Criterion for the functional dissipativity of the Lamé operator

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Dedicated to Natasha and Sasha Movchan on the occasion of their jubilee

Abstract. After introducing the concept of functional dissipativity of the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^N$ for systems of partial differential operators of the form $\partial_h(\mathscr{A}^{hk}(x)\partial_k)$ ($\mathscr{A}^{hk}(x)$ being $m \times m$ matrices with complex valued L^∞ entries), we find necessary and sufficient conditions for the functional dissipativity of the two-dimensional Lamé system. **As an application of our theory we provide two regularity results for the equilibrium problem for a body which is fixed along its boundary.**

1 Introduction

The concept of functional dissipativity of a linear operator was recently introduced in [8]. If A is the scalar second order partial differential operator $\nabla(\mathscr{A}\,\nabla)$, where $\mathscr{A}$ is a square matrix whose entries are complex valued L^∞-functions defined in the domain $\Omega \subset \mathbb{R}^N$, we say that A is functional dissipative with respect to a given positive function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ if

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla(\varphi(|u|)\,u)\rangle\, dx \geqslant 0$$

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for any $u \in \mathring{H}^1(\Omega)$ such that $\varphi(|u|)\, u \in \mathring{H}^1(\Omega)$.

As explained in the introduction of [8], a motivation for the study of this concept comes from the decrease of the Luxemburg norm of solutions of the Cauchy–Dirichlet problem

$$\begin{cases} u' = Au \\ u(0) = u_0 \,. \end{cases}$$

Here the Luxemburg norm is taken in the Orlicz space of functions u for which there exists $\alpha > 0$ such that

$$\int_\Omega \Phi(\alpha\, |u|)\, dx < +\infty \,,$$

where the Young function Φ is related to φ by

$$\Phi(s) = \int_0^s \sigma\, \varphi(\sigma)\, d\sigma \,.$$

The functional dissipativity is an extension of the concept of L^p-dissipativity, which is obtained taking $\varphi(t) = t^{p-2}$ $(1 < p < \infty)$. In a series of papers [3, 4, 5, 7] we have studied the problem of characterizing the L^p-dissipativity of scalar and matrix partial differential operators. In the monograph [6] this theory is considered in the more general frame of semi-bounded operators. For a short survey of our results we refer to the introduction of [8].

The aim of the present paper is to study the functional dissipativity of the two-dimensional Lamé operator

$$Eu = \nabla \cdot (\lambda(x)\ \mathrm{div}\, u\, I + \mu(x)\, (\nabla u + (\nabla u)^T)\,. \tag{1}$$

The Lamé parameters λ and μ are supposed to be real valued L^∞ functions satisfying the usual ellipticity conditions (see (32) below). Previously we have considered the case of constant Lamé parameters and proved that

$$Eu = \mu\Delta u + (\lambda + \mu)\nabla\, \mathrm{div}\, u\,, \tag{2}$$

is L^p-dissipative if and only if

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,,$$

where ν is the Poisson ratio (see [4, Th. 3, p.244]). Note that this condition can be written in terms of Lamé constants as

$$\left(1-\frac{2}{p}\right)^2 \leqslant 1-\left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2.$$

In the first part of the present paper we study the functional dissipativity of a general system of partial differential operators of the form

$$A=\partial_h(\mathscr{A}^{hk}(x)\partial_k) \tag{3}$$

where $\mathscr{A}^{hk}(x)=\{a_{ij}^{hk}(x)\}$ are $m\times m$ matrices whose elements are complex valued L^∞-functions functions defined in a domain $\Omega\subset\mathbb{R}^N$ $(1\leqslant i,j\leqslant m,\ 1\leqslant h,k\leqslant N)$. The Lamé system is obtained taking

$$a_{ij}^{hk}(x)=\lambda(x)\delta_{ih}\delta_{jk}+\mu(x)(\delta_{ij}\delta_{hk}+\delta_{ik}\delta_{hj})\,.$$

Concerning the general systems (3), the operator A is functional dissipative (or L^Φ-dissipative) if

$$\mathbb{R}\mathrm{e}\int_\Omega\langle\mathscr{A}^{hk}\,\partial_k u,\partial_h(\varphi(|u|)\,u)\rangle\,dx\geqslant 0$$

for any $u\in[\mathring{H}^1(\Omega)]^m$ such that $\varphi(|u|)\,u\in[\mathring{H}^1(\Omega)]^m$. We say also that the operator A is strict functional dissipative if there exists $\kappa>0$ such that

$$\mathbb{R}\mathrm{e}\int_\Omega\langle\mathscr{A}^{hk}\,\partial_k u,\partial_h(\varphi(|u|)\,u)\rangle\,dx\geqslant\kappa\int_\Omega|\nabla(\sqrt{\varphi(|u|)}\,u)|^2dx$$

for any $u\in[\mathring{H}^1(\Omega)]^m$ such that $\varphi(|u|)\,u\in[\mathring{H}^1(\Omega)]^m$.

The last concept is strictly related to the concept of p-elliptic operator. This was considered in a series of papers by Carbonaro and Dragičević [1, 2], Dindoš and Pipher [10, 11, 12, 13], Egert [14]. It is worthwhile to remark that, if the partial differential operator has no lower order terms, the concepts of p-ellipticity and strict L^p-dissipativity coincide. Our results show that the operator A is strict L^p-dissipative, i.e. p-elliptic, if and only if there exists $\kappa>0$ such that $A-\kappa\Delta$ is L^p-dissipative (see Corollary 1 below).

Concerning Lamé system with constant Lamé parameters, in [4, Corollary 1, p.246]) we proved also that there exists $\kappa>0$ such that $E-\kappa\Delta$ is L^p-dissipative if and only if

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2<\frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,,$$

i.e.

$$\left(1-\frac{2}{p}\right)^2 < 1-\left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2 .$$

As remarked before, this is equivalent to say that E is strict L^p-dissipative, i.e. E is p-elliptic. The last result was recently extended to variable Lamé parameters by Dindoš, Li and Pipher [9]. It must be pointed out that these Authors introduce an auxiliary function $r(x)$ (see [9, formula (85)]) which generates some first order terms in the partial differential operator. In the definition of p-ellipticity these terms do not play any role, while they have some role in the dissipativity. Therefore our and their results do not seem to be completely equivalent.

The main result of the present paper is that, assuming that the BMO seminorm of the function $\mu^2\,(\lambda+3\mu)^{-1}$ is sufficiently small, elasticity operator (1) is strict functional dissipative if and only if

$$\Lambda_\infty^2 < 1-\operatorname*{ess\,sup}_{x\in\Omega}\left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2 ,$$

where $\Lambda_\infty^2=\sup_{t>0}\Lambda^2(t)$ and Λ is the function defined by the relation

$$\Lambda\left(s\sqrt{\varphi(s)}\right) = -\frac{s\,\varphi'(s)}{s\,\varphi'(s)+2\,\varphi(s)} .$$

For the theory of BMO functions we refer to Stein [21, Chapter IV].

This paper is organized as follows. In Section 2 we specify the class of functions φ we are going to consider, introduce some related functions and recall some results obtained in [8].

Section 3 is devoted to prove necessary and sufficient conditions for the functional dissipativity of the general system of the second order in divergence form (3). Specifically we prove the equivalence between the functional dissipativity (strict functional dissipativity) of such an operator and the positiveness (strict positiveness) of the real part of a certain form in $[\mathring{H}^1(\Omega)]^m$.

In Section 4 we give algebraic necessary conditions for the functional dissipativity and the strict functional dissipativity of a general system when $N=2$. We remark that we prove these results under the additional assumption that the function $|s\,\varphi'(s)/\varphi(s)|$ is not decreasing.

The main result concerning the strict functional dissipativity of two-dimensional elasticity operator is proved in Section 5.

As an application of our theory, in the last Section 6 we provide two regularity results for the energy solution of the Dirichlet problem for Lamé

system with zero data on the boundary. This represents the equilibrium problem in linear elasticity for a body which is fixed along its boundary.

2 Preliminaries

In this Section we recall some definitions and results obtained in [8].

Let Ω be an open set in $\mathbb{R}^N$. As usual, by $\mathring{C}^\infty(\Omega)$ we denote the space of complex valued C^∞ functions having compact support in Ω and by $\mathring{H}^1(\Omega)$ the closure of $\mathring{C}^\infty(\Omega)$ in the norm

$$\int_\Omega (|u|^2 + |\nabla u|^2) dx,$$

∇u being the gradient of the function u.

The inner product either in $\mathbb{C}^N$ or in $\mathbb{C}$ is denoted by $\langle \cdot, \cdot \rangle$ and the bar denotes complex conjugation.

From now on we assume that φ is a positive function satisfying the following conditions

(i) $\varphi \in C^1((0, +\infty))$;

(ii) $(s\,\varphi(s))' > 0$ for any $s > 0$;

(iii) the range of the strictly increasing function $s\,\varphi(s)$ is $(0, +\infty)$;

(iv) there exist two positive constants C_1, C_2 and a real number $r > -1$ such that

$$C_1 s^r \leqslant (s\varphi(s))' \leqslant C_2\, s^r, \qquad s \in (0, s_0)$$

for a certain $s_0 > 0$. If $r = 0$ we require more restrictive conditions: there exists the finite limit $\lim_{s\to 0^+} \varphi(s) = \varphi_+(0) > 0$ and $\lim_{s\to 0^+} s\,\varphi'(s) = 0$.

(v) There exists $s_1 > s_0$ such that

$$\varphi'(s) \geqslant 0 \text{ or } \varphi'(s) \leqslant 0 \qquad \forall\, s \geqslant s_1.$$

The condition (iv) prescribes the behaviour of the function φ in a neighborhood of the origin, while (v) concerns the behaviour for large s.

Let us denote by $t\,\psi(t)$ the inverse function of $s\,\varphi(s)$. The functions

$$\Phi(s) = \int_0^s \sigma\,\varphi(\sigma)\, d\sigma, \qquad \Psi(s) = \int_0^s \sigma\,\psi(\sigma)\, d\sigma$$

are conjugate Young functions.

Lemma 1 ([8, Lemma 1]) *The function φ satisfies conditions (i)-(v) if and only if the function ψ satisfies the same conditions with $-r/(r+1)$ instead of r.*

We have also

$$\sqrt{\psi(|w|)}\, w = \sqrt{\varphi(|u|)}\, u\,, \tag{4}$$

where $w = \varphi(|u|)\, u$ (see [8, formula (43)]).

We need to introduce also some other functions.

Let $\zeta(t)$ be the inverse of the strictly increasing function $s\sqrt{\varphi(s)}$., i.e. $\zeta(t) = \left(s\sqrt{\varphi(s)}\right)^{-1}$. The range of $s\sqrt{\varphi(s)}$ is $(0,+\infty)$ and $\zeta(t)$ belongs to $C^1((0,+\infty))$. Define

$$\Theta(t) = \zeta(t)/t; \quad \Lambda(t) = t\,\Theta'(t)/\Theta(t)\,. \tag{5}$$

One can prove (see [8, formula (6)]) that

$$\Lambda\left(s\sqrt{\varphi(s)}\right) = -\frac{s\,\varphi'(s)}{s\,\varphi'(s) + 2\,\varphi(s)}\,. \tag{6}$$

Lemma 2 ([8, Lemma 2]) *Let $\widetilde{\zeta}(t)$ the inverse function of $t\,\sqrt{\psi(t)}$ and define, as in* (5)*,*

$$\widetilde{\Theta}(t) = \widetilde{\zeta}(t)/t\,; \quad \widetilde{\Lambda}(t) = t\,\widetilde{\Theta}'(t)/\widetilde{\Theta}(t)\,.$$

We have

$$\widetilde{\Theta}(t) = \frac{1}{\Theta(t)}\,, \qquad \widetilde{\Lambda}(t) = -\Lambda(t) \tag{7}$$

for any $t > 0$.

