Gradient regularity via rearrangements for p-Laplacian type elliptic boundary value problems

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Abstract

A sharp estimate for the distribution function of the gradient of solutions to a class of nonlinear Dirichlet and Neumann elliptic boundary value problems is established under weak regularity assumptions on the domain. As a consequence, the problem of gradient bounds in norms depending on global integrability properties is reduced to one-dimensional Hardy-type inequalities. Applications to gradient estimates in Lebesgue, Lorentz, Zygmund, and Orlicz spaces are presented.

1 Introduction and main results

The main result of the present paper is a new sharp estimate for the gradient of solutions to a class of nonlinear elliptic boundary value problems in domains Ω in $\mathbb{R}^n$, with $n \geq 3$. Although our results are new even for smooth domains, weak regularity assumptions on $\partial\Omega$ are pursued.

The problems under consideration consist of a quasilinear elliptic equation of the form

$$-\mathrm{div}(a(|\nabla u|)\nabla u) = f(x) \quad \text{in } \Omega \tag{1.1}$$

coupled with either the Dirichlet condition

$$u = 0 \quad \text{on } \partial\Omega\,, \tag{1.2}$$

or the Neumann condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,. \tag{1.3}$$

Mathematics Subject Classifications: 35B45, 35J25.
Keywords: Nonlinear elliptic equations, Dirichlet problems, Neumann problems, gradient estimates, rearrangements, Lorentz spaces, Orlicz spaces.

Here, Ω is a domain, namely a connected open bounded set, and ν stands for outward unit normal to $\partial\Omega$.
We assume that $a : [0,\infty) \to [0,\infty)$ is of class $C^1(0,\infty)$, and there exist $p \in [2,n)$ and $C > 0$ such that

$$\frac{ta'(t)}{a(t)} \geq p-2 \quad \text{for } t > 0, \tag{1.4}$$

and

$$ta(t) \leq C(t^{p-1}+1) \quad \text{for } t > 0. \tag{1.5}$$

Equation (1.1) is patterned on the model

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x) \quad \text{in } \Omega\,, \tag{1.6}$$

the so called p-Laplace equation, corresponding to the choice $a(t) = t^{p-2}$ for $t > 0$.

Classical contributions on gradient regularity in the theory of elliptic partial differential equations include [Be, Di, Ev, LU1, LU2, Le, Ul, Ur, To]. Estimates for the gradient of solutions to nonlinear equations whose right-hand side suffers from weak integrability properties are contained in [ACMM, AM, AFT, BBGGPV, BG, Ch, DMOP, De, Ma1, Ma3]. Equations with non-necessarily power type growth are considered e.g. in [Si, Mar, Ko, Li3]. Gradient estimates for boundary value problems in possibly irregular domains can be found in [ACMM, An, CM2, Li1, Li2, Li4, Li5, Ma1, Ma2, Ma3, Ma6]. Precise local inequalities for the gradient of solutions to nonlinear elliptic equations in terms of nonlinear potentials have recently been developed in the series of papers [DM1, DM5, DM2, DM3, Mi1, Mi2] – see see also [Ci5] for related results.

A distinctive feature of our estimate for $|\nabla u|$ via f is its independence of specific function spaces. In this sense, the estimate is universal, and amounts to a one-dimensional pointwise inequality between the decreasing rearrangement $|\nabla u|^*$ of $|\nabla u|$, and a Hardy-type operator applied to the decreasing rearrangement f^* of f. Such an inequality can be equivalently interpreted in terms of the distribution functions of $|\nabla u|$ and f, and it is flexible enough to reduce any inequality between quasi-norms of $|\nabla u|$ and f depending only on the measure of their level sets, called rearrangement invariant quasi-norms in the literature, to considerably simpler one-dimensional Hardy-type inequalities involving the corresponding representation quasi-norms. The relevant estimate for $|\nabla u|^*$ can be regarded as a sharp analogue of well-known classical estimates for u^* [Ta1, Ta2].

