The L^p-dissipativity of first order partial differential operators

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In Memory of Vladimir I. Smirnov

Abstract. We find necessary and sufficient conditions for the L^p-dissipativity of the Dirichlet problem for systems of partial differential operators of the first order with complex locally integrable coefficients. As a by-product we obtain sufficient conditions for a certain class of systems of the second order.

1 Introduction

The goal of the present paper is to find necessary and sufficient conditions for the L^p-dissipativity for systems of partial differential equations of the first order ($1 < p < \infty$).

Previously we have considered a scalar second order partial differential operator whose coefficients are complex-valued measures [1]. For some classes of such operators we have algebraically characterized the L^p-dissipativity. The main result is that the algebraic condition

$$|p-2|\,|\langle \mathbb{I}\mathrm{m}\, \mathscr{A}\, \xi, \xi\rangle| \leqslant 2\sqrt{p-1}\, \langle \mathbb{R}\mathrm{e}\, \mathscr{A}\, \xi, \xi\rangle$$

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(for any $\xi \in \mathbb{R}^n$) is necessary and sufficient for the L^p-dissipativity of the Dirichlet problem for the differential operator $\nabla^t(\mathscr{A}\nabla)$, where $\mathscr{A}$ is a matrix whose entries are complex measures and whose imaginary part is symmetric.

We remark that conditions obtained in [1] characterizes the L^p-dissipativity individually, for each p. Previous known results in the literature dealt with the L^p-dissipativity for any $p \in [1, +\infty)$, simultaneously. In the same spirit we have studied the elasticity system and some classes of systems of partial differential operators of the second order in [2, 3].

Our results are described and considered in the more general frame of semi-bounded operators in the monograph [4].

The main result of the present paper concerns the matrix operator

$$Eu = \mathscr{B}^h(x)\partial_h u + \mathscr{D}(x)u\,,$$

where $\mathscr{B}^h(x) = \{b^h_{ij}(x)\}$ and $\mathscr{D}(x) = \{d_{ij}(x)\}$ are matrices with complex locally integrable entries defined in the domain Ω of $\mathbb{R}^n$ and $u = (u_1, \dots, u_m)$ $(1 \leqslant i, j \leqslant m,\ 1 \leqslant h \leqslant n)$. It states that, if $p \neq 2$, E is L^p-dissipative if, and only if,

$$\mathscr{B}^h(x) = b_h(x) I \text{ a.e.,} \tag{1}$$

$b_h(x)$ being real locally integrable functions, and the inequality

$$\mathbb{R}\mathrm{e}\langle (p^{-1}\partial_h \mathscr{B}^h(x) - \mathscr{D}(x))\zeta, \zeta\rangle \geqslant 0$$

holds for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$. If $p = 2$ condition (1) is replaced by the more general requirement that the matrices $\mathscr{B}^h(x)$ are self-adjoint a.e..

On combining this with the results we have previously obtained, we deduce sufficient conditions for the L^p-dissipativity of certain systems of partial differential operators of the second order.

2 Preliminaries

Let Ω be a domain of $\mathbb{R}^n$. By $C_0(\Omega)$ we denote the space of complex valued continuous functions having compact support in Ω. Let $C^1_0(\Omega)$ consist of all the functions in $C_0(\Omega)$ having continuous partial derivatives of the first order. The inner product in $\mathbb{C}^m$ is denoted by $\langle \cdot, \cdot \rangle$ and, as usual, the bar denotes complex conjugation.

In what follows, if $\mathscr{G}$ is a $m \times m$ matrix function with complex valued entries, then $\mathscr{G}^*$ is its adjoint matrix, i.e., $\mathscr{G}^* = \overline{\mathscr{G}}^t$, $\mathscr{G}^t$ being the transposed matrix of $\mathscr{G}$.

Let $\mathscr{B}^h$ and $\mathscr{C}^h$ $(h = 1, \ldots, n)$ be $m \times m$ matrices whith complex-valued entries $b^h_{ij}, c^h_{ij} \in (C_0(\Omega))^*$ $(1 \leqslant i, j \leqslant m)$. Let $\mathscr{D}$ stand for a matrix whose elements d_{ij} are complex-valued distributions in $(C^1_0(\Omega))^*$.

We adopt the summation convention over repeated indices unless otherwise stated.

We denote by $\mathscr{L}(u, v)$ the sesquilinear form

$$\mathscr{L}(u, v) = \int_\Omega \langle \mathscr{B}^h \, \partial_h u, v \rangle - \langle \mathscr{C}^h \, u, \partial_h v \rangle + \langle \mathscr{D} \, u, v \rangle$$

defined in $(C^1_0(\Omega))^m \times (C^1_0(\Omega))^m$, where $\partial_h = \partial / \partial x_h$.

The integrals appearing in this definition have to be understood in a proper way. The entries b^h_{ij} being measures, the meaning of the first term is

$$\int_\Omega \langle \mathscr{B}^h \, \partial_h u, v \rangle = \int_\Omega \overline{v_i} \, \partial_h u_j \, d b^h_{ij} \, .$$

Similar meanings have the terms involving $\mathscr{C}$. Finally, the last term is the sum of the actions of the distribution $d_{ij} \in (C^1_0(\Omega))^*$ on the functions $u_j \, \overline{v_i}$ belonging to $C^1_0(\Omega)$.

The form $\mathscr{L}$ is related to the system of partial differential operators of the first order:

$$Eu = \mathscr{B}^h \, \partial_h u + \partial_h (\mathscr{C}^h \, u) + \mathscr{D} \, u$$

Following [4], we say that the form $\mathscr{L}$ is L^p-dissipative if

$$\mathbb{R}\mathrm{e} \, \mathscr{L}(u, |u|^{p-2} u) \leqslant 0 \qquad \text{if } p \geqslant 2; \tag{2}$$

$$\mathbb{R}\mathrm{e} \, \mathscr{L}(|u|^{p'-2} u, u) \leqslant 0 \qquad \text{if } 1 < p < 2 \tag{3}$$

for all $u \in (C^1_0(\Omega))^m$.

In the present paper, saying the L^p-dissipativity of the operator E, we mean the L^p-dissipativity of the corresponding form $\mathscr{L}$, just to simplify the terminology.

Let us start with a technical lemma which is a particular case of a result in [4, p.94]. The proof is mainly included here to keep the exposition as self-contained as possible.

