eigenvalue γ.*
(i) *Assume that*

$$\lim_{s\to 0} \frac{s}{\lambda_\Omega(s)} = 0\,. \tag{1.12}$$

Then

$$u \in L^q(\Omega) \tag{1.13}$$

for every $q \in (p, \infty)$, and inequality (1.6) *holds for some constant $c = c(\Omega, p, q, \gamma)$.*
(ii) *Assume that*

$$\int_0 \left(\frac{s}{\lambda_\Omega(s)}\right)^{p'} \frac{ds}{s} < \infty, \tag{1.14}$$

where $p' = \frac{p}{p-1}$, the Hölder conjugate of p. Then

$$u \in L^\infty(\Omega), \tag{1.15}$$

and inequality (1.5) *holds for some constant $c = c(\Omega, p, \gamma)$.*

Assumptions (1.12) and (1.14) are optimal, in a sense specified in the next theorem, for (1.13) and (1.15), respectively, to hold in classes of sets Ω with a prescribed decay of λ_Ω at 0. In particular, the gap between condition (1.14), ensuring $L^\infty(\Omega)$ bounds for eigenfunctions, and condition (1.12), just yielding $L^q(\Omega)$ bounds for any $q < \infty$, cannot be essentially filled. This can be demonstrated, for instance, via manifolds $\mathbb{M}$ of revolution as in Figure 1,

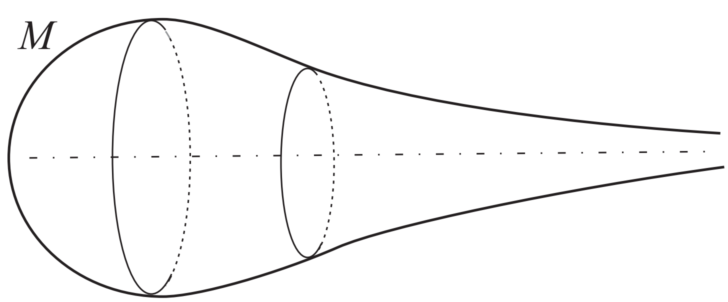

Figure 1: A manifold of revolution

whose isoperimetric function is equivalent to a function λ such that

$$\frac{\lambda(s)}{s^{\frac{n-1}{n}}} \approx \text{a non-decreasing function near } 0. \tag{1.16}$$

Such an assumption is consistent with the fact that, as noticed above, λ_Ω decays as in (1.8) in the best possible case.

Theorem 1.2. **[Sharpness of bounds via λ_Ω]** *Let $n \geq 2$ and $p > 1$.*
(i) *Given any $q \in (p, \infty)$, there exists an n-dimensional Riemannian manifold $\mathbb{M}$, with $\mathcal{H}^n(\mathbb{M}) < \infty$, such that*

$$\lambda_{\mathbb{M}}(s) \approx s \qquad \textit{near } 0, \tag{1.17}$$

and the p-Laplacian on $\mathbb{M}$ has an eigenfunction $u \notin L^q(\mathbb{M})$.
(ii) *Let $\lambda : [0, \infty) \to [0, \infty)$ be any non-decreasing function, vanishing only at 0, such that*

$$\lim_{s \to 0} \frac{s}{\lambda(s)} = 0\,, \tag{1.18}$$

but

$$\int_0 \left(\frac{s}{\lambda(s)} \right)^{p'} \frac{ds}{s} = \infty\,. \tag{1.19}$$

Assume in addition that condition (1.16) is in force. Then, there exists an n-dimensional Riemannian manifold $\mathbb{M}$, with $\mathcal{H}^n(\mathbb{M}) < \infty$, fulfilling

$$\lambda_{\mathbb{M}}(s) \approx \lambda(s) \qquad \textit{near } 0, \tag{1.20}$$

and such that the p-Laplacian on $\mathbb{M}$ has an unbounded eigenfunction.

Although the isocapacitary function has a less transparent geometric meaning than the isoperimetric function, and its behaviour can be more difficult to detect, it is in a sense more appropriate in the framework at hand. Its use provides yet finer conditions for L^q and L^∞ estimates of eigenfunctions, which are exhibited in Theorem 1.3 below. Indeed, not only are these conditions optimal in classes of sets Ω whose isocapacitary function $\nu_{\Omega,p}$ has an assigned decay at 0 (see the subsequent Theorem 1.4), but there also exist specific sets and entire manifolds where the criteria of Theorem 1.3 apply, whereas those of Theorem 1.1 fail. Typically, this may happen in presence of complicated geometric configurations. Instances of this kind of manifolds are those depicted in Figure 2, and discussed in Subsection 5.2 of the last section.

Theorem 1.3. **[Bounds for eigenfunctions via $\nu_{\Omega,p}$]** *Assume that $n \geq 2$ and $p > 1$. Let Ω be an open subset of an n-dimensional Riemannian manifold $\mathbb{M}$, such that $\mathcal{H}^n(\Omega) < \infty$. Let u be any eigenfunction of problem (1.4) associated with any eigenvalue γ.*
(i) *Assume that*

$$\lim_{s \to 0} \frac{s}{\nu_{\Omega,p}(s)} = 0\,. \tag{1.21}$$

Then

$$u \in L^q(\Omega)$$

for every $q \in (p, \infty)$, and inequality (1.6) holds for some constant $c = c(\Omega, p, q, \nu_{\Omega,p})$.
(ii) *Assume that*

$$\int_0 \left(\frac{s}{\nu_{\Omega,p}(s)} \right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty\,. \tag{1.22}$$

Then

$$u \in L^\infty(\Omega),$$

and inequality (1.5) holds for some constant $c = c(\Omega, p, \nu_{\Omega,p})$.

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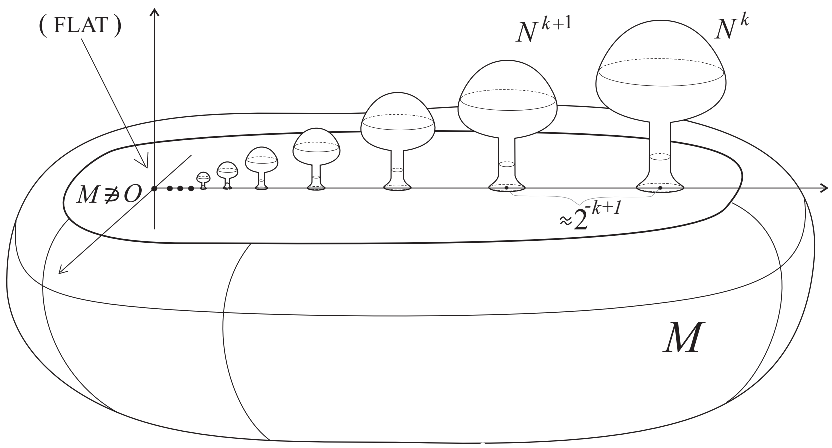

Figure 2: A manifold with a family of clustering submanifolds

The sharpness of condition (1.22) will be exhibited via manifolds $\mathbb{M}$ whose isocapacitary function is equivalent to a function ν such that either

$$1 < p < n \text{ and } \frac{\nu(s)}{s^{\frac{n-p}{n}}} \approx \text{a non-decreasing function near } 0, \tag{1.23}$$

or

$$p \geq n \text{ and } \frac{\nu(s)}{s\nu'(s)} \approx \text{a non-decreasing function near } 0. \tag{1.24}$$

These requirements reflect the fact that the behaviour of $\nu_{\Omega,p}$ given by (1.10) is the slowest possible.

Theorem 1.4. **[Sharpness of bounds via $\nu_{\Omega,p}$]** *Let $n \geq 2$ and $p > 1$.*
(i) *Given any $q \in (p, \infty)$, there exists an n-dimensional Riemannian manifold $\mathbb{M}$, with $\mathcal{H}^n(\mathbb{M}) < \infty$, such that*

$$\nu_{\mathbb{M},p}(s) \approx s \qquad \textit{near } 0, \tag{1.25}$$

and the p-Laplacian on $\mathbb{M}$ has an eigenfunction $u \notin L^q(\mathbb{M})$.
(ii) *Let $\nu : [0, \infty) \to [0, \infty)$ be any non-decreasing, continuosly differentiable function, vanishing only at 0, such that*

$$\lim_{s \to 0} \frac{s}{\nu(s)} = 0\,, \tag{1.26}$$

but

$$\int_0 \left(\frac{s}{\nu(s)}\right)^{\frac{1}{p-1}} \frac{ds}{s} = \infty\,. \tag{1.27}$$

Assume in addition that ν satisfies the Δ_2-condition near 0, and either of conditions (1.23) and (1.24) is in force. Then, there exists an n-dimensional Riemannian manifold $\mathbb{M}$, with $\mathcal{H}^n(\mathbb{M}) < \infty$, fulfilling

$$\nu_{\mathbb{M},p}(s) \approx \nu(s) \qquad \text{near } 0, \tag{1.28}$$

and such that the p-Laplacian on $\mathbb{M}$ has an unbounded eigenfunction.

Although the existence of eigenfunctions is not a main focus of this paper, we conclude this section by pointing out that it is ensured under the assumptions of Theorem 1.1, Part (i), and even under those of Theorem 1.3, Part (i).

Theorem 1.5. [Existence of eigenfunctions] *Assume that $n \geq 2$ and $p > 1$. Let Ω be an open subset of an n-dimensional Riemannian manifold $\mathbb{M}$, such that $\mathcal{H}^n(\Omega) < \infty$. Assume that the p-isocapacitary function of Ω fuflills condition (1.21). Then there exists $\gamma > 0$ such that problem (1.4) admits an eigenfunction u. In particular, the same conclusion holds if the isoperimeric function of Ω fulfills condition (1.12).*

Theorem 1.5 follows from an application of the Ljusternik-Schnirelman variational principle as in [Le], thanks to the compactness of the embedding $W^{1,p}(\Omega) \to L^p(\Omega)$, which holds under assumption (1.21) or (1.12). The compactness of this embedding is proved in [CiMa3, Theorem 2.4] when $p = 2$. The proof in the general case is completely analogous and is omitted.

2 Background

Let Ω be an open set in an n-dimensional Riemmanian manifold $\mathbb{M}$ – possibly $\Omega = \mathbb{M}$ – and let E be a measurable subset of $\mathbb{M}$. The perimeter $P(E;\Omega)$ of E relative to Ω can be defined as

$$P(E;\Omega) = \mathcal{H}^{n-1}(\Omega \cap \partial^* E),$$

where $\partial^* E$ stands for the essential boundary of E in the sense of geometric measure theory, and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure on $\mathbb{M}$ induced by its Riemannian metric. Recall that $\partial^* E$ agrees with the topological boundary ∂E of E when the latter is sufficiently regular – a Lipschitz domain with compact closure, for instance.
Assume that $\mathcal{H}^n(\Omega) < \infty$. The isoperimetric function $\lambda_\Omega : \left[0, \frac{\mathcal{H}^n(\Omega)}{2}\right] \to [0,\infty)$ of Ω is defined as

$$\lambda_\Omega(s) = \inf\left\{P(E;\Omega) : E \subset \Omega, s \leq \mathcal{H}^n(E) \leq \tfrac{\mathcal{H}^n(\Omega)}{2}\right\} \qquad \text{for } s \in \left[0, \tfrac{\mathcal{H}^n(\Omega)}{2}\right]. \tag{2.1}$$

Obviously, the function λ_Ω is non-decreasing. The isoperimetric inequality relative to Ω is a straightforward consequence of the definition of λ_Ω and has the form (1.7).
In particular, if Ω is connected, then

$$\lambda_\Omega(s) > 0 \qquad \text{for } s \in \left(0, \tfrac{\mathcal{H}^n(\Omega)}{2}\right], \tag{2.2}$$

as shown via an analogous argument as in [Ma7, Lemma 3.2.4].

The Sobolev space $W^{1,p}(\Omega)$ is defined, for $p \in [1,\infty]$, as

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \ u \text{ is weakly differentiable in } \Omega \text{ and } |\nabla u| \in L^p(\Omega)\},$$

and is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

Here, ∇ stands for the gradient operator, namely covariant differentiation on $\mathbb{M}$. We denote by $W^{1,p}_0(\Omega)$ the closure in $W^{1,p}(\Omega)$ of the set of continuously differentiable compactly supported functions in Ω.
The homogeneous Sobolev space $V^{1,p}(\Omega)$ is defined as

$$V^{1,p}(\Omega) = \{u : u \text{ is weakly differentiable in } \Omega \text{ and } |\nabla u| \in L^p(\Omega)\}.$$

If the set Ω is connected, and ω is an open set such that $\overline{\omega}$ is compact and $\overline{\omega} \subset \Omega$, then $V^{1,p}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{V^{1,p}(\Omega)} = \|u\|_{L^p(\omega)} + \|\nabla u\|_{L^p(\Omega)}. \tag{2.3}$$

Note that, replacing ω by another set with the same properties results in an equivalent norm.

