Sobolev embeddings into Orlicz spaces and isocapacitary inequalities

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Abstract

Sobolev embeddings into Orlicz spaces on domains in the Euclidean space or, more generally, on Riemannian manifolds are considered. The situation when the domains are so irregular that the degree of integrability of a function may be lower than the one of its gradient is focused. A necessary and sufficient condtion for the validity of the relevant embeddings is established in terms of the isocapacitary function of the domain. Compact embeddings are also discussed. As a consequence, sufficient conditions involving the isoperimetric function of the domain are derived. These criteria fill in a missing case in a frame of Sobolev embeddings into Lebesgue and Orlicz spaces.

1 Introduction

A standard version of the classical Sobolev embedding theorem asserts that, if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and $p \in [1, \infty]$, then

$$V^{1,p}(\Omega) \to \begin{cases} L^{\frac{np}{n-p}}(\Omega) & \text{if } 1 \leq p < n \\ L^q(\Omega) & \text{if } p = n \\ L^\infty(\Omega) & \text{if } p > n. \end{cases} \tag{1.1}$$

Here, $V^{1,p}(\Omega)$ denotes the homogenous Sobolev space of those weakly differentiable functions on Ω such that $\nabla u \in L^p(\Omega)$. The same is true when Ω is an open subset, with a Lipschitz continuous boundary and a compact closure, of an n-dimensional Riemannian manifold $\mathbb{M}$, or when $\Omega = \mathbb{M}$, if the latter is compact.

Embeddings (1.1) continue to hold under somewhat weaker regularity assumptions on Ω. This is the case, for instance, if Ω has the cone condition, or satisfies a relative isoperimetric inequality, or if it is an (ε, δ)-domain. If the irregularity of the domain Ω is too severe, Sobolev embeddings can still hold, but into larger target spaces. The (ir)regularity of Ω can be prescribed in diverse ways.

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In the papers [Ma1, Ma2], the isoperimetric and the isocapacitary function of a domain Ω were introduced, and the idea that they can allow for a characterization of Sobolev embeddings in Ω was launched. They are the largest functions of the measure of subsets of Ω which can be bounded by the perimeter or the condenser capacity of the relevant subsets, respectively. Precise definitions are recalled in Section 2. The quality of the domain Ω is reflected in the asymptotic behaviour of the isoperimetric and of the isocapacitary function at 0. In a sense, decreasing the regularity of Ω causes these functions to decay faster to 0 when their argument tends to 0.

Diverse aspects of the theory of Sobolev functions and of partial differential equations, which exploit the isoperimetric and the isocapacitary function, have later been developed in various contributions, including [Ma3, Ma4, Ma5, Ma6, MaNe]. They also constitute the core of the monograph [Ma8] (the latest updated edition of [Ma7]) and of [MaPo]. Over the years, the isocapacitary and, especially, the isoperimetric function, both in the Euclidean and in the Riemmanian setting, have been the subject of a number of investigations – see e.g. [Ba, BC, BuZa, CMS, CF, Ci1, Ci2, CGL, Co, CoMa, Ga, GP, GiPi, HK, HHN, KM, Kl, Ku, La, LaMa, MJ, NaPa, Pa, Pi, Ps, Ri1, Ri2, StZu].

Generally speaking, the use of the isoperimetric function λ_Ω of Ω yields necessary and sufficient conditions for embeddings of the space $V^{1,1}(\Omega)$. Necessary and sufficient conditions for embeddings of the space $V^{1,p}(\Omega)$, for $p > 1$, can be formulated via the p-isocapacitary function $\nu_{\Omega,p}$ of Ω, which turns out to agree with λ_Ω only if $p = 1$.

Specifically, assume that $\mathcal{H}^n(\Omega) < \infty$, where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on $\mathbb{M}$, namely the volume measure on $\mathbb{M}$ induced by its Riemannian metric. Then the criterion for the validity of the embedding

$$V^{1,p}(\Omega) \to L^q(\Omega) \tag{1.2}$$

for $p \geq 1$, takes a different form, according to whether $q \geq p \geq 1$ or $1 \leq q < p$. By [MaPo, Theorem 8.5.2], if $q \geq p \geq 1$, then embedding (1.2) holds if and only if

$$\sup_{s \in (0, \frac{\mathcal{H}^n(\Omega)}{2})} \frac{s^{\frac{p}{q}}}{\nu_{\Omega,p}(s)} < \infty. \tag{1.3}$$

If, instead, $1 \leq q < p$, then [MaPo, Theorem 8.5.3] tells us that embedding (1.2) holds if and only if

$$\int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} \left(\frac{s}{\nu_{\Omega,p}(s)}\right)^{\frac{q}{p-q}} ds < \infty. \tag{1.4}$$

In fact, these results are established in [MaPo] with a definition of the isocapacitary function that slightly differs from the one adopted in this paper. Their poofs carry over to the setting at hand via minor changes.

Let us notice that condition (1.3) only applies when the function $\nu_{\Omega,p}(s)$ does not decay faster than s as $s \to 0^+$. By contrast, condition (1.4) is relevant in the complementary circumstance when Ω is so irregular that $\nu_{\Omega,p}(s)$ is not bounded from below by (a mutilple of) s. Hence, criterion (1.4) is crucial when very irregular domains Ω are in question, in which case embedding (1.2) can only hold for $q < p$.

Conditions (1.3) or (1.4) are sharp for embeddings of the form (1.2), involving Lebesgue target spaces. On the other hand, for certain domains Ω and powers p, the relevant embeddings can be improved if the class of admissible targets is enlarged as to include Orlicz spaces. This is well known to be the case, for instance, for the borderline exponent $p = n$ in (1.1). Indeed, one has that

$$V^{1,p}(\Omega) \to \exp L^{n'}(\Omega), \tag{1.5}$$

where $\exp L^{n'}(\Omega)$ denotes the Orlicz space associated with the Young function $e^{t^{n'}}-1$, and $n'=\frac{n}{n-1}$, the Hölder conjugate of n [Po, Tr, Yu]. More generally, Orlicz spaces naturally come into play as optimal targets in embeddings in domain whose isocapacitary function does not have a power type decay at 0.

The embeddings in question have the form

$$V^{1,p}(\Omega)\to L^B(\Omega), \tag{1.6}$$

where B is a Young function and $L^B(\Omega)$ is the Orlicz space associated with B. Embedding (1.6) is equivalent to the Sobolev-Poincaré type inequality

$$\|u-\mathrm{m}(u)\|_{L^B(\Omega)}\leq C\|\nabla u\|_{L^p(\Omega)} \tag{1.7}$$

for some constant C and for every function $u\in V^{1,p}(\Omega)$, where $\mathrm{m}(u)$ denotes the median of u – see Lemma 2.6. The same assertion holds if $\mathrm{m}(u)$ is replaced by the mean value of u over Ω, since such a replacement results in an equivalent quantity on the left-hand side of inequality (1.7), up to multiplicatve constants depending on B and $\mathcal{H}^n(\Omega)$.

In [Ma8, Theorem 2.3.3/1] a criterion for embeddings into Orlicz spaces $L^B(\Omega)$ of the subspace of $V^{1,p}(\Omega)$ of functions vanishing on $\partial\Omega$ is established. This criterion generalizes (1.3) and applies to Young functions B such that

$$\frac{B(t)}{t^p}\quad\text{is non-decreasing.} \tag{1.8}$$

Indeed, condition (1.8) amounts to requiring that $q\geq p$ when restricted to power type Young functions $B(t)=t^q$, in which case $L^B(\Omega)=L^q(\Omega)$. In fact, a mildly stronger assumption on the function B appears in [Ma8, Theorem 2.3.3/1]. A version of the relevant result in the spirit of the present paper is stated in the following theorem, .which can be established via minor modifications in the proofs of [Ma8, Theorem 2.3.3/1], on exploiting methods that will introduced in what follows.

Theorem A *Let $\mathbb{M}$ be an n-dimensional Riemannian manifold and let Ω be a connected open set in $\mathbb{M}$ such that $\mathcal{H}^n(\Omega)<\infty$. Let $p\geq 1$ and let B be a Young function fulfilling condition* (1.8).
(i) Assume that

$$\sup_{s\in(0,\frac{\mathcal{H}^n(\Omega)}{2})}\frac{1}{\nu_{\Omega,p}(s)B^{-1}(1/s)^p}<\infty. \tag{1.9}$$

Then embedding (1.6) *holds, and inequality* (1.7) *holds with a constant C equal to the supremum in* (1.16) *times a constant depending on p.*
(ii) Assume that embedding (1.6) *or, equivalently, inequality* (1.7) *holds. Then condition* (1.16) *is fulfilled.*

The purpose of the present paper is to offer a characterization of embedding (1.6) in the case opposite to (1.8), namely when

$$\frac{B(t)}{t^p}\quad\text{is non-increasing.} \tag{1.10}$$

In view of the discussion above, Orlicz spaces built upon Young functions B fulfilling (1.10) are of critical use when detecting optimal targets in embeddings in very irregular domains.

A condition for embedding (1.6) to hold under assumption (1.10) is exhibited in Theorem 1.1 below. The role of the power function with exponent $\frac{q}{p-q}$ in (1.4) is performed by the function M, associated, according to the following construction, with a Young function B fulfilling assumption (1.10). Define the (non-decreasing) function $E : (0, \infty) \to [0, \infty)$ as

$$E(s) = \frac{s}{B(s^{\frac{1}{p}})} \quad \text{for } s > 0. \tag{1.11}$$

Then the function $M : (0, \infty) \to [0, \infty)$ is given by

$$M(t) = \frac{E^{-1}(t)}{t} \quad \text{for } t > 0, \tag{1.12}$$

where E^{-1} denotes the (generalized) left-continuous inverse of E. It is easily verified that the function M is non-decreasing.

The condition on the function B, which extends (1.4), appearing in Theorem 1.1 is also necessary for embedding (1.6), provided that it is complemented by an additional assumption, which identifies a class of Young functions B fulfilling (1.10), but not satisfying (1.8). This enables to rule out borderline situations: for instance, the function $B(t) = t^p$ satisfies both (1.8) and (1.10). The additional assumption in question reads

$$\lim_{r \to 0^+} \left(\liminf_{t \to \infty} \frac{B(rt)}{r^p\, B(t)} \right) = \infty. \tag{1.13}$$

In particular, the couple of conditions (1.10) and (1.13) singles out the range of powers $1 \leq q < p$ in the special case when $B(t) = t^q$.

Theorem 1.1 *Let $\mathbb{M}$ be an n-dimensional Riemannian manifold and let Ω be a connected open set in $\mathbb{M}$ such that $\mathcal{H}^n(\Omega) < \infty$. Assume that $p > 1$ and let B be a Young function fulfilling condition* (1.10). *Let M be the function defined by* (1.12).
(i) Assume that

$$\int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} M\left(\frac{s}{\gamma\, \nu_{\Omega,p}(s)}\right) ds < \infty \quad \textit{for some } \gamma > 0. \tag{1.14}$$

Then embedding (1.6) *holds and there exists a constant $c = c(p)$ such that inequality* (1.7) *holds with*

$$C = c \inf_{\gamma > 0} \gamma^{\frac{1}{p}} \left(1 + \int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} M\left(\frac{s}{\gamma\, \nu_{\Omega,p}(s)}\right) ds\right). \tag{1.15}$$

(ii) Assume in addition that condition (1.13) *is in force. If embedding* (1.6), *or, equivalently, inequality* (1.7) *holds, then condition* (1.14) *is fulfilled.*

The compactness of embedding (1.6) can also be deduced from information on the asymptotic behaviour of the isocapacitary function $\nu_{\Omega,p}$. The following result applies in the regime 1.8 for the function B. It can be deduced via an analogous argument as in the proof of [CiMa1, Theorem 2.4].