An important consequence of our result is that it translates verbatim the linear theory of integrability of $|\nabla u|$ for solutions to homogeneous boundary value problems for the Laplace equation to the theory of integrability of $|\nabla u|^{p-1}$ for solutions to nonlinear problems involving any equation of the form (1.1).

As mentioned above, weak regularity is imposed on the domain Ω. We require that $\partial\Omega$ belongs to the class $W^2L^{n-1,1}$ of those domains Ω which are locally the subgraph of a function of $n-1$ variables whose second-order distributional derivatives belong to the Lorentz space $L^{n-1,1}$. Alternatively, our results hold under the assumption that Ω is just convex, without any additional regularity on $\partial\Omega$. Either of these assumptions on Ω is essentially indispensable, as will be shown by suitable examples.

The notion of solution u either to (1.1) & (1.2), or to (1.1) & (1.3), has to be formulated in a suitable generalized sense. We shall comment on this later in the present section. Precise definitions are given in Section 2.2 below.

Our rearrangement gradient inequality reads as follows.

Theorem 1.1 *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in W^2L^{n-1,1}$. Assume that $f \in L^1(\Omega)$, and let u be a solution either to the Dirichlet problem* (1.1) *&* (1.2) *or to the Neumann problem* (1.1) *&* (1.3). *Then there exists a constant $C = C(p, \Omega)$ such that*

$$|\nabla u|^*(s)^{p-1} \leq C \int_s^{|\Omega|} f^{**}(r) r^{-\frac{1}{n'}} dr \quad \text{for } s \in (0, |\Omega|). \tag{1.7}$$

Here, $n' = \frac{n}{n-1}$, the Hölder conjugate of n, and $f^{**}(r) = \frac{1}{r}\int_0^r f^*(\rho)\, d\rho$ for $r \in (0, |\Omega|)$.

The reduction of norms estimates for $|\nabla u|$ to one-dimensional inequalities, to which we alluded above, is the content of the next result.

Corollary 1.2 *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in W^2L^{n-1,1}$. Let $X(\Omega)$ and $Y(\Omega)$ be rearrangement invariant quasi-normed spaces on Ω, and let $\overline{X}(0, |\Omega|)$ and $\overline{Y}(0, |\Omega|)$, respectively, be their representation spaces. Assume that there exists a constant C such that*

$$\left\| \int_s^{|\Omega|} \varphi(r) r^{-\frac{1}{n'}} dr \right\|_{\overline{Y}(0,|\Omega|)} \leq C \|\varphi\|_{\overline{X}(0,|\Omega|)} \tag{1.8}$$

and

$$\left\| s^{-\frac{1}{n'}} \int_0^s \varphi(r) dr \right\|_{\overline{Y}(0,|\Omega|)} \leq C \|\varphi\|_{\overline{X}(0,|\Omega|)} \tag{1.9}$$

for every non-decreasing function $\varphi \in \overline{X}(0, |\Omega|)$. If $f \in X(\Omega)$, and u is a solution either to the Dirichlet problem (1.1) *&* (1.2) *or to the Neumann problem* (1.1) *&* (1.3), *then there exists a constant C' such that*

$$\| \, |\nabla u|^{p-1} \, \|_{Y(\Omega)} \leq C' \|f\|_{X(\Omega)}. \tag{1.10}$$

Corollary 1.2 immediately follows via inequality (1.7) and basic properties of rearrangement invariant quasi-norms.
Applications of Corollary 1.2 to the special cases when $X(\Omega)$ and $Y(\Omega)$ are either Lebesgue, or Lorentz, or Orlicz spaces are presented in Section 4. The conclusions that are derived recover various estimates available in the literature, and yield new results in a unified framework.

The remaining part of this section is devoted to some comments on Theorem 1.1 and Corollary 1.2.