We write also two equalities given in [8]:

$$\Theta^2(t)\,\varphi[\zeta(t)] = 1\,, \quad \forall\, t > 0, \tag{8}$$

and

$$\Theta(t)\,\varphi'[\zeta(t)]\,[t\,\Theta'(t) + \Theta(t)] + \Theta'(t)\,\varphi[\zeta(t)] = -\Theta'(t)/\Theta^2(t)\,, \quad \forall\, t > 0. \tag{9}$$

Finally we note the following Lemma, proved in the scalar case in [8, Lemma 3]. The extension to vector valued functions is immediate.

Lemma 3 *If $u \in [H^1(\Omega)]^m$ $([\mathring{H}^1(\Omega)]^m)$ is such that $\varphi(|u|)\, u \in [H^1(\Omega)]^m$ $([\mathring{H}^1(\Omega)]^m)$, then $\sqrt{\varphi(|u|)}\, u$ belongs to $[H^1(\Omega)]^m$ $([\mathring{H}^1(\Omega)]^m)$.*

3 Necessary and sufficient conditions for the functional dissipativity of general systems

Let Ω be a domain of $\mathbb{R}^N$ and let A be the operator

$$A = \partial_h(\mathscr{A}^{hk}(x)\partial_k) \tag{10}$$

where $\partial_k = \partial/\partial x_k$ and $\mathscr{A}^{hk}(x) = \{a_{ij}^{hk}(x)\}$ are $m \times m$ matrices whose elements are complex valued L^∞-functions functions defined in Ω $(1 \leqslant i, j \leqslant m,\ 1 \leqslant h, k \leqslant N)$. Here and in the sequel, we adopt the standard summation convention on repeated indices.

The operator A is said to be L^Φ-dissipative or functional dissipative if

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}^{hk}\, \partial_k u, \partial_h(\varphi(|u|)\, u)\rangle\, dx \geqslant 0 \tag{11}$$

for any $u \in [\mathring{H}^1(\Omega)]^m$ such that $\varphi(|u|)\, u \in [\mathring{H}^1(\Omega)]^m$.

We say that the operator A is strict L^Φ-dissipative if there exists $\kappa > 0$ such that

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}^{hk}\, \partial_k u, \partial_h(\varphi(|u|)\, u)\rangle\, dx \geqslant \kappa \int_\Omega |\nabla(\sqrt{\varphi(|u|)}\, u)|^2 dx \tag{12}$$

for any $u \in [\mathring{H}^1(\Omega)]^m$ such that $\varphi(|u|)\, u \in [\mathring{H}^1(\Omega)]^m$. We remark that in view of Lemma 3 the right hand side is finite.

We have the following Lemma

Lemma 4 *If the operator A is strict L^Φ-dissipative then*

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}^{hk}\, \partial_k u, \partial_h(\varphi(|u|)\, u)\rangle\, dx \geqslant \frac{\kappa}{4} \int_\Omega \varphi(|u|)\, |\nabla u|^2 dx$$

for any $u \in [\mathring{H}^1(\Omega)]^m$ such that $\varphi(|u|)\, u \in [\mathring{H}^1(\Omega)]^m$, where κ is the constant in (12).

Proof. A direct computation shows that

$$|\nabla(\sqrt{\varphi(|u|)}\, u)|^2 = \left(\frac{(\varphi'(|u|)^2 |u|^2}{4\,\varphi(|u|)} + \varphi'(|u|)\, |u|\right) |\nabla|u||^2 + \varphi(|u|)\, |\nabla u|^2 \tag{13}$$

on the set where $u \neq 0$. We can write

$$
\begin{aligned}
&|\nabla(\sqrt{\varphi(|u|)}\,u)|^2 \\
&= \left(\frac{(\varphi'(|u|)^2|u|^2}{4\,\varphi(|u|)} + \varphi'(|u|)\,|u| + \varphi(|u|) \right) |\nabla|u||^2 \\
&\quad +\varphi(|u|)\,(|\nabla u|^2 - |\nabla|u||^2).
\end{aligned}
\tag{14}
$$

On the other hand condition (ii) implies

$$
\frac{t\varphi'(t) + 2\varphi(t)}{\varphi(t)} = \frac{t\varphi'(t) + \varphi(t)}{\varphi(t)} + 1 \geqslant 1
$$

for any $t > 0$ and then

$$
\left(\frac{t\varphi'(t)}{2\varphi(t)} + 1 \right)^2 \geqslant \frac{1}{4}\,.
$$

From (14) it follows that

$$
|\nabla(\sqrt{\varphi(|u|)}\,u)|^2 \geqslant \frac{\varphi(|u|)}{4}\,|\nabla|u||^2 + \varphi(|u|)\,(|\nabla u|^2 - |\nabla|u||^2)
$$

and this gives

$$
|\nabla(\sqrt{\varphi(|u|)}\,u)|^2 \geqslant \frac{\varphi(|u|)}{4}\,|\nabla u|^2, \tag{15}
$$

which proves the Lemma. □

In the particular case $\varphi(t) = t^{p-2}$, Dindoš and Pipher [9] (see also [11] for the scalar case) proved that $|\nabla(|u|^{(p-2)/2}u)|^2$ and $|u|^{p-2}|\nabla u|^2$ are equivalent. It is natural to ask if this is still true for a general φ.

The answer is negative. While (15) is always valid, the opposite inequality

$$
|\nabla(\sqrt{\varphi(|u|)}\,u)|^2 \leqslant C\,\varphi(|u|)\,|\nabla u|^2 \tag{16}
$$

could fail. An example is given by $\varphi(t) = \exp(t^2)$. It satisfies condition (i)-(v), but (16) cannot hold. Indeed, in view of (13), this inequality for such a function φ can be written as

$$
(|u|^4 + 2\,|u|)|\nabla|u||^2 + |\nabla u|^2 \leqslant C\,|\nabla u|^2,
$$

which is impossible to hold for any $u \in [\mathring{C}^\infty(\Omega)]^m$. A sufficient condition is given in the next Lemma.

Lemma 5 *If the function $t\varphi'(t)/\varphi(t)$ is bounded on $(0,+\infty)$, then inequality* (16) *holds.*

Proof. Assuming $|t\varphi'(t)/\varphi(t)| \leqslant K$ ($t>0$), (13) implies

$$|\nabla(\sqrt{\varphi(|u|)}\,u)|^2 \leqslant (K^2/4+K)\,\varphi(|u|)\,|\nabla|u||^2 + \varphi(|u|)\,|\nabla u|^2 \leqslant \\ (K^2/4+K+1)\,\varphi(|u|)\,|\nabla u|^2\,.$$

□

Remark 1 Lemma 5 and inequality (15) show that if $t\varphi'(t)/\varphi(t)$ is bounded, then $|\nabla(\sqrt{\varphi(|u|)}\,u)|^2$ and $\varphi(|u|)\,|\nabla u|^2$ are equivalent.

The next results of this section extend some of the results obtained in [4] in the case of L^p-dissipativity, i.e. when $\varphi(t)=t^{p-2}$. If $\mathscr{A}$ is a matrix, by $\mathscr{A}^*$ we denote the adjoint matrix of $\mathscr{A}$, i.e. $\mathscr{A}^* = \overline{\mathscr{A}}^t$, $\mathscr{A}^t$ being the transposed matrix of $\mathscr{A}$.

Lemma 6 *Let Ω be a domain in $\mathbb{R}^N$. The operator* (10) *is L^Φ-dissipative if and only if*

$$\mathbb{R}\mathrm{e}\int_\Omega \Big(\langle \mathscr{A}^{hk}\,\partial_k v, \partial_h v\rangle + \Lambda(|v|)\,|v|^{-2}\langle\big(\mathscr{A}^{hk} - (\mathscr{A}^{kh})^*\big)\,v, \partial_h v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle \\ -\Lambda^2(|v|)\,|v|^{-4}\langle \mathscr{A}^{hk}\,v, v\rangle\,\mathbb{R}\mathrm{e}\langle v,\partial_k v\rangle\,\mathbb{R}\mathrm{e}\langle v,\partial_h v\rangle\Big)dx \geqslant 0 \tag{17}$$

for any $v\in[\mathring{H}^1(\Omega)]^m$. Here and in the sequel the integrand is extended by zero on the set where v vanishes.

Proof. *Sufficiency.* Suppose $r\geqslant 0$. Let $u\in[\mathring{H}^1(\Omega)]^m$ such that $\varphi(|u|)\,u\in[\mathring{H}^1(\Omega)]^m$ and define $v=\sqrt{\varphi(|u|)}\,u$. In view of Lemma 3 we have that v belongs to $[\mathring{H}^1(\Omega)]^m$.

Since $|u|=\zeta(|v|)$ and $|v|^{-1}v=|u|^{-1}u$, we get $u=|v|^{-1}v\,\zeta(|v|)=\Theta(|v|)\,v$ (see (5)). Moreover from $\varphi(|u|)=|u|^{-2}|v|^2=[\Theta(|v|)]^{-2}$ we deduce $\varphi(|u|)\,\overline{u}=[\Theta(|v|)]^{-1}\overline{v}$. Therefore

$$\langle \mathscr{A}^{hk}\,\partial_k u, \partial_h(\varphi(|u|)\,u)\rangle = \langle \mathscr{A}^{hk}\,\partial_k(\Theta(|v|)\,v), \partial_h([\Theta(|v|)]^{-1}v)\rangle \\ = \langle \mathscr{A}^{hk}(\Theta'(|v|)v\partial_k|v| + \Theta(|v|)\partial_k v, -\Theta'(|v|)[\Theta(|v|)]^{-2}v\partial_h|v| + [\Theta(|v|)]^{-1}\partial_h v\rangle \\ = -(\Theta'(|v|)[\Theta(|v|)]^{-1})^2\langle \mathscr{A}^{hk}\,v,v\rangle\,\partial_k|v|\,\partial_h|v| \\ +\Theta'(|v|)[\Theta(|v|)]^{-1}(\langle \mathscr{A}^{hk}\,v,\partial_h v\rangle\partial_k|v| - \langle \mathscr{A}^{hk}\,\partial_k v, v\rangle\partial_h|v|) + \langle \mathscr{A}^{hk}\,\partial_k v,\partial_h v\rangle.$$

From the identities

$$
\begin{gathered}
\partial_k|v| = |v|^{-1}\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle\,, \\
\langle \mathscr{A}^{hk}\,\partial_k v, v\rangle\,\partial_h|v| = \langle \mathscr{A}^{kh}\,\partial_h v, v\rangle\,\partial_k|v| = \overline{\langle (\mathscr{A}^{kh})^* v, \partial_h v\rangle}\,\partial_k|v|\,,
\end{gathered}
\tag{18}
$$

it follows

$$
\begin{gathered}
\mathbb{R}\mathrm{e}\langle \mathscr{A}^{hk}\,\partial_k u, \partial_h(\varphi(|u|)\,u)\rangle = \mathbb{R}\mathrm{e}\,\big(\langle \mathscr{A}^{hk}\,\partial_k v, \partial_h v\rangle \\
+\Lambda(|v|)|v|^{-2}\langle (\mathscr{A}^{hk} - (\mathscr{A}^{kh})^*)v, \partial_h v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle \\
-\Lambda^2(|v|)|v|^{-4}\langle \mathscr{A}^{hk}\,v, v\rangle\;\mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle\big)
\end{gathered}
$$

on the set $\{x\in\Omega \mid u(x)\neq 0\} = \{x\in\Omega \mid v(x)\neq 0\}$.

Inequality (17) implies

$$
\mathbb{R}\mathrm{e}\int_\Omega \langle \mathscr{A}^{hk}\,\partial_k u, \partial_h(\varphi(|u|)\,u)\rangle\,dx \geqslant 0
$$

and the sufficiency is proved when $r \geqslant 0$.

If $-1 < r < 0$, setting $w = \varphi(|u|)\,u$, i.e. $u = \psi(|w|)\,w$, we can write condition (11) as

$$
\mathbb{R}\mathrm{e}\int_\Omega \langle (\mathscr{A}^{kh})^*\partial_k w, \partial_h(\psi(|w|)\,w)\rangle\,dx \geqslant 0
$$

for any $w\in[\mathring{H}^1(\Omega)]^m$ such that $\psi(|w|)\,w\in[\mathring{H}^1(\Omega)]^m$.