The isocapacitary function $\nu_{\Omega,p}$ of Ω is defined in analogy with (2.1), provided that the perimeter of a set E relative to Ω is replaced by its condenser capacity. Specifically, recall that the standard p-capacity of a set $E \subset \mathbb{M}$ can be defined, for $p \geq 1$, as

$$C_p(E) = \inf\left\{\int_{\mathbb{M}} |\nabla u|^p\, dx : u \in W^{1,p}_0(\mathbb{M}), u \geq 1 \text{ in some neighbourhood of } E\right\}. \tag{2.4}$$

Each function $u \in V^{1,p}(\Omega)$ has a representative $\widetilde{u}$, called the precise representative, enjoying the property that for every $\varepsilon > 0$, there exists a set $E \subset \Omega$, with $C_p(E) < \varepsilon$, such that $\widetilde{u}$ restricted to $\Omega \setminus E$ is continuous. The function $\widetilde{u}$ is unique, up to subsets of p-capacity zero. A pointwise property which holds up to sets of p-capacity zero is said to hold p-quasi everywhere.
In view of a classical result in potential theory (see e.g. [MaZi, Corollary 2.25]), we adopt the following definition of capacity of a condenser. Let $E \subset G \subset \Omega$. Then we set

$$\mathrm{cap}_p(E,G) = \inf\left\{\int_{\Omega} |\nabla u|^p\, d\mathcal{H}^n : u \in V^{1,p}(\Omega), \widetilde{u} \geq 1 \text{ in } E, \widetilde{u} = 0 \text{ in } \Omega \setminus G \;\; p\text{-quasi everywhere}\right\}. \tag{2.5}$$

Also, we define

$$\mathrm{cap}_p(E) = \inf\left\{\mathrm{cap}_p(E,G) : E \subset G, \mathcal{H}^n(G) \leq \tfrac{\mathcal{H}^n(\Omega)}{2}\right\} \tag{2.6}$$

for every measurable set $E \subset \Omega$ such that $\mathcal{H}^n(E) \leq \frac{\mathcal{H}^n(\Omega)}{2}$.
The p-isocapacitary function $\nu_{\Omega,p} : \left[0, \frac{\mathcal{H}^n(\Omega)}{2}\right] \to [0,\infty)$ of Ω is then given by

$$\nu_{\Omega,p}(s) = \inf\left\{\mathrm{cap}_p(E) : E \subset \Omega, s \leq \mathcal{H}^n(E) \leq \tfrac{\mathcal{H}^n(\Omega)}{2}\right\} \qquad \text{for } s \in \left[0, \tfrac{\mathcal{H}^n(\Omega)}{2}\right]. \tag{2.7}$$

The function $\nu_{\Omega,p}$ is clearly non-decreasing. The isocapacitary inequality (1.9)on Ω is a consequence of the very definition (2.7).
If $p > 1$, then the function λ_Ω is related to $\nu_{\Omega,p}$ via the inequality

$$\nu_{\Omega,p}(s) \geq \left(\int_s^{\frac{\mathcal{H}^n(\Omega)}{2}} \frac{dr}{\lambda_\Omega(r)^{p'}}\right)^{1-p} \qquad \text{for } s \in \left(0, \tfrac{\mathcal{H}^n(\Omega)}{2}\right). \tag{2.8}$$

Hence, in particular,

$$\nu_{\Omega,p}(s) > 0 \qquad \text{for } s \in \left(0, \tfrac{\mathcal{H}^n(\Omega)}{2}\right], \tag{2.9}$$

provided that Ω is connected.
When $p = 1$, one has that

$$\nu_{\Omega,1} = \lambda_\Omega \tag{2.10}$$

as shown by an analogous argument as in [Ma7, Lemma 2.2.5].

For any measurable function u on Ω, we define the distribution function $\mu_u : \mathbb{R} \to [0, \infty)$ as

$$\mu_u(t) = \mathcal{H}^n(\{x \in \Omega : u(x) \geq t\}) \qquad \text{for } t \in \mathbb{R}.$$

Note that here μ_u is defined in terms of u, instead of $|u|$ as customary. The signed decreasing rearrangement $u^\circ : [0, \mathcal{H}^n(\Omega)] \to [-\infty, \infty]$ of u is given by

$$u^\circ(s) = \sup\{t \in \mathbb{R} : \mu_u(t) \geq s\} \qquad \text{for } s \in \ [0, \mathcal{H}^n(\Omega)].$$

The median of u is defined by

$$\text{med}(u) = u^\circ\big(\tfrac{\mathcal{H}^n(\Omega)}{2}\big)\,. \tag{2.11}$$

Since u and u° are equimeasurable functions, one has that

$$\|u^\circ\|_{L^q(0,\mathcal{H}^n(\Omega))} = \|u\|_{L^q(\Omega)} \tag{2.12}$$

for every $q \in [1, \infty]$. Moreover, if $u \in W^{1,p}(\Omega)$ for some $p \in [1, \infty]$, then

$$u^\circ \text{ is locally absolutely continuous in } (0, \mathcal{H}^n(\Omega)). \tag{2.13}$$

Given $u \in W^{1,p}(\Omega)$, we define the function $\psi_u : \mathbb{R} \to [0, \infty)$ as

$$\psi_u(t) = \int_0^t \left(\int_{\{u=\tau\}} |\nabla u|^{p-1}\, d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{1-p}} d\tau, \tag{2.14}$$

where the representative of u appearing on right-hand side is the one which renders the coarea formula true. On making use of (a version on manifolds) of [Ma7, Lemma 2.2.2/1], one can deduce that, if

$$\text{med}(u) = 0\,, \tag{2.15}$$

then

$$\nu_{\Omega,p}(s) \leq \psi_u(u^\circ(s))^{1-p} \qquad \text{for } s \in \big(0, \tfrac{\mathcal{H}^n(\Omega)}{2}\big). \tag{2.16}$$

3 Proofs of Theorems 1.1 and 1.3

Our main task in the present section is to establish Theorem (1.3). Theorem 1.1 will then easily follow, thanks to inequality (2.8).

The proof of Part (ii) of Theorem (1.3) relies upon an analysis of an integral equation fulfilled by the signed rearrangement of any eigenfunction of problem (1.4). This integral equation is derived after obtaining an equation involving integrals of the eigenfunction over its level sets. Assumption (1.22) is the piece of information which, through inequality (2.16), enables us to deduce the existence and uniqueness of solutions to the integral equation in suitable spaces.

The proof of of Part (i) of the same theorem makes use of an iteration argument, which, in turn rests on the Sobolev type inequality contained in the following lemma. The inequality in question is standard in regular domains. The objective of the lemma is to show that it is also supported by domains which merely satisfy assumption (1.21).

Lemma 3.1. *Let $\mathbb{M}$ be an n-dimensional Riemannian manifold and let Ω be a connected open set in $\mathbb{M}$ such that $\mathcal{H}^n(\Omega) < \infty$. Assume that condition (1.21) holds for some $p > 1$. Then for every $\varepsilon > 0$ there exists a constant c such that*

$$\|u\|_{L^p(\Omega)} \leq \varepsilon \|\nabla u\|_{L^p(\Omega)} + c\|u\|^p_{L^1(\Omega)}\,, \tag{3.1}$$

for every $u \in W^{1,p}(\Omega)$.

Proof. Fix any $s \in (0, \mathcal{H}^n(\Omega)/2)$, and let E be any compact set in Ω such that $\mathcal{H}^n(\Omega \setminus E) < s$ (such a set E certainly exists since Ω is, in particular, a locally compact, separable topological space with a countable basis). Let ξ be any continuously differentiable compactly supported function on Ω such that $0 \leq \xi \leq 1$ and $\xi = 1$ in E. Denote by U the support of ξ. Consider the precise representative u of any function in $W^{1,p}(\Omega)$. We have that

$$\|u\|_{L^p(\Omega)} \leq \|(1-\xi)u\|_{L^p(\Omega)} + \|\xi u\|_{L^p(\Omega)}\,. \tag{3.2}$$

Let us set

$$v = (1-\xi)u.$$

Clearly, $v \in W^{1,p}(\Omega)$, and the support of v is contained in $\Omega \setminus E$. Thus, $\{x \in \Omega : |v| \geq t\} = \{x \in \Omega \setminus E : |v| \geq t\}$, and $\mathcal{H}^n\{x \in \Omega : |v| \geq t\}) \leq s \leq \mathcal{H}^n(\Omega)/2$ for every $t > 0$. Hence, by inequality (1.9),

$$\int_\Omega |v|^p \, d\mathcal{H}^n = \int_0^\infty \mathcal{H}^n(\{|v| \geq t\}) d(t^p) \leq \left(\sup_{r \leq s} \frac{r}{\nu_{\Omega,p}(r)}\right) \int_0^\infty C_p(\{|v| \geq t\}, \Omega \setminus E)\, d(t^p). \tag{3.3}$$

Owing to the monotonicity of capacity,

$$\int_0^\infty C_p(\{|v| \geq t\}, \Omega \setminus E)\, d(t^p) \leq 3 \sum_{k \in \mathbb{Z}} 2^{pk} C(\{|v| \geq 2^k\}, \Omega \setminus E)\,. \tag{3.4}$$

Let $\Psi : \mathbb{R} \to [0,1]$ be the function given by $\Psi(t) = 0$ if $t \leq 0$, $\Psi(t) = 1$ if $t \geq 1$, and $\Psi(t) = t$ if $t \in (0,1)$. Define $v_k : \Omega \to [0,1]$ as

$$v_k = \Psi(2^{1-k}|v| - 1)$$

for $k \in \mathbb{Z}$. Note that $v_k \in W^{1,p}(\Omega)$ for $k \in \mathbb{Z}$, since Ψ is Lipschitz continuous, and $v_k = 1$ in $\{|v| \geq 2^k\}$ and $v_k = 0$ in $\{|v| \leq 2^{k-1}\}$. In particular, $v_k = 0$ on $E = \Omega \setminus (\Omega \setminus E)$. Hence, by the very definition of capacity of a condenser,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} 2^{pk} C(\{|v| \geq 2^k\}, \Omega \setminus E) &\leq \sum_{k \in \mathbb{Z}} 2^{pk} \int_\Omega |\nabla v_k|^p d\mathcal{H}^n \\ &= 2^p \sum_{k \in \mathbb{Z}} \int_{\{2^{k-1} \leq |v| < 2^k\}} |\nabla v|^p d\mathcal{H}^n = 2^p \int_\Omega |\nabla v|^p d\mathcal{H}^n\,. \end{aligned} \tag{3.5}$$

From inequalities (3.3)–(3.5) one can infer that there exists a constant c such that

$$\int_\Omega v^p \, d\mathcal{H}^n \leq c \sup_{r \leq s} \frac{r}{\nu_{\Omega,p}(r)} \int_\Omega |\nabla v|^p d\mathcal{H}^n\,. \tag{3.6}$$

Consequently,

$$\begin{aligned} \|(1-\xi)u\|_{L^p(\Omega)} &\leq \left(c \sup_{r \leq s} \frac{r}{\nu_{\Omega,p}(r)}\right)^{1/p} \|\nabla((1-\xi)u)\|_{L^p(\Omega)} \\ &\leq \left(c \sup_{r \leq s} \frac{r}{\nu_{\Omega,p}(r)}\right)^{1/p} \Big(\|\nabla u\|_{L^p(\Omega)} + \|\nabla \xi\|_{L^\infty(\Omega)} \|u\|_{L^p(U)}\Big)\,, \end{aligned} \tag{3.7}$$

and, trivially,

$$\|\xi u\|_{L^p(\Omega)} \leq \|u\|_{L^p(U)} . \tag{3.8}$$

Inequalities (3.2), (3.7) and (3.8) imply that

$$\|u\|_{L^p(\Omega)} \leq c\left(\sup_{r\leq s} \frac{r}{\nu_{\Omega,p}(r)}\right)^{1/p} \|\nabla u\|_{L^p(\Omega)} + c\|u\|_{L^p(U)} \tag{3.9}$$

for some constant c.
Now, let Ω' be a Lipschitz domain such that $\overline{\Omega'}$ is compact, $\overline{\Omega'} \subset \Omega$, and $U \subset \Omega'$. Our assumptions on Ω' ensure that a version of the standard Sobolev inequality holds, which tells us that

$$\|u\|_{L^p(\Omega')} \leq \varepsilon\|\nabla u\|_{L^p(\Omega')} + c\|u\|^p_{L^1(\Omega')} , \tag{3.10}$$

for some constant c and for every $u \in W^{1,p}(\Omega')$. This follows, for instance, via an analogous argument as in [Ma7, Proof of Theorem 1.4.6/1]. Inequality (3.1) follows from inequalities (3.9) and (3.10). □

We are now in a position to prove our criteria for bounds of eigenfunctions to problem (1.4).

Proof of Theorem 1.3 *Part (i).* Let u be an eigenfunction of problem (1.4) and let $t, \alpha > 0$. Define the function $T_t : \mathbb{R} \to [0.\infty)$ as $T_t(s) = \min\{|s|, t\}$ for $s \in \mathbb{R}$. Choose the test function $\phi = T_t(u)^\alpha u$ in equation (1.4). Note that this choice is admissible, since $\phi \in W^{1,p}(\Omega)$, by classical results on truncations of Sobolev functions. One obtains that

$$\int_\Omega \left(T_t(u)^\alpha + \alpha|u|^\alpha \chi_{\{|u|<t\}}\right) |\nabla u|^p \, d\mathcal{H}^n = \gamma \int_\Omega T_t(u)^\alpha |u|^p d\mathcal{H}^n . \tag{3.11}$$

Trivially,

$$\int_\Omega \left(T_t(u)^\alpha + \alpha|u|^\alpha \chi_{\{|u|<t\}}\right) |\nabla u|^p \, d\mathcal{H}^n \geq \int_\Omega T_t(u)^\alpha |\nabla u|^p d\mathcal{H}^n. \tag{3.12}$$

On the other hand,

$$\begin{aligned} \int_\Omega \left|\nabla(T_t(u)^{\frac{\alpha}{p}} u)\right|^p d\mathcal{H}^n &\leq 2^{p-1} \int_\Omega \Big(T_t(u)^\alpha + \left(\tfrac{\alpha}{p}\right)^p |u|^\alpha \chi_{\{|u|<t\}}\Big) |\nabla u|^p d\mathcal{H}^n \\ &\leq 2^{p-1} \Big(1 + \left(\tfrac{\alpha}{p}\right)^p\Big) \int_\Omega T_t(u)^\alpha |\nabla u|^p d\mathcal{H}^n. \end{aligned} \tag{3.13}$$

Thus,

$$\int_\Omega \left|\nabla(T_t(u)^{\frac{\alpha}{p}} u)\right|^p d\mathcal{H}^n \leq c_1 \int_\Omega \left(T_t(u)^\alpha + \alpha|u|^\alpha \chi_{\{|u|<t\}}\right) |\nabla u|^p \, d\mathcal{H}^n \tag{3.14}$$

for some constant $c_1 = c_1(\alpha, p)$. From inequality (3.14), via inequality (3.1) applied to the function $T_t(u)^{\frac{\alpha}{p}} u$, one deduces that

$$\int_\Omega T_t(u)^\alpha |u|^p \, d\mathcal{H}^n - c\left(\int_\Omega T_t(u)^{\frac{\alpha}{p}} |u| \, d\mathcal{H}^n\right)^p \leq \varepsilon c_1 \int_\Omega \left(T_t(u)^\alpha + \alpha|u|^\alpha \chi_{\{|u|<t\}}\right) |\nabla u|^p \, d\mathcal{H}^n. \tag{3.15}$$

In order to estimate the right-hand side of equation (3.11), we observe that

$$\gamma \int_\Omega T_t(u)^\alpha |u|^p \, d\mathcal{H}^n = \gamma \left(\int_{\{|u|<t\}} |u|^{\alpha+p} \, d\mathcal{H}^n + t^\alpha \int_{\{|u|\geq t\}} |u|^p \, d\mathcal{H}^n\right) . \tag{3.16}$$

