On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces

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Abstract

The article is concerned with the Bourgain, Brezis and Mironescu theorem on the asymptotic behaviour of the norm of the Sobolev type embedding operator: $\mathcal{W}^{s,p} \to L^{pn/(n-sp)}$ as $s \uparrow 1$ and $s \uparrow n/p$. Their result is extended to all values of $s \in (0,1)$ and is supplied with an elementary proof. The relation

$$\lim_{s\downarrow 0} s \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy = 2p^{-1}|S^{n-1}|\,\|u\|^p_{L^p(\mathbf{R}^n)}$$

is proved.

1 Introduction

Let $s \in (0,1)$ and let $p \geq 1$. We introduce the space $\mathcal{W}_0^{s,p}(\mathbf{R}^n)$ as the completion of $C_0^\infty(\mathbf{R}^n)$ in the norm

$$\left(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{1/p}.$$

We also need the space $\mathcal{W}_\perp^{s,p}(Q)$ of functions defined on the cube $Q = \{x \in \mathbf{R}^n : |x_i| < 1/2, 1 \leq i \leq n\}$ which are orthogonal to 1 and have the finite norm

$$\left(\int_Q\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{1/p}.$$

The main result of the recent paper by J.Bourgain, H.Brezis, and P.Mironescu [BBM$_1$] is the inequality

$$\|u\|^p_{L^q(Q)} \leq c(n)\frac{1-s}{(n-sp)^{p-1}}\|u\|^p_{\mathcal{W}_\perp^{s,p}(Q)}, \tag{1}$$

where $u \in \mathcal{W}_\perp^{s,p}(Q)$, $1/2 \leq s < 1$, $sp < n$, $q = pn/(n-sp)$, and $c(n)$ depends only on n.

The present article is a direct outgrowth of this result. Figuring out a similar estimate for functions in $\mathcal{W}_0^{s,p}(\mathbf{R}^n)$, valid for the whole interval $0 < s < 1$, one could anticipate the appearance of the factor $s(1-s)$ in the right-hand side, since the norm in $\mathcal{W}_0^{s,p}(\mathbf{R}^n)$ blows up both as $s \uparrow 1$ and $s \downarrow 0$. The following theorem shows that this is really the case.

Theorem 1. *Let $n \geq 1$, $p \geq 1$, $0 < s < 1$, and $sp < n$. Then, for an arbitrary function $u \in \mathcal{W}_0^{s,p}(\mathbf{R}^n)$, there holds*

$$\|u\|^p_{L^q(\mathbf{R}^n)} \leq c(n,p)\frac{s(1-s)}{(n-sp)^{p-1}}\|u\|^p_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}, \tag{2}$$

where $q = pn/(n-sp)$ and $c(n,p)$ is a function of n and p.

From Theorem 1, one can derive inequality (1) for all $s \in (0,1)$ with a constant c depending both on n and p (Corollary 2). In the case $s \geq 1/2$ considered in [BBM$_1$], one has $1 < p < 2n$ and therefore the dependence of the constant c on p can be eliminated. Thus, we arrive at the Bourgain-Brezis-Mironescu result and extend it to the values $s < 1/2$.

The proof given in [BBM$_1$] relies upon some advanced harmonic analysis and is quite complicated. Our proof of (2) is straightforward and rather simple. It is based upon an estimate of the best constant in a Hardy type inequality for the norm in $\mathcal{W}_0^{s,p}(\mathbf{R}^n)$, which is obtained in Theorem 2 and is of independent interest.

In Theorem 3 we derive a formula for $\lim_{s\downarrow 0} s\|u\|^p_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}$ which complements an analogous formula for $\lim_{s\uparrow 1}(1-s)\|u\|^p_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}$ found in [BBM$_2$].

2 Hardy type inequalities

Theorem 2. *Let $n \geq 1$, $p \geq 1$, $0 < s < 1$, and $sp < n$. Then, for an arbitrary function $u \in \mathcal{W}_0^{s,p}(\mathbf{R}^n)$, there holds*

$$\int_{\mathbf{R}^n} |u(x)|^p \frac{dx}{|x|^{sp}} \leq c(n,p)\frac{s(1-s)}{(n-sp)^p}\|u\|^p_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}. \tag{3}$$