Lemma 1 *The operator E is L^p-dissipative in Ω if, and only if,*

$$\int_\Omega \Big((1-2/p)|v|^{-2}\,\mathbb{R}\mathrm{e}\langle \mathscr{B}^h\, v, v\rangle\, \mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle \ \ - \mathbb{R}\mathrm{e}\langle \mathscr{B}^h\, \partial_h v, v\rangle + $$
$$(1-2/p)|v|^{-2}\,\mathbb{R}\mathrm{e}\langle \mathscr{C}^h\, v, v\rangle\, \mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle \ \ + \mathbb{R}\mathrm{e}\langle \mathscr{C}^h\, v, \partial_h v\rangle - \mathbb{R}\mathrm{e}\langle \mathscr{D}\, v, v\rangle \Big) \geqslant 0 \tag{4}$$

for any $v \in (C_0^1(\Omega))^m$. Here and in the sequel the integrand is extended by zero on the set where v vanishes.

Proof. *Sufficiency.* First suppose $p \geqslant 2$. Let $u \in (C_0^1(\Omega))^m$ and set $v = |u|^{(p-2)/2}u$. We have $v \in (C_0^1(\Omega))^m$ and $u = |v|^{(2-p)/p}v$, $|u|^{p-2}u = |v|^{(p-2)/p}v$. From the identities

$$\langle \mathscr{B}^h\, \partial_h u, |u|^{p-2}u\rangle = -(1-2/p)|v|^{-1}\langle \mathscr{B}^h\, v, v\rangle \partial_h |v| + \langle \mathscr{B}^h\, \partial_h v, v\rangle,$$
$$\langle \mathscr{C}^h\, u, \partial_h(|u|^{p-2}u)\rangle = (1-2/p)|v|^{-1}\langle \mathscr{C}^h\, v, v\rangle \partial_h |v| + \langle \mathscr{C}^h\, v, \partial_h v\rangle,$$
$$\langle \mathscr{D}\, u, |u|^{p-2}u\rangle = \langle \mathscr{D}\, v, v\rangle, \qquad \partial_h |v| = |v|^{-1}\,\mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle$$

we see that the left hand side in (4) is equal to $-\mathscr{L}(u, |u|^{p-2})$. Then (2) is satisfied for any $u \in (C_0^1(\Omega))^m$.

If $1 < p < 2$ we may write (3) as

$$\mathbb{R}\mathrm{e} \int_\Omega \big(\langle (\mathscr{B}^h)^* u, \partial_h(|u|^{p'-2}u)\rangle - \langle (\mathscr{C}^h)^* \partial_h u, |u|^{p'-2}u\rangle + \langle \mathscr{D}^*\, u, |u|^{p'-2}u\rangle \big) \leqslant 0$$

for any $u \in (C_0^1(\Omega))^m$. The first part of the proof shows that

$$\int_\Omega \big(-(1-2/p')|v|^{-2}\,\mathbb{R}\mathrm{e}\langle (\mathscr{B}^h)^* v, v\rangle\, \mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle \ \ - \mathbb{R}\mathrm{e}\langle (\mathscr{B}^h)^* v, \partial_h v\rangle + $$
$$-(1-2/p')|v|^{-2}\,\mathbb{R}\mathrm{e}\langle (\mathscr{C}^h)^* v, v\rangle\, \mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle \ \ + \mathbb{R}\mathrm{e}\langle (\mathscr{C}^h)^* \partial_h v, v\rangle - \mathbb{R}\mathrm{e}\langle \mathscr{D}^*\, v, v\rangle \big)$$
$$\geqslant 0 \tag{5}$$

for any $v \in (C_0^1(\Omega))^m$. Since $1 - 2/p' = -(1-2/p)$, the last inequality coincides with (4).

Necessity. Let $p \geqslant 2$ and set

$$g_\varepsilon = (|v|^2 + \varepsilon^2)^{1/2}, \quad u_\varepsilon = g_\varepsilon^{2/p-1} v\,,$$

where $v \in (C^1_0(\Omega))^m$. We have

$$\langle \mathscr{B}^h\, \partial_h u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon\rangle =$$
$$-(1-2/p) g_\varepsilon^{-p}|v|^{p-2}\langle \mathscr{B}^h\, v,v\rangle\, \mathbb{R}\mathrm{e}\langle v,\partial_h v\rangle + g_\varepsilon^{-p+2}|v|^{p-2}\langle \mathscr{B}^h\, \partial_h v, v\rangle,$$
$$\langle \mathscr{C}^h\, u_\varepsilon, \partial_h(|u_\varepsilon|^{p-2}u_\varepsilon)\rangle = g_\varepsilon^{-p}|v|^{p-4}((1-2/p)(1-p)|v|^2+$$
$$+(p-2)g_\varepsilon^2)\langle \mathscr{C}^h\, v,v\rangle\, \mathbb{R}\mathrm{e}\langle v,\partial_h v\rangle + g_\varepsilon^{2-p}|v|^{p-2}\langle \mathscr{C}^h\, v, \partial_h v\rangle,$$
$$\langle \mathscr{D}\, u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon\rangle = g_\varepsilon^{-p+2}|v|^{p-2}\langle \mathscr{D}\, v,v\rangle,$$

on the set $F=\{x\in\Omega \mid |v(x)|>0\}$. The inequality $g_\varepsilon^a \leqslant |v|^a$ for $a\leqslant 0$, shows that the right hand sides are majorized by L^1 functions. Since $g_\varepsilon \to |v|$ pointwise as $\varepsilon\to 0^+$, an application of dominated convergence theorem gives

$$\lim_{\varepsilon\to 0^+}\int_\Omega \langle \mathscr{B}^h\, \partial_h u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon\rangle dx =$$
$$\int_\Omega (-(1-2/p)|v|^{-2}\langle \mathscr{B}^h\, v,v\rangle\, \mathbb{R}\mathrm{e}\langle v,\partial_h v\rangle + \langle \mathscr{B}^h\, \partial_h v,v\rangle)dx,$$
$$\lim_{\varepsilon\to 0^+}\int_\Omega \langle \mathscr{C}^h\, u_\varepsilon, \partial_h(|u_\varepsilon|^{p-2}u_\varepsilon)\rangle dx = \tag{6}$$
$$\int_\Omega ((1-2/p)|v|^{-2}\langle \mathscr{C}^h\, v,v\rangle\, \mathbb{R}\mathrm{e}\langle v,\partial_h v\rangle + \langle \mathscr{C}^h\, v,\partial_h v\rangle)dx,$$
$$\lim_{\varepsilon\to 0^+}\int_\Omega \langle \mathscr{D}\, u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon\rangle dx = \int_\Omega \langle \mathscr{D}\, v,v\rangle dx.$$

These formulas show that the limit

$$\lim_{\varepsilon\to 0^+}(-\,\mathbb{R}\mathrm{e}\,\mathscr{L}(u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon))$$

is equal to the left-hand side of (4). The functions u_ε being in $(C^1_0(\Omega))^m$, (2) implies (4).