Multiplying through inequality (3.11) by εc_1, and making use of equation (3.15) tell us that

$$\varepsilon c_1\gamma \int_\Omega T_t(u)^\alpha |u|^p d\mathcal{H}^n \geq \int_\Omega T_t(u)^\alpha |u|^p \, d\mathcal{H}^n - c\left(\int_\Omega T_t(u)^{\frac{\alpha}{p}} |u| \, d\mathcal{H}^n\right)^p .$$

Hence, owing to equation (3.16),

$$(1-\varepsilon c_1\gamma)\left(\int_{\{|u|<t\}} |u|^{\alpha+p} \, d\mathcal{H}^n + t^\alpha \int_{\{|u|\geq t\}} |u|^p \, d\mathcal{H}^n\right) \leq c\left(\int_\Omega T_t(u)^{\frac{\alpha}{p}} |u| \, d\mathcal{H}^n\right)^p . \tag{3.17}$$

Choose ε in a such a way that $(1-\varepsilon c_1\gamma) > \frac{1}{2}$. With this choice, inequality (3.17) yields

$$\tfrac{1}{2}\int_{\{|u|<t\}} |u|^{\alpha+p} d\mathcal{H}^n \leq c\left(\int_\Omega |u|^{\frac{\alpha}{p}+1} d\mathcal{H}^n\right)^p \tag{3.18}$$

for some constant $c = c(\Omega, \gamma, \alpha, p)$. Now, apply inequality (3.18) with $\alpha = p^2 - p$. This results in

$$\int_{\{|u|<t\}} |u|^{p^2} d\mathcal{H}^n \leq c_1 \left(\int_\Omega |u|^p d\mathcal{H}^n\right)^p$$

for some constant $c_1 = c_1(\Omega, \gamma, p)$. Letting $t \to \infty$ yields

$$\|u\|_{p^2} \leq c_1^{\frac{1}{p^2}} \|u\|_p .$$

This shows that $u \in L^{p^2}(\Omega)$. Next, choose $\alpha = p^3 - p$ in (3.18). Hence,

$$\int_{\{|u|<t\}} |u|^{p^3} d\mathcal{H}^n \leq c_2 \left(\int_\Omega |u|^{p^2} d\mathcal{H}^n\right)^p$$

for some constant $c_2 = c_2(\Omega, \gamma, p)$. Passing to the limit as $t \to \infty$ implies that

$$\|u\|_{p^3} \leq c_2^{\frac{1}{p^3}} \|u\|_{p^2} .$$

Thus $u \in L^{p^3}(\Omega)$. Iterating this argument, with $\alpha = p^k - p$ for $k \in \mathbb{N}$, shows that $u \in L^{p^k}(\Omega)$ for every $k \in \mathbb{N}$. Hence, $u \in L^q(\Omega)$, and inequality (1.6) holds for all $q > p$.

Part (ii). Assume that u is an eigenfunction of problem (1.4), and choose its representative which supports the coarea formula for Sobolev functions. Given $s \in (0, \mathcal{H}^n(\Omega))$ and $h > 0$, let ϕ be the test function in equation (1.4) given by

$$\phi(x) = \begin{cases} 0 & \text{if } u(x) < u^\circ(s+h) \\ u(x) - u^\circ(s+h) & \text{if } u^\circ(s+h) \leq u(x) \leq u^\circ(s) \\ u^\circ(s) - u^\circ(s+h) & \text{if } u^\circ(s) < u(x) \end{cases} \tag{3.19}$$

for $x \in \Omega$. Notice that $\phi \in W^{1,p}(\Omega)$ by standard results on truncations of Sobolev functions. One obtains that

$$\begin{aligned} \int_{\{u^\circ(s+h)<u<u^\circ(s)\}} |\nabla u|^p \, d\mathcal{H}^n(x) = \gamma \int_{\{u^\circ(s+h)\leq u\leq u^\circ(s)\}} |u(x)|^{p-2} u(x)\big(u(x) - u^\circ(s+h)\big) \, d\mathcal{H}^n(x) \\ + \gamma\big(u^\circ(s) - u^\circ(s+h)\big) \int_{\{u>u^\circ(s)\}} |u(x)|^{p-2} u(x) \, d\mathcal{H}^n(x) . \end{aligned} \tag{3.20}$$

Consider the function $V : (0, \mathcal{H}^n(\Omega)) \to [0, \infty)$ defined as

$$V(s) = \int_{\{u \le u^\circ(s)\}} |\nabla u|^p \, d\mathcal{H}^n(x) \qquad \text{for } s \in (0, \mathcal{H}^n(\Omega)). \tag{3.21}$$

As recalled in equation (2.13), the function u° is locally absolutely continuous in $(0, \mathcal{H}^n(\Omega))$. Moreover, the function

$$(0, \infty) \ni t \mapsto \int_{\{u \le t\}} |\nabla u|^p \, d\mathcal{H}^n(x)$$

is locally absolutely continuous, since, thanks to the coarea formula,

$$\int_{\{u \le t\}} |\nabla u|^p \, d\mathcal{H}^n(x) = \int_{-\infty}^{t} \int_{\{u = \tau\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x) \, d\tau \qquad \text{for } t \in \mathbb{R}. \tag{3.22}$$

Being the composition of monotone locally absolutely continuous functions, the function V is also locally absolutely continuous, and

$$V'(s) = -{u^\circ}'(s) \int_{\{u = u^\circ(s)\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } s \in (0, \mathcal{H}^n(\Omega)). \tag{3.23}$$

Here, and in what follows, the apex " $'$ " denotes differentiation. Therefore, dividing through by h in (3.20), and passing to the limit as $h \to 0^+$ tell us that

$$-{u^\circ}'(s) \int_{\{u = u^\circ(s)\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x) = -\gamma {u^\circ}'(s) \int_{\{u > u^\circ(s)\}} |u|^{p-2} u \, d\mathcal{H}^n(x) \tag{3.24}$$

for a.e. $s \in (0, \mathcal{H}^n(\Omega))$. On the other hand, inasmuch as the functions u and u° are equimeasurable,

$$\int_{\{u > u^\circ(s)\}} |u(x)|^{p-2} u(x) \, d\mathcal{H}^n(x) = \int_0^s |u^\circ(r)|^{p-2} u^\circ(r) \, dr \qquad \text{for a.e. } s \in (0, \mathcal{H}^n(\Omega)). \tag{3.25}$$

From equations (3.24) and (3.25) one infers that

$$-{u^\circ}'(r) = -\gamma \, {u^\circ}'(r) \left(\psi_u'(u^\circ(r))\right)^{p-1} \int_0^r |u^\circ(\varrho)|^{p-2} u^\circ(\varrho) \, d\varrho \quad \text{for a.e. } r \in (0, \mathcal{H}^n(\Omega)), \tag{3.26}$$

where ψ_u is the function defined by (2.14). Hence

$$-{u^\circ}'(r) = -\gamma^{\frac{1}{p-1}} \, {u^\circ}'(r) \psi_u'(u^\circ(r)) \left(\int_0^r |u^\circ(\varrho)|^{p-2} u^\circ(\varrho) \, d\varrho \right)^{\frac{1}{p-1}} \quad \text{for a.e. } r \in (0, \mathcal{H}^n(\Omega)). \tag{3.27}$$

Let $0 < s < \varepsilon < \mathcal{H}^n(\Omega)$. Integrating both sides of equation (3.27) over the interval (s, ε) yields

$$u^\circ(s) = u^\circ(\varepsilon) + \gamma^{\frac{1}{p-1}} \int_s^\varepsilon \big(-\psi_u(u^\circ(r))\big)' \left(\int_0^r |u^\circ(\varrho)|^{p-2} u^\circ(\varrho) d\varrho \right)^{\frac{1}{p-1}} dr \quad \text{for } s \in (0, \varepsilon). \tag{3.28}$$

Set

$$w = u - \operatorname{med}(u), \tag{3.29}$$

and observe that

$$\operatorname{med}(w) = 0 \,, \;\; w^\circ = u^\circ - \operatorname{med}(u) \,, \tag{3.30}$$

and

$$\big(\psi_u(u^\circ(s))\big)' = \big(\psi_w(w^\circ(s))\big)' \qquad \text{for } s \in (0, \mathcal{H}^n(\Omega)). \tag{3.31}$$

On setting, for simplicity, $\varpi(s) = \big(-\psi_w(w^\circ(s))\big)'$, equation reads

$$u^\circ(s) = u^\circ(\varepsilon) + \gamma^{\frac{1}{p-1}} \int_s^\varepsilon \varpi(r) \bigg(\int_0^r |u^\circ(\varrho)|^{p-2} u^\circ(\varrho) d\varrho \bigg)^{\frac{1}{p-1}} dr \quad \text{for } s \in (0, \varepsilon). \tag{3.32}$$

Let us choose $\varepsilon \in (0, \mathcal{H}^n(\Omega)/2]$ so small that $u^\circ(r) > 0$ in $(0, \varepsilon]$. Define the operator

$$T_u f(s) = u^\circ(\varepsilon) + \gamma^{\frac{1}{p-1}} \int_s^\varepsilon \bigg(\int_0^r |f(\varrho)|^{p-1} d\varrho \bigg)^{\frac{1}{p-1}} \varpi(r)\, dr \quad \text{for } s \in (0, \varepsilon) \tag{3.33}$$

for $f \in L^p(0, \varepsilon)$.
Our aim is now to prove that the equation

$$T_u f(s) = f(s) \qquad \text{for } s \in (0, \varepsilon) \tag{3.34}$$

has a solution $f \in L^\infty(0, \varepsilon)$. In order to establish this fact, define the sequence of functions $\{f_k\}$ by iteration as

$$\begin{cases} f_0 = u^\circ(\varepsilon) \\ f_k = T_u f_{k-1} \quad \text{for } k \in \mathbb{N}. \end{cases} \tag{3.35}$$

We preliminarily observe that, by Fubini's theorem and inequality (2.16),

$$\begin{aligned} \gamma^{\frac{1}{p-1}} \int_0^\varepsilon \varpi(r) r^{\frac{1}{p-1}}\, dr &= \gamma^{\frac{1}{p-1}} (p-1) \int_0^\varepsilon \varpi(r) \left(\int_0^r \varrho^{\frac{1}{p-1}-1}\, d\varrho \right) dr \\ &= \gamma^{\frac{1}{p-1}} (p-1) \int_0^\varepsilon \varrho^{\frac{1}{p-1}-1} \left(\int_\varrho^\varepsilon \varpi(r)\, dr \right) d\varrho \\ &= \gamma^{\frac{1}{p-1}} (p-1) \int_0^\varepsilon \varrho^{\frac{1}{p-1}-1} \left(\psi_v(v^\circ(\varrho)) - \psi_v(v^\circ(\varepsilon)) \right) d\varrho \\ &\le \gamma^{\frac{1}{p-1}} (p-1) \int_0^\varepsilon \varrho^{\frac{1}{p-1}-1} \nu_{\Omega,p}(\varrho)^{-\frac{1}{p-1}}\, d\varrho \\ &= \gamma^{\frac{1}{p-1}} (p-1) \int_0^\varepsilon \left(\frac{\varrho}{\nu_{\Omega,p}(\varrho)} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho}. \end{aligned} \tag{3.36}$$

Hence, owing to assumption (1.22), given $\delta \in (0,1)$, there exists ε as above such that, in addition,

$$\gamma^{\frac{1}{p-1}} \int_0^\varepsilon \varpi(r) r^{\frac{1}{p-1}}\, dr < \delta. \tag{3.37}$$

We claim that

$$\|f_k\|_{L^\infty(0,\varepsilon)} \le u^\circ(\varepsilon) \sum_{h=0}^k \delta^h \tag{3.38}$$

for $k \in \mathbb{N}\cup\{0\}$, whence $f_k \in L^\infty(0,\varepsilon)$. Inequality (3.38) can be verified by induction. Clearly $\|f_0\|_{L^\infty(0,\varepsilon)} = u^\circ(\varepsilon)$. Assume now that inequality (3.38) holds for some $k \in \mathbb{N}\cup\{0\}$. From inequality (3.37) one then deduces that

$$\begin{aligned}\|f_{k+1}\|_{L^\infty(0,\varepsilon)} &= f_{k+1}(0) = u^\circ(\varepsilon) + \gamma^{\frac{1}{p-1}}\int_0^\varepsilon \left(\int_0^r f_k(\varrho)^{p-1}d\varrho\right)^{\frac{1}{p-1}}\varpi(r)\,dr \\ &\le u^\circ(\varepsilon)\left(1+\left(\sum_{h=0}^{k}\delta^h\right)\gamma^{\frac{1}{p-1}}\int_0^\varepsilon r^{\frac{1}{p-1}}\varpi(r)\,dr\right) \le u^\circ(\varepsilon)\sum_{h=0}^{k+1}\delta^h,\end{aligned}\tag{3.39}$$

namely inequality (3.38) with k replaced by $k+1$. Hence, our claim follows. In particular, we have that

$$\|f_k\|_{L^\infty(0,\varepsilon)} \le \frac{u^\circ(\varepsilon)}{1-\delta}\,.\tag{3.40}$$

We next distinguish the cases when $p \ge 2$ or $1 < p < 2$.
Assume first that $p \ge 2$. Under this assumption, one has that

$$\|f_k - f_{k-1}\|_{L^\infty(0,\varepsilon)} \le u^\circ(\varepsilon)\delta^k\tag{3.41}$$

for $k \in \mathbb{N}$. Inequality (3.41) can be shown by induction again. Inequality (3.37) guarantees that

$$\|f_1 - f_0\|_{L^\infty(0,\varepsilon)} = u^\circ(\varepsilon)\gamma^{\frac{1}{p-1}}\int_0^\varepsilon r^{\frac{1}{p-1}}\varpi(r)\,dr \le u^\circ(\varepsilon)\delta\,.\tag{3.42}$$