Proof. Let

$$\psi(h) = |S^{n-1}|^{-1} n(n+1)(1-|h|)_+ \ ,$$

where $h \in \mathbf{R}^n$ and plus stands for the nonnegative part of a real-valued function. We introduce the standard extension of u onto $\mathbf{R}^{n+1}_+ = \{(x,z) : x \in \mathbf{R}^n, z > 0\}$

$$U(x,z) := \int_{\mathbf{R}^n} \psi(h) u(x+zh)dh.$$

A routine majoration implies

$$|\nabla U(x,z)| \leq \frac{n(n+1)(n+2)}{z\ |S^{n-1}|}\int_{|h|<1} |u(x+zh)-u(x)|dh.$$

Hence and by Hölder's inequality one has

$$\int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x,z)|^p dxdz$$

$$\leq \frac{n}{|S^{n-1}|}(n+1)^p(n+2)^p \int_0^\infty z^{-1-ps} \int_{|h|<1} \int_{\mathbf{R}^n} |u(x+zh)-u(x)|^p dx dh dz. \qquad (4)$$

Setting $\eta = zh$ and changing the order of integration, one can rewrite (4) as

$$\int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x,z)|^p dx dz \leq$$

$$\frac{n(n+1)^p(n+2)^p}{|S^{n-1}|(sp+n)} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx dy. \qquad (5)$$

By Hardy's inequality,

$$\int_0^{|x|} z^{-1-sp} \Big| \int_0^z \varphi(\tau) d\tau \Big|^p dz \leq s^{-p} \int_0^{|x|} z^{-1+p(1-s)} |\varphi(z)|^p dz$$

one has

$$\frac{|u(x)|^p}{|x|^{sp}} = p(1-s) \int_0^{|x|} z^{-1+p(1-s)} dz \frac{|u(x)|^p}{|x|^p} \leq$$

$$p(1-s) \int_0^{|x|} z^{-1-sp} dz \Big(\int_0^z \Big(\Big| \frac{\partial U}{\partial \tau}(x,\tau) \Big| + \frac{|U(x,\tau)|}{|x|} \Big) d\tau \Big)^p \leq$$

$$\frac{p(1-s)}{s^p} \int_0^{|x|} z^{-1+p(1-s)} \Big(\Big| \frac{\partial U}{\partial z}(x,z) \Big| + \frac{|U(x,z)|}{|x|} \Big)^p dz.$$

Now, the integration over $\mathbf{R}^n$ and Minkowski's inequality imply

$$\int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \leq \frac{p(1-s)}{s^p} \Big(\Big(\int_{\mathbf{R}^n} \int_0^\infty z^{-1+p(1-s)} \Big| \frac{\partial U}{\partial z}(x,z) \Big|^p dz dx \Big)^{1/p} + A \Big)^p, \qquad (6)$$

where

$$A := \Big(\int_{\mathbf{R}^n} \int_0^{|x|} z^{-1+p(1-s)} |x|^{-p} |U(x,z)|^p dz dx \Big)^{1/p}.$$

Clearly,

$$A^p \leq 2^{p/2} \int_{\mathbf{R}^n} dx \int_0^\infty z^{-1+p(1-s)} \frac{|U(x,z)|^p}{(x^2+z^2)^{p/2}} dz dx,$$

which does not exceed

$$2^{p/2} \int_{S^n_+} (\cos\theta)^{-1+p(1-s)} \int_0^\infty |U|^p \rho^{n-1-sp} d\rho d\sigma, \qquad (7)$$

where $\rho = (x^2+z^2)^{1/2}$, $\cos\theta = z/\rho$, $d\sigma$ is an element of the surface area on the unit sphere S^n, and S^n_+ is the upper half of S^n. Using Hardy's inequality

$$\int_0^\infty |U|^p \rho^{n-1-sp} d\rho \leq \Big(\frac{p}{n-sp} \Big)^p \int_0^\infty \Big| \frac{\partial U}{\partial \rho} \Big|^p \rho^{n-1+p(1-s)} d\rho,$$

one arrives at the estimate

$$A^p \leq \Big(\frac{2^{1/2} p}{n-sp} \Big)^p \int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x,z)|^p dx dz.$$

Combining this with (6), one obtains

$$\int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \le \frac{p(1-s)}{s^p} \Big(1 + \frac{2^{1/2} p}{n-sp}\Big)^p \int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x,z)|^p dx dz$$

which, along with (5), gives

$$\int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \le \frac{(1-s)}{(n-sp)^p} \frac{p(n+2p)^{3p}}{|S^{n-1}| s^p} \|u\|^p_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}. \tag{8}$$