If $1<p<2$, from (6) it follows that the limit

$$\lim_{\varepsilon\to 0^+}(-\,\mathbb{R}\mathrm{e}\,\mathscr{L}(|u_\varepsilon|^{p'-2}u_\varepsilon, u_\varepsilon))$$

coincides with the left-hand side of (5). This shows that (3) implies (5) and the proof is complete. □

3 A result for a system of ordinary differential equations of the first order

The aim of this section is to obtain an auxiliary result (see Theorem 1 below) concerning a particular system of ordinary differential equations of the first order.

We start with an elementary result, which we prove for the sake of completeness.

Lemma 2 *Let α, β, γ and δ be real constants such that*

$$\int_I (\alpha \cos^2 x + \beta \cos x \sin x + \gamma \sin^2 x)(\varphi^2(x))' dx = \int_I \delta \cos^2 x \, \varphi^2(x) dx \qquad (7)$$

for any real valued $\varphi \in C_0^1(I)$. Then $\alpha = \gamma$ and $\beta = \delta = 0$.

Proof. Setting

$$A = \alpha \cos^2 x + \beta \cos x \sin x + \gamma \sin^2 x, \ B = \delta \cos^2 x$$

we may write (7) as

$$\int_I A(\varphi^2)' dx = \int_I B\varphi^2 dx, \qquad \forall \varphi \in C_0^1(I).$$

By an integration by parts we get

$$\int_I (B + A')\varphi^2 dx = 0, \qquad \forall \varphi \in C_0^1(I).$$

Thanks to the arbitrariness of φ, we find $A' = -B$, i.e.

$$(\gamma - \alpha) \sin(2x) + (\beta + \delta/2) \cos(2x) = -\delta/2$$

for any $x \in I$. This implies $\gamma - \alpha = \beta + \delta/2 = -\delta/2 = 0$ and this gives the result. □

The next Theorem provides a criterion for the L^p-dissipativity of one-dimensional operators with complex constant coefficients and no lower order terms.

Theorem 1 *Let $I \subset \mathbb{R}$ be an open interval and $\mathscr{B}$ a constant complex matrix. We have that the operator $Eu = \mathscr{B}\, u'$ is L^p-dissipative if, and only if,*

$$\mathscr{B} = b\, I,\ b \in \mathbb{R}, \qquad \text{if } p \neq 2 \tag{8}$$

$$\mathscr{B} = \mathscr{B}^*, \qquad \text{if } p = 2. \tag{9}$$

Proof. *Sufficiency.* Let $p = 2$. We have to show that

$$-\operatorname{\mathbb{R}e} \int_I \langle \mathscr{B}\, v', v\rangle\, dx \geqslant 0$$

for any $v \in (C^1(I))^m$. The left hand side vanishes because

$$\int_I \langle \mathscr{B}\, v', v\rangle\, dx = \int_I \langle v', \mathscr{B}\, v\rangle\, dx = -\int_I \langle v, \mathscr{B}\, v'\rangle\, dx = -\int_I \overline{\langle \mathscr{B}\, v', v\rangle}\, dx\, .$$

If $p \neq 2$, in view of Lemma 1 we have to show that

$$\int_I \big((1-2/p)|v|^{-2} \operatorname{\mathbb{R}e}\langle \mathscr{B}\, v, v\rangle \operatorname{\mathbb{R}e}\langle v, v'\rangle\ - \operatorname{\mathbb{R}e}\langle \mathscr{B}\, v', v\rangle\big)\, dx \geqslant 0 \tag{10}$$

for any $v \in (C^1(I))^m$. Condition (8) implies $\langle \mathscr{B}\, v, v\rangle = b\,|v|^2$ and $\langle \mathscr{B}\, v', v\rangle = b\,\langle v', v\rangle$, the constant b being real. Therefore the left hand side of (10) is equal to

$$-2b/p \operatorname{\mathbb{R}e} \int_I \langle v, v'\rangle dx = b/p \int_I (|v|^2)' dx = 0$$

and the sufficiency is proved.

Necessity. In view of Lemma 1 we have that E is L^p-dissipative if, and only if,

$$\int_I (1-2/p)|v|^{-2} \operatorname{\mathbb{R}e}\langle \mathscr{B}\, v, v\rangle \operatorname{\mathbb{R}e}\langle v, v'\rangle\, dx - \int_I \operatorname{\mathbb{R}e}\langle \mathscr{B}\, v', v\rangle\, dx \geqslant 0 \tag{11}$$

for any $v \in (C^1_0(I))^m$.

Writing the condition (11) for the function $v(-x)$ we find

$$\int_I (1-2/p)|v|^{-2} \operatorname{\mathbb{R}e}\langle \mathscr{B}\, v, v\rangle \operatorname{\mathbb{R}e}\langle v, v'\rangle\, dx - \int_I \operatorname{\mathbb{R}e}\langle \mathscr{B}\, v', v\rangle\, dx \leqslant 0$$

and then

$$\int_I (1-2/p)|v|^{-2} \operatorname{\mathbb{R}e}\langle \mathscr{B}\, v, v\rangle \operatorname{\mathbb{R}e}\langle v, v'\rangle\, dx - \int_I \operatorname{\mathbb{R}e}\langle \mathscr{B}\, v', v\rangle\, dx = 0 \tag{12}$$

for any $v \in (C_0^1(I))^m$.