Inequality (3.41) is thus verified for $k = 1$. Suppose that inequality (3.41) holds for some $k \in \mathbb{N}$. Then

$$\begin{aligned}\|f_{k+1} - f_k\|_{L^\infty(0,\varepsilon)} &\le \gamma^{\frac{1}{p-1}}\int_0^\varepsilon \big|\|f_k\|_{L^{p-1}(0,r)} - \|f_{k-1}\|_{L^{p-1}(0,r)}\big|\,\varpi(r)\,dr \\ &\le \gamma^{\frac{1}{p-1}}\int_0^\varepsilon \|f_k - f_{k-1}\|_{L^{p-1}(0,r)}\varpi(r)\,dr \\ &\le \|f_k - f_{k-1}\|_{L^\infty(0,r)}\gamma^{\frac{1}{p-1}}\int_0^\varepsilon r^{\frac{1}{p-1}}\varpi(r)\,dr \le u^\circ(\varepsilon)\delta^{k+1},\end{aligned}\tag{3.43}$$

namely inequality (3.41) with k replaced by $k+1$.
Assume now that $1 < p < 2$. We shall make use of the inequalities

$$\big|r^{\frac{1}{p-1}} - s^{\frac{1}{p-1}}\big| \le c_1|r-s|\big(r^{\frac{2-p}{p-1}} + s^{\frac{2-p}{p-1}}\big) \quad \text{for } r,s>0,\tag{3.44}$$

$$\big|r^{p-1} - s^{p-1}\big| \le c_2\,\frac{|r-s|}{r^{2-p}+s^{2-p}} \quad \text{for } r,s>0\tag{3.45}$$

for some constants $c_1 = c_1(p)$ and $c_2 = c_2(p)$.
In this case, one has that

$$\|f_k - f_{k-1}\|_{L^\infty(0,\varepsilon)} \le \frac{(c_1c_2)^{k-1}u^\circ(\varepsilon)\delta^k}{(1-\delta)^{(2-p)(k-1)}}\tag{3.46}$$

for $k \in \mathbb{N}$. Inequality (3.46) holds with $k = 1$ thanks to (3.42), which is still valid even if $1 < p < 2$. Arguing by induction again, assume now that inequality (3.46) holds for some $k \in \mathbb{N}$. Then

(3.47)
$$\begin{aligned}
\|f_{k+1} - f_k\|_{L^\infty(0,\varepsilon)} &\le \gamma^{\frac{1}{p-1}} \int_0^\varepsilon \left| \left(\int_0^r f_k(\varrho)^{p-1} d\varrho \right)^{\frac{1}{p-1}} - \left(\int_0^r f_{k-1}(\varrho)^{p-1} d\varrho \right)^{\frac{1}{p-1}} \right| \varpi(r)\, dr \\
&\le c_1 \gamma^{\frac{1}{p-1}} \int_0^\varepsilon \left| \int_0^r \left(f_k^{p-1}(\varrho) - f_{k-1}(\varrho)^{p-1} \right) d\varrho \right| \left(\left(\int_0^r f_k(\varrho)^{p-1} d\varrho \right)^{\frac{2-p}{p-1}} + \left(\int_0^r f_{k-1}(\varrho)^{p-1} d\varrho \right)^{\frac{2-p}{p-1}} \right) \varpi(r)\, dr \\
&\le 2c_1 \gamma^{\frac{1}{p-1}} \left(\frac{u^\circ(\varepsilon)}{1-\delta} \right)^{2-p} \int_0^\varepsilon \left| \int_0^r \left(f_k(\varrho)^{p-1} - f_{k-1}(\varrho)^{p-1} \right) d\varrho \right| r^{\frac{2-p}{p-1}} \varpi(r)\, dr \\
&\le 2c_1 \gamma^{\frac{1}{p-1}} \left(\frac{u^\circ(\varepsilon)}{1-\delta} \right)^{2-p} \int_0^\varepsilon \left(\int_0^r \frac{c_2 |f_k(\varrho) - f_{k-1}(\varrho)|}{f_k(\varrho)^{2-p} + f_{k-1}(\varrho)^{2-p}} d\varrho \right) r^{\frac{2-p}{p-1}} \varpi(r)\, dr \\
&\le 2c_1 \gamma^{\frac{1}{p-1}} \left(\frac{u^\circ(\varepsilon)}{1-\delta} \right)^{2-p} \frac{c_2 \|f_k - f_{k-1}\|_{L^\infty(0,\varepsilon)}}{2u^\circ(\varepsilon)^{2-p}} \int_0^\varepsilon r^{\frac{2-p}{p-1}+1} \varpi(r)\, dr \\
&\le \frac{c_1 c_2}{(1-\delta)^{(2-p)}} \frac{(c_1c_2)^{k-1} u^\circ(\varepsilon) \delta^k}{(1-\delta)^{(2-p)(k-1)}} \gamma^{\frac{1}{p-1}} \int_0^\varepsilon r^{\frac{1}{p-1}} \varpi(r)\, dr \le \frac{(c_1c_2)^k u^\circ(\varepsilon) \delta^{k+1}}{(1-\delta)^{(2-p)k}}\,,
\end{aligned}$$

where the second inequality holds by (3.44), the third by (3.40), the fourth by (3.45), the fifth by the fact that $f_k(\varrho) \ge u^\circ(\varepsilon)$, the sixth by (3.46), and the last one by (3.37).
Inequality (3.41) when $p \ge 2$ and inequality (3.46) when $1 < p < 2$ ensure that, if ε is sufficiently small, then the sequence $\{f_k\}$ converges in $L^\infty(0,\varepsilon)$ to some function f, which solves equation (3.34).

Now, we already know that equation (3.34) admits a solution $f = u^\circ \in L^p(0,\varepsilon)$. Our next purpose is to show that such a function is the unique solution in $L^p(0,\varepsilon)$. Assume, by contradiction, that f and g are distinct functions in $L^p(0,\varepsilon)$ satisfying equation (3.34). Let us again distinugish the cases when $p \ge 2$ or $1 < p < 2$.
If $p \ge 2$ then

$$\begin{aligned}
|f(s) - g(s)| &= \gamma^{\frac{1}{p-1}} \left| \int_s^\varepsilon \varpi(r) \left(\|f\|_{L^{p-1}(0,r)} - \|g\|_{L^{p-1}(0,r)} \right) dr \right| \\
&\le \gamma^{\frac{1}{p-1}} \int_s^\varepsilon \varpi(r) \|f-g\|_{L^{p-1}(0,r)}\, dr \le \gamma^{\frac{1}{p-1}} \|f-g\|_{L^p(0,\varepsilon)} \int_s^\varepsilon \varpi(r) r^{\frac{1}{p(p-1)}}\, dr \quad \text{for } s \in (0,\varepsilon).
\end{aligned}$$

Hence, owing to Minkowski's integral inequality and to inequality (3.37),

$$\begin{aligned}
\|f-g\|_{L^p(0,\varepsilon)} &\le \gamma^{\frac{1}{p-1}} \|f-g\|_{L^p(0,\varepsilon)} \left[\int_0^\varepsilon \left(\int_s^\varepsilon r^{\frac{1}{p(p-1)}} \varpi(r)\, dr \right)^p ds \right]^{\frac{1}{p}} \\
&\le \gamma^{\frac{1}{p-1}} \|f-g\|_{L^p(0,\varepsilon)} \int_0^\varepsilon r^{\frac{1}{p(p-1)}} \varpi(r) \left(\int_0^r ds \right)^{\frac{1}{p}} dr \\
&\le \gamma^{\frac{1}{p-1}} \|f-g\|_{L^p(0,\varepsilon)} \int_0^\varepsilon r^{\frac{1}{p-1}} \varpi(r)\, dr < \delta \|f-g\|_{L^p(0,\varepsilon)},
\end{aligned} \tag{3.48}$$

a contradiction, since $\delta \in (0,1)$.
Suppose next that $1 < p < 2$. Without loss of generality, we may assume that $u^\circ(\varepsilon) = 1$. Indeed, if f solves equation (3.34), then the function $\frac{f}{u^\circ(\varepsilon)}$ solves equation (3.34) with $u^\circ(\varepsilon) = 1$. Under this assumption, one has that

$$f(r) \ge 1 \quad \text{and} \quad g(r) \ge 1 \quad \text{for } r \in (0,\varepsilon).$$

We begin by showing that

$$\|f\|_{L^p(0,r)} \leq \left(\frac{1+\delta}{1-\delta}\right) f(r) r^{\frac{1}{p}} \quad \text{for } r \in (0,\varepsilon). \tag{3.49}$$

Actually, given any $r \in (0,\varepsilon)$, one has that

$$\begin{aligned} \|f\|_{L^p(0,r)} \leq & r^{\frac{1}{p}} + \gamma^{\frac{1}{p-1}} \left(\int_0^r \left(\int_s^\varepsilon \left(\int_0^\varrho f(\sigma)^{p-1}\, d\sigma\right)^{\frac{1}{p-1}} \varpi(\varrho)\, d\varrho\right)^p ds\right)^{\frac{1}{p}} \\ \leq & r^{\frac{1}{p}} + \gamma^{\frac{1}{p-1}} \left(\int_0^r \left(\int_s^\varepsilon \|f\|_{L^p(0,\varrho)} \varrho^{\frac{1}{p(p-1)}} \varpi(\varrho)\, d\varrho\right)^p ds\right)^{\frac{1}{p}}. \end{aligned} \tag{3.50}$$

On the other hand, Minkowski's integral inequality tells us that

$$\begin{aligned} & \gamma^{\frac{1}{p-1}} \left(\int_0^r \left(\int_s^\varepsilon \|f\|_{L^p(0,\varrho)} \varrho^{\frac{1}{p(p-1)}} \varpi(\varrho)\, d\varrho\right)^p ds\right)^{\frac{1}{p}} \\ & \leq \gamma^{\frac{1}{p-1}} \int_0^r \|f\|_{L^p(0,\varrho)} \varrho^{\frac{1}{p(p-1)}+\frac{1}{p}} \varpi(\varrho)\, d\varrho \ + \gamma^{\frac{1}{p-1}} \int_r^\varepsilon \|f\|_{L^p(0,\varrho)} \varrho^{\frac{1}{p(p-1)}} r^{\frac{1}{p}} \varpi(\varrho)\, d\varrho \\ & \leq \|f\|_{L^p(0,r)} \gamma^{\frac{1}{p-1}} \int_0^r \varrho^{\frac{1}{p-1}} \varpi(\varrho)\, d\varrho + \gamma^{\frac{1}{p-1}} r^{\frac{1}{p}} \int_r^\varepsilon \|f\|_{L^p(0,\varrho)} \varrho^{\frac{1}{p(p-1)}} . \varpi(\varrho)\, d\varrho \end{aligned} \tag{3.51}$$

Moreover,

$$\|f\|_{L^p(0,\varrho)} \leq \|f\|_{L^p(0,r)} + \|f\|_{L^p(r,\varrho)} \leq \|f\|_{L^p(0,r)} + f(r)\varrho^{\frac{1}{p}}$$

for $\varrho \in (r,\varepsilon)$. Thus,

$$\begin{aligned} & \gamma^{\frac{1}{p-1}} r^{\frac{1}{p}} \int_r^\varepsilon \|f\|_{L^p(0,\varrho)} \varrho^{\frac{1}{p(p-1)}} \varpi(\varrho)\, d\varrho \\ & \qquad \leq \|f\|_{L^p(0,r)} \gamma^{\frac{1}{p-1}} \int_r^\varepsilon \varrho^{\frac{1}{p-1}} \varpi(\varrho)\, d\varrho + f(r) r^{\frac{1}{p}} \gamma^{\frac{1}{p-1}} \int_0^r \varrho^{\frac{1}{p-1}} \varpi(\varrho)\, d\varrho. \end{aligned} \tag{3.52}$$

Coupling inequality (3.51) with (3.52) yields

$$\begin{aligned} \gamma^{\frac{1}{p-1}} \left(\int_0^r \left(\int_s^\varepsilon \|f\|_{L^p(0,\varrho)} \varrho^{\frac{1}{p(p-1)}} \varpi(\varrho)\, d\varrho\right)^p ds\right)^{\frac{1}{p}} \leq & (\|f\|_{L^p(0,r)} + f(r) r^{\frac{1}{p}}) \gamma^{\frac{1}{p-1}} \int_0^\varepsilon \varrho^{\frac{1}{p-1}} \varpi(\varrho)\, d\varrho \\ \leq & \delta(\|f\|_{L^p(0,r)} + f(r) r^{\frac{1}{p}}). \end{aligned} \tag{3.53}$$

From inequalities (3.50) and (3.53) one infers that

$$(1-\delta)\|f\|_{L^p(0,r)} \leq (1+\delta f(r)) r^{\frac{1}{p}} \leq (1+\delta) f(r) r^{\frac{1}{p}},$$

whence inequality (3.49) follows.
Next, observe that

$$\|f-g\|_{L^p(0,\varepsilon)} \leq \gamma^{\frac{1}{p-1}} \left[\int_0^\varepsilon \left(\int_s^\varepsilon \left| \left(\int_0^r f(\varrho)^{p-1} d\varrho\right)^{\frac{1}{p-1}} - \left(\int_0^r g(\varrho)^{p-1} d\varrho\right)^{\frac{1}{p-1}} \right| \varpi(r)\, dr\right)^p ds\right]^{\frac{1}{p}}. \tag{3.54}$$

The following chain holds:

$$
\begin{aligned}
(3.55)\qquad & \left|\left(\int_0^r f(\varrho)^{p-1}d\varrho\right)^{\frac{1}{p-1}} - \left(\int_0^r g(\varrho)^{p-1}d\varrho\right)^{\frac{1}{p-1}}\right| \\
&\leq c_1 \left|\int_0^r \left(f(\varrho)^{p-1} - g(\varrho)^{p-1}\right)\, d\varrho\right| \left[\left(\int_0^r f(\varrho)^{p-1}d\varrho\right)^{\frac{2-p}{p-1}} + \left(\int_0^r g(\varrho)^{p-1}d\varrho\right)^{\frac{2-p}{p-1}}\right] \\
&\leq c_1 \int_0^r |f(\varrho)^{p-1} - g(\varrho)^{p-1}|\, d\varrho \left[\left(\int_0^r f(\varrho)^p\, d\varrho\right)^{\frac{2-p}{p'(p-1)}} + \left(\int_0^r g(\varrho)^p\, d\varrho\right)^{\frac{2-p}{p'(p-1)}}\right] r^{\frac{2-p}{p(p-1)}} \\
&\leq c_1c_2 \left(\|f\|^{2-p}_{L^p(0,r)} + \|g\|^{2-p}_{L^p(0,r)}\right) r^{\frac{2-p}{p(p-1)}} \int_0^r \frac{|f(\varrho) - g(\varrho)|}{f(\varrho)^{2-p} + g(\varrho)^{2-p}}\, d\varrho \\
&\leq c_1c_2 \left(\frac{1+\delta}{1-\delta}\right)^{2-p} (f(r)^{2-p} + g(r)^{2-p}) r^{\frac{2-p}{p} + \frac{2-p}{p(p-1)}} \cdot \frac{\|f-g\|_{L^p(0,r)} r^{\frac{p-1}{p}}}{f(r)^{2-p} + g(r)^{2-p}} \\
&= c_1c_2 \left(\frac{1+\delta}{1-\delta}\right)^{2-p} \|f-g\|_{L^p(0,r)} r^{\frac{1}{p(p-1)}},
\end{aligned}
$$

where the first inequality is due to (3.44), the second to Hölder's inequality, the third to (3.45) and the fourth to (3.49). Combining inequalities (3.54) and (3.55), and an application of Minkowski's integral inequality as in (3.48), enable one to deduce that

$$
\begin{aligned}
(3.56)\qquad \|f-g\|_{L^p(0,\varepsilon)} &\leq c_1c_2\gamma^{\frac{1}{p-1}} \left(\frac{1+\delta}{1-\delta}\right)^{2-p} \|f-g\|_{L^p(0,\varepsilon)} \left[\int_0^\varepsilon \left(\int_s^\varepsilon r^{\frac{1}{p(p-1)}} \varpi(r)\, dr\right)^p ds\right]^{\frac{1}{p}} \\
&\leq c_1c_2\gamma^{\frac{1}{p-1}} \left(\frac{1+\delta}{1-\delta}\right)^{2-p} \delta \|f-g\|_{L^p(0,\varepsilon)}\,.
\end{aligned}
$$

Inequality (3.56) yields a contradiction, provided that δ is chosen small enough.