In order to justify (3) we need to improve (8) for small values of s. Clearly,

$$\frac{|S^{n-1}|}{2^{sp} sp} \int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx = \int_{\mathbf{R}^n} \int_{|x-y|>2|x|} \frac{dy}{|x-y|^{n+sp}} |u(x)|^p dx.$$

Since $|x-y| > 2|x|$ implies $2|y|/3 < |x-y| < 2|y|$, we obtain

$$\Big(\frac{|S^{n-1}|}{2^{sp} sp} \int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx\Big)^{1/p} \le \Big(\int_{\mathbf{R}^n} \int_{|x-y|>|x|} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx dy\Big)^{1/p} +$$

$$\Big(|S^{n-1}| \frac{3^{sp}-1}{2^{sp} sp} \int_{\mathbf{R}^n} \frac{|u(y)|^p}{|y|^{sp}} dy\Big)^{1/p}.$$

Hence,

$$\Big(\frac{|S^{n-1}|}{2^{sp} sp}\Big)^{1/p} (1 - (3^{sp}-1)^{1/p}) \Big(\int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx\Big)^{1/p} \le 2^{-1/p} \|u\|_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}.$$

Let δ be an arbitrary number in $(0,1)$. If $s \le (4p)^{-1}\delta^p$, we conclude

$$\int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \le \frac{2^{sp-1} sp}{|S^{n-1}|(1-\delta)^p} \|u\|^p_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}. \tag{9}$$

Setting $\delta = 2^{-1}$ and comparing this inequality with (8), we arrive at (3) with $c(n,p) = |S^{n-1}|^{-1}(n+2p)^{3p} p^{p+2} 2^{(n+1)(n+2)}$. The proof is complete.

From Theorem 2, we shall deduce an inequality, analogous to (3), for functions defined on the cube Q. Unlike (3), this inequality contains no factor s in the right-hand side, which is not surprising, because, for smooth u, the norm $\|u\|_{\mathcal{W}_\perp^{s,p}(Q)}$ tends to a finite limit as $s \downarrow 0$.

Corollary 1. *Let $n \ge 1$, $p \ge 1$, $0 < s < 1$, and $sp < n$. Then any function $u \in \mathcal{W}_\perp^{s,p}(Q)$ satisfies*

$$\int_Q |u(x)|^p \frac{dx}{|x|^{sp}} \le c(n,p) \frac{1-s}{(n-sp)^p} \|u\|^p_{\mathcal{W}_\perp^{s,p}(Q)}. \tag{10}$$

Proof. Let us preserve the notation u for the mirror extension of $u \in \mathcal{W}_\perp^{s,p}(Q)$ to the cube $3Q$, where aQ stands for the cube obtained from Q by dilation with the coefficient a. We choose a cut-off function η, equal to 1 on Q and vanishing outside $2Q$, say, $\eta(x) = \prod_{i=1}^n \min\{1, 2(1-x_i)_+\}$. By Theorem 2, it is enough to prove that

$$\|\eta u\|^p_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)} \le s^{-1} c(n,p) \|u\|^p_{\mathcal{W}_\perp^{s,p}(Q)}. \tag{11}$$

Clearly, the norm in the left-hand side is majorized by

$$\Big(\int_{3Q}\int_{3Q}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,\eta(y)^p dy\Big)^{1/p}+\Big(\int_{3Q}\int_{3Q}\frac{|\eta(x)-\eta(y)|^p}{|x-y|^{n+sp}}dx\,|u(y)|^p dy\Big)^{1/p}+$$

$$\Big(2\int_{3Q}\int_{\mathbf{R}^n\setminus 3Q}\frac{dy}{|x-y|^{n+sp}}|(\eta u)(x)|^p dx\Big)^{1/p}.$$

The first term does not exceed $6^{n/p}\|u\|_{\mathcal{W}_\perp^{s,p}(Q)}$; the second term is not greater than

$$2n^{1/2}\Big(\int_{3Q}\int_{3Q}\frac{dy}{|x-y|^{n-p(1-s)}}|u(y)|^p dy\Big)^{1/p}\le n3^{2+n/p}\Big(\frac{|S^{n-1}|}{p(1-s)}\Big)^{1/p}\|u\|_{L^p(Q)},$$

and the third one is dominated by

$$\Big(2\int_{2Q}\int_{|x-y|>1/2}\frac{dy}{|x-y|^{n+sp}}|u(x)|^p dx\Big)^{1/p}\le\Big(\frac{2^{n+1+p}}{sp}\Big)^{1/p}\|u\|_{L^p(Q)}.$$