Suppose now $p \neq 2$. Fix $1 \leqslant j \leqslant m$ and consider the vector $v = (v_1, \dots, v_m)$ in which $v_k = 0$ for $k \neq j$. Equality (12) reduces to

$$(1/2 - 1/p)\, \mathbb{R}\mathrm{e}\, b_{jj} \int_I (|v_j|^2)' dx - \int_I \mathbb{R}\mathrm{e}(b_{jj} v_j' \overline{v_j}) dx = 0,$$

(without summation convention) and since the first integral vanishes, we get

$$\mathbb{I}\mathrm{m}\, b_{jj} \int_I \mathbb{I}\mathrm{m}(v_j' \overline{v_j}) dx = 0.$$

The arbitrariness of v_j leads to

$$\mathbb{I}\mathrm{m}\, b_{jj} = 0 \qquad (j = 1, \dots, m). \tag{13}$$

Fix $1 \leqslant h, j \leqslant m$ with $h \neq j$ and consider the vector $v = (v_1, \dots, v_m)$ in which $v_k = 0$ for $k \neq h, j$. In view of (12) we have (without summation convention)

$$\begin{gathered}(1/2 - 1/p) \int_I (|v_h|^2 + |v_j|^2)^{-1}\, \mathbb{R}\mathrm{e}(b_{hh}|v_h|^2 + b_{hj} v_j \overline{v_h} + b_{jh} v_h \overline{v_j} + b_{jj}|v_j|^2) \times \\ (|v_h|^2 + |v_j|^2)' dx + \\ - \int_I \mathbb{R}\mathrm{e}(b_{hh} v_h' \overline{v_h} + b_{hj} v_j' \overline{v_h} + b_{jh} v_h' \overline{v_j} + b_{jj} v_j' \overline{v_j}) dx = 0.\end{gathered} \tag{14}$$

In particular, taking $v_h = \alpha$ and $v_j = \beta$, with α and β real valued functions, integrating by parts in the last integral and taking into account (13), we find

$$\begin{gathered}(1/2 - 1/p) \int_I (\alpha^2 + \beta^2)^{-1} (b_{hh}\alpha^2 + \mathbb{R}\mathrm{e}(b_{hj} + b_{jh})\alpha\beta + b_{jj}\beta^2)(\alpha^2 + \beta^2)' dx + \\ - \mathbb{R}\mathrm{e}(b_{hj} - b_{jh}) \int_I \alpha \beta' dx = 0\end{gathered}$$

Taking now $\alpha(x) = \varphi(x)\cos x$, $\beta(x) = \varphi(x)\sin x$, $\varphi \in C_0^1(I)$, we obtain

$$\begin{gathered}(1/2 - 1/p) \int_I (b_{hh}\cos^2 x + b_{jj}\sin^2 x + \\ (\mathbb{R}\mathrm{e}(b_{hj} + b_{jh}) - p/(p-2)\, \mathbb{R}\mathrm{e}(b_{hj} - b_{jh})) \sin x \cos x)(\varphi^2(x))' dx = \\ \mathbb{R}\mathrm{e}(b_{hj} - b_{jh}) \int_I \cos^2 x\, \varphi^2(x) dx\,.\end{gathered}$$

By Lemma 2 we get

$$b_{jj} = b_{hh}, \qquad \mathbb{R}\mathrm{e}\, b_{hj} = 0 \ (h \neq j). \tag{15}$$

Take now $v_h = \alpha$, $v_j = i\beta$ in (14). On account of (13), we have

$$(1/2 - 1/p)\, \mathbb{R}\mathrm{e}(i(b_{hj} - b_{jh})) \int_I (\alpha^2 + \beta^2)^{-1} \alpha\beta(\alpha^2 + \beta^2)' dx +$$
$$- \mathbb{R}\mathrm{e}(i(b_{hj} + b_{jh})) \int_I \alpha\beta' dx = 0\,.$$

The same reasoning as before leads to

$$\mathbb{I}\mathrm{m}\, b_{hj} = 0 \ (h \neq j).$$

Together with (13) and (15), this implies the result for $p \neq 2$.

An inspection of the proof just given, shows that, if $p = 2$, we have only:

$$\mathbb{I}\mathrm{m}\, b_{jj} = 0, \qquad (j = 1, \ldots, m);$$
$$\mathbb{R}\mathrm{e}(b_{hj} - b_{jh}) = 0, \quad \mathbb{I}\mathrm{m}(b_{hj} + b_{jh}) = 0 \quad (j \neq h),$$

and (9) is proved.

□

4 L^p-dissipativity of systems of partial differential operators of the first order

Let us consider the system of partial differential operators of the first order

$$Eu = \mathscr{B}^h(x)\partial_h u + \mathscr{D}(x)u\,. \tag{16}$$

From now on $\mathscr{B}^h(x) = \{b^h_{ij}(x)\}$ and $\mathscr{D}(x) = \{d_{ij}(x)\}$ are matrices with complex locally integrable entries defined in the domain Ω of $\mathbb{R}^n$ $(1 \leqslant i, j \leqslant m,\ 1 \leqslant h \leqslant n)$. Moreover we suppose that also $\partial_h \mathscr{B}^h$ (where the derivatives are in the sense of distributions) is a matrix with complex locally integrable entries.

Theorem 2 *The operator* (16) *is L^p – dissipative if, and only if, the following conditions are satisfied:*

1.

$$\mathscr{B}^h(x) = b_h(x)\, I, \qquad \textit{if } p \neq 2, \tag{17}$$

$$\mathscr{B}^h(x) = (\mathscr{B}^h)^*(x), \qquad \textit{if } p = 2, \tag{18}$$

for almost any $x \in \Omega$ and $h = 1, \ldots, n$. Here b_h are real locally integrable functions ($1 \leqslant h \leqslant n$).

2.

$$\mathbb{R}\mathrm{e}\langle (p^{-1}\partial_h \mathscr{B}^h(x) - \mathscr{D}(x))\zeta, \zeta\rangle \geqslant 0 \tag{19}$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$.

Proof. *Sufficiency.* In view of Lemma 1 we have to show that

$$\begin{aligned}\int_\Omega \Big((1-2/p)|v|^{-2}\, \mathbb{R}\mathrm{e}\langle \mathscr{B}^h\, v, v\rangle\, \mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle\ + \\ -\, \mathbb{R}\mathrm{e}\langle \mathscr{B}^h\, \partial_h v, v\rangle - \mathbb{R}\mathrm{e}\langle \mathscr{D}\, v, v\rangle \Big) dx \geqslant 0\end{aligned} \tag{20}$$

holds for any $v \in (C_0^1(\Omega))^m$.

Let $p = 2$. In view of the self-adjointness of $\mathscr{B}$, we have

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{B}^h\, \partial_h v, v\rangle dx = -\,\mathbb{R}\mathrm{e} \int_\Omega \langle (\partial_h \mathscr{B}^h) v, v\rangle dx - \mathbb{R}\mathrm{e} \int_\Omega \overline{\langle \mathscr{B}^h\, \partial_h v, v\rangle} dx$$

and then

$$2\, \mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{B}^h\, \partial_h v, v\rangle dx = -\,\mathbb{R}\mathrm{e} \int_\Omega \langle (\partial_h \mathscr{B}^h) v, v\rangle dx\, .$$

Consequently

$$\begin{gathered}-\,\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{B}^h\, \partial_h v, v\rangle dx - \mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{D}\, v, v\rangle dx = \\ \int_\Omega (2^{-1}\, \mathbb{R}\mathrm{e}\langle (\partial_h \mathscr{B}^h) v, v\rangle - \mathbb{R}\mathrm{e}\langle \mathscr{D}\, v, v\rangle) dx\end{gathered}$$

and the last integral is greater than or equal to zero because of (19).