We have therefore shown that, for sufficiently small ε, the function u° is the unique solution to equation (3.34) in $L^p(0,\varepsilon)$, and that a solution also exists in $L^\infty(0,\varepsilon)$. As a consequence, $u^\circ \in L^\infty(0,\varepsilon)$. The same argument, applied to $-u$, implies that $u^\circ \in L^\infty(\mathcal{H}^n(\Omega)-\varepsilon, \mathcal{H}^n(\Omega))$. Altogether, since the function u° is non-increasing, we conclude that $u^\circ \in L^\infty(0,\mathcal{H}^n(\Omega))$.

It remains to prove inequality (1.5). To this purpose, from equation (3.32) one can deduce that, for every $\varepsilon \in (0,\mathcal{H}^n(\Omega))$,

$$
(3.57)\qquad \|u^\circ\|_{L^\infty(0,\varepsilon)} \leq |u^\circ(\varepsilon)| + \gamma^{\frac{1}{p-1}}\delta\|u^\circ\|_{L^\infty(0,\varepsilon)}\,.
$$

Choose $\varepsilon \in \left(0, \frac{\mathcal{H}^n(\Omega)}{2}\right]$ so small that the number δ, defined by (3.37), fulfills the inequality

$$
1 - \gamma^{\frac{1}{p-1}}\delta \geq \tfrac{1}{2}.
$$

Therefore

$$
(3.58)\qquad \|u^\circ\|_{L^\infty(0,\varepsilon)} \leq \frac{|u^\circ(\varepsilon)|}{1-\gamma^{\frac{1}{p-1}}\delta} \leq 2|u^\circ(\varepsilon)|.
$$

Owing to the monotonicity of u°, if $u^\circ(\varepsilon) > 0$, then

$$
|u^\circ(\varepsilon)| \leq \varepsilon^{-\frac{1}{p}}\|u^\circ\|_{L^p(0,\varepsilon)},
$$

whereas, if $u^\circ(\varepsilon) < 0$, then
$$|u^\circ(\varepsilon)| \leq (\mathcal{H}^n(\Omega) - \varepsilon)^{-\frac{1}{p}} \|u^\circ\|_{L^p(\varepsilon,\mathcal{H}^n(\Omega))}.$$
Since $\varepsilon \in \big(0, \frac{\mathcal{H}^n(\Omega)}{2}\big]$, we hence obtain that

$$\|u^\circ\|_{L^\infty(0,\varepsilon)} \leq 2\varepsilon^{-\frac{1}{q}} \|u^\circ\|_{L^q(0,\mathcal{H}^n(\Omega))}. \tag{3.59}$$

The same argument, applied to $-u$, yields the parallel inequality

$$\|u^\circ\|_{L^\infty(\mathcal{H}^n(\Omega)-\varepsilon,\mathcal{H}^n(\Omega))} \leq 2\varepsilon^{-\frac{1}{p}} \|u^\circ\|_{L^p(0,\mathcal{H}^n(\Omega))}. \tag{3.60}$$

Altogether, we conclude that

$$\|u^\circ\|_{L^\infty(0,\mathcal{H}^n(\Omega))} \leq 2\varepsilon^{-\frac{1}{q}} \|u^\circ\|_{L^p(0,\mathcal{H}^n(\Omega))}, \tag{3.61}$$

whence inequality (1.5) follows. □

Proof of Theorem 1.1. *Part (i).* By assumption (1.12), for every $\varepsilon > 0$, there exists $s_\varepsilon \in \big(0, \frac{\mathcal{H}^n(\Omega)}{2}\big)$ such that $\frac{s^{p'}}{\lambda_\Omega(s)^{p'}} < \varepsilon$ if $s \in (0, s_\varepsilon)$. Thereby, thanks to inequality (2.8),

$$\begin{aligned}\frac{s}{\nu_{\Omega,p}(s)} \leq s\left(\int_s^{\frac{\mathcal{H}^n(\Omega)}{2}} \frac{dr}{\lambda_M(r)^{p'}}\right)^{p-1} &\leq \varepsilon^{p-1} s\left(\int_s^{s_\varepsilon} \frac{dr}{\lambda_M(r)^{p'}}\right)^{p-1} + s\left(\int_{s_\varepsilon}^{\frac{\mathcal{H}^n(\Omega)}{2}} \frac{dr}{\lambda_M(r)^{p'}}\right)^{p-1} \\ &\leq \varepsilon^{p-1}(p-1)^{p-1} + s\left(\int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} \frac{dr}{\lambda_M(r)^{p'}}\right)^{p-1}\end{aligned} \tag{3.62}$$

Owing to the arbitrariness of ε, passing to the limit as $s \to 0^+$ in inequality (3.62) yields equation (1.21). The conclusion hence follows via Theorem 1.3, Part (i).

Part (ii). Inequality (2.8) and Fubini's theorem ensure that

$$\begin{aligned}\int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} \left(\frac{s}{\nu_{\Omega,p}(s)}\right)^{\frac{1}{p-1}} \frac{ds}{s} &\leq \int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} s^{\frac{1}{p-1}-1} \int_s^{\mathcal{H}^n(\Omega)/2} \frac{dr}{\lambda_M(r)^{p'}}\, ds \\ &= (p-1)\int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} \left(\frac{s}{\lambda_\Omega(s)}\right)^{p'} \frac{ds}{s}.\end{aligned} \tag{3.63}$$

Thereby, assumption (1.14) implies that equation (1.22) is fulfilled as well. The conclusion hence follows via Theorem 1.3, Part (ii). □

4 Sharpness

The sharpness of the results from Theorems 1.1 and 1.3 will be demonstrated in our proofs of Theorems 1.2 and 1.4 via model "manifolds of revolution", patterned as in Figure 1 of Section 1, and defined as follows.

Let $\varphi : [0,\infty) \to [0,\infty)$ be any function in $C^1([0,\infty))$, such that

$$\varphi(r) > 0 \quad \text{for } r \in (0,\infty), \tag{4.1}$$

$$\varphi(0) = 0\,, \qquad \text{and} \qquad \varphi'(0) = 1\,. \tag{4.2}$$

Given $n \geq 2$, we call n-dimensional manifold of revolution $\mathbb{M}$ built upon φ the space $\mathbb{R}^n$ parametrized, in polar coordinates, as $\{(r,\omega) : r \in [0,\infty), \omega \in \mathbb{S}^{n-1}\}$ and equipped with the Riemannian metric

$$ds^2 = dr^2 + \varphi(r)^2 d\omega^2 \,. \tag{4.3}$$

Here, $d\omega^2$ denotes the standard metric on $\mathbb{S}^{n-1}$. Our assumptions on φ ensure that the metric (4.3) is of class $C^1(\mathbb{M})$. Observe that

$$\int_{\mathbb{M}} u \, d\mathcal{H}^n = \int_{\mathbb{S}^{n-1}} \int_0^\infty u \, \varphi(r)^{n-1} \, dr \, d\mathcal{H}^{n-1} \,, \tag{4.4}$$

for any integrable function $u : \mathbb{M} \to \mathbb{R}$. In particular, $\mathcal{H}^n(\mathbb{M}) < \infty$ if and only if

$$\int_0^\infty \varphi(r)^{n-1} \, dr < \infty. \tag{4.5}$$

We shall make use of functions $u : \mathbb{M} \to \mathbb{R}$ depending only on r, which, with some abuse of notation, will simply be denoted by $u = u(r)$. For functions of this kind, one has that

$$|\nabla u| = |u'(r)| \quad \text{for } r \in [0,\infty).$$

Moreover, the p-Laplace operator takes the form

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \varphi(r)^{1-n}\big(\varphi(r)^{n-1}|u'(r)|^{p-2}u'(r)\big)' \,. \tag{4.6}$$

Thus, equation (1.2) on $\mathbb{M}$ reduces to the ordinary differential equation

$$\big(\varphi^{n-1} \left|u'\right|^{p-2} u'\big)' + \gamma \varphi^{n-1}|u|^{p-2}u = 0 \qquad \text{in } (0,\infty). \tag{4.7}$$

The membership of u in $W^{1,p}(\mathbb{M})$ reads

$$\int_0^\infty \big(|u(r)|^p + |u'(r)|^p\big)\varphi(r)^{n-1} dr < \infty \,. \tag{4.8}$$

Thus, u is an eigenfunction of problem (1.4) if it satifies condition (4.8) and

$$\int_0^\infty \big(\left|u'\right|^{p-2} u'\phi' - \gamma|u|^{p-2}u\phi\big)\varphi^{n-1} \, dr = 0$$

for every locally absolutely continuous function $\phi : (0,\infty) \to \mathbb{R}$ such that $\int_0^\infty \big(|\phi(r)|^p + |\phi'(r)|^p\big)\varphi(r)^{n-1} dr < \infty$.

It will be convenient to perform a change of variables, in order to get rid of the coefficient φ^{n-1} in the differential operator in (4.7). To this puprose, define the function $\psi : (0,\infty) \to \mathbb{R}$ as

$$\psi(r) = \int_{r_0}^r \frac{d\rho}{\varphi(\rho)^{\frac{n-1}{p-1}}} \qquad \text{for } r \in (0,\infty), \tag{4.9}$$

where r_0 is any number in $(0,\infty)$ if $p \leq n$, and $r_0 = 0$ if $p > n$. Note that

$$\psi(0) = -\infty \;\text{ if }\; 1 < p \leq n, \quad \text{ and } \quad \psi(0) = 0 \;\text{ if }\; p > n,$$

where we have set $\psi(0) = \lim_{r \to 0^+} \psi(r)$ when $1 < p \leq n$. Under the change of variables

$$s = \psi(r),$$

$$v(s) = u(\psi^{-1}(s)),$$

and

$$\eta(s) = \varphi\big(\psi^{-1}(s)\big)^{\frac{p(n-1)}{p-1}}, \tag{4.10}$$

equation (4.7) turns into

$$(|v'|^{p-2}v')' + \gamma\,\eta|v|^{p-2}v = 0 \qquad \text{in } (\psi(0), \psi(\infty)), \tag{4.11}$$

where we have set $\psi(\infty) = \lim_{s\to\infty} \psi(s)$. Moreover, condition (4.8) reads

$$\int_{\psi(0)}^{\psi(\infty)} \big(|v(s)|^p\, \eta(s) + |v'(s)|^p\big)\, ds < \infty. \tag{4.12}$$

A locally absolutely continuous function $v : (\psi(0), \psi(\infty)) \to \mathbb{R}$ is a solution to problem (4.11) if it satisfies condition (4.12) and

$$\int_{\psi(0)}^{\psi(\infty)} \big(|v'|^{p-2}v'\phi' - \gamma\,|v|^{p-2}v\phi\,\eta\big)\, ds = 0 \tag{4.13}$$

for every locally absolutely continuous function $\phi : (\psi(0), \psi(\infty)) \to \mathbb{R}$ such that $\int_{\psi(0)}^{\psi(\infty)}(|\phi(s)|^p\, \eta(s) + |\phi'(s)|^p)\, ds < \infty$.

We introduce now a few notations to be employed in what follows. Let I be an interval of the form $I = (a, \infty)$, where either $a \in \mathbb{R}$ or $a = -\infty$, and let $\eta : I \to [0, \infty)$ be a function such that $\eta \in L^1(I)$. We define, for $p \in [1, \infty]$, the weighted Lebesgue space

$$L^p(I, \eta) = \Big\{ v \text{ is measurable in } I : \int_I |v(s)|^p\, \eta(s)\, ds < \infty \Big\},$$

endowed with the norm

$$\|v\|_{L^p(I,\eta)(I)} = \Big(\int_I |v(s)|^p\, \eta(s) ds \Big)^{\frac{1}{p}}.$$

Moreover, we define the Sobolev space

$$W^{1,p}(I, \eta) = \Big\{ v \text{ is locally absolutely continuous in } I : \int_I \big(|v(s)|^p\, \eta(s) + |v'(s)|^p\big)\, ds < \infty \Big\},$$

equipped with the norm

$$\|v\|_{W^{1,p}(I,\eta)(I)} = \Big(\int_I |v(s)|^p\, \eta(s) ds \Big)^{\frac{1}{p}} + \Big(\int_I |v'(s)|^p ds \Big)^{\frac{1}{p}}.$$

Conditions on the weight function η for the embedding

$$W^{1,p}(I) \to L^p(I, \eta) \tag{4.14}$$

to be compact are given in the following proposition.