Theorem B *Let $\mathbb{M}$ be an n-dimensional Riemannian manifold and let Ω be a connected open set in $\mathbb{M}$ such that $\mathcal{H}^n(\Omega) < \infty$. Let $p \geq 1$ and let B be a Young function fulfilling condition* (1.8).
(i) Assume that

$$\lim_{s \to 0^+} \frac{1}{\nu_{\Omega,p}(s) B^{-1}(1/s)^p} = 0. \tag{1.16}$$

Then embedding (1.6) *is compact.*

The compactness of embedding (1.6) for functions B fulfilling condition (1.10) is the subject of the next theorem. It requires that the expression on the right-hand side of equation (1.15) decays to 0 when $\frac{\mathcal{H}^n(\Omega)}{2}$ is replaced by an upper limit of integration which is sent to 0. Observe that such an assumption is always satified whenever the function B, and hence the function M, is a power.

Theorem 1.2 *Let Ω, p, B and M be as in Theorem 1.1. Assume that*

$$\lim_{\kappa\to 0^+}\left(\inf_{\gamma>0}\gamma^{\frac{1}{p}}\left(1+\int_0^\kappa M\Big(\frac{s}{\gamma\,\nu_{\Omega,p}(s)}\Big)\,ds\right)\right)=0. \tag{1.17}$$

Then embedding (1.6) *is compact.*

Sufficient conditions, in terms of the isoperimetric function λ_Ω, for the validity of embedding (1.6) and for its compactness can be derived from Theorem 1.1, thanks to the inequality

$$\nu_{\Omega,p}(s)\geq\left(\int_s^{\frac{\mathcal{H}^n(\Omega)}{2}}\frac{dr}{\lambda_\Omega(r)^{p'}}\right)^{1-p}\qquad\text{for } s\in\big(0,\tfrac{\mathcal{H}^n(\Omega)}{2}\big). \tag{1.18}$$

This inequality holds for every $p>1$ and every connected open set $\Omega\subset\mathbb{M}$ with $\mathcal{H}^n(\Omega)<\infty$, as can be shown via the same proof as [Ma8, Proposition 6.3.5/1].

Corollary 1.3 *Let Ω, p and B be as in Theorem 1.1. Assume that*

$$\int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} M\left(\frac{s}{\gamma}\left(\int_s^{\frac{\mathcal{H}^n(\Omega)}{2}}\frac{dr}{\lambda_\Omega(r)^{p'}}\right)^{p-1}\right)ds<\infty\quad\textit{for some } \gamma>0. \tag{1.19}$$

Then embedding (1.6) *holds, and there exists a constant $c=c(p)$ such that inequality* (1.7) *holds with*

$$C=c\inf_{\gamma>0}\gamma^{\frac{1}{p}}\left(1+\int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} M\left(\frac{s}{\gamma}\left(\int_s^{\frac{\mathcal{H}^n(\Omega)}{2}}\frac{dr}{\lambda_\Omega(r)^{p'}}\right)^{p-1}\right)ds\right).$$

Moreover, if

$$\lim_{\kappa\to 0^+}\left(\inf_{\gamma>0}\gamma^{\frac{1}{p}}\left(1+\int_0^{\kappa} M\left(\frac{s}{\gamma}\left(\int_s^{\frac{\mathcal{H}^n(\Omega)}{2}}\frac{dr}{\lambda_\Omega(r)^{p'}}\right)^{p-1}\right)ds\right)\right)=0, \tag{1.20}$$

then embedding (1.6) *is compact.*

Remark 1.4 The monotoncity assumptions (1.8) in Theorem A, and (1.10) in Theorems 1.1–1.2 and in Corollary 1.3 can be weakened by requiring that they just hold near infinity. Indeed, in either case, the function B can be replaced, if necessary, by a Young function equivalent near infinity for which the monotonicity in question holds in the whole of $(0,\infty)$. If $\mathcal{H}^n(\Omega)<\infty$, such a replacemente leaves the Orlicz space $L^B(\Omega)$ unchanged, up to equivalent norms.

2 Background and preliminaries

In this section we collect some definitions and standard results in the theory of function spaces entering our analysis, as well as a few technical lemmas, to be used in the proofs of our main results.

2.1 Young functions

A function $A : [0, \infty) \to [0, \infty]$ is called a Young function if it is convex (non trivial), left-continuous and $A(0) = 0$. If, in addition, A is finite-valued, strictly increasing and

$$(2.1) \qquad \lim_{t\to 0^+} \frac{A(t)}{t} = 0 \qquad \lim_{t\to\infty} \frac{A(t)}{t} = \infty,$$

then it is said to be an N-function.
Assume that A is a Young function. Then

$$(2.2) \qquad A(t) + A(\tau) \leq A(t+\tau) \quad \text{for } t, \tau \geq 0.$$

Moreover,

$$(2.3) \qquad k\, A(t) \leq A(kt) \quad \text{for } k \geq 1 \text{ and } t \geq 0.$$

The Young conjugate of A is is the Young function $\widetilde{A}$ defined by

$$\widetilde{A}(t) = \sup\{\tau t - A(\tau) : \tau \geq 0\} \quad \text{for } t \geq 0.$$

In particular, if A is an N-function, then $\widetilde{A}$ is an N-function as well.
One can show that

$$(2.4) \qquad s \leq A^{-1}(s)\widetilde{A}^{-1}(s) \leq 2s \qquad \text{for } s \geq 0.$$

Here, inverses are defined as to be right-continuous.
Assume that the function $\mathcal{A} : (0, \infty) \to [0, \infty)$ is such that $\frac{\mathcal{A}(t)}{t}$ is increasing. Then the function $A : [0, \infty) \to [0, \infty)$ given by

$$(2.5) \qquad A(t) = \int_0^t \frac{\mathcal{A}(\tau)}{\tau}\, d\tau \quad \text{for } t \geq 0,$$

is a Young function, and

$$(2.6) \qquad A(t) \leq \mathcal{A}(t) \leq A(2t) \quad \text{for } t \geq 0.$$

Let A be a Young function and let $q \in (1, \infty)$. Then

$$(2.7) \qquad \frac{A(t)}{t^q} \nearrow \quad \text{if and only if} \quad \frac{\widetilde{A}(t)}{t^{q'}} \searrow .$$

Here, and in what follows, the notations "$\nearrow$" and "$\searrow$" stand for "non-decreasing" and "non-increasing", respectively.
A Young function A is said to dominate another Young function B near infinity if there exist positive constants t_0 and c such that

$$(2.8) \qquad B(t) \leq A(ct) \quad \text{for } t \geq t_0.$$

The functions A and B are called equivalent near infinity if they dominate each other near infinity. If inequality (2.8) holds with $t_0 = 0$, then A is said to dominate B globally. Global equivalence of Young functions is defined accordingly. The relation

$$A \approx B$$

will be used to denote equivalence of functions in this sense.

The following technical lemmas concern properties of auxiliary functions built upon Young functions B fulfilling property (1.10), which will be needed in the proofs of our main results.

Lemma 2.1 *Let $p > 1$ and let $B : [0,\infty) \to [0,\infty)$ be a function fuflilling condition* (1.8)*. Let $\underline{B} : [0,\infty) \to [0,\infty)$ be the function obeying*

$$\underline{B}^{-1}(t) = \left(\int_0^t \frac{B^{-1}(s)^p}{s}\, ds\right)^{\frac{1}{p}} \qquad \text{for } t \geq 0. \tag{2.9}$$

Then $\underline{B}$ is equivalent to B, with absolute equivalence constants, and $\big(\underline{B}^{-1}\big)^p$ is a Young function.

Proof. As a consequence of assumption (1.10), the function $B^{-1}(s)^p/s$ is non-decreasing. Hence, the function

$$\int_0^t \frac{B^{-1}(s)^p}{s}\, ds, \tag{2.10}$$

which agrees with $\big(\underline{B}^{-1}(t)\big)^p$ for $t > 0$, is actually a Young function. It remains to show that $\underline{B}$ is equivalent to B. As a consequence of property (2.6), one has that

$$B^{-1}(t/2)^p \leq \underline{B}^{-1}(t)^p \leq B^{-1}(t)^p \quad \text{for } t \geq 0.$$

Therefore,

$$B(t) \leq \underline{B}(t) \leq 2B(t) \leq B(2t) \quad \text{for } t \geq 0.$$

Hence, the equivalence of B and $\underline{B}$ follows. □

Lemma 2.2 *Let $p > 1$ and let $B : [0,\infty) \to [0,\infty)$ be a function such that the function $H : [0,\infty) \to [0,\infty)$ defined by*

$$H(t) = B^{-1}(t)^p \qquad \text{for } t \geq 0, \tag{2.11}$$

is a Young function fulfilling

$$\lim_{t\to\infty} \frac{H(t)}{t} = \infty. \tag{2.12}$$

Define the function $Q : (0,\infty) \to [0,\infty)$ as

$$Q(t) = \frac{\widetilde{H}(t)}{t} \qquad \text{for } t > 0. \tag{2.13}$$

Then,

$$Q(t) \quad \nearrow\ , \tag{2.14}$$

and

$$\frac{Q(t^{p-1})}{t} \quad \nearrow \quad \text{in } (0,\infty). \tag{2.15}$$

Moreover, the function $N : [0,\infty) \to [0,\infty)$, given by

$$N(t) = \int_0^t \frac{Q(\tau^{p-1})}{\tau} \qquad \text{for } t \geq 0, \tag{2.16}$$

is a Young function such that

$$N(t) \leq Q(t^{p-1}) \leq N(2t) \qquad \text{for } t > 0, \tag{2.17}$$

and

$$Q((t-\tau)^{p-1}) \leq N(2t) - N(2\tau) \qquad \text{if } 0 < \tau < t. \tag{2.18}$$

Proof. Since H is Young function, then $\widetilde{H}$ is also Young function, whence property (2.14) follows. In particular, observe that, thanks to assumption (2.12), the functions $\widetilde{H}$ and Q are finite-valued and continuous. As for (2.15), one has that

$$\begin{aligned} \frac{Q(t^{p-1})}{t} \nearrow \quad \text{if and only if} \quad \frac{\widetilde{H}(t)}{t^{p'}} \nearrow \quad \text{if and only if} \quad \frac{H(t)}{t^p} \searrow \\ \text{if and only if} \quad \left(\frac{B^{-1}(t)}{t}\right)^p \searrow \quad \text{if and only if} \quad \left(\frac{t}{B(t)}\right)^p \searrow . \end{aligned} \tag{2.19}$$

Note that the second equivalence holds owing to property (2.7). Since B is a Young function, the last property of chain (2.19) holds. Hence, property (2.15) follows.
Inequalities (2.17) are consequences of property (2.6).
Finally, inequality (2.18) holds owing to the second inequality in (2.17) and property (2.2). □

Lemma 2.3 *Let $p > 1$, let B be a function as in Lemma 2.2 and let E be the function given by* (1.11). *Let H be the Young function defined by* (2.11) *and let $\underline{B}$ be the Young function defined via* (2.9). *Set*

$$\underline{H}(t) = \underline{B}^{-1}(t)^p \quad \textit{for } t \geq 0. \tag{2.20}$$

Then

$$\widetilde{\underline{H}}(t) \approx \widetilde{H}(t) \approx E^{-1}(t) \quad \textit{for } t \geq 0, \tag{2.21}$$

with absolute equivalence constants.