We begin by discussing the notion of solution to problems (1.1) & (1.2) and (1.1) & (1.3) employed in Theorem 1.1 and Corollary 1.2. If f is assumed to belong to some function space of measurable functions contained in the dual of the Sobolev space $W^{1,p}_0(\Omega)$, in particular if $f \in L^{\frac{np}{np-n+p}}(\Omega)$, then weak solutions $u \in W^{1,p}_0(\Omega)$ to the Dirichlet problem (1.1) & (1.2) are well defined, and are known to exist and be unique. An analogous conclusion holds for solutions $u \in W^{1,p}(\Omega)$ to the Neumann problem (1.1) & (1.3) if f belongs to some function space of measurable functions contained in the dual of the Sobolev space $W^{1,p}(\Omega)$ and fulfills the compatibility condition

$$\int_\Omega f(x)dx = 0, \tag{1.11}$$

and Ω is sufficiently regular, say with $\partial\Omega \in \mathrm{Lip}$. In this standard framework, $|\nabla u|$ belongs to the standard energy space $L^p(\Omega)$, by the very definition of weak solution. However, if f is so poorly

integrable that it need not belong to the dual of the natural Sobolev space associated with the boundary value problem – a mere function in $L^1(\Omega)$ in the worst case – then solutions can only be defined in a generalized sense. Several notions of solutions have been introduced in the literature in this connection [BG, BBGGPV, DaA, DM, LM, M1, M2], which, a posteriori, turn out to be equivalent. Simple examples show that gradients of these solutions do not belong to $L^p(\Omega)$ in general, but enjoy weaker integrability properties. On the other hand, extra integrability of f beyond membership to the Lebesgue space $L^{\frac{np}{np-n+p}}(\Omega)$ results in higher integrability, possibly even boundedness, of $|\nabla u|$. This requires, though, higher regularity on $\partial\Omega$ than just Lipschitz continuity.
In order to cover the whole range of possible gradient estimates via a unified approach, all the requirements outlined above have to be simultaneously met. In Theorem 1.1 and Corollary 1.2 we thus need both to work with generalized solutions, in order to allow right-hand sides affected by lack of integrability, and to require a qualified regularity on $\partial\Omega$, to be able to achieve strong gradient regularity in presence of highly integrable right-hand sides.

Theorem 1.1 should be compared with other estimates in rearrangement form available in the literature on elliptic equations. Rearrangement estimates for solutions u to Dirichlet problems go back to [Ta1, Ta2]; Neumann problems are treated in [Ci1, MS1, MS2]. These estimates hold for classes of non smooth elliptic operators with a more general structure than those appearing in (1.1). Moreover, no regularity at all on Ω is needed when the Dirichlet homogeneous boundary datum is imposed, and weaker – for instance Lipschitz – regularity on Ω is required in case of Neumann boundary conditions. The relevant estimates tell us that

$$u^*(s) \le C \int_s^{|\Omega|} f^{**}(r)^{\frac{1}{p-1}} r^{-1+\frac{p'}{n}}\, dr \quad \text{for } s \in (0, |\Omega|), \tag{1.12}$$

where the constant C depends either just on n and $|\Omega|$, or on Ω, according to whether Dirichlet or Neumann conditions are imposed. Moreover, u is normalized in such a way that its median vanishes in the latter case.
A rearrangement estimate for $|\nabla u|$ is also known for the same class of equations and domains. This estimate reads

$$|\nabla u|^*(s)^p \le \frac{C}{s^p} \int_{s/2}^{|\Omega|} f^{**}(r)^{p'} r^{\frac{p'}{n}}\, dr \quad \text{for } s \in (0, |\Omega|), \tag{1.13}$$