Proposition 4.1. *Let $\eta : I \to [0,\infty)$ be such that $\eta \in L^1(I)$. Assume that η is essentially bounded in every bounded subset of I, and that*

$$\lim_{s\to\infty} s\left(\int_s^\infty \eta(t)dt\right)^{\frac{1}{p-1}} = 0\,. \tag{4.15}$$

If $a=-\infty$, assume in addition that

$$\lim_{s\to-\infty} s\left(\int_{-\infty}^s \eta(t)dt\right)^{\frac{1}{p-1}} = 0\,. \tag{4.16}$$

Then embedding (4.14) *is compact.*

Proof Assume that $a=-\infty$, namely that $I=\mathbb{R}$, the proof when $a\in\mathbb{R}$ being analogous.
Fix $\varepsilon>0$. By assumptions (4.15) and (4.16), there exists $\ell>0$ such that

$$\sup_{\ell<t<\infty}\left(\int_t^\infty \eta(\rho)\,d\rho\right)^{\frac{1}{p}}\left(\int_\ell^t ds\right)^{\frac{1}{p'}} < \varepsilon \tag{4.17}$$

and

$$\sup_{-\infty<t<-\ell}\left(\int_{-\infty}^t \eta(\rho)\,d\rho\right)^{\frac{1}{p}}\left(\int_t^{-\ell} ds\right)^{\frac{1}{p'}} < \varepsilon. \tag{4.18}$$

Pick a compactly supported continuously differentiable function $\xi:\mathbb{R}\to[0,1]$ such that $\xi_{|(-\ell,\ell)}=1$, $\xi_{|(-\infty,-\ell-1]\cup[\ell+1,\infty)}=0$. Given $u\in W^{1,p}(\mathbb{R},\eta)$, define the function $v:\mathbb{R}\to\mathbb{R}$ as

$$v(s)=(1-\xi(s))u(s)\quad \text{for } s\in\mathbb{R}. \tag{4.19}$$

On setting $A_\ell=(-\ell-1,\ell+1)$ and $B_\ell=\mathbb{R}\setminus(-\ell,\ell)$, we have that

$$\|u\|_{L^p(\mathbb{R},\eta)} \le \|v\chi_{\mathbb{R}\setminus A_\ell,}\|_{L^p(\mathbb{R},\eta)}+\|u\chi_{A_\ell}\|_{L^p(\mathbb{R},\eta)} \le \|v\chi_{B_\ell}\|_{L^p(\mathbb{R},\eta)}+\|u\chi_{A_\ell}\|_{L^p(\mathbb{R},\eta)}. \tag{4.20}$$

Since $v(-\ell)=v(\ell)=0$, one has that $v(s)=\int_\ell^t v'(t)dt$ for $s>\ell$ and $v(s)=\int_{-\ell}^s v'(t)dt$ for $s<-\ell$. Thus, as a consequence of standard weighted Hardy type inequalities – see e.g. [Ma2, Theorems 1.3.2/2 and 1.3.2/3] – inequalities (4.17) and (4.18) ensure that there exists a constant $c=c(p)$ such that

$$\|v\chi_{(\ell,\infty)}\|_{L^p(\mathbb{R},\eta)} = \left(\int_\ell^\infty\left|\int_\ell^s v'(t)dt\right|^p \eta(s)ds\right)^{\frac{1}{p}} \le c\varepsilon\|v'\|_{L^p(\ell,\infty)}$$

and

$$\|v\chi_{(-\infty,-\ell)}\|_{L^p(\mathbb{R},\eta)} = \left(\int_{-\infty}^{-\ell}\left|\int_s^{-\ell} v'(t)dt\right|^p \eta(s)ds\right)^{\frac{1}{p}} \le c\varepsilon\|v'\|_{L^p(-\infty,\ell)}\,.$$

Hence,

$$\|v\chi_{B_\ell}\|_{L^p(\mathbb{R},\eta)} \le 2c\varepsilon\|v'\|_{L^p(B_\ell)}. \tag{4.21}$$

Now, consider any bounded sequence $\{u_k\}$ in $W^{1,p}(\mathbb{R},\eta)$. Thereby, $\|u_k'\|_{L^p(\mathbb{R})}\le C$ and $\|u_k\|_{L^p(\mathbb{R},\eta)}\le C$ for some constant $C>0$ and every $k\in\mathbb{N}$. By inequality (4.21), applied with u replaced by u_k-u_m in the definition of v, one has that

$$\begin{aligned}\|(u_k-u_m)\chi_{\mathbb{R}\setminus A_\ell}\|_{L^p(\mathbb{R},\eta)} &\le \|(u_k-u_m)(1-\xi)\chi_{B_\ell}\|_{L^p(\mathbb{R},\eta)} \le 2c\varepsilon\|((u_k-u_m)(1-\xi))'\|_{L^p(B_\ell)}\\ &\le 2c\varepsilon\big(\|u_k'-u_m'\|_{L^p(\mathbb{R})}+c'\|(u_k-u_m)\chi_{(-\ell-1,-\ell)\cup(\ell,\ell+1)}\|_{L^p(\mathbb{R})}\big) \le c''\varepsilon.\end{aligned} \tag{4.22}$$

for some constants c, c', c''.
On the other hand, owing to the compactness of the embedding $W^{1,p}(A_\ell) \to L^p(A_\ell)$, the sequence $\{u_k\}$, restricted to A_ℓ, admits a Cauchy subsequence, still denoted by $\{u_k\}$, in $L^p(A_\ell)$. Our assumptions on the function η entail that $\operatorname{ess\,sup}_{A_\ell} \eta < \infty$, a property which guarantees that $\{u_k \chi_{A_\ell}\}$ is also a Cauchy sequence in $L^p(\mathbb{R}, \eta)$. This piece of information, combined with inequality (4.22), tells us that $\{u_k\}$ is a Cauchy sequence in the Banach space $L^p(\mathbb{R}, \eta)$, and hence converges to some function $u \in L^p(\mathbb{R}, \eta)$. □

The following theorem extends the results of [CiMa4, Theorem 4.1 and Corollary 4.2] to the case when $p \neq 2$, with analogous proof. The details are omitted for brevity.

Theorem 4.2. *Let $\varphi : [0, \infty) \to [0, \infty)$ be a function in $C^1([0, \infty))$ fulfilling* (4.1), (4.2) *and such that:*

$$\lim_{r \to \infty} \varphi(r) = 0; \tag{4.23}$$

(4.24) *there exists $L_0 > 0$ such that φ is decreasing and convex in (L_0, ∞);*

$$\int_0^\infty \varphi(\rho)^{n-1}\, d\rho < \infty. \tag{4.25}$$

Set $\omega_{n-1} = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$. Let $\Phi : (0, \infty) \to [0, \infty)$ be the function defined as

$$\Phi(r) = \omega_{n-1} \int_r^\infty \varphi(\rho)^{n-1} d\rho \quad \text{for } r > 0, \tag{4.26}$$

and let $\lambda : \big(0, \omega_{n-1} \int_0^\infty \varphi(\rho)^{n-1} d\rho\big) \to [0, \infty)$ be the function defined as

$$\lambda(s) = \omega_{n-1} \varphi(\Phi^{-1}(s))^{n-1} \quad \text{for } s \in \Big(0, \omega_{n-1} \int_{L_0}^{\infty} \varphi(\rho)^{n-1} d\rho\Big), \tag{4.27}$$

and such that $\lambda(s) = \lambda\big(\omega_{n-1} \int_{L_0}^\infty \varphi(\rho)^{n-1} d\rho\big)$ for $s \in \big(\omega_{n-1} \int_{L_0}^\infty \varphi(r)^{n-1} dr, \omega_{n-1} \int_0^\infty \varphi(r)^{n-1} dr\big)$.
Part 1. *The metric of the n-dimensional manifold of revolution $\mathbb{M}$ built upon φ is of class $C^1(\mathbb{M})$, and $\mathcal{H}^n(\mathbb{M}) < \infty$. Moreover,*

$$\lambda_{\mathbb{M}}(s) \approx \lambda(s) \qquad \text{near } 0\,, \tag{4.28}$$

and

$$\nu_{\mathbb{M},p}(s) \approx \left(\int_s^{\mathcal{H}^n(\Omega)/2} \frac{dr}{\lambda(r)^{p'}} \right)^{1-p} \qquad \text{near } 0. \tag{4.29}$$

Part 2. *The following conditions are equivalent:*

$$\lim_{s \to 0} \frac{s}{\nu_{\mathbb{M},p}(s)} = 0\,,$$

$$\lim_{s \to 0} \frac{s}{\lambda_{\mathbb{M}}(s)} = 0,$$

$$\lim_{r\to\infty}\left(\int_{r_0}^{r}\frac{d\varrho}{\varphi(\varrho)^{\frac{n-1}{p-1}}}\right)\left(\int_{r}^{\infty}\varphi(\varrho)^{n-1}d\varrho\right)^{\frac{1}{p-1}}=0 \quad \textit{for any } r_0\in(0,\infty).$$

Part 3. *The following conditions are equivalent:*

$$\int_0\left(\frac{s}{\nu_{\mathbb{M},p}(s)}\right)^{\frac{1}{p-1}}\frac{ds}{s}<\infty\,,$$

$$\int_0\left(\frac{s}{\lambda_{\mathbb{M}}(s)}\right)^{p'}\frac{ds}{s}<\infty\,,$$

$$\int^{\infty}\left(\frac{1}{\varphi(r)^{n-1}}\int_r^{\infty}\varphi(\rho)^{n-1}d\rho\right)^{\frac{1}{p-1}}dr<\infty.$$

The construction of the manifolds of revolution provided by the following propostition relies on Theorem 4.2.

Proposition 4.3. *Let $n\geq 2$, and let $\nu:[0,\infty)\to[0,\infty)$ be a function as in Theorem 1.4. Then there exists a function $\varphi:[0,\infty)\to[0,\infty)$ as in the statement of Theorem 4.2, such that the n-dimensional manifold of revolution $\mathbb{M}$ built upon φ enjoys property* (4.28) *and*

$$\nu(s)\approx\nu_{\mathbb{M},p}(s)\approx\left(\int_s^{\mathcal{H}^n(\Omega)/2}\frac{dr}{\lambda_M(r)^{p'}}\right)^{1-p} \qquad \textit{near } 0\,. \tag{4.30}$$

Proof To begin with, recall that the Δ_2-condtion near zero fulfilled by the function ν ensures that there exists a constant $c>0$ such that

$$\nu(2s)\leq c\,\nu(s) \qquad \text{near } 0. \tag{4.31}$$

Let $1<p<n$. Assumption (1.23) ensures that there exists a function $\vartheta:[0,\infty)\to[0,\infty)$, which is non-decreasing near 0, and positive constants c_1 and c_2 such that

$$c_1\vartheta(c_1s)\leq\frac{\nu(s)}{s^{\frac{n-p}{n}}}\leq c_2\vartheta(c_2s) \quad \text{near} \;\; 0. \tag{4.32}$$

Hence,

$$\nu(s)\approx s^{\frac{n-p}{n}}\vartheta(s) \quad \text{near } 0. \tag{4.33}$$

Define the function $\nu_1:[0,\infty)\to[0,\infty)$ as

$$\nu_1(s)=\left(\int_0^s\vartheta(r)^{\frac{n}{n-p}}dr\right)^{\frac{n-p}{n}} \qquad \text{for } s\geq 0. \tag{4.34}$$

Owing to the monotonicity of the function ϑ, we have that

$$\nu_1(s)\approx\nu(s) \quad \text{near } 0. \tag{4.35}$$

Moreover,

$$\nu_1'(s)=\frac{n-p}{n}\nu_1(s)^{-\frac{p}{n-p}}\vartheta(s)^{\frac{n}{n-p}}\;. \tag{4.36}$$

Hence, via equations (4.35) and (4.33) and the Δ_2-condition near 0 for ν we deduce that there exist constants c_3 and c_4 such that

$$c_3\frac{\nu_1(s)}{s} \le \nu_1'(s) \le c_4\frac{\nu_1(s)}{s} \quad \text{near } 0. \tag{4.37}$$

Define now the function $\lambda : (0,\infty) \to (0,\infty)$ as

$$\lambda(s) = \frac{\nu_1(s)}{\nu_1'(s)^{\frac{1}{p'}}} \quad \text{for } s > 0. \tag{4.38}$$

From equations (4.37) and (4.35) we obtain that

$$\frac{\lambda(s)}{s^{\frac{1}{n'}}} = O\bigg(\Big(\frac{\nu_1(s)}{s^{\frac{n-p}{n}}}\Big)^{\frac{1}{p}}\bigg) \quad \text{near } 0. \tag{4.39}$$

When $p \ge n$ we instead define the function λ as

$$\lambda(s) = \frac{\nu(s)}{\nu'(s)^{\frac{1}{p'}}} \quad \text{for } s > 0, \tag{4.40}$$

and infer that

$$\frac{\lambda(s)}{s^{\frac{1}{n'}}} = \left(\frac{\nu(s)}{\nu'(s)s}\right)^{\frac{1}{p'}} \nu(s)^{\frac{1}{p}} s^{\frac{1}{p'}-\frac{1}{n'}} \quad \text{near } 0\,. \tag{4.41}$$

From either equations (4.39) and (1.23), or equations (4.41) and (1.24), one deduces that, for every $p > 1$,

$$\frac{\lambda(s)}{s^{\frac{n-1}{n}}} \approx \text{a non-decreasing function near } 0. \tag{4.42}$$

Furthermore, since

$$\frac{\nu_1'(s)}{\nu_1(s)^{p'}} = \frac{1}{\lambda(s)^{p'}} \quad \text{near 0 if } 1 < p < n, \quad \text{and} \quad \frac{\nu'(s)}{\nu(s)^{p'}} = \frac{1}{\lambda(s)^{p'}} \quad \text{near 0 if } p \ge n,$$

property (4.35) ensures that

$$\nu(s) = O\bigg(\Big(\int_s^{s_0} \frac{dr}{\lambda(r)^{p'}}\Big)^{1-p}\bigg) \quad \text{near } 0, \tag{4.43}$$

for any given $s_0 \in (0,\infty)$.
Let us next notice that

$$\int_0 \frac{dr}{\lambda(r)} = \infty. \tag{4.44}$$

Indeed, if $p < n$, then by (4.38), (4.37) and (4.35), condition (4.44) is equivalent to

$$\int_0 \left(\frac{s}{\nu(s)}\right)^{\frac{1}{p}} \frac{ds}{s} = \infty, \tag{4.45}$$

and the latter holds, owing to assumptions (1.26) and (1.27).
If $p \geq n$, then

$$\frac{\nu(s)}{\nu'(s)s} = O(\varsigma(s)) \quad \text{near } 0$$

for some non-decreasing function $\varsigma : (0,\infty) \to (0,\infty)$. Therefore, owing to equation (4.40), condition (4.44) is equivalent to

$$\int_0 \left(\frac{s}{\nu(s)}\right)^{\frac{1}{p}} \frac{ds}{s\varsigma(s)^{\frac{1}{p'}}} = \infty, \tag{4.46}$$

which holds thanks to assumptions (1.26) and (1.27), and to monotonicity of the function ς. Consequently, equation (4.44) holds for every $p > 1$.
Owing to equations (4.42)–(4.44), the conclusion follows from [CiMa4, Proposition 4.3] and Theorem 4.2. □

Proof of Theorem 1.4 *Part (i).* Given $q > p$ and $n \geq 2$ we shall produce an $n-$dimensional manifold of revolution $\mathbb{M}$, as defined at the beginning of this section, fulfilling property (1.25) and such that problem (1.4), with $\Omega = \mathbb{M}$, has an eignefunction $u \notin L^q(\mathbb{M})$. The eigenfunction to be detected will depend only on the coordinate r. It thus sufficies to exhibit a solution v to equation (4.11) for some function η having the form (4.10), with φ as in the definition of the manifold $\mathbb{M}$.