Proof. To begin with, observe that $\underline{H}$ is a Young function, since it agrees with the function appearing in equation (2.10). Next, by Lemma 2.1,

$$\frac{s}{\underline{H}^{-1}(s)} = \frac{s}{\underline{B}(s^{\frac{1}{p}})} \approx \frac{s}{B(s^{\frac{1}{p}})} = \frac{s}{H^{-1}(s)} = E(s) \quad \text{for } s > 0, \tag{2.22}$$

with absolute equivalence constants. Here H^{-1} and $\underline{H}^{-1}$ stand for right-continuous inverses. From equations (2.22) and (2.4), we deduce that

$$E(s) \leq \widetilde{H}^{-1}(s) \leq 2E(s) \quad \text{and} \quad E(s/c) \leq \widetilde{\underline{H}}^{-1}(s) \leq 2E(cs) \quad \text{for } s > 0, \tag{2.23}$$

for some absolute constant $c > 0$. Hence, equation (2.21) follows, since both $\widetilde{H}$ and $\widetilde{\underline{H}}$ are Young functions. □

Lemma 2.4 *Let p and B be as in Lemma 2.2, and let Q be the function defined by* (2.13). *Then*

$$B^{-1}(Q(t))^p \leq tQ(t) \qquad \textit{for } t > 0. \tag{2.24}$$

Proof. By the very definition (2.11) of the function H,

$$H^{-1}(t) = B\big(t^{\frac{1}{p}}\big) \quad \text{for } t \geq 0. \tag{2.25}$$

Hence, owing to property (2.4),

$$\widetilde{H}^{-1}(t) \geq \frac{t}{H^{-1}(t)} = \frac{t}{B\big(t^{\frac{1}{p}}\big)} \quad \text{for } t > 0. \tag{2.26}$$

Inequality (2.24) is equivalent to

$$B^{-1}(Q(t))^p \le \widetilde{H}(t) \qquad \text{for } t > 0. \tag{2.27}$$

Since $\widetilde{H}^{-1}(\widetilde{H}(t)) \ge t$ for $t > 0$, inequality (2.27) will follow if we show that

$$B^{-1}(Q(\widetilde{H}^{-1}(t)))^p \le t \qquad \text{for } t > 0. \tag{2.28}$$

Inasmuch as

$$B^{-1}(Q(\widetilde{H}^{-1}(t)))^p = B^{-1}\left(\frac{t}{\widetilde{H}^{-1}(t)}\right)^p \qquad \text{for } t > 0, \tag{2.29}$$

and $\widetilde{H}(\widetilde{H}^{-1}(t)) \le t$ for $t > 0$, inequality (2.28) will in turn follow if we show that

$$B^{-1}\left(\frac{t}{\widetilde{H}^{-1}(t)}\right)^p \le t \quad \text{for } t > 0. \tag{2.30}$$

The latter inequality can be rewritten as

$$\frac{t}{\widetilde{H}^{-1}(t)} \le B\big(t^{\frac{1}{p}}\big) \quad \text{for } t > 0, \tag{2.31}$$

which holds, owing to (2.26). □

2.2 Orlicz and Sobolev spaces

Let Ω be a measurable subset of an n-dimensional manifold $\mathbb{M}$, and let A be a Young function. The Orlicz space $L^A(\Omega)$ is the Banach space of those measurable functions $u : \Omega \to \mathbb{R}$ for which the Luxemburg norm

$$\|u\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A\left(\frac{|u|}{\lambda}\right) d\mathcal{H}^n \le 1\right\}$$

is finite. In particular, $L^A(\Omega) = L^p(\Omega)$ if $A(t) = t^p$ for some $p \in [1,\infty)$, and $L^A(\Omega) = L^\infty(\Omega)$ if $A(t) = 0$ for $t \in [0,1]$ and $A(t) = \infty$ for $t > 0$.

When convenient for specific choices of A, we shall also adopt the notation $A(L)(\Omega)$ to denote the Orlicz space $L^A(\Omega)$.

If $\mathcal{H}^n(\Omega) < \infty$ and A dominates B near infinity, then there exists a constant $c = c(A, B, \mathcal{H}^n(\Omega))$ such that

$$\|u\|_{L^B(\Omega)} \le C\|u\|_{L^A(\Omega)} \tag{2.32}$$

for every $u \in L^A(\Omega)$. In particular, if A is equivalent to B near infinity, then $L^A(\Omega) = L^B(\Omega)$, up to equivalent norms.

Assume now that Ω is open. The Sobolev space $W^{1,p}(\Omega)$ is defined, for $p \in [1,\infty]$, as

$$W^{1,p}(\Omega) = \{u \in L^p(M) : \ u \text{ is weakly differentiable on } \Omega \text{ and } |\nabla u| \in L^p(\Omega)\ \},$$

and is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

We denote by $W^{1,p}_0(\Omega)$ the closure in $W^{1,p}(\Omega)$ of the set of smooth compactly supported functions on Ω.
The homogenous Sobolev space $V^{1,p}(\Omega)$ is defined as

$$V^{1,p}(\Omega) = \{u : u \text{ is weakly differentiable in } \Omega \text{ and } |\nabla u| \in L^p(\Omega)\}\ .$$

If the set Ω is connected, and ω is an open set such that $\overline{\omega}$ is compact and $\overline{\omega} \subset \Omega$, then $V^{1,p}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{V^{1,p}(\Omega)} = \|u\|_{L^p(\omega)} + \|\nabla u\|_{L^p(\Omega)}. \tag{2.33}$$

Note that, replacing ω by another set with the same properties results in an equivalent norm.

Lemma 2.5 *Let Ω be a connected open set on an n-dimensional Riemannian manifold $\mathbb{M}$, such that $\mathcal{H}^n(\Omega) < \infty$. Let ω be an open set such that $\overline{\omega}$ is compact and $\overline{\omega} \subset \Omega$, and let $\kappa \in (0, \mathcal{H}^n(\Omega))$. Assume that $p \in [1, \infty]$. Then there exists a constant c such that*

$$\|u\|_{L^p(\omega)} \le c\|\nabla u\|_{L^p(\Omega)} \tag{2.34}$$

for every function $u \in V^{1,p}(\Omega)$ such that $\mathcal{H}^n(\{|u| > 0\}) \le \kappa$.

Proof. We make use of a contradiction argument along the lines of [Zi, Theorem 4.1.1]. Assume that inequality (2.34) fails for every constant c. Thus, there exist a sequence of functions $\{u_k\} \subset V^{1,p}(\Omega)$ and a sequence of measurable sets $\{E_k\}$ such that $E_k \subset \Omega$, $\mathcal{H}^n(E_k) \ge \kappa$, $u_k = 0$ a.e. in E_k, and

$$\|u_k\|_{L^p(\omega)} \ge k\|\nabla u_k\|_{L^p(\Omega)} \tag{2.35}$$

for $k \in \mathbb{N}$. On replacing u_k by $u_k/\|u_k\|_{L^p(\omega)}$, if necessary, we may assume that

$$\|u_k\|_{L^p(\omega)} = 1 \tag{2.36}$$

for $k \in \mathbb{N}$. Thus, the sequence $\{u_k\}$ is bounded in the space $V^{1,p}(\Omega)$, equipped with the norm defined as in (2.33). Let Ω' be an open set, with a smooth boundary, such that $\overline{\Omega'}$ is compact and $\overline{\omega} \subset \Omega' \subset \overline{\Omega'} \subset \Omega$ and $\mathcal{H}^n(\Omega') > \mathcal{H}^n(\Omega) - \kappa$. The embedding

$$V^{1,p}(\Omega') \to L^p(\Omega')$$

is compact, as a consequence of the classical Reillich-Kondrachov theorem applied to the intersection of Ω' with each chart of a finite covering of Ω'. Thus, there exists a subsequence of $\{u_k\}$, still indexed by k and a function $u \in L^p(\Omega)$, such that $u_k \to u$ in $L^p(\Omega')$ and $u_k \to u$ a.e. in Ω'. From equations (2.35) and (2.36) one infers that $\nabla u_k \to 0$ in $L^p(\Omega')$. Hence, $u \in W^{1,p}(\Omega')$ and $\nabla u = 0$ in Ω'. Consequently, there exists a constant c such that $u = c$ in Ω'. Thanks to equation (2.36), one has that

$$\|u\|_{L^p(\omega)} = 1,$$

whence $c \ne 0$. Inasmuch as $u_k \to u$ a.e. in Ω', we have that

$$u = 0 \quad \mathcal{H}^n\text{-a.e. in the set} \quad \cap_{k=1}^\infty \cup_{h=k}^\infty (E_k \cap \Omega'). \tag{2.37}$$

Since

$$\mathcal{H}^n(E_k \cup \Omega') + \mathcal{H}^n(E_k \cap \Omega') = \mathcal{H}^n(E_k) + \mathcal{H}^n(\Omega') \ge \kappa + \mathcal{H}^n(\Omega'),$$

one has that

$$\mathcal{H}^n(E_k \cap \Omega') \ge \kappa + \mathcal{H}^n(\Omega') - \mathcal{H}^n(E_k \cup \Omega') \ge \kappa + \mathcal{H}^n(\Omega') - \mathcal{H}^n(\Omega) > 0 \tag{2.38}$$

for $k \in \mathbb{N}$. Therefore,

$$\mathcal{H}^n\big(\cap_{k=1}^\infty \cup_{h=k}^\infty (E_k \cap \Omega')\big)\kappa + \mathcal{H}^n(\Omega') - \mathcal{H}^n(\Omega) > 0. \tag{2.39}$$

Equations (2.37) and (2.39) contradict the fact that $u = c \ne 0$ in Ω'. $\square$

The proofs of the following two lemmas can be accomplished along the same lines as those of [CiMa1, Lemma 2.6] and [CiMa1, Lemma 2.5], respectively. The details are omitted for brevity.

Lemma 2.6 *Let Ω be a connected open set on an n-dimensional Riemannian manifold $\mathbb{M}$, such that $\mathcal{H}^n(\Omega) < \infty$. Let $p \in [1, \infty]$ and let B be a Young function. Then there exists a constant c such that*

$$\|u - \mathrm{med}(u)\|_{L^B(\Omega)} \leq c \|\nabla u\|_{L^p(\Omega)} \tag{2.40}$$

for every function $u \in V^{1,p}(\Omega)$ if and only if

$$V^{1,p}(\Omega) \to L^B(\Omega). \tag{2.41}$$

Lemma 2.7 *Let Ω be a connected open set on an n-dimensional Riemannian manifold $\mathbb{M}$, such that $\mathcal{H}^n(\Omega) < \infty$. Let $p \in [1, \infty]$ and let B be a Young function. Then the embedding*

$$V^{1,p}(\Omega) \to L^B(\Omega) \tag{2.42}$$

is compact if and only if the embedding

$$V^{1,p}(\Omega) \cap L^B(\Omega) \to L^B(\Omega). \tag{2.43}$$

is compact.