Recalling Lemma 1, what we have already proved for $r \geqslant 0$ shows that this inequality holds if

$$
\begin{gathered}
\mathbb{R}\mathrm{e}\int_\Omega \Big(\langle (\mathscr{A}^{kh})^*\partial_k v, \partial_h v\rangle + \widetilde{\Lambda}(|v|)\,|v|^{-2}\langle \big((\mathscr{A}^{kh})^* - \mathscr{A}^{hk}\big)\,v, \partial_h v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle \\
-\widetilde{\Lambda}^2(|v|)\,|v|^{-4}\langle (\mathscr{A}^{kh})^* v, v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle\Big)dx \geqslant 0
\end{gathered}
\tag{19}
$$

for any $v\in[\mathring{H}^1(\Omega)]^m$. Since $\widetilde{\Lambda}(|v|) = -\Lambda(|v|)$ (see (7)), conditions (19) coincides with (17) and the sufficiency is proved also for $-1 < r < 0$.

Necessity. Let $v\in[\mathring{C}^1(\Omega)]^m$ and define $u_\varepsilon = \Theta(g_\varepsilon)\,v$, where $g_\varepsilon = \sqrt{|v|^2+\varepsilon^2}$.

The function u_ε and $\varphi(|u_\varepsilon|)\, u_\varepsilon$ belong to $[\mathring{C}^1(\Omega)]^m$ and we have

$$
\begin{gathered}
\langle \mathscr{A}^{hk}\, \partial_k u_\varepsilon, \partial_h(\varphi(|u_\varepsilon|)\, u_\varepsilon\rangle \\
= \varphi(|u_\varepsilon|)\, \langle \mathscr{A}^{hk}\, \partial_k u_\varepsilon, \partial_h u_\varepsilon\rangle + \varphi'(|u_\varepsilon|)\langle \mathscr{A}^{hk}\, \partial_k u_\varepsilon, u_\varepsilon\, \partial_h(|u_\varepsilon|)\rangle \\
= \varphi[\Theta(g_\varepsilon)\,|v|]\, \langle \mathscr{A}^{hk}(\Theta'(g_\varepsilon)\, v\, \partial_k g_\varepsilon + \Theta(g_\varepsilon)\partial_k v), \Theta'(g_\varepsilon)\, v\, \partial_h g_\varepsilon + \Theta(g_\varepsilon)\partial_h v\rangle \\
+\varphi'[\Theta(g_\varepsilon)\,|v|] \\
\times\langle \mathscr{A}^{hk}(\Theta'(g_\varepsilon)\, v\, \partial_k g_\varepsilon + \Theta(g_\varepsilon)\partial_k v), \Theta(g_\varepsilon)\, v\, (\Theta'(g_\varepsilon)\,|v|\, \partial_h g_\varepsilon + \Theta(g_\varepsilon)\partial_h|v|)\rangle \\
= \varphi[\Theta(g_\varepsilon)\,|v|]\big\{[\Theta'(g_\varepsilon)]^2\langle \mathscr{A}^{hk}\, v, v\rangle\, \partial_k g_\varepsilon\, \partial_h g_\varepsilon \\
+\Theta'(g_\varepsilon)\,\Theta(g_\varepsilon)\,[\langle \mathscr{A}^{hk}\, v, \partial_h v\rangle\, \partial_k g_\varepsilon + \langle \mathscr{A}^{hk}\, \partial_k v, v\rangle\, \partial_h g_\varepsilon] + \Theta^2(g_\varepsilon)\, \langle \mathscr{A}^{hk}\, \partial_k v, \partial_h v\rangle\big\} \\
+\varphi'[\Theta(g_\varepsilon)\,|v|]\big\{\Theta(g_\varepsilon)[\Theta'(g_\varepsilon)]^2|v|\langle \mathscr{A}^{hk}\, v, v\rangle\, \partial_k g_\varepsilon\, \partial_h g_\varepsilon \\
+\Theta^2(g_\varepsilon)\,\Theta'(g_\varepsilon)[\langle \mathscr{A}^{hk}\, v, v\rangle\, \partial_k g_\varepsilon\, \partial_h|v| + |v|\langle \mathscr{A}^{hk}\, \partial_k v, v\rangle\, \partial_h g_\varepsilon] \\
+\Theta^3(g_\varepsilon)\langle \mathscr{A}^{hk}\, \partial_k v, v\rangle\, \partial_h|v|\big\}\,.
\end{gathered} \tag{20}
$$

Letting $\varepsilon \to 0^+$ the right hand side tends to

$$
\begin{gathered}
\varphi[\Theta(|v|)\,|v|]\Big\{[\Theta'(|v|)]^2\langle \mathscr{A}^{hk}\, v, v\rangle\, \partial_k|v|\partial_h|v| \\
+\Theta'(|v|)\Theta(|v|)[\langle \mathscr{A}^{hk}\, v, \partial_h v\rangle\partial_k|v| + \langle \mathscr{A}^{hk}\, \partial_k v, v\rangle\partial_h|v|] \\
+\Theta^2(|v|)\langle \mathscr{A}^{hk}\, \partial_k v, \partial_h v\rangle\Big\} \\
+\varphi'[\Theta(|v|)\,|v|]\Big\{\Theta(|v|)[\Theta'(|v|)]^2|v|\langle \mathscr{A}^{hk}\, v, v\rangle\, \partial_k|v|\, \partial_h|v| \\
+\Theta^2(|v|)\,\Theta'(|v|)[\langle \mathscr{A}^{hk}\, v, v\rangle\, \partial_k|v|\, \partial_h|v| + |v|\langle \mathscr{A}^{hk}\, \partial_k v, v\rangle\, \partial_h|v|] \\
+\Theta^3(|v|)\langle \mathscr{A}^{hk}\, \partial_k v, v\rangle\, \partial_h|v|\Big\} \\
= \varphi[\Theta(|v|)\,|v|]\,\Theta^2(|v|)\, \langle \mathscr{A}^{hk}\, \partial_k v, \partial_h v\rangle \\
+\varphi[\Theta(|v|)\,|v|]\,\Theta'(|v|)\,\Theta(|v|)\, \langle \mathscr{A}^{hk}\, v, \partial_h v\rangle\partial_k|v| \\
+\Theta(|v|)\big\{\varphi[\Theta(|v|)\,|v|]\,\Theta'(|v|) \\
+\varphi'[\Theta(|v|)\,|v|]\,\Theta(|v|)\,[\Theta'(|v|)\,|v| + \Theta(|v|)]\big\}\langle \mathscr{A}^{hk}\, \partial_k v, v\rangle\, \partial_h|v| \\
+\Theta'(|v|)\big\{\varphi[\Theta(|v|)\,|v|]\,\Theta'(|v|) \\
+\varphi'[\Theta(|v|)\,|v|]\,\Theta(|v|)\,[\Theta'(|v|)\,|v| + \Theta(|v|)]\big\}\langle \mathscr{A}^{hk}\, v, v\rangle\, \partial_k|v|\, \partial_h|v|
\end{gathered} \tag{21}
$$

on the set $\Omega_0 = \{x \in \Omega \mid v(x) \neq 0\}$.

In view of (8) and (9) we have

$$\varphi[\Theta(|v|)\,|v|]\,\Theta^2(|v|)=1,\ \varphi[\Theta(|v|)\,|v|]\,\Theta'(|v|)\,\Theta(|v|)=\Theta'(|v|)/\Theta(|v|),$$
$$\varphi[\Theta(|v|)\,|v|]\,\Theta'(|v|)+\varphi'[\Theta(|v|)\,|v|]\,\Theta(|v|)\,[\Theta'(|v|)\,|v|+\Theta(|v|)]$$
$$=-\Theta'(|v|)/\Theta^2(|v|).$$

Substituting these equalities in (21) and keeping in mind (18) and (20), we find

$$\begin{aligned}&\lim_{\varepsilon\to0^+}\langle\mathscr{A}^{hk}\,\partial_k u_\varepsilon,\partial_h(\varphi(|u_\varepsilon|)\,u_\varepsilon\rangle\\
&=\langle\mathscr{A}^{hk}\,\partial_k v,\partial_h v\rangle+\Theta'(|v|)/\Theta(|v|)\,[\langle\mathscr{A}^{hk}\,v,\partial_h v\rangle\,\partial_k|v|-\langle\mathscr{A}^{hk}\,\partial_k v,v\rangle\,\partial_h|v|]\\
&\quad-(\Theta'(|v|)/\Theta(|v|))^2\,\langle\mathscr{A}^{hk}\,v,v\rangle\,\partial_k|v|\,\partial_h|v|\\
&=\langle\mathscr{A}^{hk}\,\partial_k v,\partial_h v\rangle+\Lambda(|v|)\,|v|^{-2}\left(\langle\mathscr{A}^{hk}\,v,\partial_h v\rangle-\overline{\langle(\mathscr{A}^{kh})^*v,\partial_h v\rangle}\right)\mathbb{R}\mathrm{e}\langle v,\partial_k v\rangle\\
&\quad-\Lambda^2(|v|)\,|v|^{-4}\langle\mathscr{A}^{hk}\,v,v\rangle\,\mathbb{R}\mathrm{e}\langle v,\partial_k v\rangle\,\mathbb{R}\mathrm{e}\langle v,\partial_h v\rangle\end{aligned}$$

on Ω_0.

As in [8, (58)] one can prove that each term in the last expression of (20) can be majorized by L^1 functions not depending on ε. By the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}\lim_{\varepsilon\to0^+}\mathbb{R}\mathrm{e}\int_\Omega\langle\mathscr{A}^{hk}\,\partial_k u_\varepsilon,\partial_h(\varphi(|u_\varepsilon|)\,u_\varepsilon\rangle dx=\mathbb{R}\mathrm{e}\int_\Omega\Big(\langle\mathscr{A}^{hk}\,\partial_k v,\partial_h v\rangle\\
+\Lambda(|v|)\,|v|^{-2}\langle\big(\mathscr{A}^{hk}-(\mathscr{A}^{kh})^*\big)\,v,\partial_h v\rangle\,\mathbb{R}\mathrm{e}\langle v,\partial_k v\rangle\\
-\Lambda^2(|v|)\,|v|^{-4}\langle\mathscr{A}^{hk}\,v,v\rangle\,\mathbb{R}\mathrm{e}\langle v,\partial_k v\rangle\,\mathbb{R}\mathrm{e}\langle v,\partial_h v\rangle\Big)dx\,.\end{aligned}\tag{22}$$

The left hand side being non negative (see (11)), inequality (17) holds for any $v\in[\mathring{C}^1(\Omega)]^m$.