Consider first the case when $1 < p \leq n$. We are going to construct a function $\eta : \mathbb{R} \to (0,\infty)$ such that $\eta \in C^1(\mathbb{R})$, $\lim_{r\to-\infty} \eta(r) = 0$, $\lim_{r\to\infty} \eta(r) = 0$,

$$\int_{-\infty} \eta(\varrho)^{\frac{1}{p}} d\varrho < \infty, \text{ and } \int_{-\infty}^{\infty} \eta(\varrho)^{\frac{1}{p}} d\varrho = \infty\,. \tag{4.47}$$

The function η is defined as follows. Let $s_1 < -1 < 1 < s_2$ to be fixed later, and set

$$\eta(s) = s^{-p} \quad \text{for } s \geq s_2. \tag{4.48}$$

Let $0 < \gamma < \left(\frac{p-1}{p}\right)^p$. One can verify that there exists $\alpha = \alpha(\gamma, p) \in (0, \frac{p-1}{p})$ such that the function

$$v(s) = s^\alpha$$

solves equation (4.11) in $[s_2, \infty)$. Also, $\alpha \to \frac{p-1}{p}$ as $\gamma \to (\frac{p-1}{p})^p$. For $s \in (-\infty, s_1]$, we define

$$\eta(s) = \begin{cases} \dfrac{(-s)^{\frac{p(n-1)}{p-n}}}{\left[\left(\frac{n-p}{p-1}\right)^{\frac{p(n-1)}{(p-1)(n-p)}} - \gamma^{\frac{1}{p-1}} \frac{(n-p)^{\frac{p}{p-1}}}{p} \left(\frac{1}{n(p-1)}\right)^{\frac{1}{p-1}} (-s)^{\frac{p}{p-n}}\right]^{p-1}} & \text{if } 1 < p < n \\[2ex] \dfrac{n^n e^{ns}}{\gamma(n-1)^{n-1}(1-e^{\frac{n}{n-1}s})^{n-1}} & \text{if } p = n. \end{cases} \tag{4.49}$$

Thus the function v, defined as

$$v(s) = \begin{cases} \left(\frac{n-p}{p-1}\right)^{\frac{p(n-1)}{(p-1)(n-p)}} - \gamma^{\frac{1}{p-1}} \frac{(n-p)^{\frac{p}{p-1}}}{p} \left(\frac{1}{n(p-1)}\right)^{\frac{1}{p-1}} (-s)^{\frac{p}{p-n}} & \text{if } 1 < p < n \\ 1 - e^{\frac{n}{n-1}s} & \text{if } p = n, \end{cases} \tag{4.50}$$

solves equation (4.11) in $(-\infty, s_1]$. Next, given $\beta > 0$ and disjoint neighborhoods I_{-1} and I_1 of -1 and 1, respectively, let η be defined in $I_1 \cup I_1$ as

$$\eta(s) = \begin{cases} \frac{(p+1)p(p-1)}{\gamma} \frac{(\beta-(p+1)|s-1|^p)^{p-2}}{(\beta-|s-1|^p)^{p-1}} & \text{for } s \in I_1 \\[2ex] \frac{(p+1)p(p-1)}{\gamma} \frac{(\beta-(p+1)|s+1|^p)^{p-2}}{(\beta-|s+1|^p)^{p-1}} & \text{for } s \in I_{-1}. \end{cases} \tag{4.51}$$

Hence the function v, given by

$$(4.52)\qquad v(s)=\begin{cases}(s-1)\left(\beta-|s-1|^{p}\right) & \text{for } s\in I_1\\ -(s+1)\left(\beta-|s+1|^{p}\right) & \text{for } s\in I_{-1},\end{cases}$$

is a solution to (4.11) in $I_{-1}\cup I_1$. Moreover, v is convex in a left neighborhood of 1 and in a right neighborhood of -1, whereas it is concave in a right neighborhood of 1 and in a left neighborhood of -1. Finally, in a neighborhood I_0 of 0, define

$$(4.53)\qquad \eta(s)=\frac{(p')^{p-1}}{\gamma}(k-|s|^{p'})^{1-p}\quad \text{for } s\in I_0,$$

for $k>0$. Then the function v, given by

$$(4.54)\qquad v(s)=\tfrac{1}{p'}(|s|^{p'}-k)\quad \text{for } s\in I_0,$$

is a convex solution to (4.11) in I_0.
One can verify that, if β is sufficiently large, s_2 and $-s_1$ are sufficiently large depending on β, and I_1, I_{-1} and I_0 are sufficiently small, then v can be continued to the whole of $\mathbb{R}$ in such a way that:

$$v\in W^{1,p}(\mathbb{R},\eta);$$

$$(|v'|^{p-2}v')'=(p-1)|v'|^{p-2}v''\le -C \text{ and } v\ge C \text{ in } \mathbb{R}\setminus(I_{-1}\cup(-1,1)\cup I_1), \text{ for some positive constant C;}$$

$$(|v'|^{p-2}v')'=(p-1)|v'|^{p-2}v''\ge C \text{ and } v\le -C \text{ in } (-1,1)\setminus(I_{-1}\cup I_1), \text{ for some positive constant C.}$$

Thereby, the function η can be continued to the whole of $\mathbb{R}$ as a positive function in $C^1(\mathbb{R})$, fulfilling conditions (4.47), in such a way that v is a solution to equation (4.11) in $\mathbb{R}$. Also, the function v satisfies condition (4.12).
One can verify that, if $q>p$ and $\frac{p-1}{q}<\alpha<\frac{p-1}{p}$, then

$$v\notin L^q(\mathbb{R},\eta).$$

Now, define the function $F:\mathbb{R}\to(0,\infty)$ as

$$(4.55)\qquad F(r)=\int_{-\infty}^{r}\eta(\varrho)^{\frac{1}{p}}d\varrho\quad \text{for } r\in\mathbb{R},$$

and the function $\psi:(0,\infty)\to\mathbb{R}$ as

$$(4.56)\qquad \psi(s)=F^{-1}(s)\qquad \text{for } s>0.$$

Thus, $\psi(0)=-\infty$ and $\psi(\infty)=\infty$. Next, define the function $\varphi:[0,\infty)\to[0,\infty)$ by

$$(4.57)\qquad \varphi(r)=\begin{cases}\eta\left(\psi(r)\right)^{\frac{p-1}{p(n-1)}} & \text{if } r>0\\ 0 & \text{if } r=0.\end{cases}$$

One has that $\lim_{r\to0^+}\varphi(r)=0$ and $\lim_{r\to\infty}\varphi(r)=0$. Furthermore,
(4.58)

$$\varphi'(r)=\frac{p-1}{p(n-1)}\eta\left(\psi(r)\right)^{\frac{p-1}{p(n-1)}-1}\eta'(\psi(r))\psi'(r)=\frac{p-1}{p(n-1)}\eta\left(\psi(r)\right)^{\frac{p-1}{p(n-1)}-1-\frac{1}{p}}\eta'(\psi(r))\quad \text{for } r>0.$$

Let us show that the function φ satisfies assumptions (4.1), (4.2) and (4.23) – (4.25). Assumptions (4.1) and (4.23) are satisfied by the very definition of φ. This definition also tells us that $\varphi(0)=0$. From equations (4.55), (4.58) and (4.49) one can deduce that $\varphi'(0) = 1$. Assumption (4.2) is hence fulfilled. Equations (4.55), (4.48) and (4.57) imply that

$$\varphi(r) = O(e^{-\frac{p-1}{n-1}r}) \quad \text{for } r \geq r_2. \tag{4.59}$$

Therefore, (4.24) and (4.25) hold as well, and

$$\lambda_{\mathbb{M}}(s) \approx s \quad \text{near } 0, \tag{4.60}$$

and

$$\nu_{M,p}(s) \approx s \quad \text{near } 0. \tag{4.61}$$

Assume next that $p > n$. Let $0 < s_1 < s_2$ to be chosen later. The functions v and η are defined in the interval $[s_2, \infty)$ in the same fashion as above. For $s \in [0, s_1]$, we set

$$\eta(s) = \frac{s^{\frac{p(n-1)}{p-n}}}{\left[\left(\frac{p-n}{p-1}\right)^{\frac{p(n-1)}{(p-1)(p-n)}} - \gamma^{\frac{1}{p-1}} \frac{(p-n)^{\frac{p}{p-1}}}{p} \left(\frac{1}{n(p-1)}\right)^{\frac{1}{p-1}} s^{\frac{p}{p-n}}\right]^{p-1}}. \tag{4.62}$$

Hence, in the same interval the function v, given by

$$v(s) = \left(\frac{p-n}{p-1}\right)^{\frac{p(n-1)}{(p-1)(p-n)}} - \gamma^{\frac{1}{p-1}} \frac{(p-n)^{\frac{p}{p-1}}}{p} \left(\frac{1}{n(p-1)}\right)^{\frac{1}{p-1}} s^{\frac{p}{p-n}}, \tag{4.63}$$

solves equation (4.11). In the interval (s_1, s_2) on can define the functions v and η in an way analogous to the case when $1 < p \leq n$, just suitably translating the neighbours I_{-1}, I_1 and I_0.
The function $F : [0, \infty) \to [0, \infty)$ is now given by

$$F(r) = \int_0^r \eta(\varrho)^{\frac{1}{p}} d\varrho \quad \text{for } r \geq 0, \tag{4.64}$$

and the function $\psi : [0, \infty) \to [0, \infty)$ by

$$\psi(s) = F^{-1}(s) \qquad \text{for } s \geq 0. \tag{4.65}$$

The function $\varphi : [0, \infty) \to [0, \infty)$ is still defined as in (4.57). The conclusion then follows as in the case when $1 < p \leq n$. The details are omitted for brevity.
Part (ii). Let φ be a function as in the definition of manifolds of revolution introduced at the beginning of the present section. By Proposition 4.3, if ν is as in the statement, then the function φ can be chosen in such a way that the associated n-dimensional manifold of revolution $\mathbb{M}$ fulfils (1.28), and hence

$$\lim_{s\to 0} \frac{s}{\nu_M(s)} = \lim_{s\to 0} \frac{s}{\nu(s)} = 0 \tag{4.66}$$

and

$$\int_0 \left(\frac{s}{\nu_{M,p}(s)}\right)^{\frac{1}{p-1}} \frac{ds}{s} = \int_0 \left(\frac{s}{\nu(s)}\right)^{\frac{1}{p-1}} \frac{ds}{s} = \infty\,. \tag{4.67}$$

Now, recall that the function φ satisfies condition (4.2). Hence,

$$\lim_{r\to 0} \left(\int_r^1 \frac{d\rho}{\varphi(\rho)^{\frac{n-1}{p-1}}}\right)^{p-1} \left(\int_0^r \varphi(\rho)^{n-1} d\rho\right) = 0\,. \tag{4.68}$$

Moreover,

$$\int_1^\infty \frac{dr}{\varphi(r)^{\frac{n-1}{p-1}}} = \infty\,, \tag{4.69}$$

since $\lim_{r\to\infty} \varphi(r) = 0$ by property (i) of Theorem 4.2.
Owing to Theorem 4.2, Part 2, condition (4.66) is equivalent to

$$\lim_{r\to\infty} \Big(\int_1^r \frac{d\varrho}{\varphi(\varrho)^{\frac{n-1}{p-1}}} \Big)^{p-1} \Big(\int_r^\infty \varphi(\varrho)^{n-1} d\varrho \Big) = 0\,, \tag{4.70}$$

and, by Part 3 of the same theorem, condition (4.67) is equivalent to

$$\int^\infty \Big(\frac{1}{\varphi(r)^{n-1}} \int_r^\infty \varphi(\rho)^{n-1} d\rho \Big)^{\frac{1}{p-1}} dr = \infty\,. \tag{4.71}$$