Summing up these estimates, one obtains

$$\|\eta u\|_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}\le 6^{n/p}\|u\|_{\mathcal{W}_0^{s,p}(Q)}+n3^{2+n/p}p^{-1/p}(s^{-1/p}+(1-s)^{-1/p})\|u\|_{L^p(Q)}. \quad (12)$$

We preserve the notation u for the mirror extention of u onto $\mathbf{R}^n$. Recalling that $u\perp 1$ on Q, we have

$$\int_Q|u(x)|^p dx\le\int_Q\int_Q|u(x)-u(y)|^p dxdy\le\int_{2Q}dh\int_Q|u(x+h)-u(x)|^p dx \quad (13)$$

Let U be the extension of u to $\mathbf{R}_+^{n+1}$. For any $z>0$ and $h\in 2Q$

$$\|u(\cdot+h)-u(\cdot)\|_{L^p(Q)}\le\Big\|\int_0^z\frac{\partial U}{\partial\tau}(\cdot+h,\tau)d\tau\Big\|_{L^p(Q)}+\Big\|\int_0^z\frac{\partial U}{\partial\tau}(\cdot,\tau)d\tau\Big\|_{L^p(Q)}+$$

$$\|U(\cdot+h,z)-U(\cdot,z)\|_{L^p(Q)}\le 2\int_0^z\Big\|\frac{\partial U}{\partial\tau}(\cdot,\tau)\Big\|_{L^p(3Q)}d\tau+\|U(\cdot+h,z)-U(\cdot,z)\|_{L^p(Q)}.$$

Hence

$$p^{-1/p}(1-s)^{-1/p}|h|^{1-s}\|u(\cdot+h)-u(\cdot)\|_{L^p(Q)}=$$

$$\Big(\int_0^{|h|}\|u(\cdot+h)-u(\cdot)\|^p_{L^p(Q)}\,z^{-1+p(1-s)}dz\Big)^{1/p}\le\alpha+\beta, \quad (14)$$

where

$$\alpha^p=(2|h|)^p\int_0^{|h|}\Big(\int_0^z\Big\|\frac{\partial U}{\partial\tau}(\cdot,\tau)\Big\|_{L^p(3Q)}d\tau\Big)^p z^{-1-ps}dz$$

and

$$\beta^p=\int_0^{|h|}\|U(\cdot+h,z)-U(\cdot,z)\|^p_{L^p(Q)}\,z^{-1+p(1-s)}dz.$$

Using Hardy's inequality already referred to at the beginning of the proof of Theorem 2, we arrive at

$$\alpha^p\le\Big(\frac{2|h|}{s}\Big)^p\int_0^{|h|}\Big\|\frac{\partial U}{\partial z}(\cdot,z)\Big\|^p_{L^p(3Q)}z^{-1+p(1-s)}dz.$$

The trivial inequality

$$\int_Q |U(x+h,z)-U(x,z)|^p dx \le |h|^p \int_{3nQ} |\nabla U(x,z)|^p dx$$

implies

$$\beta^p \le |h|^p \int_0^{|h|} \|\nabla U(\cdot,z)\|^p_{L^p(3nQ)} z^{-1+p(1-s)} dz.$$

We put the estimates for α and β just obtained into (14) and deduce

$$\|u(\cdot+h)-u(\cdot)\|^p_{L^p(Q)} \le p(1-s)\Big(\frac{2}{s}+1\Big)^p |h|^{ps} \int_0^{|h|} \|\nabla U(\cdot,z)\|^p_{L^p(2nQ)} z^{-1+p(1-s)} dz.$$

Noting that $|h| \le \sqrt{n}$ for $h \in 2Q$, we find

$$|h|^{-ps} \|u(\cdot+h)-u(\cdot)\|^p_{L^p(Q)}$$

$$\le p(1-s)\Big(\frac{2}{s}+1\Big)^p \int_0^{\sqrt{n}} \int_{3nQ} |\nabla U(x,z)|^p dx\, z^{-1+p(1-s)} dz. \tag{15}$$