2.3 Isoperimetric and isocapacitary functions

Let Ω be an open set in an n-dimensional Riemmanian manifold $\mathbb{M}$ (possibly $\Omega = \mathbb{M}$) and let E be a measurable subset of $\mathbb{M}$. The perimeter $P(E; \Omega)$ of E relative to Ω can be defined as

$$P(E; \Omega) = \mathcal{H}^{n-1}(\Omega \cap \partial^* E)\,,$$

where $\partial^* E$ stands for the essential boundary of E in the sense of geometric measure theory, and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure on M induced by its Riemannian metric ([Ma8, Zi]). Recall that $\partial^* E$ agrees with the topological boundary ∂E of E when the latter is sufficiently regular – an open subset of M with a smooth boundary, for instance.
Assume that $\mathcal{H}^n(\Omega) < \infty$. The isoperimetric function $\lambda_\Omega : \left[0, \frac{\mathcal{H}^n(\Omega)}{2}\right] \to [0, \infty)$ of Ω is defined as

$$\lambda_\Omega(s) = \inf\left\{P(E; \Omega) : E \subset \Omega, s \leq \mathcal{H}^n(E) \leq \tfrac{\mathcal{H}^n(\Omega)}{2}\right\} \qquad \text{for } s \in \left[0, \tfrac{\mathcal{H}^n(\Omega)}{2}\right]. \tag{2.44}$$

Obviously, the function λ_Ω is non-decreasing. The isoperimetric inequality relative to Ω is a straightforward consequence of the definition of λ_Ω, and reads:

$$\lambda_\Omega(\mathcal{H}^n(E)) \leq P(E; \Omega) \tag{2.45}$$

for every measurable set $E \subset \Omega$ with $\mathcal{H}^n(E) \leq \frac{\mathcal{H}^n(\Omega)}{2}$.
In particular, if Ω is connected, then

$$\lambda_\Omega(s) > 0 \qquad \text{for } s \in \left(0, \tfrac{\mathcal{H}^n(\Omega)}{2}\right], \tag{2.46}$$

as shown via an analogous argument as in [Ma2, Lemma 3.2.4].
Let us emhasize that, in view of our applications, only the asymptotic behaviour of the isoperimetric

function λ_Ω near 0 is relevant. For instance, if Ω has a Lipschitz continuous boundary and $\overline{\Omega}$ is compact, or $\Omega = \mathbb{M}$ and the latter is compact, then

$$\lambda_\Omega(s) = O(s^{\frac{1}{n'}}) \quad \text{as} \quad s \to 0^+. \tag{2.47}$$

The isocapacitary function $\nu_{\Omega,p}$ of Ω is defined in analogy with (2.44), provided that the perimeter of a set E relative to Ω is repleced by its condenser capacity. To be more specific, let us begin by recalling that the standard p-capacity of a set $E \subset \Omega$ can be defined, for $p \geq 1$, as

$$C_p(E) = \inf\left\{\int_\Omega |\nabla u|^p\, dx : u \in W^{1,p}_0(\Omega), u \geq 1 \text{ in some neighbourhood of } E\right\}. \tag{2.48}$$

Each function $u \in V^{1,p}(\Omega)$ has a representative $\widetilde{u}$, called the precise representative, which is C_p-quasi continuous, in the sense that for every $\varepsilon > 0$, there exists a set $E \subset M$, with $C_p(E) < \varepsilon$, such that $\widetilde{u}$ restricted to $\Omega \setminus E$ is continuous. The function $\widetilde{u}$ is unique, up to subsets of p-capacity zero. In what follows, we assume that any function $u \in V^{1,p}(\Omega)$ agrees with its precise representative.

In view of a classical result in potential theory (see e.g. [MZ, Corollary 2.25]), we adopt the following definition of capacity of a condenser. Let $E \subset G \subset \Omega$. Then we set

$$\operatorname{cap}_p(E, G) = \inf\left\{\int_\Omega |\nabla u|^p\, d\mathcal{H}^n : u \in V^{1,p}(\Omega), u \geq 1 \text{ in } E, u = 0 \text{ in } G \text{ (up to sets of } p\text{-capacity zero)}\right\}. \tag{2.49}$$

Also, if $\mathcal{H}^n(E) \leq \frac{\mathcal{H}^n(\Omega)}{2}$, we define

$$\operatorname{cap}_p(E) = \inf\left\{\operatorname{cap}_p(E, G) : E \subset G, \mathcal{H}^n(G) \leq \tfrac{\mathcal{H}^n(\Omega)}{2}\right\}. \tag{2.50}$$

The p-isocapacitary function $\nu_{\Omega,p} : [0, \frac{\mathcal{H}^n(\Omega)}{2}] \to [0, \infty)$ of Ω is then defined as

$$\nu_{\Omega,p}(s) = \inf\left\{\operatorname{cap}_p(E) : E \subset \Omega, s \leq \mathcal{H}^n(E) \leq \tfrac{\mathcal{H}^n(\Omega)}{2}\right\} \qquad \text{for } s \in [0, \tfrac{\mathcal{H}^n(\Omega)}{2}]. \tag{2.51}$$

The function $\nu_{\Omega,p}$ is clearly non-decreasing. In the light of definition (2.51), the isocapacitary inequality on Ω takes the form

$$\nu_{\Omega,p}(\mathcal{H}^n(E)) \leq \operatorname{cap}_p(E, G) \tag{2.52}$$

for every measurable sets $E \subset G \subset \Omega$ such that $\mathcal{H}^n(G) \leq \frac{\mathcal{H}^n(\Omega)}{2}$.
If $p > 1$, the function λ_Ω is related to $\nu_{\Omega,p}$ via inequality (1.18). Hence, in particular,

$$\nu_{\Omega,p}(s) > 0 \qquad \text{for } s \in \big[0, \tfrac{\mathcal{H}^n(\Omega)}{2}\big], \tag{2.53}$$

provided that Ω is connected.
When $p = 1$, one has that

$$\nu_{\Omega,1} = \lambda_\Omega \tag{2.54}$$

as shown by an analogous argument as in [Ma2, Lemma 2.2.5]. In particluar, if Ω has a regular boundary and $\overline{\Omega}$ is compact, or $\Omega = \mathbb{M}$ and the latter is compact, then

$$\nu_{\Omega,p}(s) = \begin{cases} O\big(s^{\frac{n-p}{n}}\big) & \text{if } 1 \leq p < n, \\ O\big(\big(\log \frac{1}{s}\big)^{1-n}\big) & \text{if } p = n, \\ O(1) & \text{if } p > n, \end{cases} \qquad \text{as} \quad s \to 0^+. \tag{2.55}$$

Irregular domains whose isocapacitary function has a different (faster) decay at 0 are presented in the examples discussed in Section 4.

A generalized version of the definitions of cap_p and $\nu_{\Omega,p}$ will also be exploited. Given $\kappa \in (0, \mathcal{H}^n(\Omega))$, and a set $E \subset \Omega$ such that $\mathcal{H}^n(E) \leq \kappa$, we set

$$\mathrm{cap}_{p,\kappa}(E) = \inf \Big\{ \mathrm{cap}_p(E, G) : E \subset G, \mathcal{H}^n(G) \leq \kappa \Big\}, \tag{2.56}$$

and define the function $\nu_{\Omega,p,\kappa} : [0, \kappa] \to [0, \infty)$ of Ω as

$$\nu_{\Omega,p,\kappa}(s) = \inf \Big\{ \mathrm{cap}_{p,\kappa}(E) : E \subset \Omega, s \leq \mathcal{H}^n(E) \leq \kappa \Big\} \qquad \text{for } s \in [0, \kappa]. \tag{2.57}$$

Plainly, $\nu_{\Omega,p,\frac{\mathcal{H}^n(\Omega)}{2}} = \nu_{\Omega,p}$, and, if $0 < \kappa < \frac{\mathcal{H}^n(\Omega)}{2}$, then

$$\nu_{\Omega,p,\kappa}(s) \geq \nu_{\Omega,p}(s) \qquad \text{for } s \in [0, \kappa]. \tag{2.58}$$

3 Proof of the main results

This section is devoted to the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1 Part (i). Assume that condition (1.14) is in force. Also, it suffices to prove inequality (1.7) under the assumption that $\mathrm{m}(u) = 0$. Hence,

$$\mathcal{H}^n(\{u > 0\}) \leq \tfrac{\mathcal{H}^n(\Omega)}{2} \quad \text{ and } \quad \mathcal{H}^n(\{u < 0\}) \leq \tfrac{\mathcal{H}^n(\Omega)}{2}. \tag{3.1}$$

To begin with, observe that, if $\lim_{t\to\infty} \frac{B(t)}{t^p} > 0$, then, by (1.10), $B(t)$ is in fact equivalent to t^p near infinity. The function E thus has a finte limit at infinity, and hence $M(t) = \infty$ for large t. As a consequence, condition (1.14) forces the function $\frac{s}{\nu_{M,p}(s)}$ to be bounded. Hence the conclusion follows from Theorem A, which holds under assumption (1.8), since condtion (1.16) turns out to be satified with $B(t) = t^p$.
We may thus assume that

$$\lim_{t\to\infty} \frac{B(t)}{t^p} = 0. \tag{3.2}$$

By Lemma 2.1, on replacing, if necessary, the function B by the equivalent function $\overline{B}$ obeying (2.9), we may assume, without loss of generality, that the function $\underline{H}$, defined as in (2.11) with B replaced ny $\underline{B}$, is a Young function. Since condition (3.2) still holds with B replaced by $\underline{B}$, we thus have that $\lim_{t\to\infty} \frac{\underline{H}(t)}{t} = \infty$: Consequently, the assumption of Lemma 2.2 are fulfilled after these replacements. Also, by Lemma 2.3, the function Q given by (2.13) is equivalent to the function M, and the same is true for the the function $\underline{Q}$ defined as in (2.13), with $\underline{H}$ instead of H after the replacement of B by $\underline{B}$. Let us assume in what follows that all these replacements are in force and let us drop, for simplicity of notation, the underlines in the notations of $\underline{B}$, $\underline{H}$ and $\underline{Q}$. In particular, we can thus write

$$Q(t) \approx M(t) \qquad \text{for } t > 0. \tag{3.3}$$

Observe that, in view of Lemmas 2.1 and 2.3, all the constants involved in the equivalences mentioned above are absolute.