with the same dependence of the constant C is as in (1.12) [ACMM] (see also [AFT] for an earlier slightly weaker result). Let us also mention that suitable versions of (1.12) and (1.13) are available for solutions to Neumann problems in irregular domains [ACMM, CM1].
Inequality (1.13) is much weaker than (1.7), which requires stronger structure assumptions on the elliptic operator and on the domain Ω. In fact, no bound for norms of $|\nabla u|$ stronger than L^p, the natural norm associated with the nonlinearity of the problems at hand, can be derived via (1.13).
An estimate of the form (1.7) for the gradient of solutions to the Laplace equation

$$-\Delta u = f(x) \qquad \text{in } \Omega, \tag{1.14}$$

under either homogeneous Dirichlet, or Neumann boundary conditions, can be established via classical tools, provided that $\partial\Omega \in C^\infty$. Such an estimate entails that

$$|\nabla u|^*(s) \le C \int_s^{|\Omega|} f^{**}(r) r^{-\frac{1}{n'}}\, dr \quad \text{for } s \in (0, |\Omega|), \tag{1.15}$$

and follows via a representation formula for ∇u in terms of the Green function (see e.g. [MP, Section 3]), combined with a rearrangement inequality for convolutions [On]. Inequality (1.15) is recovered from (1.7), with $p = 2$, the exponent associated with the Laplace operator. Also, inequality (1.7) tells us that, for every $p \geq 2$, the expression $|\nabla u|^*(s)^{p-1}$ admits exactly the same kind of estimate in terms of f^*. In this sense, in the light of Corollary 1.2, estimates for $|\nabla u|^{p-1}$ in nonlinear problems take the same form as estimates for $|\nabla u|$ for the Laplace equation.

Let us now focus the assumptions on the domain Ω. As mentioned above, the following statements parallel to Theorem 1.1 and Corollary 1.2 hold for boundary value problems in convex domains.

Theorem 1.3 *Let Ω be a convex domain in $\mathbb{R}^n$, $n \geq 3$. Then inequality* (1.7) *holds under the same assumptions on f and u as in Theorem 1.1.*

Corollary 1.4 *Let Ω be a convex domain in $\mathbb{R}^n$, $n \geq 3$. Then inequality* (1.10) *holds under the same assumptions on f, u, $X(\Omega)$ and $Y(\Omega)$ as in Theorem 1.1.*

The sharpness of assumption $\partial\Omega \in W^2L^{n-1,1}$ in Theorem 1.1 and Corollary 1.2 will now be illustrated by a couple of examples. The second example also demonstrates how the conclusions of Theorem 1.3 and Corollary 1.4 may fail for domains which are very close to being convex.
Let $2 \leq p \leq n-1$. Consider the Dirichlet problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.16}$$

where Ω is any domain whose boundary contains 0, and it is smooth outside a neighborhood of 0, where it agrees with

$$\{x = (x', x_n) : x_n = -L|x'|\}$$

for some number $L > 0$. Here, $x' = (x_1, \dots, x_{n-1})$. One can easily verify that $\partial\Omega \in W^2L^q$ for every $q < n-1$, and, in fact, $\partial\Omega \in W^2L^{q,1}$ for every $q < n-1$, but $\partial\Omega \notin W^2L^{n-1,1}$.
Now, the function f can be chosen in such a way that it is smooth, vanishes in a neighborhood of 0, and the solution u to (1.16) satisfies

$$u(x) \approx |x|^{\alpha(L)} F\big(x_n/|x|\big) \quad \text{as } x \to 0, \tag{1.17}$$

for some smooth function $F : \mathbb{R} \to \mathbb{R}$, and some exponent $\alpha(L) > 0$ such that

$$\lim_{L\to\infty} \alpha(L) \to 0$$

[KM]. In (1.17), the notation $\approx$ means that the two sides are bounded by each other up to multiplicative constants independent of x. Thus, given any $q > n$, $|\nabla u| \notin L^q(\Omega)$, provided that L is sufficiently large, even if f is very smooth. On the contrary, if $\partial\Omega \in W^2L^{n-1,1}$, Theorem 1.1, or its Corollary 1.2, entail that $|\nabla u| \in L^\infty(\Omega)$ provided that $f \in L^{n,1}(\Omega)$ (see Theorem 4.2, Section 4), and hence, in particular, if $f \in L^\infty(\Omega)$.
Consider now the Neumann problem for the Laplace equation