Let now $v\in[\mathring{H}^1(\Omega)]^m$ and $v_n\in[\mathring{C}^\infty(\Omega)]^m$ such that $v_n\to v$ almost everywhere in Ω and in H^1 norm. Reasoning as in [8, Lemma 5] one can prove that

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{R}\mathrm{e}\int_\Omega\Big(\langle\mathscr{A}^{hk}\,\partial_k v_n,\partial_h v_n\rangle\\
&+\Lambda(|v_n|)\,|v_n|^{-2}\langle\big(\mathscr{A}^{hk}-(\mathscr{A}^{kh})^*\big)\,v_n,\partial_h v_n\rangle\,\mathbb{R}\mathrm{e}\langle v_n,\partial_k v_n\rangle\\
&-\Lambda^2(|v_n|)\,|v_n|^{-4}\langle\mathscr{A}^{hk}\,v_n,v_n\rangle\,\mathbb{R}\mathrm{e}\langle v_n,\partial_k v_n\rangle\,\mathbb{R}\mathrm{e}\langle v_n,\partial_h v_n\rangle\Big)dx\\
&=\mathbb{R}\mathrm{e}\int_\Omega\Big(\langle\mathscr{A}^{hk}\,\partial_k v,\partial_h v\rangle+\Lambda(|v|)\,|v|^{-2}\langle\big(\mathscr{A}^{hk}-(\mathscr{A}^{kh})^*\big)\,v,\partial_h v\rangle\,\mathbb{R}\mathrm{e}\langle v,\partial_k v\rangle\\
&-\Lambda^2(|v|)\,|v|^{-4}\langle\mathscr{A}^{hk}\,v,v\rangle\,\mathbb{R}\mathrm{e}\langle v,\partial_k v\rangle\,\mathbb{R}\mathrm{e}\langle v,\partial_h v\rangle\Big)dx\,,\end{aligned}$$

and the result follows. $\square$

We have also

Lemma 7 *Let Ω be a domain in $\mathbb{R}^N$. The operator* (10) *is strict L^Φ-dissipative if and only if there exists $\kappa > 0$ such that*

$$\mathbb{R}\mathrm{e} \int_\Omega \Big(\langle \mathscr{A}^{hk}\, \partial_k v, \partial_h v\rangle + \Lambda(|v|)\, |v|^{-2} \langle \big(\mathscr{A}^{hk} - (\mathscr{A}^{kh})^*\big)\, v, \partial_h v\rangle \, \mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle \\ - \Lambda^2(|v|)\, |v|^{-4} \langle \mathscr{A}^{hk}\, v, v\rangle \, \mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle \, \mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle \Big) dx \geqslant \kappa \int_\Omega |\nabla v|^2 dx \tag{23}$$

for any $v \in [\mathring{H}^1(\Omega)]^m$.

Proof. *Sufficiency.* As in the proof of the previous Lemma, suppose $r \geqslant 0$ and take $v = \sqrt{\varphi(|u|)}\, u$, where $u \in [\mathring{H}^1(\Omega)]^m$ is such that $\varphi(|u|)\, u \in [\mathring{H}^1(\Omega)]^m$. In Lemma 6 we showed that the left hand side of (12) coincides with the left hand side of (23). The right hand sides being equal, the sufficiency is proved when $r \geqslant 0$.

If $-1 < r < 0$, setting $w = \varphi(|u|)\, u$, i.e. $u = \psi(|w|)\, w$, and recalling (4), we can write condition (12) as

$$\mathbb{R}\mathrm{e} \int_\Omega \langle (\mathscr{A}^{kh})^* \partial_k w, \partial_h(\psi(|w|)\, w)\rangle \, dx \geqslant \kappa \int_\Omega |\nabla(\sqrt{\psi(|w|)}\, w)|^2 dx$$

for any $w \in [\mathring{H}^1(\Omega)]^m$ such that $\psi(|w|)\, w \in [\mathring{H}^1(\Omega)]^m$. As in the previous Lemma this implies (23) for any $v \in [\mathring{H}^1(\Omega)]^m$.

Necessity. As in the proof of Necessity in Lemma 6, let $v \in [\mathring{C}^1(\Omega)]^m$ and define $u_\varepsilon = \Theta(g_\varepsilon)\, v$. Let us consider the integral

$$\int_\Omega |\nabla(\sqrt{\varphi(|u_\varepsilon|)}\, u_\varepsilon)|^2 dx \, .$$

Let us write $\sqrt{\varphi(|u_\varepsilon|)}\, u_\varepsilon$ as $\varrho_\varepsilon v$, where $\varrho_\varepsilon = \sqrt{\varphi(|u_\varepsilon|)}\, \Theta(g_\varepsilon)$. Keeping in mind (8), we find

$$\lim_{\varepsilon \to 0^+} \varrho_\varepsilon = \lim_{\varepsilon \to 0^+} \sqrt{\varphi[\Theta(g_\varepsilon)|v|]}\, \Theta(g_\varepsilon) = \sqrt{\varphi[\Theta(|v|)|v|]}\, \Theta(|v|) = 1$$

on $\Omega_0 = \{x \in \Omega \mid v(x) \neq 0\}$. Moreover

$$\lim_{\varepsilon\to 0^+} \partial_h \varrho_\varepsilon = \lim_{\varepsilon\to 0^+} \Big(\frac{1}{2}\, \frac{\varphi'[\Theta(g_\varepsilon)|v|]}{\sqrt{\varphi[\Theta(g_\varepsilon)|v|]}}\, (|v|\, \Theta'(g_\varepsilon)\, \partial_h g_\varepsilon + \Theta(g_\varepsilon)\, \partial_h |v|)\Theta(g_\varepsilon) \\ + \sqrt{\varphi[\Theta(g_\varepsilon\, |v|]}\, \Theta'(g_\varepsilon)\, \partial_h g_\varepsilon\Big) \\ = \frac{\varphi'[\Theta(|v|)|v|]\, \Theta(|v|)[\Theta'(|v|)|v| + \Theta(|v|)] + 2\varphi[\Theta(|v|)|v|]\, \Theta'(|v|)}{2\sqrt{\varphi[\Theta(|v|)|v|]}}\, .$$

Equality (9) shows that the numerator in the last expression can be written as

$$-\Theta'(|v|)/\Theta^2(|v|) \; + \varphi[\Theta(|v|)|v|]\, \Theta'(|v|) \\ = \Theta'(|v|)(-1 + \varphi[\Theta(|v|)|v|]\, \Theta^2(|v|))/\Theta^2(|v|) = 0$$

(see also (8)) and then

$$\lim_{\varepsilon\to 0^+} \partial_h \varrho_\varepsilon = 0$$

on Ω_0. This implies that

$$\lim_{\varepsilon\to 0^+} \partial_h(\sqrt{\varphi(|u_\varepsilon|)}\, u_\varepsilon) = \lim_{\varepsilon\to 0^+} \partial_h(\varrho_\varepsilon v) = \lim_{\varepsilon\to 0^+} (v\partial_h \varrho_\varepsilon + \varrho_\varepsilon \partial_h v) = \partial_h v$$

on Ω_0.

By Fatou's Lemma we get

$$\int_\Omega |\nabla v|^2 dx \leqslant \liminf_{\varepsilon\to 0^+} \int_\Omega |\nabla(\sqrt{\varphi(|u_\varepsilon|)}\, u_\varepsilon)|^2 dx\, .$$

On the other hand we know that (22) holds and therefore the inequality

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}^{hk}\, \partial_k u_\varepsilon, \partial_h(\varphi(|u_\varepsilon|)\, u_\varepsilon)\rangle\, dx \geqslant \kappa \int_\Omega |\nabla(\sqrt{\varphi(|u_\varepsilon|)}\, u_\varepsilon)|^2 dx$$

implies (23) for any $v \in [\mathring{C}^1(\Omega)]^m$.

The result for any $v \in [\mathring{H}^1(\Omega)]^m$ follows by approximating v by a sequence $v_n \in [\mathring{C}^\infty(\Omega)]^m$ (as in the previous Lemma). □

We conclude this Section with the following Corollary concerning the strict L^Φ-dissipativity of the operator (10).

Corollary 1 *Suppose*

$$\sup_{t>0} \Lambda^2(t) < 1\,. \tag{24}$$

The operator A is strict L^Φ-dissipative if and only if there exists $\kappa > 0$ such that $A - \kappa\Delta$ is L^Φ-dissipative.

Proof. If the operator A is strict L^Φ-dissipative, (23) holds. This implies

$$\begin{gathered}\mathbb{R}\mathrm{e}\int_\Omega \Big(\langle \mathscr{A}^{hk}\,\partial_k v, \partial_h v\rangle \\ +\Lambda(|v|)\,|v|^{-2}\langle\big(\mathscr{A}^{hk} - (\mathscr{A}^{kh})^*\big)\,v, \partial_h v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle \\ -\Lambda^2(|v|)\,|v|^{-4}\langle \mathscr{A}^{hk}\,v, v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle\Big)dx \geqslant \\ \kappa\int_\Omega \Big(|\nabla v|^2 - \Lambda^2(|v|)\,|\nabla|v||^2\Big)dx\end{gathered} \tag{25}$$

for any $v \in [\mathring{H}^1(\Omega)]^m$. Observing that if $\{a_{ij}^{hk}\} = \{\delta_{hk}\delta_{ij}\}$ we have

$$|v|^{-4}\langle \mathscr{A}^{hk}\,v, v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle = |v|^{-2}\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle = |\nabla|v||^2,$$

Lemma 6 shows that $A - \kappa\Delta$ is L^Φ-dissipative.

Viceversa, if $A - \kappa\Delta$ is L^Φ-dissipative, inequality (25) holds for any $v \in [\mathring{H}^1(\Omega)]^m$. Since

$$|\nabla|v||^2 \leqslant |\nabla v|^2, \tag{26}$$

we find

$$\int_\Omega \Big(|\nabla v|^2 - \Lambda^2(|v|)\,|\nabla|v||^2\Big)dx \geqslant (1 - \sup_{t>0}\Lambda^2(t))\int_\Omega |\nabla v|^2 dx\,.$$

Thanks to (24), the inequality (23) holds with κ replaced by the positive constant $\kappa(1 - \sup_{t>0}\Lambda^2(t))$. □

We remark that in the case of L^p-dissipativity, i.e. $\varphi(t) = t^{p-2}$ $(1 < p < \infty)$, condition (24) is satisfied, because $\Lambda^2(t) = (1 - 2/p)^2$.

4 A necessary condition for the functional dissipativity when $N = 2$

From now on we require also the following condition on the function φ:

(vi) the function
$$|s\,\varphi'(s)/\varphi(s)|$$
is not decreasing.

A first consequence of condition (vi) is the following

Lemma 8 *The function* $\Lambda^2(t)$ *is not decreasing on* $(0,+\infty)$.

Proof. Since the function $s\sqrt{\varphi(s)}$ is stricly increasing and its range is $(0,+\infty)$, the function $\Lambda^2(t)$ is not decreasing if and only if the function $\Lambda^2(s\sqrt{\varphi(s)})$ is not decreasing. Define $\gamma(s)=s\,\varphi'(s)/\varphi(s)$. Note that the monotoniciy of $|\gamma(s)|$ (see condition (vi)) implies that if there exists $\sigma>0$ such that $\gamma(\sigma)=0$, then $\gamma(s)=0$ for any $0<s\leqslant\sigma$. If there exists $s>0$ such that $\gamma(s)=0$ put
$$\sigma_0=\inf\{\sigma>0\mid\gamma(\sigma)=0\}\,.$$
If $\gamma(s)\neq 0$ for any $s>0$, put $\sigma_0=0$. Note that, in any case, $\gamma(s)\neq 0$ for any $s>\sigma_0$. Recalling (6), we have that $\Lambda^2(s\sqrt{\varphi(s)})=0$ for any $s\leqslant\sigma_0$ and $\Lambda^2(s\sqrt{\varphi(s)})>0$ for any $s>\sigma_0$. Then it is it suffices to prove that $\Lambda^2(s\sqrt{\varphi(s)})\leqslant\Lambda^2(t\sqrt{\varphi(t)})$ for any $\sigma_0<s<t$. Defining
$$\gamma(s)=\frac{s\varphi'(s)}{s\varphi'(s)+2\varphi(s)}$$
and observing that condition (ii) implies $\gamma(s)+2>0$ (for any $s>0$), we have that $\Lambda^2(s\sqrt{\varphi(s)})\leqslant\Lambda^2(t\sqrt{\varphi(t)})$ means
$$|\gamma(s)|\,(\gamma(t)+2)\leqslant|\gamma(t)|\,(\gamma(s)+2)\,.$$

Since $|\gamma(s)|\,\gamma(t)=|\gamma(t)|\,\gamma(s)$, the last inequality is equivalent to $|\gamma(s)|\leqslant|\gamma(t)|$, which is true in view of condition (vi). □

We remark that, since the function Λ does not change the sign, the previous result implies the monotonicity of the bounded function $\Lambda(s)$ and then the existence of the finite limit
$$\Lambda_\infty=\lim_{t\to+\infty}\Lambda(t)\,. \tag{27}$$

We have also
$$\Lambda^2_\infty=\sup_{t>0}\Lambda^2(t)\,. \tag{28}$$

The next theorem provides a necessary condition for the L^Φ-dissipativity of operator A when $N=2$.