Now, let U be the same extension of u onto $\mathbf{R}^{n+1}_+$ as in the beginning of the proof of Theorem 2. Repeating with obvious changes the standard argument in the proof of Theorem 2, which leads to (5), we conclude that the integral over $(0,\sqrt{n}) \times 3nQ$ in the right-hand side of (15) is majorized by

$$c_0 \int_0^{\sqrt{n}} \int_{|\chi|<1} \int_{3nQ} |u(x+z\chi)-u(x)|^p dx d\chi z^{-1-ps} dz \le$$

$$\frac{c_0}{n+ps} \int_{|\eta|<\sqrt{n}} |\eta|^{-n-ps} \int_{3nQ} |u(x+\eta)-u(x)|^p dx d\eta,$$

where

$$c_0 = \frac{n}{|S^{n-1}|}(n+1)^p (n+2)^p.$$

Therefore,

$$|h|^{-ps} \|u(\cdot+h)-u(\cdot)\|^p_{L^p(Q)} \le c_0 3^{n+2p} n^{2n} \frac{1-s}{s^p} \int_Q \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dx dy. \tag{16}$$

Let $s > 1/2$. It follows from (13) that

$$\int_Q |u(x)|^p dx \le n^{(n+ps)/2} \int_Q \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dx dy.$$

This inequality together with (16) shows that for all $s \in (0,1)$

$$\|u\|_{L^p(Q)} \le (4n)^{4n} (1-s)^{1/p} \|u\|_{\mathcal{W}^{s,p}_\perp(Q)}.$$

Combining this inequality with (12), we arrive at (11) and hence complete the proof.

Corollary 2. *Let $0 < s < 1$ and $p \ge 1$. Then there holds*

$$\sup |h|^{-s} \|u(\cdot+h)-u(\cdot)\|_{L^p(Q)} \le c(n,p)(1-s)^{1/p} \|u\|_{\mathcal{W}^{s,p}_\perp(Q)}.$$

Proof. The result follows from the well-known imbedding $B^s_p(Q) \subset B^s_{p,\infty}(Q)$ if $s \le 1/2$ and from (16) if $s > 1/2$.

3 Sobolev embeddings

Proof of Theorem 1. It is well known that the fractional Sobolev norm of order $s \in (0,1)$ is nonincreasing with respect to symmetric rearrangement of functions decaying to zero at infinity (see [W], [AL], Theorem 9.2, and [Ci]). Let $v(|x|)$ denote the rearrangement of $|u(x)|$. Then

$$\|u\|_{L^q(\mathbf{R}^n)} = \Big(\frac{|S^{n-1}|}{n}\int_0^\infty v(r)^q d(r^n)\Big)^{1/q}, \tag{17}$$

where $|S^{n-1}|$ is the area of the unit sphere S^{n-1}. Recalling that an arbitrary nonnegative nonincreasing function f on the semi-axis $(0,\infty)$ satisfies

$$\int_0^\infty f(t)^\lambda d(t^\lambda) \le \lambda \int_0^\infty \Big(\int_0^t f(\tau)d\tau\Big)^{\lambda-1} f(t)dt = \Big(\int_0^\infty f(t)dt\Big)^\lambda, \quad \lambda \ge 1$$

(see [HLP]), one finds that the right-hand side in (17) does not exceed

$$\Big(\frac{|S^{n-1}|}{n}\Big)^{1/q}\Big(\int_0^\infty v(r)^p d(r^{n-sp})\Big)^{1/p} = \frac{(n-sp)^{1/p}}{n^{1/q}|S^{n-1}|^{s/n}}\Big(\int_{\mathbf{R}^n} v(|x|)^p \frac{dx}{|x|^{sp}}\Big)^{1/p}.$$

We now see that (2) results from inequality (3).

Corollary 3. *Let $n \ge 1$, $p \ge 1$, $0 < s < 1$, and $sp < n$. Then any function $u \in \mathcal{W}^{s,p}_\perp(Q)$ satisfies*

$$\|u\|^p_{L^q(Q)} \le c(n,p)\frac{1-s}{(n-sp)^{p-1}}\|u\|^p_{\mathcal{W}^{s,p}_\perp(Q)}.$$

Proof. Let η be the same cut-off function as in Corollary 1. The result follows by combining inequality (11) with Theorem 1 where u is replaced by ηu.