Denote by $b : [0, \infty) \to [0, \infty)$ be the left-continuous derivative of the function B. Let us set

$$\mu(t) = \mathcal{H}^n(\{u > t\}) \text{ for } t > 0 \quad \text{ and } \quad \mu_j = \mu(2^{-j}) \text{ for } j \in \mathbb{Z}.$$

We have that

$$\int_{\{u>0\}} B(u)\,d\mathcal{H}^n = \int_0^\infty b(t)\mu(t)\,dt = \sum_{j\in\mathbb{Z}} \int_{2^{-j}}^{2^{-j+1}} b(t)\mu(t)\,dt \tag{3.4}$$
$$\leq \sum_{j\in\mathbb{Z}} \mu_j \int_{2^{-j}}^{2^{-j+1}} b(t)\,dt \leq \sum_{j\in\mathbb{Z}} \mu_j B(2^{-j+1}).$$

Define

$$\Omega_j = \{u > 2^{-j}\} \quad \text{ and } \quad \sigma_j = \operatorname{cap}_p(\Omega_j, \Omega_{j+1}) \text{ for } j \in \mathbb{Z}.$$

Owing to equation (3.4), the following chain holds for every $\gamma > 0$:

$$\int_{\{u>0\}} B(u)\,d\mathcal{H}^n \leq \sum_{j\in\mathbb{Z}} \frac{\mu_j}{\gamma\sigma_j} B(2^{-j+1})\gamma\sigma_j \leq \gamma \sum_{j\in\mathbb{Z}} H\big(B(2^{-j+1})\big)\sigma_j + \gamma \sum_{j\in\mathbb{Z}} \widetilde{H}\Big(\frac{\mu_j}{\gamma\sigma_j}\Big)\sigma_j \tag{3.5}$$
$$= \gamma 2^p \sum_{j\in\mathbb{Z}} 2^{-jp}\sigma_j + \gamma \sum_{j\in\mathbb{Z}} \widetilde{H}\Big(\frac{\mu_j}{\gamma\sigma_j}\Big)\sigma_j.$$

On the other hand,

$$\int_{\{u>0\}} |\nabla u|^p\,d\mathcal{H}^n = \sum_{j\in\mathbb{Z}} \int_{\Omega_{j+1}\setminus\Omega_j} |\nabla u|^p\,d\mathcal{H}^n \geq \sum_{j\in\mathbb{Z}} (2^{-j} - 2^{-j-1})^p \sigma_j = 2^{-p} \sum_{j\in\mathbb{Z}} 2^{-jp}\sigma_j. \tag{3.6}$$

Notice that the inequality in (3.6) is a consequence of the definition of condenser capacity and of the fact that the function u_j defined as $u_j = \frac{\max\{u-2^{-j-1},0\}}{2^{-j}-2^{-j-1}}$ for $j \in \mathbb{Z}$ enjoys the following properties: $u_j \in W^{1,p}_0(\Omega)$, $u_j \geq 1$ in Ω_j, $u_j = 0$ in $\Omega \setminus \Omega_{j+1}$.
Coupling equation (3.5) with (3.6) tells us that

$$\int_{\{u>0\}} B(u)\,d\mathcal{H}^n \leq \gamma 4^p \int_{\{u>0\}} |\nabla u|^p\,d\mathcal{H}^n + \gamma \sum_{j\in\mathbb{Z}} \widetilde{H}\Big(\frac{\mu_j}{\gamma\sigma_j}\Big)\sigma_j. \tag{3.7}$$

Set $U = \{u > 0\}$, and define

$$\theta_j = \operatorname{cap}_p(\Omega_j, U) \qquad \text{for } j \in \mathbb{Z}. \tag{3.8}$$

Hence,

$$\theta_j \geq \nu_{\Omega,p}(\mu_j) \qquad \text{for } j \in \mathbb{Z}. \tag{3.9}$$

Since $\Omega_j \subset \Omega_{j+1} \subset U$ for $j \in \mathbb{Z}$, by [Ma8, Lemma 2.3.8]

$$\operatorname{cap}_p(\Omega_j, \Omega_{j+1})^{\frac{1}{1-p}} + \operatorname{cap}_p(\Omega_{j+1}, U)^{\frac{1}{1-p}} \leq \operatorname{cap}_p(\Omega_j, U)^{\frac{1}{1-p}}, \tag{3.10}$$

namely

$$\sigma_j^{-1} \leq \Big(\theta_j^{\frac{1}{1-p}} - \theta_{j+1}^{\frac{1}{1-p}}\Big)^{p-1} \qquad \text{for } j \in \mathbb{Z}. \tag{3.11}$$

Let N be the function given by (2.16). The following chain holds for every $m \in \mathbb{N}$:

$$\gamma \sum_{j=-m}^{m} \widetilde{H}\Big(\frac{\mu_j}{\gamma\sigma_j}\Big)\sigma_j = \sum_{j=-m}^{m} \mu_j Q\Big(\frac{\mu_j}{\gamma\sigma_j}\Big) \leq \sum_{j=-m}^{m} \mu_j Q\Big(\Big[\Big(\frac{\mu_j}{\lambda}\Big)^{\frac{1}{p-1}} \Big(\theta_j^{\frac{1}{1-p}} - \theta_{j+1}^{\frac{1}{1-p}}\Big)\Big]^{p-1}\Big) \tag{3.12}$$

$$
\begin{aligned}
&\le \sum_{j=-m}^{m} \mu_j \Big[N\Big(2\Big(\frac{\mu_j}{\gamma}\Big)^{\frac{1}{p-1}}\theta_j^{\frac{1}{1-p}}\Big) - N\Big(2\Big(\frac{\mu_j}{\gamma}\Big)^{\frac{1}{p-1}}\theta_{j+1}^{\frac{1}{1-p}}\Big)\Big]\\
&= \sum_{j=-m-1}^{m-1} \mu_{j+1} N\Big(2\Big(\frac{\mu_{j+1}}{\gamma}\Big)^{\frac{1}{p-1}}\theta_{j+1}^{\frac{1}{1-p}}\Big) - \sum_{j=-m}^{m} \mu_j N\Big(2\Big(\frac{\mu_j}{\gamma}\Big)^{\frac{1}{p-1}}\theta_{j+1}^{\frac{1}{1-p}}\Big)\\
&= \sum_{j=-m}^{m-1} \Big[\mu_{j+1} N\Big(2\Big(\frac{\mu_{j+1}}{\gamma}\Big)^{\frac{1}{p-1}}\theta_{j+1}^{\frac{1}{1-p}}\Big) - \mu_j N\Big(2\Big(\frac{\mu_j}{\gamma}\Big)^{\frac{1}{p-1}}\theta_{j+1}^{\frac{1}{1-p}}\Big)\Big]\\
&\qquad + \mu_{-m} N\Big(2\Big(\frac{\mu_{-m}}{\gamma}\Big)^{\frac{1}{p-1}}\theta_{-m}^{\frac{1}{1-p}}\Big) - \mu_m N\Big(2\Big(\frac{\mu_m}{\gamma}\Big)^{\frac{1}{p-1}}\theta_m^{\frac{1}{1-p}}\Big)
\end{aligned}
$$

Notice that the first inequality holds owing to (3.11) and to the fact that the function Q is non-decreasing, and the second inequality holds owing to inequality (2.18). Define the function $P : [0,\infty) \to [0,\infty)$ as

$$P(t) = tN\big(t^{\frac{1}{p-1}}\big) \qquad \text{for } t \ge 0. \tag{3.13}$$

Observe that, by the first inequality in (2.17),

$$P'(t) = N\big(t^{\frac{1}{p-1}}\big) + \frac{1}{p-1}Q(t) \le p'Q(t) \qquad \text{for } t > 0. \tag{3.14}$$

Moreover, thanks to inequality (3.9) and the monotonicity of the function $\nu_{\Omega,p}$,

$$\frac{1}{\theta_{j+1}} \le \frac{1}{\nu_{\Omega,p}(\mu_{j+1})} \le \frac{1}{\nu_{\Omega,p}(s)} \qquad \text{for } s \in (\mu_j, \mu_{j+1}) \text{ and } j \in \mathbb{Z}. \tag{3.15}$$

Therefore, by equations (3.14) and (3.15),

$$
\begin{aligned}
&\mu_{j+1} N\Big(2\Big(\frac{\mu_{j+1}}{\gamma}\Big)^{\frac{1}{p-1}}\theta_{j+1}^{\frac{1}{1-p}}\Big) - \mu_j N\Big(2\Big(\frac{\mu_j}{\gamma}\Big)^{\frac{1}{p-1}}\theta_{j+1}^{\frac{1}{1-p}}\Big) = \gamma \int_{\mu_j/\gamma}^{\mu_{j+1}/\gamma} \frac{d}{dt}\Big[tN\Big(2t^{\frac{1}{p-1}}\theta_{j+1}^{\frac{1}{1-p}}\Big)\Big]\,dt\\
&= \frac{\gamma\,\theta_{j+1}}{2^{p-1}} \int_{\mu_j/\gamma}^{\mu_{j+1}/\gamma} \frac{d}{dt}\Big[P(2^{p-1}t\theta_{j+1}^{-1})\Big]\,dt = \gamma \int_{\mu_j/\gamma}^{\mu_{j+1}/\gamma} P'(2^{p-1}t\theta_{j+1}^{-1})\,dt\\
&\le \gamma\, p' \int_{\mu_j/\gamma}^{\mu_{j+1}/\gamma} Q(2^{p-1}t\theta_{j+1}^{-1})\,dt = p' \int_{\mu_j}^{\mu_{j+1}} Q\Big(2^{p-1}\frac{s}{\gamma}\theta_{j+1}^{-1}\Big)\,ds\\
&\le p' \int_{\mu_j}^{\mu_{j+1}} Q\Big(\frac{2^{p-1}}{\gamma}\frac{s}{\nu_{\Omega,p}(s)}\Big)\,ds \qquad \text{for } m \in \mathbb{Z}.
\end{aligned} \tag{3.16}
$$

A completely analogous chain yields the inequality

$$\mu_{-m} N\Big(2\Big(\frac{\mu_{-m}}{\gamma}\Big)^{\frac{1}{p-1}}\theta_{-m}^{\frac{1}{1-p}}\Big) \le p' \int_0^{\mu_{-m}} Q\Big(\frac{2^{p-1}}{\gamma}\frac{s}{\nu_{\Omega,p}(s)}\Big)\,ds \qquad \text{for } m \in \mathbb{Z}. \tag{3.17}$$

Combining inequalities (3.12), (3.16) and (3.17) implies that

(3.18)

$$\gamma \sum_{j=-m}^{m} \widetilde{H}\Big(\frac{\mu_j}{\gamma\sigma_j}\Big)\sigma_j \le \sum_{j=-m}^{m-1} p' \int_{\mu_j}^{\mu_{j+1}} Q\Big(\frac{2^{p-1}}{\gamma}\frac{s}{\nu_{\Omega,p}(s)}\Big)\,ds + p' \int_0^{\mu_{-m}} Q\Big(\frac{2^{p-1}}{\gamma}\frac{s}{\nu_{\Omega,p}(s)}\Big)\,ds$$

$$= p' \int_0^{\mu_m} Q\Big(\frac{2^{p-1}}{\gamma} \frac{s}{\nu_{\Omega,p}(s)}\Big)\, ds \le p' \int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} Q\Big(\frac{2^{p-1}}{\gamma} \frac{s}{\nu_{\Omega,p}(s)}\Big)\, ds \qquad \text{for } m \in \mathbb{Z}.$$

On letting $m \to \infty$ in inequality (3.18), we deduce from (3.7) that

$$\int_{\{u>0\}} B(u)\, d\mathcal{H}^n \le \gamma 4^p \int_{\{u>0\}} |\nabla u|^p \, d\mathcal{H}^n + p' \int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} Q\Big(\frac{2^{p-1}}{\gamma} \frac{s}{\nu_{\Omega,p}(s)}\Big)\, ds. \tag{3.19}$$

A parallel inequality holds for $-u$, namely

$$\int_{\{u<0\}} B(-u)\, d\mathcal{H}^n \le \gamma 4^p \int_{\{u<0\}} |\nabla u|^p \, d\mathcal{H}^n + p' \int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} Q\Big(\frac{2^{p-1}}{\gamma} \frac{s}{\nu_{\Omega,p}(s)}\Big)\, ds. \tag{3.20}$$