Theorem 1 *Let Ω be a domain of $\mathbb{R}^2$. If the operator* (10) *is L^Φ-dissipative we have*

$$\begin{gathered}\mathbb{R}\mathrm{e}\Big(\langle(\mathscr{A}^{hk}(x)\xi_h\xi_k)\eta,\eta\rangle-\Lambda_\infty^2\,\langle(\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega,\omega\rangle(\mathbb{R}\mathrm{e}\langle\eta,\omega\rangle)^2\\ +\Lambda_\infty\,(\langle(\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega,\eta\rangle-\langle(\mathscr{A}^{hk}(x)\xi_h\xi_k)\eta,\omega\rangle)\,\mathbb{R}\mathrm{e}\langle\eta,\omega\rangle\Big)\geqslant 0\end{gathered}\tag{29}$$

for almost every $x\in\Omega$ and for any $\xi\in\mathbb{R}^2$, η, $\omega\in\mathbb{C}^m$, $|\omega|=1$. Here the constant Λ_∞ is given by (27)*.*

Proof. As in [4, Theorem 2], let us assume first that $\mathscr{A}^{hk}$ are constant matrices and that $\Omega=\mathbb{R}^2$. Let us fix $\omega\in\mathbb{C}^m$ with $|\omega|=1$ and take $v(x)=w(x)\,g(\log|x|/\log R)$, where

$$w(x)=\mu\omega+\psi(x),$$

μ, $R\in\mathbb{R}^+$, $R>1$, $\psi\in(\mathring{C}^\infty(\mathbb{R}^2))^m$, $g\in C^\infty(\mathbb{R})$, $g(t)=1$ if $t\leqslant 1/2$ and $g(t)=0$ if $t\geqslant 1$.

Put the function v in (17) and let $R\to+\infty$. Using the same arguments as in the first part of the proof of [4, Theorem 2] and observing that Λ is continuous and $|\Lambda(t)|<1$ (see [8, (32)]), we find

$$\begin{gathered}\mathbb{R}\mathrm{e}\int_{B_\delta(0)}\big(\langle\mathscr{A}^{hk}\,\partial_k w,\partial_h w\rangle\\ -\Lambda^2(|w|)\,|w|^{-4}\langle\mathscr{A}^{hk}\,w,w\rangle\,\mathbb{R}\mathrm{e}\langle w,\partial_k w\rangle\,\mathbb{R}\mathrm{e}\langle w,\partial_h w\rangle\\ +\Lambda(|w|)\,|w|^{-2}\langle\big(\mathscr{A}^{hk}-(\mathscr{A}^{kh})^*\big)\,w,\partial_h w\rangle\,\mathbb{R}\mathrm{e}\langle w,\partial_k w\rangle\big)dx\geqslant 0,\end{gathered}\tag{30}$$

where $\delta>0$ is such that $\operatorname{spt}\psi\subset B_\delta(0)$.

We have also

$$\begin{gathered}\mathbb{R}\mathrm{e}\langle\mathscr{A}^{hk}\,\partial_k w,\partial_h w\rangle=\mathbb{R}\mathrm{e}\langle\mathscr{A}^{hk}\,\partial_k\psi,\partial_h\psi\rangle,\\ \Lambda^2(|w|)|w|^{-4}\,\mathbb{R}\mathrm{e}\langle\mathscr{A}^{hk}\,w,w\rangle\,\mathbb{R}\mathrm{e}\langle w,\partial_k w\rangle\,\mathbb{R}\mathrm{e}\langle w,\partial_h w\rangle\\ =\Lambda^2(|\mu\omega+\psi|)|\mu\omega+\psi|^{-4}\\ \times\mathbb{R}\mathrm{e}\langle\mathscr{A}^{hk}(\mu\omega+\psi),\mu\omega+\psi\rangle\,\mathbb{R}\mathrm{e}\langle\mu\omega+\psi,\partial_k\psi\rangle\,\mathbb{R}\mathrm{e}\langle\mu\omega+\psi,\partial_h\psi\rangle,\\ \Lambda(|w|)\,|w|^{-2}\,\mathbb{R}\mathrm{e}\langle\big(\mathscr{A}^{hk}-(\mathscr{A}^{kh})^*\big)\,w,\partial_h w\rangle\,\mathbb{R}\mathrm{e}\langle w,\partial_k w\rangle\\ =\Lambda(|\mu\omega+\psi|)|\mu\omega+\psi|^{-2}\,\mathbb{R}\mathrm{e}(\langle(\mathscr{A}^{hk}-(\mathscr{A}^{kh})^*)(\mu\omega+\psi),\partial_h\psi\rangle\,\mathbb{R}\mathrm{e}\langle\mu\omega+\psi,\partial_k\psi\rangle.\end{gathered}$$

Letting $\mu \to +\infty$ in (30), we obtain

$$\begin{aligned}\mathbb{R}\mathrm{e}\int_{\mathbb{R}^2} \big(\langle \mathscr{A}^{hk}\, \partial_k\psi, \partial_h\psi\rangle - \Lambda_\infty^2 \langle \mathscr{A}^{hk}\, \omega,\omega\rangle\, \mathbb{R}\mathrm{e}\langle\omega,\partial_k\psi\rangle\, \mathbb{R}\mathrm{e}\langle\omega,\partial_h\psi\rangle \\ +\Lambda_\infty((\langle(\mathscr{A}^{hk} - (\mathscr{A}^{kh})^*)\omega, \partial_h\psi\rangle\, \mathbb{R}\mathrm{e}\langle\omega,\partial_k\psi\rangle\big)\, dx \geqslant 0.\end{aligned} \tag{31}$$

Putting in (31)

$$\psi(x) = \eta\, \varphi(x)\, e^{i\mu\langle\xi,x\rangle}$$

where $\eta \in \mathbb{C}^m$, $\varphi \in \mathring{C}^\infty(\mathbb{R}^2)$ and μ is a real parameter, by standard arguments (see, e.g., [15, p.107–108]), we find (29).

If the matrices $\mathscr{A}^{hk}$ are not constant and defined in Ω, take

$$v(x) = w((x-x_0)/\varepsilon)$$

where $x_0 \in \Omega$ is a fixed point, $w \in [\mathring{C}^\infty(B_1(0))]^m$ and $0 < \varepsilon < \operatorname{dist}(x_0, \partial\Omega)$. Putting this particular v in (17) we find

$$\begin{aligned}
0 \leqslant \frac{1}{\varepsilon^2}\,\mathbb{R}\mathrm{e}\int_\Omega \Big(&\langle \mathscr{A}^{hk}(x)\partial_k w((x-x_0)/\varepsilon), \partial_h w((x-x_0)/\varepsilon)\rangle \\
&+\Lambda(|w((x-x_0)/\varepsilon)|)|w((x-x_0)/\varepsilon)|^{-2} \\
&\times\langle(\mathscr{A}^{hk}(x) - (\mathscr{A}^{kh})^*(x))w((x-x_0)/\varepsilon), \partial_h w((x-x_0)/\varepsilon)\rangle \\
&\times \mathbb{R}\mathrm{e}\langle w((x-x_0)/\varepsilon), \partial_k w((x-x_0)/\varepsilon)\rangle \\
&-\Lambda^2(|w((x-x_0)/\varepsilon)|)\,|w((x-x_0)/\varepsilon)|^{-4}\langle\mathscr{A}^{hk}(x)w((x-x_0)/\varepsilon), w((x-x_0)/\varepsilon)\rangle \\
&\times \mathbb{R}\mathrm{e}\langle w((x-x_0)/\varepsilon), \partial_k w((x-x_0)/\varepsilon)\rangle\, \mathbb{R}\mathrm{e}\langle w((x-x_0)/\varepsilon), \partial_h w((x-x_0)/\varepsilon)\rangle\Big)dx \\
= \mathbb{R}\mathrm{e}\int_\Omega \Big(&\langle\mathscr{A}^{hk}(x_0+\varepsilon y)\partial_k w(y), \partial_h w(y)\rangle \\
&+\Lambda(|w((y)|)|w(y)|^{-2}\langle(\mathscr{A}^{hk}(x_0+\varepsilon y) - (\mathscr{A}^{kh})^*(x_0+\varepsilon y))w(y), \partial_h w((y)\rangle \\
&\times \mathbb{R}\mathrm{e}\langle w(y), \partial_k w(y)\rangle - \Lambda^2(|w(y)|)\,|w(y)|^{-4}\langle\mathscr{A}^{hk}(x_0+\varepsilon y)w(y), w(y)\rangle \\
&\times \mathbb{R}\mathrm{e}\langle w(y), \partial_k w(y)\rangle\, \mathbb{R}\mathrm{e}\langle w(y), \partial_h w(y)\rangle\Big)dy\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbb{R}\mathrm{e}\int_\Omega \Big(\langle \mathscr{A}^{hk}(x_0)\partial_k w(y), \partial_h w(y)\rangle \\
&+\Lambda(|w((y)|)|w(y)|^{-2}\langle(\mathscr{A}^{hk}(x_0)-(\mathscr{A}^{kh})^*(x_0 y))w(y), \partial_h w((y)\rangle \\
&\times \mathbb{R}\mathrm{e}\langle w(y), \partial_k w(y)\rangle - \Lambda^2(|w(y)|)\,|w(y)|^{-4}\langle \mathscr{A}^{hk}(x_0)w(y), w(y)\rangle \\
&\times \mathbb{R}\mathrm{e}\langle w(y), \partial_k w(y)\rangle\, \mathbb{R}\mathrm{e}\langle w(y), \partial_h w(y)\rangle\Big) dy \\
&= \lim_{\varepsilon\to 0^+} \mathbb{R}\mathrm{e}\int_\Omega \Big(\langle \mathscr{A}^{hk}(x_0+\varepsilon y)\partial_k w(y), \partial_h w(y)\rangle \\
&+\Lambda(|w((y)|)|w(y)|^{-2}\langle(\mathscr{A}^{hk}(x_0+\varepsilon y)-(\mathscr{A}^{kh})^*(x_0+\varepsilon y))w(y), \partial_h w((y)\rangle \\
&\times \mathbb{R}\mathrm{e}\langle w(y), \partial_k w(y)\rangle - \Lambda^2(|w(y)|)\,|w(y)|^{-4}\langle \mathscr{A}^{hk}(x_0+\varepsilon y)w(y), w(y)\rangle \\
&\times \mathbb{R}\mathrm{e}\langle w(y), \partial_k w(y)\rangle\, \mathbb{R}\mathrm{e}\langle w(y), \partial_h w(y)\rangle\Big) dy \geqslant 0
\end{aligned}$$

for almost any $x_0 \in \Omega$. The arbitrariness of $w \in [\mathring{C}^\infty(\mathbb{R}^2)]^m$ and what we have already obtained for constant matrices give the result. □

With the same proof we have

Theorem 2 *Let Ω be a domain of $\mathbb{R}^2$. If the operator* (10) *is strict L^Φ-dissipative, there exists $\kappa > 0$ such that*

$$\begin{aligned}
&\mathbb{R}\mathrm{e}\Big(\langle(\mathscr{A}^{hk}(x)\xi_h\xi_k)\eta,\eta\rangle - \Lambda_\infty^2\,\langle(\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega,\omega\rangle(\mathbb{R}\mathrm{e}\langle\eta,\omega\rangle)^2 \\
&+\Lambda_\infty\,(\langle(\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega,\eta\rangle - \langle(\mathscr{A}^{hk}(x)\xi_h\xi_k)\eta,\omega\rangle)\,\mathbb{R}\mathrm{e}\langle\eta,\omega\rangle\Big) \geqslant \kappa\,|\xi|^2|\eta|^2
\end{aligned}$$

for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^2$, η, $\omega \in \mathbb{C}^m$, $|\omega| = 1$.