Let now $p \neq 2$. Keeping in mind (17) we get

$$\begin{gathered}(1-2/p) \int_\Omega |v|^{-2}\, \mathbb{R}\mathrm{e}\langle \mathscr{B}^h\, v, v\rangle\, \mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle dx + \\ -\int_\Omega \mathbb{R}\mathrm{e}\langle \mathscr{B}^h\, \partial_h v, v\rangle dx - \int_\Omega \mathbb{R}\mathrm{e}\langle \mathscr{D}\, v, v\rangle dx = \\ p^{-1} \int_\Omega (\partial_h b_h)\, |v|^2 dx - \int_\Omega \mathbb{R}\mathrm{e}\langle \mathscr{D}\, v, v\rangle dx\, .\end{gathered}$$

Condition (19) gives the result.

Necessity. Denote by B_1 the open ball $\{y \in \mathbb{R}^n \mid |y| < 1\}$, take $\psi \in (C^1_0(B_1))^m$ and define

$$v(x) = \psi((x - x_0)/\varepsilon)$$

where x_0 is a fixed point in Ω and $0 < \varepsilon < \mathrm{dist}(x_0, \partial\Omega)$.

Putting this particular v in (20) and making a change of variables, we obtain

$$\int_{B_1} (1 - 2/p)|\psi|^{-2}\, \mathbb{R}\mathrm{e}\langle \mathscr{B}^h(x_0 + \varepsilon y)\psi, \psi\rangle\, \mathbb{R}\mathrm{e}\langle \psi, \partial_h\psi\rangle dy$$
$$- \int_{B_1} \mathbb{R}\mathrm{e}\langle \mathscr{B}^h(x_0 + \varepsilon y)\partial_h\psi, \psi\rangle dy - \varepsilon \int_{B_1} \mathbb{R}\mathrm{e}\langle \mathscr{D}(x_0 + \varepsilon y)\psi, \psi\rangle dy \geqslant 0.$$

Letting $\varepsilon \to 0^+$ we find

$$\begin{aligned} &\int_{B_1} (1 - 2/p)|\psi|^{-2}\, \mathbb{R}\mathrm{e}\langle \mathscr{B}^h(x_0)\psi, \psi\rangle\, \mathbb{R}\mathrm{e}\langle \psi, \partial_h\psi\rangle dy + \\ &\qquad - \int_{B_1} \mathbb{R}\mathrm{e}\langle \mathscr{B}^h(x_0)\partial_h\psi, \psi\rangle dy \geqslant 0 \end{aligned} \tag{21}$$

for almost any $x_0 \in \Omega$ and for any $\psi \in (C^1_0(B_1))^m$.

Fix now $1 \leqslant k \leqslant n$. Take $\alpha \in (C^1_0(\mathbb{R}))^m$ and $\beta \in C^1_0(\mathbb{R}^{n-1})$. Consider

$$\psi_\varepsilon(x) = \alpha((x_k - (x_0)_k)/\varepsilon)\, \beta(y_k),$$

where y_k denotes the $(n-1)$-dimensional vector $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$. Choose ε, α and β in such a way $\mathrm{spt}\, \psi_\varepsilon \subset \Omega$.

We have

$$\sum_{h=1}^{n} \int_\Omega |\psi_\varepsilon|^{-2}\, \mathbb{R}\mathrm{e}\langle \mathscr{B}^h(x_0)\psi_\varepsilon, \psi_\varepsilon\rangle\, \mathbb{R}\mathrm{e}\langle \psi_\varepsilon, \partial_h\psi_\varepsilon\rangle dx =$$
$$\int_{\mathbb{R}} |\alpha(t)|^{-2}\, \mathbb{R}\mathrm{e}\langle \mathscr{B}^k(x_0)\, \alpha(t), \alpha(t)\rangle\, \mathbb{R}\mathrm{e}\langle \alpha(t), \alpha'(t)\rangle dt \int_{\mathbb{R}^{n-1}} |\beta(y_k)|^2 dy_k +$$
$$\varepsilon \sum_{\substack{h=1 \\ h \neq k}}^{n} \int_{\mathbb{R}} \mathbb{R}\mathrm{e}\langle \mathscr{B}^h(x_0)\, \alpha(t), \alpha(t)\rangle dt \int_{\mathbb{R}^{n-1}} \mathbb{R}\mathrm{e}(\beta(y_k)\partial_h\overline{\beta(y_k)}) dy_k\,,$$

$$
\begin{gathered}
\int_\Omega \mathrm{Re}\langle \mathscr{B}^h(x_0)\partial_h\psi_\varepsilon, \psi_\varepsilon\rangle dx = \\
\int_{\mathbb{R}} \mathrm{Re}\langle \mathscr{B}^k(x_0)\alpha'(t), \alpha(t)\rangle dt \int_{\mathbb{R}^{n-1}} |\beta(y_k)|^2 dy_k + \\
\varepsilon \sum_{\substack{h=1\\ h\neq k}}^n \int_{\mathbb{R}} \mathrm{Re}\langle \mathscr{B}^h(x_0)\, \alpha(t), \alpha(t)\rangle dt \int_{\mathbb{R}^{n-1}} \mathrm{Re}(\overline{\beta(y_k)}\, \partial_h\beta(y_k))\, dy_k \,.
\end{gathered}
$$