From inequalities (3.19) and (3.20) one infers that

$$\int_{\Omega} B(|u|)\, d\mathcal{H}^n \le \gamma 4^p \int_{\Omega} |\nabla u|^p \, d\mathcal{H}^n + 2p' \int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} Q\Big(\frac{2^{p-1}}{\gamma} \frac{s}{\nu_{\Omega,p}(s)}\Big)\, ds. \tag{3.21}$$

An application of this inequality with u replaced by u/λ, with $\lambda > 0$, yields

$$\int_{\Omega} B\Big(\frac{|u|}{\lambda}\Big)\, d\mathcal{H}^n \le \frac{\gamma 4^p}{\lambda^p} \int_{\Omega} |\nabla u|^p \, d\mathcal{H}^n + 2p' \int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} Q\Big(\frac{2^{p-1}}{\gamma} \frac{s}{\nu_{\Omega,p}(s)}\Big)\, ds. \tag{3.22}$$

The choice $\lambda = 4\gamma^{\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}$ then results in

$$\int_{\Omega} B\bigg(\frac{|u|}{4\gamma^{\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}}\bigg)\, d\mathcal{H}^n \le 1 + 2p' \int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} Q\Big(\frac{2^{p-1}}{\gamma} \frac{s}{\nu_{\Omega,p}(s)}\Big)\, ds. \tag{3.23}$$

Now, recall that the functions B and Q have been replaced at the beginning of this proof by equivalent functions, with absolute equivlance constants. Therefore, owing to equation (3.3), there exist absolute constants c_1 and c_2 such that

$$\int_{\Omega} B\bigg(\frac{c_1|u|}{\gamma^{\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}}\bigg)\, d\mathcal{H}^n \le 1 + 2p' \int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} M\Big(\frac{2^{p-1}c_2}{\gamma} \frac{s}{\nu_{\Omega,p}(s)}\Big)\, ds. \tag{3.24}$$

On denoting by K the right-hand side of inequality (3.24) we hence deduce, via inequality (2.3), that

$$\int_{\Omega} B\bigg(\frac{c_1|u|}{K\gamma^{\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}}\bigg)\, d\mathcal{H}^n \le 1. \tag{3.25}$$

Therefore, by the definition of Luxemburg norm,

$$\|u\|_{L^B(\Omega)} \le \frac{K\gamma^{\frac{1}{p}}}{c_1} \|\nabla u\|_{L^p(\Omega)}. \tag{3.26}$$

Consequently, inequality (1.7) holds with

$$C = \inf_{\gamma>0} \frac{K\gamma^{\frac{1}{p}}}{c_1} \bigg(1 + 2p' \int_0^{\frac{\mathcal{H}^n(\Omega)}{2}} M\Big(\frac{2^{p-1}c_2}{\gamma} \frac{s}{\nu_{\Omega,p}(s)}\Big)\, ds\bigg),$$

and hence with C of the form (1.15).

Part (ii). Assume that the additional assumption (1.13) is in force. Then the limit (3.2) holds as well, otherwise $B(t)$ would be equivalent to t^p near infinity, and hence $\liminf_{t\to\infty}\frac{B(rt)}{B(t)}$ would also be equivalent to r^p, up to multiplicative positive constant, thus contradicting (1.13). Thus, if necessary, the functions B, Q and M can be replaced (without changing notation) by equivalent Young functions, with absolute equivalence constants, as in Part (i).

Assume now that inequality (1.7) holds. By the definition of the isocapacitary function $\nu_{\Omega,p}$, given any $j \in \mathbb{Z}$ such that

$$2^j \le \tfrac{\mathcal{H}^n(\Omega)}{2}, \tag{3.27}$$

there exists a set $G_j \subset \Omega$ such that

$$2^j \le \mathcal{H}^n(G_j) \le \tfrac{\mathcal{H}^n(\Omega)}{2}, \tag{3.28}$$

and

$$\mathrm{cap}_p(G_j) \le 2\nu_{\Omega,p}(2^j), \tag{3.29}$$

and a function $u_j \in V^{1,p}(\Omega)$ such that

$$u_j \ge 1 \quad C_p\text{-q.e. on } G_j, \qquad \mathcal{H}^n(\{|u_j| > 0\}) \le \tfrac{\mathcal{H}^n(\Omega)}{2}, \tag{3.30}$$

and

$$\int_\Omega |\nabla u_j|^p \, d\mathcal{H}^n \le 4\nu_{\Omega,p}(2^j). \tag{3.31}$$

Let $h \in \mathbb{Z}$ be such that

$$2^h \le \tfrac{\mathcal{H}^n(\Omega)}{2} < 2^{h+1}. \tag{3.32}$$

Set, for $j \in \mathbb{Z}$,

$$\eta_j = B^{-1}\Big(Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big)\Big), \tag{3.33}$$

and define, for $k \in \mathbb{Z}$, $k \le h$, the function $v_k : \Omega \to \mathbb{R}$ as

$$v_k(x) = \max_{k\le j\le h} \eta_j u_j(x) \qquad \text{for } x \in \Omega. \tag{3.34}$$

A standard property of Sobolev functions ensures that $v_k \in V^{1,p}(\Omega)$ for every $k \le h$, and, owing to inequality (3.31),

$$\int_\Omega |\nabla v_k|^p \, d\mathcal{H}^n \le \sum_{j=k}^h \eta_j^p \int_\Omega |\nabla u_j|^p \, d\mathcal{H}^n \le 4\sum_{j=k}^h \eta_j^p \nu_{\Omega,p}(2^j). \tag{3.35}$$

Moreover, given any open set ω as in Lemma 2.5, by inequalities (2.34) and (3.31), there exists a constant c such that

$$\int_\omega |v_k|^p \, d\mathcal{H}^n \le \sum_{j=k}^h \eta_j^p \int_\omega |u_j|^p \, d\mathcal{H}^n \le c\sum_{j=k}^h \eta_j^p \int_\Omega |\nabla u_j|^p \, d\mathcal{H}^n \le 4c\sum_{j=k}^h \eta_j^p \nu_{\Omega,p}(2^j) \quad \text{for } k \le h. \tag{3.36}$$

Hence,

$$\int_\omega |v_k|^p \, d\mathcal{H}^n + \int_\Omega |\nabla v_k|^p \, d\mathcal{H}^n \le 4(c+1) \sum_{j=k}^h \eta_j^p \nu_{\Omega,p}(2^j) \quad \text{for } k \le h. \tag{3.37}$$

As a consequence of inequality (1.7) and Lemma 2.6, there exists a constant c such that

$$\|v\|_{L^B(\Omega)} \le c\big(\|v\|_{L^p(\omega)} + \|\nabla v\|_{L^p(\Omega)}\big)$$

for every function $v \in V^{1,p}(\Omega)$. An application of this inequality to the function v_k tells us that

$$\int_0^{\mathcal{H}^n(\Omega)} B\bigg(\frac{v_k^*(s)}{c\big(\|v_k\|_{L^p(\omega)} + \|\nabla v_k\|_{L^p(\Omega)}\big)}\bigg) \, ds = \int_\Omega B\bigg(\frac{|v_k|}{c\big(\|v_k\|_{L^p(\omega)} + \|\nabla v_k\|_{L^p(\Omega)}\big)}\bigg) \, d\mathcal{H}^n \le 1 \tag{3.38}$$

for $k \le h$. Here, $v_k^* : [0,\infty) \to [0,\infty)$ denotes the decreasing rearrangement of v_k, defined as

$$v_k^*(s) = \inf\big\{t \ge 0 : \mathcal{H}^n(\{x \in \mathbb{M} : |v_k(x)| > t\}) \le s\big\} \qquad \text{for } s \in [0, \mathcal{H}^n(\Omega)].$$

Next, observe that

$$v_k(x) \ge \eta_j \quad \text{for } x \in G_j \text{ and } k \le j \le h.$$

Therefore, by the first inequality in (3.28),

$$v_k^*(s) \ge \eta_j \qquad \text{if } s \in (0, 2^j) \quad \text{and} \quad k \le j \le h. \tag{3.39}$$

Inequalities (3.37)–(3.39) and inequality (2.24) imply that there exist constants c and c' such that

(3.40)

$$\begin{aligned} 1 &\ge \sum_{j=k}^h \int_{2^{j-1}}^{2^j} B\bigg(\frac{v_k^*(s)}{c\big(\|v_k\|_{L^p(\omega)} + \|\nabla v_k\|_{L^p(\Omega)}\big)}\bigg) \, ds \ge \sum_{j=k}^h 2^{j-1} B\bigg(\frac{\eta_j}{c\big(\|v_k\|_{L^p(\omega)} + \|\nabla v_k\|_{L^p(\Omega)}\big)}\bigg) \\ &\ge \sum_{j=k}^h 2^{j-1} B\bigg(\frac{\eta_j}{c'\big(\sum_{j=k}^h \eta_j^p \nu_{\Omega,p}(2^j)\big)^{\frac{1}{p}}}\bigg) = \frac{1}{2} \sum_{j=k}^h 2^j B\Bigg(\frac{B^{-1}\big(Q\big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\big)\big)}{c\Big(\sum_{j=k}^h 2^j B^{-1}\big(Q\big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\big)\big)^p \frac{\nu_{\Omega,p}(2^j)}{2^j}\Big)^{\frac{1}{p}}}\Bigg) \\ &\ge \frac{1}{2} \sum_{j=k}^h 2^j B\Bigg(\frac{B^{-1}\big(Q\big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\big)\big)}{c\Big(\sum_{j=k}^h 2^j Q\big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\big)\Big)^{\frac{1}{p}}}\Bigg) \end{aligned}$$

The monotonicity of the functions $\nu_{\Omega,p}$ and Q implies that

$$\begin{aligned} \sum_{j=-\infty}^h 2^j Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big) &\ge \sum_{j=-\infty}^h 2^{j-1} Q\Big(\frac{2^{j-1}}{\nu_{\Omega,p}(2^{j-1})}\Big) = \sum_{j=-\infty}^h \int_{2^{j-1}}^{2^j} Q\Big(\frac{2^j}{2\nu_{\Omega,p}(2^{j-1})}\Big) \, ds \\ &\ge \sum_{j=-\infty}^h \int_{2^{j-1}}^{2^j} Q\Big(\frac{s}{2\nu_{\Omega,p}(s)}\Big) \, ds = \int_0^{2^h} Q\Big(\frac{s}{2\nu_{\Omega,p}(s)}\Big) \, ds. \end{aligned} \tag{3.41}$$

Our purpose is now to show that

$$\sum_{j=-\infty}^h 2^j Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big) < \infty. \tag{3.42}$$

Assume, by contradiction that inequality fails, namely that

$$\sum_{j=-\infty}^{h} 2^j Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big) = \infty. \tag{3.43}$$

By assumption (1.13), for every $L > 0$ there exist $r_L > 0$ and $t_L > 0$ such that

$$\frac{B(rt)}{r^p B(t)} \geq L \quad \text{if } t \geq t_L \text{ and } 0 < r \leq r_L. \tag{3.44}$$

Set

$$J = \Big\{ j \in \mathbb{N} : B^{-1}\Big(Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big)\Big) \geq t_L \Big\}. \tag{3.45}$$

Since

$$\begin{aligned} \sum_{j=-\infty}^{h} 2^j Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big) &= \sum_{j=-\infty,\, j\in J}^{h} 2^j Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big) + \sum_{j=-\infty,\, j\notin J}^{h} 2^j Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big) \\ &\leq \sum_{j=-\infty,\, j\in J}^{h} 2^j Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big) + B(t_L) \sum_{j=-\infty}^{h} 2^j, \end{aligned} \tag{3.46}$$

and the last series is convergent, assumption (3.43) entails that

$$\sum_{j=-\infty,\, j\in J}^{h} 2^j Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big) = \infty. \tag{3.47}$$