5 Elasticity

In this section we consider the two-dimensional linear system of elasticity (1). The Lamé coefficients λ, μ are supposed to be measurable essentially bounded real valued functions such that

$$\operatorname*{ess\,inf}_{x\in\Omega} \mu(x) > 0\,;\ \operatorname*{ess\,inf}_{x\in\Omega}(\lambda(x)+2\mu(x)) > 0. \tag{32}$$

The next Theorems provide necessary conditions for the L^Φ-dissipativity and the strict L^Φ-dissipativity of the elasticity operator.

Theorem 3 *If the operator* (1) *is L^Φ-dissipative, then*

$$\Lambda_\infty^2 \leqslant 1 - \operatorname*{ess\,sup}_{x\in\Omega} \left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2 . \tag{33}$$

Proof. In view of Theorem 1, condition (29) holds. We have

$$\langle(\mathscr{A}^{hk}\,\xi_h\xi_k)\eta,\eta\rangle = \mu\,|\xi|^2|\eta|^2 + (\lambda+\mu)\langle\xi,\eta\rangle^2,$$
$$\langle(\mathscr{A}^{hk}\,\xi_h\xi_k)\omega,\omega\rangle = \mu\,|\xi|^2 + (\lambda+\mu)\langle\xi,\omega\rangle^2$$

for any ξ, η, $\omega \in \mathbb{R}^2$, $|\omega| = 1$. Since $(\mathscr{A}^{kh})^* = \mathscr{A}^{hk}$, condition (29) can be written as

$$\mu\,|\xi|^2|\eta|^2 + (\lambda+\mu)\langle\xi,\eta\rangle^2 - \Lambda_\infty^2[\mu\,|\xi|^2 + (\lambda+\mu)\langle\xi,\omega\rangle^2]\langle\eta,\omega\rangle^2 \geqslant 0 \tag{34}$$

for almost any $x\in\Omega$ and for any ξ, η, $\omega \in \mathbb{R}^2$, $|\omega| = 1$. Fix $x \in \Omega$ and rewrite (34) as

$$|\xi|^2|\eta|^2 + \mu^{-1}(\lambda+\mu)\langle\xi,\eta\rangle^2 - \Lambda_\infty^2[|\xi|^2 + \mu^{-1}(\lambda+\mu)\langle\xi,\omega\rangle^2]\langle\eta,\omega\rangle^2 \geqslant 0\,. \tag{35}$$

Reasoning as in [4, p.244–245] (just replace $(1-2\nu)^{-1}$ and $(1-2/p)$ in [4] by $\mu^{-1}(\lambda+\mu)$ and Λ_∞, respectively) we find that (35) implies

$$\Lambda_\infty^2 \leqslant 1 - \left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2 .$$

Taking the infimum of the right hand side, we get (33). □

Theorem 4 *If the operator* (1) *is strict L^Φ-dissipative, then*

$$\Lambda_\infty^2 < 1 - \operatorname*{ess\,sup}_{x\in\Omega} \left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2 . \tag{36}$$

Proof. Theorem 2 shows that

$$\mu\,|\xi|^2|\eta|^2 + (\lambda+\mu)\langle\xi,\eta\rangle^2 - \Lambda_\infty^2[\mu\,|\xi|^2 + (\lambda+\mu)\langle\xi,\omega\rangle^2]\langle\eta,\omega\rangle^2 \geqslant \kappa\,|\xi|^2|\eta|^2$$

for almost any $x\in\Omega$ and for any ξ, η, $\omega\in\mathbb{R}^2$, $|\omega|=1$.

This implies that, for any $0 < h \leqslant \kappa$ we have

$$(\mu-h)\,|\xi|^2|\eta|^2 + (\lambda+\mu)\langle\xi,\eta\rangle^2 - \Lambda_\infty^2[(\mu-h)\,|\xi|^2 + (\lambda+\mu)\langle\xi,\omega\rangle^2]\langle\eta,\omega\rangle^2 \geqslant 0$$

for almost any $x \in \Omega$ and for any ξ, η, $\omega \in \mathbb{R}^2$, $|\omega| = 1$.

Setting $\widetilde{\mu} = \mu - h$, $\widetilde{\lambda} = \lambda + h$, we can rewrite the last inequalty as

$$\widetilde{\mu}\,|\xi|^2|\eta|^2 + (\widetilde{\lambda} + \widetilde{\mu})\langle \xi, \eta\rangle^2 - \Lambda_\infty^2[\widetilde{\mu}\,|\xi|^2 + (\widetilde{\lambda} + \widetilde{\mu})\langle \xi, \omega\rangle^2]\langle \eta, \omega\rangle^2 \geqslant 0\,.$$

The proof of Theorem 3 shows that this implies

$$\Lambda_\infty^2 \leqslant 1 - \operatorname*{ess\,sup}_{x\in\Omega} \left(\frac{\widetilde{\lambda} + \widetilde{\mu}}{\widetilde{\lambda} + 3\widetilde{\mu}}\right)^2 ,$$

i.e.

$$\Lambda_\infty^2 \leqslant 1 - \operatorname*{ess\,sup}_{x\in\Omega} \left(\frac{\lambda + \mu}{\lambda + 3\mu - 2h}\right)^2 . \tag{37}$$

If h is sufficiently small, we have

$$\operatorname*{ess\,sup}_{x\in\Omega} \left(\frac{\lambda + \mu}{\lambda + 3\mu}\right)^2 < \operatorname*{ess\,sup}_{x\in\Omega} \left(\frac{\lambda + \mu}{\lambda + 3\mu - 2h}\right)^2$$

and then (37) implies (36). □

The next Theorem provides a sufficient conditions for the strict L^Φ-dissipativity of the elasticity operator.

Theorem 5 *Assume that the* BMO *seminorm of the function* $\mu^2\,(\lambda + 3\mu)^{-1}$ *is sufficiently small. If* (36) *holds, then the elasticity operator* (1) *is strict* L^Φ*-dissipative.*

Proof. We note that condition (36) implies $\Lambda_\infty^2 < 1$. In view of (28) and Corollary (1), if we prove that there exists $h > 0$ such that $E - h\Delta$ is L^Φ-dissipative, the assertion follows.

Let δ be a real constant such that

$$0 < \delta < 1 - \operatorname*{ess\,sup}_{x\in\Omega} \left(\frac{\lambda + \mu}{\lambda + 3\mu}\right)^2 - \Lambda_\infty^2 \tag{38}$$

and, taking into account (32), choose κ such that

$$0 < \kappa < \frac{\delta}{2(1 - \Lambda_\infty^2)} \min\Big\{ \operatorname*{ess\,inf}_{x\in\Omega} \mu(x)\,; \operatorname*{ess\,inf}_{x\in\Omega} (\lambda(x) + 2\mu(x))\Big\}. \tag{39}$$

Let $v \in [\mathring{H}^1(\Omega)]^2$. For elasticity operator the left hand side of (23) becomes

$$\int_\Omega \Big((\mu-\kappa)|\nabla v|^2 + \lambda(\operatorname{div} v)^2 + \mu \sum_{k,j} \partial_k v_j\, \partial_j v_k \\ -\Lambda^2(|v|)\left[(\mu-\kappa)|\nabla|v||^2 + (\lambda+\mu)|v|^{-2}|v_h\partial_h|v||^2\right]\Big) dx\,. \tag{40}$$

Following the ideas used in [4], given $v \in [\mathring{H}^1(\Omega)]^2$, we define

$$X_1 = |v|^{-1}(v_1\partial_1|v| + v_2\partial_2|v|), \quad X_2 = |v|^{-1}(v_2\partial_1|v| - v_1\partial_2|v|)$$
$$Y_1 = |v|[\partial_1(|v|^{-1}v_1) + \partial_2(|v|^{-1}v_2)], \quad Y_2 = |v|[\partial_1(|v|^{-1}v_2) - \partial_2(|v|^{-1}v_1)]$$

on the set $\Omega_0 = \{x \in \Omega \mid v(x) \neq 0\}$. We have $|v|^{-2}|v_h\partial_h|v||^2 = X_1^2$ and, as it was proved in [4, p.245],

$$|\nabla v|^2 = X_1^2 + X_2^2 + Y_1^2 + Y_2^2; \quad (\operatorname{div} v)^2 = (X_1+Y_1)^2;$$
$$|\nabla|v||^2 = X_1^2 + X_2^2.$$

We have also

$$\sum_{k,j} \partial_k v_j \partial_j v_k = (\operatorname{div} v)^2 + 2(\partial_1 v_2 \partial_2 v_1 - \partial_1 v_1 \partial_2 v_2) = (X_1+Y_1)^2 - 2(X_1Y_1 + X_2Y_2). \tag{41}$$

By means of these equalities, the integral (40) can be written as

$$\int_{\Omega_0} \Big((\lambda+2\mu-\kappa)[1-\Lambda^2(|v|)]X_1^2 + 2\lambda X_1Y_1 + (\lambda+2\mu-\kappa)Y_1^2\Big) dx \\ + \int_{\Omega_0} \Big((\mu-\kappa)[1-\Lambda^2(|v|)]X_2^2 - 2\mu X_2Y_2 + (\mu-\kappa)Y_2^2\Big) dx\,. \tag{42}$$

Define

$$\gamma(x) = \mu(x)\frac{\lambda(x)+\mu(x)}{\lambda(x)+3\mu(x)}$$

and rewrite (42) as

$$\int_{\Omega_0} \Big((\mu-\kappa)[1-\Lambda^2(|v|)]X_2^2 - 2\gamma X_2Y_2 + (\mu-\kappa)Y_2^2\Big) dx \\ +2\int_{\Omega_0} (\gamma-\mu)(X_1Y_1 + X_2Y_2)\, dx \\ + \int_{\Omega_0} \Big((\lambda+2\mu-\kappa)[1-\Lambda^2(|v|)]X_1^2 + 2(\lambda+\mu-\gamma)X_1Y_1 + (\lambda+2\mu-\kappa)Y_1^2\Big) dx. \tag{43}$$

We claim that

$$\gamma^2 = \mu^2 \left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2 < (\mu-\kappa)^2(1-\Lambda_\infty^2) \quad \text{a.e.}\,. \tag{44}$$

Indeed (39) leads to $2\kappa(1-\Lambda_\infty^2) < \delta\mu$ a.e., which implies

$$(2\mu\kappa - \kappa^2)(1-\Lambda_\infty^2) < \delta\mu^2 \qquad \text{a.e.}\,.$$

Since in view of (38)

$$\mu^2 \left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2 < \mu^2(1-\Lambda_\infty^2-\delta) \qquad \text{a.e.}\,,$$

inequality (44) follows. By similar arguments one can prove that

$$(\lambda+\mu-\gamma)^2 = (\lambda+2\mu)^2 \left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2 < (\lambda+2\mu-\kappa)^2(1-\Lambda_\infty^2) \quad \text{a.e.}\,. \tag{45}$$

Inequalities (44) and (45) show that

$$(\mu-\kappa)[1-\Lambda^2(|v|)]X_2^2 - 2\gamma X_2 Y_2 + (\mu-\kappa)Y_2^2 \geqslant 0 \quad \text{a.e.}$$

for any X_2, Y_2 and

$$(\lambda+2\mu-\kappa)[1-\Lambda^2(|v|)]X_1^2 + 2(\lambda+\mu-\gamma)X_1Y_1 + (\lambda+2\mu-\kappa)Y_1^2 \geqslant 0 \quad \text{a.e.}$$

for any X_1, Y_1. Therefore, keeping in mind (43), we can write

$$\begin{gathered}\int_\Omega \Big((\mu-\kappa)|\nabla v|^2 + \lambda(\operatorname{div} v)^2 + \mu \sum_{k,j} \partial_k v_j\, \partial_j v_k \\ -\Lambda^2(|v|)\left[(\mu-\kappa)|\nabla|v||^2 + (\lambda+\mu)|v|^{-2}|v_h\partial_h|v||^2\right] \Big) dx \\ \geqslant 2\int_{\Omega_0} (\gamma-\mu)(X_1Y_1 + X_2Y_2)\, dx\,. \end{gathered} \tag{46}$$