Therefore

$$
\begin{gathered}
\lim_{\varepsilon\to 0^+} \Big(\int_\Omega (1-2/p)|\psi_\varepsilon|^{-2}\, \mathrm{Re}\langle \mathscr{B}^h(x_0)\psi_\varepsilon, \psi_\varepsilon\rangle\, \mathrm{Re}\langle \psi_\varepsilon, \partial_h\psi_\varepsilon\rangle dx + \\
- \int_\Omega \mathrm{Re}\langle \mathscr{B}^h(x_0)\partial_h\psi_\varepsilon, \psi_\varepsilon\rangle dx \Big) = \\
\int_{\mathbb{R}} \Big((1-2/p)|\alpha(t)|^{-2}\, \mathrm{Re}\langle \mathscr{B}^h(x_0)\, \alpha(t), \alpha(t)\rangle\, \mathrm{Re}\langle \alpha(t), \alpha'(t)\rangle dt + \\
- \mathrm{Re}\langle \mathscr{B}^k(x_0)\alpha'(t), \alpha(t)\rangle \Big) dt \int_{\mathbb{R}^{n-1}} |\beta(y_k)|^2 dy_k
\end{gathered}
$$

and from (21) it follows

$$
\begin{gathered}
(1-2/p) \int_{\mathbb{R}} |\alpha(t)|^{-2}\, \mathrm{Re}\langle \mathscr{B}^k(x_0)\, \alpha(t), \alpha(t)\rangle\, \mathrm{Re}\langle \alpha(t), \alpha'(t)\rangle dt + \\
- \int_{\mathbb{R}} \mathrm{Re}\langle \mathscr{B}^k(x_0)\alpha'(t), \alpha(t)\rangle dt \geqslant 0.
\end{gathered}
$$

The arbitrariness of α shows that the operator with constant coefficients $\mathscr{B}^k(x_0)u'$ is L^p-dissipative. Theorem 1 applies and (17)-(18) is satisfied.

As we already proved in the sufficiency part, (17)-(18) implies that inequality (20) can be written as

$$
\int_\Omega (p^{-1}\, \mathrm{Re}\langle (\partial_h\, \mathscr{B}^h)v, v\rangle - \mathrm{Re}\langle \mathscr{D}\, v, v\rangle) dx \geqslant 0, \tag{22}
$$

for any $v \in (C^1_0(\Omega))^m$. Take

$$
v_\varepsilon(x) = \varepsilon^{-n/2}\zeta\, \varphi((x-x_0)/\varepsilon)
$$

where $x_0 \in \Omega$, $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$, φ is a real scalar function in $C^1_0(\mathbb{R}^n)$, $\operatorname{spt}\varphi$ is contained in the unit ball, ε is sufficiently small and

$$
\int_{\mathbb{R}^n} \varphi^2(x) dx = 1\,.
$$

Putting v_ε in (22) and letting $\varepsilon \to 0^+$ we obtain (19) for almost any $x_0 \in \Omega$.

□

Let us consider now instead of (16), the operator

$$\mathscr{B}^h(x)\partial_h u + \partial_h(\mathscr{C}^h(x)u) + \mathscr{D}(x)u, \tag{23}$$

where $\mathscr{B}^h$, $\mathscr{C}^h$, $\mathscr{D}$, $\partial_h \mathscr{B}^h$ and $\partial_h \mathscr{C}^h$ are matrices with complex locally integrable entries.

Theorem 3 *The operator* (23) *is L^p-dissipative if, and only if, the following conditions are satisfied*

1.

$$\mathscr{B}^h(x) + \mathscr{C}^h(x) = b_h(x)\, I, \qquad \text{if } p \neq 2, \tag{24}$$

$$\mathscr{B}^h(x) + \mathscr{C}^h(x) = (\mathscr{B}^h)^*(x) + (\mathscr{C}^h)^*(x), \qquad \text{if } p = 2, \tag{25}$$

for almost any $x \in \Omega$ and $h = 1, \dots, n$. Here b_h are real locally integrable functions ($1 \leqslant h \leqslant n$).

2.

$$\mathbb{R}\mathrm{e}\langle (p^{-1}\partial_h \mathscr{B}^h(x) - p'^{-1}\partial_h \mathscr{C}^h(x) - \mathscr{D}(x))\zeta, \zeta\rangle \geqslant 0 \tag{26}$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$.

Proof. It is sufficient to write the operator (23) as

$$(\mathscr{B}^h(x) + \mathscr{C}^h(x))\partial_h u + \partial_h(\mathscr{C}^h(x))\, u + \mathscr{D}(x)u$$

and apply Theorem 2, observing that

$$p^{-1}\partial_h(\mathscr{B}^h + \mathscr{C}^h) - \partial_h \mathscr{C}^h = p^{-1}\partial_h \mathscr{B}^h - p'^{-1}\partial_h \mathscr{C}^h\,.$$

□

5 Sufficient conditions for the L^p-dissipativity of certain systems of partial differential operators of the second order

As a by-product of the results obtained in the previous section, we obtain now sufficient conditions for the L^p-dissipativity of a class of systems of partial differential equations of the second order.

Theorem 4 *Let E be the operator*

$$Eu = \partial_h(\mathscr{A}^h(x)\partial_h u) + \mathscr{B}^h(x)\partial_h u + \mathscr{D}(x)u, \tag{27}$$

where $\mathscr{A}^h(x) = \{a^h_{ij}(x)\}$ are $m \times m$ matrices with complex locally integrable entries and the matrices $\mathscr{B}^h(x)$, $\mathscr{D}(x)$ satisfy the hypothesis of Theorem 2. If

$$\begin{aligned}\mathbb{R}\mathrm{e}\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathbb{R}\mathrm{e}\langle \mathscr{A}^h(x)\omega,\omega\rangle(\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle)^2 \\ -(1-2/p)\,\mathbb{R}\mathrm{e}(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\,\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle \geqslant 0\end{aligned} \tag{28}$$

for almost every $x \in \Omega$ and for every $\lambda,\omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1,\dots,n$, and conditions (17)-(18) *and* (19) *are satisfied, the operator E is L^p-dissipative.*

Proof. Theorem 2 shows that the operator of the first order

$$E_1 = \mathscr{B}^h(x)\partial_h u + \mathscr{D}(x)u$$

is L^p-dissipative. Moreover, inequality (28) is necessary and sufficient for the L^p-dissipativity of the second order operator

$$E_0 = \partial_h(\mathscr{A}^h(x)\partial_h u) \tag{29}$$

(see [4, Theorem 4.20, p.115]). Since $E = E_0 + E_1$, the result follows at once. □

Consider now the operator (27) in the scalar case (i.e. $m = 1$)

$$\partial_h(a^h(x)\partial_h u) + b^h(x)\partial_h u + d(x)u$$

(a^h, b^h and d being scalar functions). In this case such an operator can be written in the form

$$Eu = \mathrm{div}(\mathscr{A}(x)\nabla u) + \mathscr{B}(x)\nabla u + d(x)\,u \tag{30}$$

where $\mathscr{A} = \{c_{hk}\}$, $c_{hh} = a^h$, $c_{hk} = 0$ if $h \neq k$ and $\mathscr{B} = \{b^h\}$. For such an operator one can show that (28) is equivalent to

$$\frac{4}{pp'}\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\xi,\xi\rangle + \langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\eta,\eta\rangle - 2(1-2/p)\langle \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\xi,\eta\rangle \geqslant 0 \qquad (31)$$

for almost any $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^n$ (see [4, Remark 4.21, p.115]). Condition (31) is in turn equivalent to the inequality:

$$|p-2|\,|\langle \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\xi,\xi\rangle \qquad (32)$$

for almost any $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$ (see [4, Remark 2.8, p.42]). We have then