Assumption (3.43) also implies that

$$\sum_{j=k,\, j\in J}^{h} 2^j Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big) \geq \frac{1}{2} \sum_{j=k}^{h} 2^{j-1} Q\Big(\frac{2^{j-1}}{\nu_{\Omega,p}(2^{j-1})}\Big) \tag{3.48}$$

provided that $-k$ is sufficiently large. Actually, on splitting the sum on the right-hand side into the sums over the indices $j \in J$ and $j \notin J$, and abosrbing the first sum to the left-hand side, inequality (3.48) can be rewritten as

$$\frac{1}{2} \sum_{j=k,\, j\in J}^{h} 2^j Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big) \geq \frac{1}{2} \sum_{j=k,\, j\notin J}^{h} 2^j Q\Big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\Big),$$

and the latter inequality certainly holds if $-k$ is large enough, since the sum on the left-hand side diverges as $k \to -\infty$, whereas the sum on the right-hand side converges.
From inequalitie (3.40), (3.48) and (**??**) we deduce that

$$\begin{aligned} 2 &\geq \sum_{j=k}^{h} 2^j B\left(\frac{B^{-1}\big(Q\big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\big)\big)}{c\Big(\sum_{j=k}^{h} 2^j Q\big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\big)\Big)^{\frac{1}{p}}} \right) \geq \sum_{j=k,\, j\in J}^{h} 2^j B\left(\frac{B^{-1}\big(Q\big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\big)\big)}{c\Big(2\sum_{j=k,\, j\in J}^{h} 2^j Q\big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\big)\Big)^{\frac{1}{p}}} \right) \\ &\geq \frac{L}{2\, c^p} \sum_{j=k,\, j\in J}^{h} \frac{2^j Q\big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\big)}{\sum_{j=k,\, j\in J}^{h} 2^j Q\big(\frac{2^j}{\nu_{\Omega,p}(2^j)}\big)} = \frac{L}{2\, c^p} \end{aligned} \tag{3.49}$$

provided that $-k$ is sufficiently large. This is impossible, thanks to the arbitrariness of L.
Inequality (3.42) is thus established. Hence, inequality (1.14) holds for some $\gamma > 0$, owing to equation (3.41) and (3.3). $\square$

Proof of Theorem 1.2 Owing to Lemma 2.7, it suffices to show that the embedding (2.43) is compact. Fix any $\kappa \in (0, \frac{\mathcal{H}^n(\Omega)}{2})$, and let E be any compact set in Ω such that $\mathcal{H}^n(\Omega \setminus E) < s$. Let η be any smooth compactly supported function on Ω such that $0 \leq \eta \leq 1$ and $\eta = 1$ in E. Set $U = \operatorname{supp} \eta$. Given any function $u \in V^{1,p}(\Omega) \cap L^B(\Omega)$, we have that

$$\|u\|_{L^B(\Omega)} \leq \|(1-\eta)u\|_{L^B(\Omega)} + \|\eta u\|_{L^B(U)}\,. \tag{3.50}$$

Let us set

$$v = (1-\eta)u.$$

Clearly, $v \in V^{1,p}(\Omega) \cap L^B(\Omega)$, and $\operatorname{supp} v \subset \Omega \setminus E$. Thus, for every $t > 0$, $\{x \in \Omega : |v| \geq t\} = \{x \in \Omega \setminus E : |v| \geq t\}$, and $\mathcal{H}^n\{x \in \Omega : |v| \geq t\}) \leq \kappa \leq \frac{\mathcal{H}^n(\Omega)}{2}$.
A close inspection of the proof of inequalty (1.7) reveals that the same argument yields the inequality

$$\|w\|_{L^B(\Omega)} \leq C\|\nabla w\|_{L^p(\Omega)} \tag{3.51}$$

for every function $w \in V^{1,p}(\Omega)$ such that $\mathcal{H}^n(\{|w| > 0\}) \leq \kappa$, with

$$C = c \inf_{\gamma>0} \gamma^{\frac{1}{p}} \left(1 + \int_0^\kappa M\Big(\frac{s}{\gamma\, \nu_{\Omega,p,\kappa}(s)}\Big)\, ds\right), \tag{3.52}$$

and $c = c(p)$, where $\mathrm{I}\nu_{\Omega,p,\kappa}$ is the generalized isocapacitary function defined as in (2.57). Inequality (2.58) implies that the integral on the right-hand side of equation (3.52) cannot decrease if the function $\nu_{\Omega,p,\kappa}$ is replaced by $\nu_{\Omega,p,\cdot}$. Thereby, thanks to assumption (1.17), given any $\varepsilon > 0$ there exists $\kappa \in (0, \frac{\mathcal{H}^n(\Omega)}{2})$ such that

$$\|w\|_{L^B(\Omega)} \leq \varepsilon\|\nabla w\|_{L^p(\Omega)} \tag{3.53}$$

for every function $w \in V^{1,p}(\Omega)$ such that $\mathcal{H}^n(\{|w| > 0\}) \leq \kappa$. From inequality (3.50) and inequality (3.53), applied with $w = v$, we deduce that

$$\|u\|_{L^B(\Omega)} \leq \varepsilon\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^B(U)}\,. \tag{3.54}$$

Let Ω' be an open bounded set in $\mathbb{M}$, with a smooth boundary, and such that $U \subset \Omega'$ and that $\overline{\Omega'}$ is compact in $\mathbb{M}$. We claim that the embedding

$$V^{1,p}(\Omega') \cap L^B(\Omega') \to L^B(\Omega') \tag{3.55}$$

is compact. Indeed, the embedding $V^{1,p}(\Omega') \cap L^1(\Omega') \to L^p(\Omega')$ is compact as a consequence of the classical Reillich-Kondrachov theorem applied to the intersection of Ω' with each chart of a finite covering of Ω'. On the other hand, $L^p(\Omega') \to L^B(\Omega')$, owing to assumption (1.10). Hence the compactness of embedding (3.55) follows, inasmuch as $V^{1,p}(\Omega') \cap L^B(\Omega') \to V^{1,p}(\Omega') \cap L^1(\Omega')$.
Now, consider any bounded sequence $\{u_k\}$ in $V^{1,p}(\Omega) \cap L^B(\Omega)$and denote by C a constant such that $\|\nabla u_k\|_{L^p(\Omega)} \leq C$ for $k \in \mathbb{N}$. By the compactness of embedding (3.55), there exists a Cauchy subsequence, still denoted by $\{u_k\}$, in $L^B(\Omega')$. Hence, given any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\|u_k - u_m\|_{L^B(\Omega')} \leq \varepsilon \tag{3.56}$$

if $k, m \geq k_0$. An application of inequality (3.54) with u replaced by $u_k - u_m$ for $k, m \in \mathbb{N}$ thus yields

$$\|u_k - u_m\|_{L^B(\Omega)} \leq \varepsilon\|\nabla u_k - \nabla u_m\|_{L^p(\Omega)} + \|u_k - u_m\|_{L^B(\Omega')} \leq (2C+1)\varepsilon \tag{3.57}$$

if $k, m \geq k_0$. Thanks to the arbitrariness of ε, this tells us that $\{u_k\}$ is Cauchy sequence in $L^B(\Omega)$. The compactness of embedding (2.43) is thus established. □

4 Examples

We conclude by offering Sobolev embeddings, in specific families of domains, which are deduced via our results. In all the examples discussed, the use of Orlicz spaces yields an augmented target space in the classical embeddings (1.2) into Lebesgue spaces, in the range $1 \leq q < p$. The last example also demonstrates that the criterion in terms of the isocapacitary function of a very irregular domain can yield stronger embeddings than the one involving its isoperimetric function.

Example 4.1 (An unbounded domain).
Let $\zeta : [0, \infty) \to (0, \infty)$ be a differentiable convex function such that $\lim_{\rho\to 0^+} \zeta'(\rho) > -\infty$ and $\lim_{\rho\to\infty} \zeta(\rho) = 0$. Consider the unbounded set

$$\Omega = \{x \in \mathbb{R}^n : x_n > 0, |x'| < \zeta(x_n)\}$$

(see Figure 1), where $x = (x', x_n)$ and $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Assume that

$$\int_0^\infty \zeta(r)^{n-1} dr < \infty\,, \tag{4.1}$$

in such a way that $\mathcal{H}^n(\Omega) < \infty$. Let $\Upsilon : [0, \infty) \to [0, \infty)$ be the function given by

$$\Upsilon(\rho) = n\omega_n \int_\rho^\infty \zeta(r)^{n-1} dr \qquad \text{for } \rho > 0.$$

By [Ma8, Example 3.3.3/2] ,

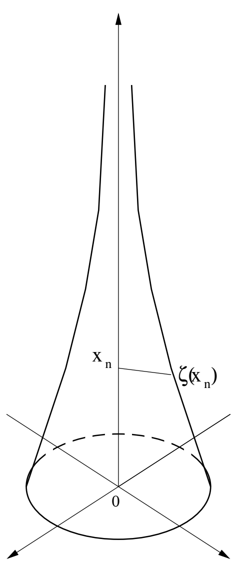

Figure 1: an unbounded domain

$$\lambda_\Omega(s) = O\big(\zeta(\Upsilon^{-1}(s))^{n-1}\big) \quad \text{as } s \to 0^+. \tag{4.2}$$

Moreover, by [Ma8, Example 4.3.5/2] if $p > 1$,

$$\nu_{\Omega,p}(s) = O\bigg(\bigg(\int_{\Upsilon^{-1}(\frac{\mathcal{H}^n(\Omega)}{2})}^{\Upsilon^{-1}(s)} \zeta(r)^{\frac{1-n}{p-1}} dr\bigg)^{1-p}\bigg) \quad \text{as } s \to 0^+. \tag{4.3}$$

Assume, for instance, that

$$\zeta(r) = e^{-r^\alpha} \quad \text{for } r > 0, \tag{4.4}$$

for some $\alpha > 0$. Equation (4.3) tells us that

$$\nu_{\Omega,p}(s) = O\big(s\big(\log(1/s)\big)^{(1-\frac{1}{\alpha})p}\big) \qquad \text{as } s \to 0^+. \tag{4.5}$$

We claim that

$$V^{1,p}(\Omega) \to L^p(\log L)^{(1-\frac{1}{\alpha})p}(\Omega). \tag{4.6}$$

Indeed, if $\alpha \geq 1$, then condition (1.8) is fulfilled with the choice $B(t) = t^p(\log(c+t))^{(1-\frac{1}{\alpha})p}$ near infinity, where c is a sufficiently large constant. Embedding (4.6) then follows from Theorem A. On the other hand, if $\alpha \in (0,1)$, then the same choice of the function B makes condition (1.10) satisfied. Embedding (4.6) can now be deduced from Theorem 1.1, since

$$M(t) \approx e^{t^{\frac{\alpha}{(1-\alpha)p}}} \qquad \text{near infinity.}$$

Note that the same conclusions can be derived on making use of the isoperimetric function of Ω, which, owing to equation (4.2), obeys

$$\lambda_\Omega(s) = O\big(s\big(\log(1/s)\big)^{1-\frac{1}{\alpha}}\big) \qquad \text{as } s \to 0^+. \tag{4.7}$$