Since (see (41))

$$2(X_1Y_1 + X_2Y_2) = (\operatorname{div} v)^2 - \sum_{k,j} \partial_k v_j \partial_j v_k$$

and

$$\gamma - \mu = -2\frac{\mu^2}{\lambda + 3\mu}\,,$$

the last integral in (46) can be written as

$$2\int_\Omega \frac{\mu^2}{\lambda+3\mu}\Big(\sum_{k,j}\partial_k v_j \partial_j v_k - (\operatorname{div} v)^2\Big)dx\,.$$

If $v \in [\mathring{C}^\infty(\Omega)]^2$ and we consider $\mu^2/(\lambda+3\mu)$ as a distribution f, we have

$$\int_\Omega f\partial_k v_j \partial_j v_k\, dx = -\int_\Omega \partial_k f\, v_j \partial_j v_k\, dx - \int_\Omega f\, v_j \partial_{kj} v_k\, dx =$$
$$-\int_\Omega \partial_k f\, v_j\partial_j v_k\, dx + \int_\Omega \partial_j f\, v_j \partial_k v_k\, dx + \int_\Omega f\, \partial_j v_j \partial_k v_k\, dx$$

and then

$$\int_\Omega f\Big(\sum_{k,j}\partial_k v_j\partial_j v_k - (\operatorname{div} v)^2\Big)dx = \sum_{k,j}\int_\Omega \partial_k f\,(v_k\partial_j v_j - v_j\partial_j v_k)dx\,.$$

Thanks to a result by Maz'ya and Verbitsky [18, Lemma 4.9, p.1315] (see also [19]) we have the commutator inequality

$$\left|\sum_{k,j}\int_\Omega \partial_k f\,(v_k\partial_j v_j - v_j\partial_j v_k)dx\right| \leqslant C_0\,\|f\|_{BMO}\,\|\nabla v_1\|\,\|\nabla v_2\|\,.$$

Therefore

$$\left|2\int_\Omega \frac{\mu^2}{\lambda+3\mu}\Big(\sum_{k,j}\partial_k v_j\partial_j v_k - (\operatorname{div} v)^2\Big)dx\right|$$
$$\leqslant C_0\,\|\mu^2(\lambda+3\mu)^{-1}\|_{BMO}\,\|\nabla v\|^2$$

for any $v\in[\mathring{C}^\infty(\Omega)]^2$. By density, the same inequality holds for any $v \in [\mathring{H}^1(\Omega)]^2$.

From (46) it follows

$$\begin{aligned}\int_\Omega \Big(\Big(\mu - \frac{\kappa}{2}\Big)|\nabla v|^2 + \lambda(\operatorname{div} v)^2 + \mu\sum_{k,j}\partial_k v_j\,\partial_j v_k \\ -\Lambda^2(|v|)\Big[\Big(\mu-\frac{\kappa}{2}\Big)|\nabla|v||^2 + (\lambda+\mu)|v|^{-2}|v_h\partial_h|v||^2\Big]\Big)dx \\ \geqslant \frac{\kappa}{2}\int_\Omega\Big(|\nabla v|^2 - \Lambda^2(|v|)\,|\nabla|v||^2\Big)dx \\ -C_0\,\|\mu^2(\lambda+3\mu)^{-1}\|_{BMO}\int_\Omega|\nabla v|^2 dx\,.\end{aligned} \tag{47}$$

If

$$\|\mu^2(\lambda+3\mu)^{-1}\|_{BMO} \leqslant \frac{\kappa}{2\,C_0}\,(1-\Lambda_\infty^2) \tag{48}$$

we have (see also (26))

$$C_0\,\|\mu^2(\lambda+3\mu)^{-1}\|_{BMO}\int_\Omega |\nabla v|^2 dx \leqslant \frac{\kappa}{2}\,(1-\Lambda_\infty^2)\int_\Omega |\nabla v|^2 dx$$
$$\leqslant \frac{\kappa}{2}\int_\Omega \Big(|\nabla v|^2 - \Lambda^2(|v|)\,|\nabla |v||^2\Big) dx$$

and the right hand side of (47) is nonnegative. This means that the operator $E-(\kappa/2)\Delta$ is L^Φ-dissipative, which proves the theorem. $\square$

Combining theorems 4 and 5, we have immediately the following necessary and sufficient condition.

Theorem 6 *Assume that the BMO seminorm of the function $\mu^2\,(\lambda+3\mu)^{-1}$ is sufficiently small. The elasticity operator* (1) *is strict L^Φ-dissipative if and only if the stric inequality* (36) *holds.*

Remark 2 If λ and μ are constant, the BMO seminorm of the function $\mu^2\,(\lambda+3\mu)^{-1}$ is zero and then the strict inequality (36) is necessary and sufficient for the strict L^Φ-dissipativity of elasticity operator (1).

Remark 3 Condition (vi) on the function φ is used only in the necessity part of Theorem 6. Therefore, if (vi) it is not satisfied, the sufficiency part of Theorem 6 is still valid, where $\Lambda_\infty^2 = \sup_{t>0}\Lambda^2(t)$.

6 Some applications

In this section we show two applications of the theory we have developed. In particular we obtain regularity results for energy solutions of Dirichlet problem for Lamé system. In these results the energy solution which *a priori* belongs to the Sobolev space $H^1(\Omega)$, actually satisfy higher integrability conditions, provided the datum satisfies a certain condition, slightly more restrictive than belonging to $L^2(\Omega)$.

6.1 The N-dimensional case ($N \geqslant 3$)

We prove the following result wich concern the N-dimensional Lamé system ($N \geqslant 3$) with constant Lamé coefficients. As usually these constants are supposed to satisfy the inequalities: $\mu > 0, \lambda + 2\mu > 0$.

First we give the following sufficient condition for the strict L^Φ-dissipativity of the Lamé operator (2).

Theorem 7 *Let Ω be a domain in $\mathbb{R}^N$ and suppose that*

$$\Lambda_\infty^2 < \begin{cases} \mu/(\lambda+2\mu), & \text{if } \lambda+\mu > 0; \\ (\lambda+2\mu)/\mu, & \text{if } \lambda+\mu \leqslant 0. \end{cases}$$

Then the Lamé operator (2) *is strictly L^Φ-dissipative.*

Proof. Choose κ such that $0 < \kappa < \min\{\mu,\, \lambda+2\mu\}$ and

$$\Lambda_\infty^2 < \begin{cases} (\mu-\kappa)/(\lambda+2\mu-\kappa), & \text{if } \lambda+\mu > 0; \\ (\lambda+2\mu-\kappa)/(\mu-\kappa), & \text{if } \lambda+\mu \leqslant 0. \end{cases}$$

Setting $\lambda' = \lambda + \kappa$, $\mu' = \mu - \kappa$, we can write

$$\Lambda_\infty^2 < \begin{cases} \mu'/(\lambda'+2\mu'), & \text{if } \lambda'+\mu' > 0; \\ (\lambda'+2\mu')/\mu', & \text{if } \lambda'+\mu' \leqslant 0. \end{cases}$$

Note that $\mu' > 0$, $\lambda' + 2\mu' > 0$. By repeating the arguments we have used in in [5, p.126] for the L^p-dissipativity, we find that the operator

$$E'u = \mu' \Delta u + (\lambda' + \mu') \nabla \operatorname{div} u\,,$$

is L^Φ-dissipative. Since the L^Φ-dissipative operator E' coincide with $E - \kappa\Delta$, Corollary 1 shows that E is strictly L^Φ-dissipative. □

Theorem 8 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $p \geqslant 2$ such that*

$$\left(1 - \frac{2}{p}\right)^2 < \begin{cases} \mu/(\lambda+2\mu), & \text{if } \lambda+\mu > 0; \\ (\lambda+2\mu)/\mu, & \text{if } \lambda+\mu \leqslant 0. \end{cases} \tag{49}$$

Consider the Dirichlet problem for the Lamé operator (2)

$$\begin{cases} u \in \mathring{H}^1(\Omega) \\ Eu = \operatorname{Div} F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{50}$$

where $F = \{F_{ij}\}$ *is a given matrix in* $[L^{\frac{Np}{N+p-2}}(\Omega)]^{N^2}$ *and* $\operatorname{Div} F$ *denotes the vector whose* j*-th component is* $\partial_i F_{ij}$*. Then the solution* u *satisfies the inequality*

$$\int_\Omega |\nabla u|^2 \, (|u|^{p-2} + 1) \, dx < +\infty \, . \tag{51}$$

In particular,

$$\int_\Omega |\nabla u|^2 \, |u|^{p-2} \, dx \leqslant C \left(\int_\Omega |F|^{\frac{Np}{N+p-2}} dx \right)^{\frac{N+p-2}{N}} \tag{52}$$

where the constant C *does not depend on* u*. Moreover the solution* u *belongs to* $L^{\frac{Np}{N-2}}(\Omega)$*.*

Proof. Saying that u is solution of problem (50) means

$$\int_\Omega [\mu \, \langle \nabla u, \nabla v \rangle \; + \; (\lambda + \mu)(\operatorname{div} u)(\operatorname{div} v)] \, dx = \int_\Omega F_{ij} \, \partial_i v_j \, dx \tag{53}$$

for any $v \in \mathring{H}^1(\Omega)$. The existence and the uniqueness of the solution $u \in \mathring{H}^1(\Omega)$ is guaranteed by classic results, because $F \in [L^{\frac{Np}{N+p-2}}(\Omega)]^{N^2} \subset [L^2(\Omega)]^{N^2}$, the number $Np/(N+p-2)$ being greater than or equal to 2.

Let now $k > 0$ and define

$$\varphi_k(t) = \begin{cases} t^{p-2}, & \text{if } 0 \leqslant t < k; \\ k^{p-2}, & \text{if } t \geqslant k. \end{cases}$$

Let us consider the functional Φ_k-dissipativity related to φ_k. The function φ_k is not C^1 - as required in condition (i) - but only Lipschitz. One can show that all of our results holds under this more general assumption. Moreover φ_k satisfies condition (ii)-(v) and then we can apply our results. Setting

$$\Lambda_k \left(s\sqrt{\varphi_k(s)} \right) = - \frac{s \, \varphi_k'(s)}{s \, \varphi_k'(s) + 2 \, \varphi_k(s)} \, ,$$

we have $\Lambda_k^2 \leqslant (1-2/p)^2$. Inequality (49) and Theorem 7 show that E is L^{Φ_k}-dissipative with the same constant κ, and then

$$\int_\Omega [\mu\, \langle \nabla v, \nabla(\varphi_k(|v|)\, v)\rangle \;+\; (\lambda+\mu)(\operatorname{div} v)(\operatorname{div}(\varphi_k(|v|)\, v))]\, dx \\ \geqslant \kappa \int_\Omega |\nabla(\sqrt{\varphi_k(|v|)}\, v)|^2 dx$$

for any $v \in \mathring{H}^1(\Omega)$. Since $\varphi_k(|u|)\, u$ belongs to $\mathring{H}^1(\Omega)$, this inequality and (53) lead to

$$\kappa \int_\Omega |\nabla(\sqrt{\varphi_k(|u|)}\, u)|^2 dx \leqslant \int_\Omega |F_{ij}|\, |\partial_i(\varphi_k(|u|)\, u_j)|\, dx. \tag{54}$$

We can write

$$\partial_i(\varphi_k(|u|)\, u_j) = \partial_i(\sqrt{\varphi_k(|u|)}\,\sqrt{\varphi_k(|u|)}\, u_j) \\ = \sqrt{\varphi_k(|u|)}\, u_j\, \partial_i(\sqrt{\varphi_k(|u|)}\,) + \sqrt{\varphi_k(|u|)}\, \partial_i(\sqrt{\varphi_k(|u|)}\, u_j)\,.$$

By Cauchy inequality we get

$$\int_\Omega |F_{ij}|\, |\partial_i(\varphi_k(|u|)\, u_j)|\, dx \\ \leqslant \left(\int_\Omega |F|^2 \varphi_k(|u|)\, dx\right)^{1/2} \left[\left(\int_\Omega |u|^2\, |\nabla(\sqrt{\varphi_k(|u|)}|^2 dx\right)^{\frac{1}{2}} \right. \\ \left. + \left(\int_\Omega |\nabla(\sqrt{\varphi_k(|u|)}\, u)|^2 dx\right)^{\frac{1}{2}} \right],$$

where $|F| = (\sum_{i,j}^{1,N} |F_{ij}|^2)^{1/2}$. Since

$$|u|^2\, |\nabla(\sqrt{\varphi_k(|u|)}\,|^2 \leqslant (1-p/2)^2 \varphi_k(|u|)|\nabla u|^2 \leqslant (p-2)^2 |\nabla(\sqrt{\varphi_k(|u|)}\, u|^2$$

(see (15)), from (54) it follows

$$\int_\Omega |\nabla v_k|^2 dx \leqslant C \int_\Omega |F|^2 \varphi_k(|u|)\, dx\,, \tag{55}$$

where $v_k = \sqrt{\varphi_k(|u|)}\, u$. Here and in the sequel the same symbol C denotes different constants which do not depend on v_k.