Theorem 5 *Let E be the scalar operator* (30) *where $\mathscr{A}$ is a diagonal matrix. If inequality* (32) *and conditions* (17)-(18) *and* (19) *are satisfied, the operator E is L^p-dissipative.*

More generally, consider the scalar operator (30) with a matrix $\mathscr{A} = \{a_{hk}\}$ not necessarily diagonal. By using [4, Theorem 2.7, p.40], we get

Theorem 6 *Let the matrix $\mathbb{I}\mathrm{m}\,\mathscr{A}$ be symmetric, i.e. $\mathbb{I}\mathrm{m}\,\mathscr{A}^t = \mathbb{I}\mathrm{m}\,\mathscr{A}$. If inequality* (32) *and conditions* (17)-(18) *and* (19) *are satisfied, the operator* (30) *is L^p-dissipative.*

Coming back to system (27), in case the main part of the operator (27) has real coefficients, i.e. the matrices $\mathscr{A}^h$ have real locally integrable entries, we have also

Theorem 7 *Let E be the operator* (27) *where $\mathscr{A}^h$ are real matrices. Let us suppose $\mathscr{A}^h = (\mathscr{A}^h)^t$ and $\mathscr{A}^h \geqslant 0$ ($h = 1,\dots,n$). If conditions* (17)-(18) *and* (19) *are satisfied and*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 (\mu_1^h(x)+\mu_m^h(x))^2 \leqslant \mu_1^h(x)\,\mu_m^h(x) \qquad (33)$$

for almost every $x \in \Omega$, $h = 1,\dots,n$, where $\mu_1^h(x)$ and $\mu_m^h(x)$ are the smallest and the largest eigenvalues of the matrix $\mathscr{A}^h(x)$ respectively, the operator E is L^p-dissipative. In the particular case $m = 2$ condition (33) *is equivalent to*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 (\mathrm{tr}\,\mathscr{A}^h(x))^2 \leqslant \det \mathscr{A}^h(x)$$

for almost every $x \in \Omega$, $h = 1,\dots,n$.

Proof. Theorem 4.22 in [4, p.116] shows that (33) holds if and only if the operator (29) is L^p-dissipative. Combining this with Theorem 2 gives the result. □

Results similar to Theorems 4, 6 and 7 holds for the operator

$$\partial_h(\mathscr{A}^h(x)\partial_h u) + \mathscr{B}^h(x)\partial_h u + \partial_h(\mathscr{C}^h(x)u) + \mathscr{D}(x)u\,.$$

We have just to replace conditions (17), (18) and (19) by (24), (25) and (26) respectively.

6 The L^p-quasi-dissipativity

The operator E is said to be L^p-quasi-dissipative if the operator $E - \omega I$ is L^p-dissipative for a suitable $\omega \geqslant 0$. This means that there exists $\omega \geqslant 0$ such that

$$\mathbb{R}\mathrm{e} \int_\Omega \langle Eu, |u|^{p-2}u\rangle dx \leqslant \omega \|u\|_p^p$$

for any u in the domain of E.

The aim of this section is to provide necessary and sufficient conditions for the L^p-quasi-dissipativity of a partial differential operator of the first order.

Lemma 3 *The operator* (16) *is L^p-quasi-dissipative if, and only if, there exists $\omega \geqslant 0$ such that*

$$\begin{aligned}(1-2/p)\int_\Omega |v|^{-2}\,\mathbb{R}\mathrm{e}\langle \mathscr{B}^h\, v, v\rangle\, \mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle dx - \mathbb{R}\mathrm{e}\int_\Omega \langle \mathscr{B}^h\, \partial_h v, v\rangle dx + \\ - \mathbb{R}\mathrm{e}\int_\Omega \langle \mathscr{D}\, v, v\rangle dx \geqslant -\omega \int_\Omega |v|^2 dx\end{aligned} \tag{34}$$

for any $v \in (C_0^1(\Omega))^m$.

Proof. The result follows immediately from Lemma 1. □

Theorem 8 *Let E be the operator* (16)*, in which the entries of $\mathscr{D}$ and the entries of $\partial_h \mathscr{B}^h$ belong to $L^\infty(\Omega)$. The operator E is L^p-quasi-dissipative if, and only if, condition* (17)-(18) *is satisfied.*

Proof. *Necessity.* Arguing as in the first part of the proof of Theorem 2, we find that (34) implies that the ordinary differential operator $\mathscr{B}^k(x_0)u'$ is L^p-dissipative, for almost any $x_0 \in \Omega$, $k = 1, \ldots, m$. As we know, this in turn implies that (17)-(18) is satisfied.

Sufficiency. As in the proof of Theorem 2, condition (17)-(18) gives

$$(1-2/p)\int_\Omega |v|^{-2}\,\mathbb{R}\mathrm{e}\langle \mathscr{B}^h\, v, v\rangle\, \mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle dx - \mathbb{R}\mathrm{e}\int_\Omega \langle \mathscr{B}^h\, \partial_h v, v\rangle dx = p^{-1}\,\mathbb{R}\mathrm{e}\int_\Omega \langle (\partial_h\, \mathscr{B}^h)v, v\rangle dx\, .$$

We define ω by setting

$$\omega = \max\left\{0,\ \operatorname*{ess\,sup}_{\substack{x\in\Omega\\ \zeta\in\mathbb{C}^m, |\zeta|=1}} \mathbb{R}\mathrm{e}\langle (\mathscr{D}(x) - p^{-1}\partial_h\, \mathscr{B}^h(x))\zeta, \zeta\rangle \right\}. \tag{35}$$

The left hand side of (34) being equal to

$$\mathbb{R}\mathrm{e}\int_\Omega \langle (p^{-1}\partial_h\, \mathscr{B}^h(x) - \mathscr{D}(x))v, v\rangle dx\, ,$$

inequality (34) follows from (35).