4 Asymptotics of the norm in $\mathcal{W}^{s,p}_0(\mathbf{R}^n)$ as $s \downarrow 0$

Theorem 3. *For any function $u \in \cup_{0<s<1}\mathcal{W}^{s,p}_0(\mathbf{R}^n)$, there exists the limit*

$$\lim_{s\downarrow 0} s\|u\|^p_{\mathcal{W}^{s,p}_0(\mathbf{R}^n)} = 2p^{-1}|S^{n-1}|\ \|u\|^p_{L^p(\mathbf{R}^n)}.$$

Proof. Since δ can be chosen arbitrarily small, inequality (9) implies

$$\liminf_{s\downarrow 0} s\|u\|^p_{\mathcal{W}^{s,p}_0(\mathbf{R}^n)} \ge 2p^{-1}|S^{n-1}|\ \|u\|^p_{L^p(\mathbf{R}^n)}. \tag{18}$$

Let us majorize the upper limit. By (18), it suffices to assume that $u \in L^p(\mathbf{R}^n)$. Clearly,

$$s\|u\|^p_{\mathcal{W}^{s,p}_0(\mathbf{R}^n)} \le 2\Big\{\Big(s\int_{\mathbf{R}^n}\int_{|y|\ge 2|x|}\frac{dy}{|x-y|^{n+sp}}|u(x)|^p dx\Big)^{1/p} +$$
$$\Big(s\int_{\mathbf{R}^n}|u(y)|^p\int_{|y|\ge 2|x|}\frac{dxdy}{|x-y|^{n+sp}}\Big)^{1/p}\Big\}^p +$$

$$2s\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy.$$

The first term in braces does not exceed

$$\Big(s\int_{\mathbf{R}^n}\int_{|y|\geq|x|}\frac{dy}{|x-y|^{n+sp}}|u(x)|^p dx\Big)^{1/p}=\frac{|S^{n-1}|^{1/p}}{p^{1/p}}\Big(\int_{\mathbf{R}^n}\frac{|u(x)|^p}{|x|^{sp}}dx\Big)^{1/p}$$

hence its $\limsup_{s\downarrow 0}$ is dominated by $|S^{n-1}|^{1/p}p^{-1/p}\|u\|_{L^p(\mathbf{R}^n)}$. The second term in braces is not greater than

$$s^{1/p}\Big(2^{n+sp}\int_{\mathbf{R}^n}\frac{|u(y)|^p}{|y|^{n+sp}}dy\int_{|x|<|y|/2}dx\Big)^{1/p}=2^s\Big(\frac{s}{p}|S^{n-1}|\Big)^{1/p}\Big(\int_{\mathbf{R}^n}\frac{|u(y)|^p}{|y|^{sp}}dy\Big)^{1/p},$$

so it tends to zero as $s\downarrow 0$.

We claim that

$$\limsup_{s\downarrow 0}s\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy=0. \tag{19}$$

By assumption of the Theorem, $u\in\mathcal{W}_0^{\tau,p}(\mathbf{R}^n)$ for a certain $\tau\in(0,1)$. Let N be an arbitrary number greater than 1 and let $s<\tau$. We have

$$2s\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy\leq$$

$$2sN^{p(\tau-s)}\int_{\mathbf{R}^n}\int_{\substack{|x|<|y|<2|x|\\|x-y|\leq N}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\tau p}}dxdy+2s\int_{\mathbf{R}^n}\int_{\substack{|x|<|y|<2|x|\\|x-y|>N}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy.$$

The first term in the right-hand side tends to zero as $s\downarrow 0$ and the second one does not exceed

$$2^{p+1}s\int_{|x|>N/3}\int_{|x-y|>N}\frac{dy}{|x-y|^{n+sp}}|u(x)|^p dx\leq c(n,p)\int_{|x|>N/3}|u(x)|^p dx,$$

which is arbitrarily small if N is sufficiently large. The proof is complete.

Remark. Since the proof of Theorem 3 holds for vector-valued functions, one can write

$$\lim_{s\downarrow 0}s\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|\nabla u(x)-\nabla u(y)|^p}{|x-y|^{n+sp}}dxdy=2p^{-1}|S^{n-1}|\int_{\mathbf{R}^n}|\nabla u(x)|^p dx$$

for any function u such that $\nabla u\in\cup_{0<s<1}\mathcal{W}_0^{s,p}(\mathbf{R}^n)$. This formula complements the following relation which was established in [BBM$_2$]:

$$\lim_{s\uparrow 1}(1-s)\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy=\int_{S^{n-1}}|\cos\theta|^p d\sigma\int_{\mathbf{R}^n}|\nabla u(x)|^p dx,$$

where θ is the angle deviation from the vertical.

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