The conclusion will follow if we exhibit a number $\gamma > 0$ and an unbounded solution $v : \mathbb{R} \to \mathbb{R}$ to equation (4.11) fulfilling (4.12).
Let ψ and η be the functions defined in terms of φ, p and n as in (4.9) and (4.10), respectively. Owing to condition (4.69), the function ψ fulfills $\psi(\infty) = \infty$. Moreover, for every $p > 1$ condition (4.70) is equivalent to (4.15). Also, if $1 < p \le n$, then condition (4.68) is equivalent to (4.16). Thus, by Proposition 4.1, the embedding

$$W^{1,p}(I) \to L^p(I, \eta) \tag{4.72}$$

is compact, where I denotes either $\mathbb{R}$ or $[0, \infty)$, according to whether $1 < p \le n$ or $p > n$. The existence of an eigenfunction of problem (4.11) could hence be established via the general Ljusternik-Schnirelman principle, as hinted at the end of Section 1. However, we also give a direct, more elementary proof, exploiting the one-dimensional nature of the problem at hand. Let J be the functional given by

$$J(v) = \frac{\int_I |v'(s)|^p ds}{\int_I |v(s)|^p\, \eta(s) ds} \tag{4.73}$$

for $v \in W^{1,p}(I)$. We claim that J achieves its minimum among all (not identically vanishing) functions $v \in W^{1,p}(I)$ such that

$$\int_I |v(s)|^{p-2} v(s) \eta(s)\, ds = 0. \tag{4.74}$$

Indeed, consider any minimizing sequence $\{v_k\}$. Owing to the homogeneity of J, the functions v_k can be normalized in such a way that $\int_I |v_k(s)|^p\, ds = 1$ for $k \in \mathbb{N}$. Hence, the sequence $\{v_k\}$ is bounded in $W^{1,p}(I)$. By the compactness of the embedding (4.72), there exists a function $v \in W^{1,p}(I)$ and a subsequence of $\{v_k\}$, still denoted by $\{v_k\}$, such that $v_k \to v$ in $L^p(I, \eta)$ and $v_k \rightharpoonup v$ weakly in $W^{1,p}(I)$. Also, the function v satisfies the constraint (4.74). Such a function is thus a miminizer for J under (4.74). It remains to show that the function v fulfills the Euler-Lagrange equation (4.13) for all test functions $\phi \in W^{1,p}(I, \eta)$. To verify this assertion, we make use of an argument reminiscent of that of [DGS, Lemma 2.4]. Observe that the function v also minimizes the functional G defined as

$$G(v) = \int_I |v'(s)|^p\, ds - \gamma \int_I |v(s)|^p\, \eta(s) ds.$$

Consider, for the time being, test functions $\phi \in W^{1,p}(I) \cap L^\infty(I)$. Given any $h \in (0,1)$, there exists $\beta_h \in \mathbb{R}$ such that the function $v + h\phi + \beta_h$ fulfills constraint (4.74). This is due to the fact that, fixing h, the function

$$\beta \mapsto \int_I |v(s) + h\phi(s) + \beta|^p \, \eta(s)\, ds$$

is convex and tends to ∞ as $\beta \to \pm\infty$. Hence, it admits a minimum point β_h, at which

$$\int_I |v(s) + h\phi(s) + \beta_h|^{p-2}(v(s) + h\phi(s) + \beta_h)\, \eta(s)\, ds = 0, \tag{4.75}$$

i.e. condition (4.74) is actually satisfied with v replaced by $v + h\phi + \beta_h$.
Next, there exists $x_h \in \mathbb{R}$ such that $\phi(x_h) + \frac{\beta_h}{h} = 0$. Indeed, if $h\phi(s) + \beta_h$ were positive [resp. negative] for every $s \in I$, then, by equation (4.74) and the monotonicity of the function $|t|^{t-2}t$, the integral in equation (4.75) would be positive [negative].
Since we are assuming that the function ϕ is bounded, there exists a sequence $\{h_k\}$ a number $c \in \mathbb{R}$ such that $\lim_{k\to\infty} \phi(x_{h_k}) = c$, whence, $\lim_{k\to\infty} \frac{\beta_{h_k}}{h_k} = -c$. As a consequence of the minimizing property of the function v, one can thus infer that

$$\begin{aligned}
0 &\le \lim_{k\to\infty} \frac{1}{h_k}\big(G(v + h_k\phi + \beta_{h_k}) - G(v)\big) \\
&= \lim_{k\to\infty} \frac{1}{h_k}\big(G(v + h_k\phi + \beta_{h_k}) - G(v + h_k\phi)\big) + \lim_{k\to\infty} \frac{1}{h_k}\big(G(v + h_k\phi) - G(v)\big) \\
&= -\gamma \lim_{k\to\infty} \frac{1}{h_k}\bigg(\int_I |v + h_k\phi|^p\, \eta ds - \int_I |v + h_k\phi + \beta_{h_k}|^p\, \eta ds\bigg) + \lim_{k\to\infty} \frac{1}{h_k}\big(G(v + h_k\phi) - G(v)\big) \\
&= -c\int_I |v|^{p-2} v\eta ds + \int_I |v'|^{p-2} v'\phi'\, ds - \gamma\int_I |v|^{p-2} v\phi\eta ds \\
&= \int_I |v'|^{p-2} v'\phi'\, ds - \gamma\int_I |v|^{p-2} v\phi\eta ds.
\end{aligned} \tag{4.76}$$

Equation (4.13) hence follows under the assumption $\phi \in W^{1,p}(I) \cap L^\infty(I)$. Next, we claim that

$$W^{1,p}(I) \cap L^\infty(I) \quad \text{is dense in} \quad W^{1,p}(I,\eta). \tag{4.77}$$

To verify this assertion, we first show that the space $W^{1,p}(I) \cap L^\infty(I)$ is dense in $W^{1,p}(I,\eta) \cap L^\infty(I)$. For every $k \in \mathbb{N}$, consider a continuously differentiable function $\xi_k : \mathbb{R} \to [0,1]$ such that $\xi_k = 1$ in $[-k,k]$, $\xi_k = 0$ in $\mathbb{R} \setminus [-2k,2k]$ and $|\xi_k'| \le \frac{c}{k}$ for some constant c. Given any function $v \in W^{1,p}(I,\eta) \cap L^\infty(I)$, define the sequence of functions $\{v_k\}$ in I as $v_k = v\,\xi_k$ for $k \in \mathbb{N}$. One has that $v_k \in W^{1,p}(I)$. Moreover,

$$\begin{aligned}
\|v - v_k\|_{W^{1,p}(I,\eta)} &= \bigg(\int_I |v' - v'\xi_k - v\xi_k'|^p\, ds\bigg)^{\frac{1}{p}} + \bigg(\int_I |v - v\xi_k|^p \eta\, ds\bigg)^{\frac{1}{p}} \\
&\le \bigg(\int_{I\setminus[-k,k]} |v'|^p\, ds\bigg)^{\frac{1}{p}} + \|v\|_{L^\infty(I)}\bigg(\int_{\{s\in I: k\le|s|\le 2k\}} \big(\tfrac{c}{k}\big)^p\, ds\bigg)^{\frac{1}{p}} + \bigg(\int_{I\setminus[-k,k]} |v|^p \eta\, ds\bigg)^{\frac{1}{p}} \\
&\le \bigg(\int_{I\setminus[-k,k]} |v'|^p\, ds\bigg)^{\frac{1}{p}} + \frac{2^{\frac{1}{p}} c}{k^{\frac{1}{p'}}}\|v\|_{L^\infty(I)} + \bigg(\int_{I\setminus[-k,k]} |v|^p \eta\, ds\bigg)^{\frac{1}{p}}.
\end{aligned}$$

Inasmuch as $v \in W^{1,p}(I,\eta)$, the rightmost side of this chain of inequalities tends to 0 as $k \to \infty$. Hence, $v_k \to v$ in $W^{1,p}(I)$. The density of the space $W^{1,p}(I) \cap L^\infty(I)$ in $W^{1,p}(I,\eta) \cap L^\infty(I)$ is thus established. On the other hand, the space $W^{1,p}(I,\eta) \cap L^\infty(I)$ is in turn dense in $W^{1,p}(I,\eta)$, as can be shown on approximating any function in the latter space by its truncations. Altogether, property (4.77) follows. It remains to show that any eigenfunction v of problem (4.11) is unbounded. By Theorem 3 of [TaYo], condition (4.15) entails that equation (4.11) is nonoscillatory at infinity, and hence that every solution has constant sign at infinity. Thus, we may assume that $v(s) > 0$ for large s. Consequently,

$$(p-1)|v'|^{p-2}v'' = (|v'|^{p-2}v')' < 0 \qquad \text{for large } s,$$

and hence v is concave near ∞. Now, assume by contradiction that v is bounded. Then $\lim_{s\to\infty} v(s)$ exists and, on denoting by $v(\infty)$ this limit, one has that $v(\infty) \in (0,\infty)$. Moreover, v is increasing for large s and

$$\lim_{s\to\infty} v'(s) = 0\,.$$

An integration of equation (4.11) and this limit yield

$$(v')^{p-1}(s) = \gamma \int_s^\infty v(t)^{p-1}\eta(t)dt \qquad \text{for large } s.$$

Hence, there exists $s_0 > 0$ such that

$$v'(s) \geq \gamma^{\frac{1}{p-1}} \frac{v(\infty)}{2} \left(\int_s^\infty \eta(t)dt\right)^{\frac{1}{p-1}} \qquad \text{for } s \geq s_0.$$

An integration of this inequality over (s_0,∞) in turn tells us that

$$v(\infty) - v(s_0) \geq \gamma^{\frac{1}{p-1}} \frac{v(\infty)}{2} \int_{s_0}^\infty \left(\int_s^\infty \eta(t)dt\right)^{\frac{1}{p-1}} ds\,.$$

This is impossible, since condition (4.71), rewritten in terms of the function η, reads

$$\int^\infty \left(\int_s^\infty \eta(r)\,dr\right)^{\frac{1}{p-1}} ds = \infty.$$

□

Proof of Theorem 1.2. *Part (i).* Let $\mathbb{M}$ be the manifold constructed in the proof of Theorem 1.4, Part (i). Since the function λ satisfies equation (4.60), owing to property (4.28) the isoperimetric function of $\mathbb{M}$ fulfills assumption (1.17). The conclusion thus holds for this manifold $\mathbb{M}$, thanks to the result of Theorem 1.4, Part (i).
Part (ii). Let $\mathbb{M}$ be the manifold constructed in the proof of Theorem 1.4, Part (ii). This manifold is defined as in Theorem 4.2, with the function λ given by (4.27) and satisfying assumptions (1.18) and (1.19). Since, by equation (4.28), $\lambda_{\mathbb{M}} \approx \lambda$ near 0, the conclusion holds for this manifold $\mathbb{M}$, thanks to the result of Theorem 1.4, Part (ii). □

5 Applications

We conclude with applications of our results to two special instances. Both of them involve families of noncompact manifolds. However, the former is less pathological, and can either be handled by exploiting isoperimetric or by isocapacitary inequalities, with the same output. That the use of the isocapacitary function can actually yield sharper conclusions than those obtained via the isoperimetric function is demonstrated by the latter example, which deals with a class of manifolds with a more complicated geometry.

5.1 A family of manifolds of revolution with borderline decay

Consider a one-parameter family of manifolds of revolution $\mathbb{M}$ as in Section 4, whose profile $\varphi : [0,\infty) \to [0,\infty)$ is such that

$$\varphi(r) = e^{-r^\alpha} \qquad \text{for large } r, \tag{5.1}$$

and fulfills the assumptions of Theorem 4.2. This theorem tells us that

$$\lambda_{\mathbb{M}}(s) \approx s\big(\log(1/s))\big)^{1-1/\alpha} \qquad \text{near } 0, \tag{5.2}$$

and

$$\nu_{\mathbb{M},p}(s) \approx \bigg(\int_s^{\mathcal{H}^n(\mathbb{M})} \frac{dr}{\lambda_{\mathbb{M}}(r)^{p'}}\bigg)^{1-p} \approx s\big(\log(1/s)\big)^{p-p/\alpha} \qquad \text{near } 0. \tag{5.3}$$

An application of Theorem 1.3 , Part (i), ensures, via (5.3), that all eigenfunctions of problem (1.4), with $\Omega = \mathbb{M}$, belong to $L^q(\mathbb{M})$, provided that

$$\alpha > 1. \tag{5.4}$$

On the other hand, from Part (ii) of the same theorem and equation (5.3) one can infer that the relevant eigenfunctions are bounded under the more stringent assumption that

$$\alpha > p. \tag{5.5}$$

The same conclusions can be derived via Theorem 1.1. Thus, as for any other manifold of revolution of the kind considered in Theorem 4.2, isoperimetric and isocapacitary methods lead to equivalent results for this family of noncompact manifolds.
In both cases, the existence of eigenfunctions is guaranteed by Theorem 1.5.

5.2 A family of manifolds with clustering submanifolds

Here, we are concerned with a class of non compact surfaces $\mathbb{M}$ in $\mathbb{R}^3$, which are shaped as in Figure 2 of Section 1, and are patterned on an example appearing in [CoHi], dealing with a planar domain. Their main feature is the presence of a sequence of mushroom-shaped submanifolds $\{N^k\}$ clustering at some point.

Let us emphasize that the submanifolds $\{N^k\}$ are not obtained just by dilation of each other. Roughly speaking, the diameter of the head and the length of the neck of N^k decay to 0 as 2^{-k} when $k \to \infty$, whereas the width of the neck of N^k decays to 0 as $\sigma(2^{-k})$, where σ is a function such that

$$\lim_{s\to 0} \frac{\sigma(s)}{s} = 0.$$

The isoperimetric and isocapacitary functions of $\mathbb{M}$ depend on the behavior of σ at 0 in a way described in the next result (Proposition 5.1). Qualitatively, a faster decay to 0 of the function $\sigma(s)$ as $s \to 0$ results in a faster decay to 0 of $\lambda_{\mathbb{M}}(s)$ and $\nu_{\mathbb{M}}(s)$, and hence in a manifold $\mathbb{M}$ with a more irregular geometry. The proof of Proposition 5.1 can be found in [CiMa4, Propositions 7.1 and 7.2], to which we also refer for a more precise definition of the manifold $\mathbb{M}$.

Proposition 5.1. *Let $\mathbb{M}$ be the 2-dimensional manifold in Figure 2 and let $1 < p \leq 2$. Assume that $\sigma : [0, \infty) \to [0, \infty)$ is an increasing function of class Δ_2 such that*

$$\frac{s^{\beta+1}}{\sigma(s)} \quad \text{is non-increasing} \tag{5.6}$$

for some $\beta > 0$.
(i) If

$$\frac{s^2}{\sigma(s)} \quad \text{is non-decreasing,}$$

then

$$\lambda_{\mathbb{M}}(s) \approx \sigma(s^{1/2}) \qquad \text{near } 0. \tag{5.7}$$

(ii) If

$$\frac{s^{p+1}}{\sigma(s)} \quad \text{is non-decreasing,}$$

then

$$\nu_{\mathbb{M},p}(s) \approx \sigma(s^{1/2}) s^{-\frac{p-1}{2}} \qquad \text{near } 0. \tag{5.8}$$

Owing to equation (5.8), one can derive the following conclusions from Theorem 1.3, involving the isocapacitary function $\nu_{\mathbb{M},p}$. Assume that

$$\lim_{s \to 0} \frac{s^{p+1}}{\sigma(s)} = 0\,. \tag{5.9}$$

Then any eigenfunction of problem (1.4), with $\Omega = \mathbb{M}$ belongs to $L^q(\mathbb{M})$ for any $q < \infty$. If (5.9) is strengthened to

$$\int_0 \left(\frac{s^2}{\sigma(s)} \right)^{\frac{1}{p-1}} ds < \infty\,, \tag{5.10}$$

then any eigenfunction is in fact bounded.
Conditions (5.9) and (5.10) are weaker than parallel conditions which are obtained from an application of Theorem 1.1 and equation (5.7), and read

$$\lim_{s \to 0} \frac{s^2}{\sigma(s)} = 0\,, \tag{5.11}$$

and

$$\int_0 \frac{s^{\frac{p+1}{p-1}}}{\sigma(s)^{p'}} \, ds < \infty\,, \tag{5.12}$$

respectively. For instance, if $b > 1$ and

$$\sigma(s) = s^b \qquad \text{for } s > 0,$$

then (5.9) and (5.10) amount to $b < p+1$, whereas (5.11) and (5.12) are equivalent to the more stringent condition that $b < 2$.

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