□

Remark. It is clear from the proof of Theorem 8 that, if

$$\lambda = \operatorname*{ess\,sup}_{\substack{x\in\Omega\\ \zeta\in\mathbb{C}^m, |\zeta|=1}} \mathbb{R}\mathrm{e}\langle (\mathscr{D}(x) - p^{-1}\partial_h\, \mathscr{B}^h(x))\zeta, \zeta\rangle < 0,$$

we have not only the L^p-dissipativity of the operator E, but also the stronger inequality

$$\mathbb{R}\mathrm{e}\int_\Omega \langle Eu, |u|^{p-2}u\rangle dx \leqslant \lambda \|u\|_p^p$$

with $\lambda < 0$.

The next result follows from similar arguments to those above.

Theorem 9 *Let E be the operator* (23)*, in which the entries of $\mathscr{D}$, $\partial_h\, \mathscr{B}^h$ and $\partial_h\, \mathscr{C}^h$ belong to $L^\infty(\Omega)$. The operator E is L^p-quasi-dissipative if, and only if, condition* (24)-(25) *is satisfied.*

7 The angle of dissipativity

Let E be the operator (16) and assume it is L^p-dissipative. The aim of this Section is to determine the angle of dissipativity of E. This means to find the set of complex values z such that zE is still L^p-dissipative.

What comes out is that the angle of dissipativity of E is always a zero-sided angle, unless E degenerates to the operator $\mathscr{D}\,u$.

We start recalling the following Lemma

Lemma 4 *Let P and Q two real measurable functions defined on a set $\Omega \subset \mathbb{R}^n$. Let us suppose that $P(x) \geqslant 0$ almost everywhere. The inequality*

$$P(x)\, \cos\vartheta - Q(x)\, \sin\vartheta \geqslant 0 \qquad (\vartheta \in [-\pi, \pi])$$

holds for almost every $x \in \Omega$ if and only if

$$\operatorname{arccot}\,[\operatorname*{ess\,inf}_{x\in\Xi}\,(Q(x)/P(x))] - \pi \leqslant \vartheta \leqslant \operatorname{arccot}\,[\operatorname*{ess\,sup}_{x\in\Xi}\,(Q(x)/P(x))]$$

where $\Xi = \{x \in \Omega \mid P^2(x) + Q^2(x) > 0\}$ and we set

$$Q(x)/P(x) = \begin{cases} +\infty & \text{if } P(x) = 0,\ Q(x) > 0 \\ -\infty & \text{if } P(x) = 0,\ Q(x) < 0. \end{cases}$$

Here $0 < \operatorname{arccot} y < \pi$, $\operatorname{arccot}(+\infty) = 0$, $\operatorname{arccot}(-\infty) = \pi$ and

$$\operatorname*{ess\,inf}_{x\in\Xi}\,(Q(x)/P(x)) = +\infty, \quad \operatorname*{ess\,sup}_{x\in\Xi}\,(Q(x)/P(x)) = -\infty$$

if Ξ has zero measure.

For a proof we refer to [2, p.236] (see also [4, p.138]).

Theorem 10 *Let E be the operator* (16) *and suppose it is L^p-dissipative. Set*

$$P(x,\zeta) = -\operatorname{\mathbb{R}e}\langle \mathscr{D}(x)\zeta, \zeta\rangle,$$
$$Q(x,\zeta) = -\operatorname{\mathbb{I}m}\langle \mathscr{D}(x))\zeta, \zeta\rangle,$$
$$\Xi = \{(x,\zeta) \in \Omega \times \mathbb{C}^m \mid |\zeta| = 1,\ P^2(x,\zeta) + Q^2(x,\zeta) > 0\}.$$

If

$$\mathscr{B}^h(x) = 0 \quad a.e. \quad (h = 1, \dots, n), \tag{36}$$

the operator zE is L^p-dissipative if, and only if,

$$\vartheta_- \leqslant \arg z \leqslant \vartheta_+ \,, \tag{37}$$

where

$$\vartheta_- = \operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\zeta)\in\Xi} Q(x,\zeta)/P(x,\zeta)\right) - \pi$$
$$\vartheta_+ = \operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\zeta)\in\Xi} Q(x,\zeta)/P(x,\zeta)\right).$$

If condition (36) *is not satisfied and there exist $\zeta \in \mathbb{C}^m$ and $x \in \Omega$ such that*

$$\mathbb{R}\mathrm{e}\langle (p^{-1}\partial_h \mathscr{B}^h(x) - \mathscr{D}(x))\zeta, \zeta\rangle > 0, \tag{38}$$

the angle of dissipativity of E is zero, i.e. zE is dissipative if, and only if, $\mathbb{I}\mathrm{m}\, z = 0$, $\mathbb{R}\mathrm{e}\, z \geqslant 0$. Finally, if condition (36) *is not satisfied and*

$$\mathbb{R}\mathrm{e}\langle (p^{-1}\partial_h \mathscr{B}^h(x) - \mathscr{D}(x))\zeta, \zeta\rangle = 0, \tag{39}$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$, the operator zE is L^p-dissipative if, and only if, $\mathbb{R}\mathrm{e}\, z = 0$.

Proof. Suppose (36) holds. It is obvious that zE satisfies condition (17)-(18) for any $z \in \mathbb{C}$. Since E is L^p-dissipative, $P(x,\zeta) \geqslant 0$ for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$ (see (19)). In view of Lemma 4 we have

$$\mathbb{R}\mathrm{e}\langle -z\, \mathscr{D}(x))\zeta, \zeta\rangle \geqslant 0$$

if, and only if, (37) holds.

Assumue now that (36) is not satisfied. Condition (17)-(18) is valid for all the matrices $z\,\mathscr{B}^h$ if, and only if, $\mathbb{I}\mathrm{m}\, z = 0$. Moreover, suppose that there exist $\zeta \in \mathbb{C}^m$ and $x \in \Omega$ such that (38) holds; therefore

$$\mathbb{R}\mathrm{e}\left(\mathbb{R}\mathrm{e}\, z\langle (p^{-1}\partial_h \mathscr{B}^h(x) - \mathscr{D}(x))\zeta, \zeta\rangle\right) \geqslant 0 \tag{40}$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$ if, and only if, $\mathbb{R}\mathrm{e}\, z \geqslant 0$.

Assume instead (39) for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$; then condition (40) holds for any $z \in \mathbb{C}$. This means that zE is L^p-dissipative if, and only if, $\mathbb{I}\mathrm{m}\, z = 0$. □

Remark. Suppose (36) is not satisfied and that (39) holds for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$. In this case we have “two” zero sided angles of dissipativity and not only one. This should not surprise. Indeef for such operators we have the L^p-dissipativity of both E and $(-E)$. This is evident, e.g., for the operators with constant coefficients considered in Theorem 1.

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