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.18}$$

where Ω is as above. Of course, we may assume that Ω is convex when $L = 0$. One can show that there exist functions f which are smooth, vanish in a neighborhood of 0, and such that solution u to (1.16) satisfies

$$u(x) \approx |x|^{\beta(L)} F\big(x_n/|x|\big) \quad \text{as } x \to 0,$$

up to multiplicative constants independent of x, for some smooth function $F : \mathbb{R} \to \mathbb{R}$. Here, $\beta(L)$ is a positive exponent such that

$$\beta(L) < 1 \quad \text{if } L \text{ is sufficiently close to } 0,$$

see e.g. [KMR, Section 2.3.2]. Thus, if L is sufficiently small, there exists $q < \infty$ such that $|\nabla u| \notin L^q(\Omega)$.
An analogous conclusion holds if the Neumann condition in (1.18) is replaced with the Dirichlet condition $u = 0$ on $\partial\Omega$.
This is another example showing that the regularity assumption on $\partial\Omega$ in Theorem 1.1 and Corollary 1.2 cannot be essentially relaxed. Indeed, boundedness, and high integrability, of $|\nabla u|$ need not be guaranteed, yet for the Laplace equation with a smooth right-hand side, even if $\partial\Omega$ is smooth everywhere, except at a single point, in a neighborhood of which $\partial\Omega$ is almost flat, and the regularity assumption $\partial\Omega \in W^2L^{n-1,1}$ is just slightly relaxed.
The same example also demonstrates that even a mild local perturbation of convexity may affect the conclusions of Theorem 1.3 and Corollary 1.4.

2 Background

2.1 Rearrangements and rearrangement invariant spaces

Let u be a measurable function in Ω. The distribution function $\mu_u : [0, \infty) \to [0, \infty)$ of u is defined as

$$\mu_u(t) = |\{x \in \Omega : |u(x)| \geq t\}| \qquad \text{for } t \geq 0. \tag{2.1}$$

The decreasing rearrangement $u^* : [0, |\Omega|] \to [0, \infty]$ of u is given by

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) \geq s\}| \qquad \text{for } s \in [0, |\Omega|], \tag{2.2}$$

and its increasing rearrangement $u_* : [0, |\Omega|] \to [0, \infty]$ by

$$u_*(s) = u^*(|\Omega| - s), \qquad \text{for } s \in [0, |\Omega|].$$

A basic property of rearrangements is the Hardy-Littlewood inequality, which tells us that

$$\int_0^{|\Omega|} u^*(s) v_*(s)\, ds \leq \int_\Omega |u(x)v(x)|\, dx \leq \int_0^{|\Omega|} u^*(s) v^*(s)\, ds \tag{2.3}$$

for any measurable functions u and v in Ω.

A quasi-normed function space $X(\Omega)$ on a measurable subset Ω of $\mathbb{R}^n$ is a linear space of measurable functions on Ω equipped with a functional $\|\cdot\|_{X(\Omega)}$, a quasi-norm, enjoying the following properties:

(i) $\|u\|_{X(\Omega)} > 0$ if $u \neq 0$;
$\|\lambda u\|_{X(\Omega)} = |\lambda| \|u\|_{X(\Omega)}$ for every $\lambda \in \mathbb{R}$ and $u \in X(\Omega)$;
$\|u + v\|_{X(\Omega)} \leq c(\|u\|_{X(\Omega)} + \|v\|_{X(\Omega)})$ for some constant $c \geq 1$ and for every $u, v \in X(\Omega)$;

(ii) $0 \le |v| \le |u|$ a.e. in Ω implies $\|v\|_{X(\Omega)} \le \|u\|_{X(\Omega)}$;

(iii) $0 \le u_k \nearrow u$ a.e. implies $\|u_k\|_{X(\Omega)} \nearrow \|u\|_{X(\Omega)}$ as $k \to \infty$;

(iv) if G is a measurable subset of Ω and $|G| < \infty$, then $\|\chi_G\|_{X(\Omega)} < \infty$;

(v) for every measurable subset G of Ω with $|G| < \infty$, there exists a constant C such that $\int_G |u|dx \le C\|u\|_{X(\Omega)}$ for every $u \in X(\Omega)$.