This follows again from Theorems A and Theorem 1.1, via inequality (1.18) (which, for this class of domains, holds in fact as an equivalence). Of course, one can directly use Corollary 1.3 in the latter case.
Another one-parameter family of domains from this class is obtained with the choice

$$\zeta(r) = (1+r)^{-\beta} \quad \text{for } r > 0, \tag{4.8}$$

for some $\beta > \frac{1}{n-1}$. In this case, formulas (4.2) and (4.3) yield

$$\lambda_\Omega(s) = O\Big(s^{\frac{\beta(n-1)}{\beta(n-1)-1}}\Big) \quad \text{as } s \to 0^+, \tag{4.9}$$

and

$$\nu_{\Omega,p}(s) = O\Big(s^{1+\frac{p}{\beta(n-1)-1}}\Big) \quad \text{as } s \to 0^+. \tag{4.10}$$

Thus, one can apply either Theorem 1.1 or Corollary 1.3, with

$$B(t) = t^{\frac{(\beta(n-1)-1)p}{\beta(n-1)+p-1}}(\log(c+t))^{-\sigma} \quad \text{near infinity.}$$

Inasmuch as

$$M(t) \approx t^{\frac{\beta(n-1)-1}{p}}(\log(c+t))^{-\frac{\sigma}{p}(\beta(n-1)+p-1)} \quad \text{near infinity,} \tag{4.11}$$

it follows that

$$V^{1,p}(\Omega) \to L^{\frac{(\beta(n-1)-1)p}{\beta(n-1)+p-1}}(\log L)^{-\sigma}(\Omega), \tag{4.12}$$

provided that $\beta > \frac{2p-1}{(p-1)(n-1)}$ and $\sigma > \frac{p}{\beta(n-1)+p-1}$.
Observe that, by contrast, Theorem A is of no use here, since assumption (1.16) is not fulfilled for any Young function B satisfying condition (1.8).
Embedding (4.12) is also compact. This follows from Theorem 1.2. Indeed, one can verify that, if $\nu_{\Omega,p}$ and M obey (4.10) and (4.11), then, on setting $\varsigma = \frac{\sigma}{p}(\beta(n-1)+p-1)$ and $\gamma(\kappa) = (\log(c+1/\kappa))^{\frac{(1-\varsigma)p}{\beta(n-1)-1}}$,

$$\begin{aligned}\inf_{\gamma>0} \gamma^{\frac{1}{p}} \left(1+\int_0^\kappa M\Big(\frac{s}{\gamma\,\nu_{\Omega,p}(s)}\Big)\,ds\right) &\le \gamma(\kappa)^{\frac{1}{p}} \left(1+\int_0^\kappa M\Big(\frac{s}{\gamma(\kappa)\,\nu_{\Omega,p}(s)}\Big)\,ds\right)\\ &\le C(\log(c1/\kappa))^{\frac{1-\varsigma}{\beta(n-1)-1}}\end{aligned} \tag{4.13}$$

for some constant C, provided that κ is sufficiently small. Hence, assumption (1.17) holds, since $\varsigma > 1$.

Example 4.2 (A noncompact Riemannian manifold of revolution)
We call n-dimensional manifold of revolution $\mathbb{M}$, with profile $\varphi : [0,\infty) \to [0,\infty)$, the space $\mathbb{R}^n$ endowed with polar coordinates $\{(r,\omega) : r \in [0,L), \omega \in \mathbb{S}^{n-1}\}$ and with the Riemannian metric

$$ds^2 = dr^2 + \varphi(r)^2 d\omega^2\,. \tag{4.14}$$

Here, $d\omega^2$ stands for the standard metric on $\mathbb{S}^{n-1}$, and the function $\varphi \in C^1([0,\infty))$ and fulfills

$$\varphi(r) > 0 \quad \text{for } r \in (0,\infty), \tag{4.15}$$

$$\varphi(0) = 0\,, \qquad \text{and} \qquad \varphi'(0) = 1\,, \tag{4.16}$$

where, φ' denotes the derivative of φ.
Consider a one-parameter family of manifolds of revolution whose profile φ is such that:
(i) $\lim_{r\to L} \varphi(r) = 0$;
(ii) there exists $L_0 \in (0,L)$ such that φ is decreasing and convex in (L_0,L);
(iii) $\int_0^L \varphi(\rho)^{n-1}\,d\rho < \infty$.

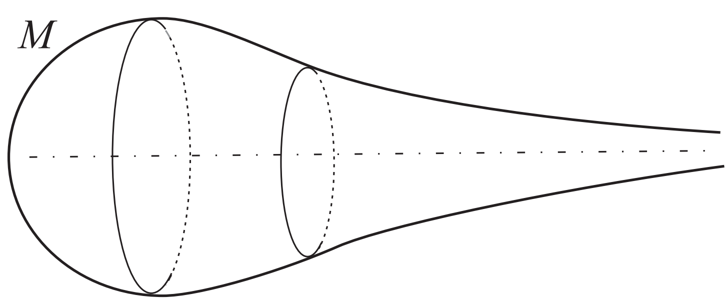

Figure 2: A manifold of revolution

Define the function $\Theta : [0,L) \to (0,\mathcal{H}^n(\mathbb{M})]$ as

$$\Theta(r) = n\omega_n \int_r^L \varphi(\rho)\,d\rho \qquad \text{for } \rho \in \big(0,\mathcal{H}^n(\mathbb{M})].$$

By [CiMa2, Theorem 4.1, Equation (4.16)],

$$\lambda_{\mathbb{M}}(s) = O\big(\varphi\big(\Theta^{-1}(s)\big)^{n-1}\big) \qquad \text{as } s \to 0^+. \tag{4.17}$$

Also, an easy variant of the proof of equation (4.17) of the same theorem tells us that, if $p > 1$, then

$$\nu_{\mathbb{M},p}(s) = O\bigg(\bigg(\int_{\Theta^{-1}(\frac{\mathcal{H}^n(\mathbb{M})}{2})}^{\Theta^{-1}(s)} \varphi(r)^{\frac{1-n}{p-1}}\,dr\bigg)^{1-p}\bigg) \quad \text{as } s \to 0^+. \tag{4.18}$$

Sobolev embeddings $V^{1,p}(\mathbb{M}) \to L^B(\mathbb{M})$ can be derived from Theorems A and 1.1, thanks to formula (4.18). The same conclusion follow from the use of (4.17), since the two sides of inequality (1.18) are, in fact, equivalent for this class of manifolds.
In particular, with the choices $L = \infty$ and $\varphi = \zeta$, where ζ is the function defined by (4.4) or (4.8), embeddings (4.6) and (4.12), respectively, hold, with Ω replaced by $\mathbb{M}$.

Example 4.3 (Nikodým domain)
We conclude with a family of highly irregular domains $\Omega \subset \mathbb{R}^2$, which was introduced by Nikodým in his study of Sobolev embeddings. These domains display two distinctive features in connection with

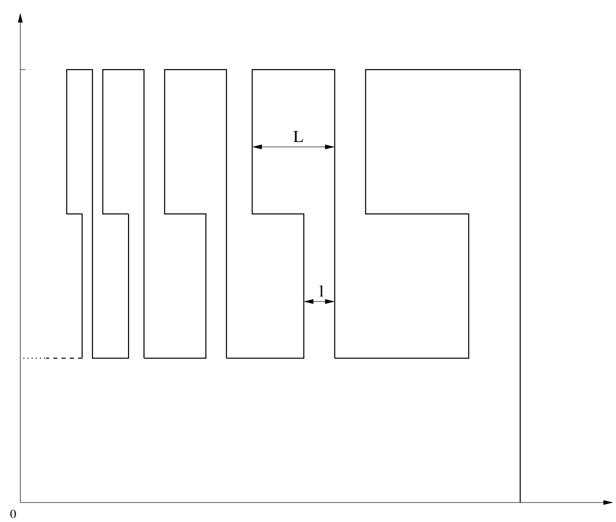

Figure 3: Nikodým example

the embeddings considered in this paper. Firstly, they are so irregular that condition (1.8) fails for every Young function B. Hence, Theorem A is of no use. In particular, no embedding of the form

$$V^{1,p}(\Omega) \to L^B(\Omega)$$

can hold if the Young function B satisfies condition (1.8). Hence, no higher (and not even equivalent) degree of integrability of a function compared to that of its gradient is guaranteed in these domains. On the other hand, Young functions B obeying (1.10) are allowed, and therefore Theorem 1.1 comes into play.
Secondly, Theorem 1.1 provides us with stronger embeddings than those which can be derived from Corollary 1.3. This demonstrates that our criterion involving the isocapacitary function can indeed be sharper than its consequence in terms of the more classical isoperimetric function.
The domains in question are shaped like the one depicted in Figure 3. With reference to this figure, set $L = 2^{-k}$ and $l = 2^{-\delta k}$, where $\delta > 1$ and $k \in \mathbb{N}$. As shown in [Ma8, Equation (6.5.28)], one has that

$$\lambda_{\Omega}(s) = O(s^{\delta}) \qquad \text{as } s \to 0^+, \tag{4.19}$$

and, if $p > 1$,

$$\nu_{\Omega,p}(s) = O(s^{\delta}) \qquad \text{as } s \to 0^+. \tag{4.20}$$

Assume that $\delta < p$. An application of Theorem 1.1, with

$$B(t) = t^{\frac{p}{\delta}} (\log(c+t))^{-\sigma} \quad \text{near infinity,}$$

where $\sigma > 1 - \frac{1}{\delta}$, so that

$$M(t) \approx t^{\frac{1}{\delta-1}} (\log(c+t))^{-\frac{\sigma\delta}{\delta-1}} \quad \text{near infinity,} \tag{4.21}$$

tells us that

$$V^{1,p}(\Omega) \to L^{\frac{p}{\delta}} (\log L)^{-\sigma}(\Omega). \tag{4.22}$$

Embedding (4.22) augments the embedding

$$V^{1,p}(\Omega) \to L^{\frac{p}{\delta}-\varepsilon}(\Omega), \tag{4.23}$$

which holds for $\varepsilon > 0$, and follows from condition (1.4) for embeddings of the form (1.2).
Notice that, embedding (4.22), as well as (4.23), cannot be derived via Corollary 1.3 and equation (4.19).
An application of Theorem 1.2 ensures that embedding (4.23) is also compact. Actually, owing to equations (4.20) and (4.21), on setting $\gamma(\kappa) = (\log(c+1/\kappa))^{\frac{(\delta-1-\sigma\delta)p}{\delta-1+p}}$ one can show that

$$\begin{aligned} \inf_{\gamma>0} \gamma^{\frac{1}{p}} \left(1 + \int_0^{\kappa} M\Big(\frac{s}{\gamma\, \nu_{\Omega,p}(s)}\Big)\, ds\right) &\le \gamma(\kappa)^{\frac{1}{p}} \left(1 + \int_0^{\kappa} M\Big(\frac{s}{\gamma(\kappa)\, \nu_{\Omega,p}(s)}\Big)\, ds\right) \\ &\le C(\log(c+1/\kappa))^{\frac{\delta-1-\sigma\delta}{\delta-1+p}}, \end{aligned} \tag{4.24}$$

for some constant C, provided that κ is small enough. Inasmuch as $\sigma > 1 - \frac{1}{\delta}$, this implies that condition (1.17) is actually fulfilled.

Compliance with Ethical Standards

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