Setting $\alpha = Np/((N-2)(p-2))$, by Hölder inequality we have

$$\int_\Omega |F|^2 \varphi_k(|u|)\,dx \leqslant \left(\int_\Omega \varphi_k(|u|)^\alpha dx\right)^{\frac{1}{\alpha}} \left(\int_\Omega |F|^{2\alpha'} dx\right)^{\frac{1}{\alpha'}}$$

where $\alpha' = Np/(2(N+p)-4)$.

Observing that $\varphi_k(|u|) \leqslant |v_k|^{2(p-2)/p}$, we find

$$\int_\Omega |F|^2 \varphi_k(|u|)\,dx \leqslant \left(\int_\Omega |v_k|^{\frac{2N}{N-2}} dx\right)^{\frac{1}{\alpha}} \left(\int_\Omega |F|^{2\alpha'} dx\right)^{\frac{1}{\alpha'}}$$

and in view of Sobolev imbedding theorem we obtain

$$\int_\Omega |F|^2 \varphi_k(|u|)\,dx \leqslant C \left(\int_\Omega |\nabla v_k|^2 dx\right)^{\frac{p-2}{p}} \left(\int_\Omega |F|^{2\alpha'} dx\right)^{\frac{1}{\alpha'}} .$$

Inequality (55) implies

$$\int_\Omega |\nabla v_k|^2 dx \leqslant C \left(\int_\Omega |F|^{\frac{Np}{N+p-2}} dx\right)^{\frac{N+p-2}{N}}$$

In view of (15) we have also

$$4\,|\nabla v_k|^2 \geqslant \varphi_k(|u|)\,|\nabla u|^2$$

and then

$$\int_\Omega |\nabla u|^2\,\varphi_k(|u|)\,dx \leqslant C \left(\int_\Omega |F|^{\frac{Np}{N+p-2}} dx\right)^{\frac{N+p-2}{N}} .$$

Letting $k \to +\infty$ we obtain (52) and (51) follows immediately. Recalling Remark 1, we have also

$$\int_\Omega |\nabla(|u|^{\frac{p-2}{2}}\, u)|^2\,dx \leqslant C \left(\int_\Omega |F|^{\frac{Np}{N+p-2}} dx\right)^{\frac{N+p-2}{N}}$$

and then u belongs to $L^{\frac{Np}{N-2}}(\Omega)$, because of the Sobolev imbedding theorems. □

6.2 The 2-dimensional case

Before giving our result for $N = 2$, we recall a couple of facts concerning Orlicz spaces. For the general theory of these spaces we refer to the monographs [16] and [20].

Let (M, N) be a complementary pair of Young's functions (see, e.g., [20, p.6]) defined for $t \geqslant 0$. The Orlicz space $\mathscr{L}_M(\Omega)$ is defined as the class of measurable functions defined in Ω such that there exists $\alpha > 0$ such that

$$\int_\Omega M(\alpha\,|u|)\,dx < +\infty.$$

In the space $\mathscr{L}_M(\Omega)$ we can introduce two norms:

$$\|u\|_{\mathscr{L}_M(\Omega)} = \sup\left\{ \int_\Omega u\,v\,dx \;\Big|\; \int_\Omega N(|v|)\,dx \leqslant 1 \right\}$$

which is called Orlicz norm, and the Luxemburg norm

$$|\!|\!|u|\!|\!|_{\mathscr{L}_M(\Omega)} = \inf\left\{ \lambda > 0 \;\Big|\; \int_\Omega M(|u|/\lambda)\,dx \leqslant 1 \right\}.$$

The two norms are equivalent, because of the inequalities

$$|\!|\!|u|\!|\!|_{\mathscr{L}_M(\Omega)} \leqslant \|u\|_{\mathscr{L}_M(\Omega)} \leqslant 2\,|\!|\!|u|\!|\!|_{\mathscr{L}_M(\Omega)} \tag{56}$$

for any $u \in \mathscr{L}_M(\Omega)$ (see, e.g., [20, p.61]). We recall also that Hölder inequality holds in the following form (see, e.g., [20, p.58])

$$\int_\Omega |u\,v|\,dx \leqslant 2\,|\!|\!|u|\!|\!|_{\mathscr{L}_M(\Omega)}\,|\!|\!|v|\!|\!|_{\mathscr{L}_N(\Omega)}\,. \tag{57}$$

Theorem 9 *Let Ω be a bounded domain in $\mathbb{R}^2$. Let E be the Lamé operator* (1) *with Lamé coefficients in $L^\infty(\Omega)$. Suppose*

$$\left(1 - \frac{2}{p}\right)^2 < 1 - \operatorname*{ess\,sup}_{x\in\Omega} \left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2$$

with $p \geqslant 2$ and that the BMO norm of the function $\mu^2/(\lambda+3\mu)$ satisfies inequality (48)*. Consider the Dirichlet problem* (50)*, where the matrix F is such that*

$$\int_\Omega |F|^2 (\log(|F|+e))^{\frac{p-2}{p}}\,dx < +\infty \tag{58}$$

Then the solution u *satisfies the inequality* (51)*. In particular,*

$$\int_\Omega |\nabla u|^2\, |u|^{p-2}\, dx \leqslant K \Big|\!\Big|\!\Big|\, |F|^2 \,\Big|\!\Big|\!\Big|^{\frac{p}{2}}_{\mathscr{L}_{\widetilde{N}}(\Omega)} \tag{59}$$

where

$$\widetilde{N}(t) = t\,(\log(t+e))^{\frac{p-2}{p}}$$

and the constant K *does not depend on* u*.*

Proof. If $p = 2$ the result is well known, since in this case (58) means $F \in [L^2(\Omega)]^2$. Suppose then $p > 2$. The assumptions on p and Lamé coefficients assure that E is stric L^p-dissipative, because of Theorem 5. As in Theorem 8 we find (55), keeping in mind also Remark 3, .

The function v_k being in $\overset{\circ}{H}{}^1(\Omega)$, Trudinger inequality [22] holds:

$$\int_\Omega e^{4\pi|v_k|^2} dx < +\infty\,.$$

Setting $w_k = \varphi_k(|u|)$ we have also

$$\int_\Omega e^{4\pi w_k^{p/(p-2)}} dx < +\infty\,,$$

because $w_k \leqslant |v_k|^{2(p-2)/p}$. Set $M(t) = e^{4\pi t^{p/(p-2)}} - 1$. Let $N(t)$ be its complementary Young's function. We have

$$N(t) = t\left(\frac{\log(t+e)}{4\pi}\right)^{\frac{p-2}{p}} (1 + o(1)) \tag{60}$$

(as $t \to +\infty$). Thanks to the Hölder inequality (57)

$$\int_\Omega |F|^2 w_k\, dx \leqslant 2 \,|\!|\!|\, w_k \,|\!|\!|_{\mathscr{L}_M(\Omega)} \Big|\!\Big|\!\Big|\, |F|^2 \,\Big|\!\Big|\!\Big|_{\mathscr{L}_N(\Omega)}\,. \tag{61}$$

Let us introduce now another Orlicz space $\mathscr{L}_{M_0}(\Omega)$, where $M_0(t) = e^{4\pi t} - 1$. Let us prove that

$$|\!|\!|\, w_k \,|\!|\!|_{\mathscr{L}_M(\Omega)} \leqslant \Big|\!\Big|\!\Big|\, |v_k|^2 \,\Big|\!\Big|\!\Big|^{\frac{p-2}{p}}_{\mathscr{L}_{M_0}(\Omega)}\,. \tag{62}$$

Take $\mu > 0$ such that

$$\int_\Omega (e^{4\pi\frac{|v_k|^2}{\mu}} - 1)\, dx \leqslant 1. \tag{63}$$

We have also

$$\int_\Omega (e^{4\pi(\frac{w_k}{\lambda})^{\frac{p}{p-2}}} - 1)\, dx \leqslant 1$$

where $\lambda = \mu^{(p-2)/p}$. By definition of Luxemburg norm

$$|||\, w_k \,|||_{\mathscr{L}_M(\Omega)} \leqslant \lambda\,,$$

i.e.

$$|||\, w_k \,|||^{\frac{p}{p-2}}_{\mathscr{L}_M(\Omega)} \leqslant \mu.$$

This being true for any μ satisfying (63), we obtain inequality (62) taking the infimum on the right hand side.

We claim now that

$$\|\, |v|^2 \,\|_{\mathscr{L}_{M_0}(\Omega)} \leqslant C \int_\Omega |\nabla v|^2 dx\,. \tag{64}$$

This inequality is a particular case of a general result proved by Maz'ya (see [17, p.158]). In view of this theorem, we can say that (64) is true for any $v \in \overset{\circ}{C}{}^\infty(\Omega)$ if and only if there exists a constant β such that

$$m(F)\, N_0^{-1}(1/m(F)) \leqslant \beta\, \text{cap}(F,\Omega) \tag{65}$$

for any compact set $F \subset \Omega$. Here $m(F)$ denotes the Lebesgue measure of F and cap(F,Ω) is the capacity of F relative to Ω, i.e.

$$\text{cap}(F,\Omega) = \inf \left\{ \int_\Omega |\nabla u|^2 dx \;\middle|\; u \in \overset{\circ}{C}{}^\infty(\Omega),\; u \geqslant 1 \text{ on } F \right\}.$$

We can write

$$N_0^{-1}(t) = 4\pi t/\log(m(\Omega)t + e)(1 + o(1))$$

and then

$$\begin{aligned} m(F)\, N_0^{-1}(1/m(F)) &= 4\pi/\log([m(\Omega)/m(F)] + e)(1 + o(1)) \\ &\leqslant 4\pi/\log([m(\Omega)/m(F)])(1 + o(1)). \end{aligned}$$

On the other hand, the following isocapacitary inequality holds (see [17, p.148])

$$\text{cap}(F,\Omega) \geqslant 4\pi/\log([m(\Omega)/m(F)])$$

and (65) is proved. Thanks to Maz'ya's result, estimate (64) is valid for any $v \in \mathring{C}^\infty(\Omega)$ and then, by density, for any $v \in \mathring{H}^1(\Omega)$.

From (55), (61), (62), (56) and (64) it follows

$$\int_\Omega |\nabla u|^2 \, \varphi_k(|u|) \, dx \leqslant C ||| \, |F|^2 \, |||_{\mathscr{L}_N(\Omega)}^{\frac{p}{2}}$$

Letting $k \to +\infty$ we obtain

$$\int_\Omega |\nabla u|^2 \, |u|^{p-2} dx \leqslant C ||| \, |F|^2 \, |||_{\mathscr{L}_N(\Omega)}^{\frac{p}{2}}$$

and (59) follows immediately from (60). □

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