The space $X(\Omega)$ is called a Banach function space if (i) holds with $c = 1$. In this case, the functional $\|\cdot\|_{X(\Omega)}$ is actually a norm which renders $X(\Omega)$ a Banach space.

A quasi-normed function space (in particular, a Banach function space) $X(\Omega)$ is called rearrangement invariant (r.i., for short) if there exists a quasi-normed function space $\overline{X}(0, |\Omega|)$ on the interval $(0, |\Omega|)$, called the representation space of $X(\Omega)$, having the property that

$$\|u\|_{X(\Omega)} = \|u^*\|_{\overline{X}(0,|\Omega|)} \tag{2.4}$$

for every $u \in X(\Omega)$. Obviously, if $X(\Omega)$ is an r.i. quasi-normed space, then

$$\|u\|_{X(\Omega)} = \|v\|_{X(\Omega)} \qquad \text{if} \quad u^* = v^*. \tag{2.5}$$

The dilation operator $D_\delta : \overline{X}(0, |\Omega|) \to \overline{X}(0, |\Omega|)$ is defined for $\delta > 0$ and $\varphi \in \overline{X}(0, |\Omega|)$ as

$$D_\delta\varphi(s) = \begin{cases} \varphi(s\delta) & \text{if } s\delta \in (0, |\Omega|) \\ 0 & \text{otherwise,} \end{cases}$$

and is bounded whenever $X(\Omega)$ is an r.i. Banach function space [BS, Chapter 3, Prop. 5.11]. Its norm is denoted by $\|D_\delta\|$, and comes into play in the definition of the Boyd index $I(X)$ of $X(\Omega)$, given by

$$I(X) = \lim_{\delta\to 0} \frac{\log \|D_\delta\|}{\log(1/\delta)}\,.$$

One has that $I(X) \in [0, 1]$ for every r.i. Banach function space $X(\Omega)$.

Let $X_1(\Omega)$ and $X_2(\Omega)$ be quasi-normed spaces. Their K–functional is defined for $u \in X_1(\Omega)+X_2(\Omega)$ as

$$K(s, u; X_1(\Omega), X_2(\Omega)) = \inf_{u=u_1+u_2} \big(\|u_1\|_{X_1(\Omega)} + s\|u_2\|_{X_2(\Omega)}\big) \qquad \text{for } s \in (0, |\Omega|).$$

Similarly, given a vector-valued measurable function $U : \Omega \to \mathbb{R}^m$, $m \ge 1$, such that $U \in (X_1(\Omega))^m + (X_2(\Omega))^m$, we set

$$K(s, U; (X_1(\Omega))^m, (X_2(\Omega))^m) = \inf_{U=U_1+U_2} \big(\|\,|U_1|\,\|_{X_1(\Omega)} + s\|\,|U_2|\,\|_{X_2(\Omega)}\big) \qquad \text{for } s \in (0, |\Omega|).$$

Clearly,

$$K(s, |U|; X_1(\Omega), X_2(\Omega)) \approx K(s, U; (X_1(\Omega))^m, (X_2(\Omega))^m) \qquad \text{for } s \in (0, |\Omega|), \tag{2.6}$$

and for $U \in (X_1(\Omega))^m + (X_2(\Omega))^m$, up to multiplicative constants depending on m. We refer to [BS] for a comprehensive treatment of r.i. spaces.

Besides the Lebesgue spaces, customary examples of r.i. normed, or quasi-normed, spaces include the Lorentz, the Lorentz-Zygmund, and the Orlicz spaces.

Given $q \in (1, \infty]$ and $k \in (0, \infty]$, or $q = 1$ and $k \in (0, 1]$, the Lorentz space $L^{q,k}(\Omega)$ is defined as the set of all measurable functions u on Ω for which the expression

$$\|u\|_{L^{q,k}(\Omega)} = \|s^{\frac{1}{q}-\frac{1}{k}} u^*(s)\|_{L^k(0,|\Omega|)} \tag{2.7}$$

is finite. In particular,

$$L^{q,q}(\Omega) = L^q(\Omega)$$

for every $q \in [1, \infty]$. Moreover, $L^{q,k_1}(\Omega) \subsetneqq L^{q,k_2}(\Omega)$ if $k_1 < k_2$, and, $L^{q_1,k_1}(\Omega) \subsetneqq L^{q_2,k_2}(\Omega)$ if $q_1 > q_2$ and k_1, k_2 are admissible exponents in $(0, \infty]$.
If $q > 1$, then

$$\|s^{\frac{1}{q}-\frac{1}{k}} u^*(s)\|_{L^k(0,|\Omega|)} \approx \|s^{\frac{1}{q}-\frac{1}{k}} u^{**}(s)\|_{L^k(0,|\Omega|)}, \tag{2.8}$$

up to multiplicative constants depending on q and k. Moreover, if either $q > 1$ and $k \in [1, \infty]$, or $q = k = 1$, then $L^{q,k}(\Omega)$ is in fact a Banach function space, up to equivalent norms.

The Lorentz-Zygmund spaces, a generalization of the Lorentz spaces, will come into play in certain borderline situations. If either $q \in (1, \infty]$, $k \in (0, \infty]$, $\beta \in \mathbb{R}$, or $q = 1$, $k \in (0, 1]$, $\beta \in [0, \infty)$, the Lorentz-Zygmund space $L^{q,k}(\log L)^\beta(\Omega)$ is defined as the set of all measurable functions u on Ω making the expression

$$\|u\|_{L^{q,k}(\log L)^\beta(\Omega)} = \|s^{\frac{1}{q}-\frac{1}{k}}(1 + \log(|\Omega|/s))^\beta u^*(s)\|_{L^k(0,|\Omega|)} \tag{2.9}$$

finite. If $k \geq 1$ and the weight multiplying $u^*(s)$ on the right-hand side of (2.9) is non-increasing, then the functional $\|u\|_{L^{q,k}(\log L)^\beta(\Omega)}$ is actually a norm, and $L^{q,k}(\log L)^\beta(\Omega)$ is an r.i. Banach function space equipped with this norm. Otherwise, this functional is only a quasi-norm. For certain values of the parameters q, k and β, it is however equivalent to an r.i. norm obtained on replacing u^* by u^{**} in the definition. A detailed analysis of Lorentz-Zygmund spaces can be found in [OP].

A Young function $A : [0, \infty) \to [0, \infty]$ is a convex function, vanishing at 0, which is neither identically equal to 0, nor to ∞. The Orlicz space $L^A(\Omega)$ associated with A is the r.i. space of those measurable functions u on Ω such that the Luxemburg norm

$$\|u\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A\Big(\frac{|u(x)|}{\lambda}\Big)\, dx \leq 1\right\}$$

is finite. The Orlicz spaces $L^A(\Omega)$ and $L^B(\Omega)$ agree, up to equivalent norms, if and only if the Young functions A and B are equivalent near infinity, in the sense that there exist positive constants c and t_0 such that $B(t/c) \leq A(t) \leq B(ct)$ for $t \geq t_0$.
In the special case when

$$A(t) \text{ is equivalent to } t^q(\log(1 + t))^\alpha \text{ near infinity,}$$

where either $q > 1$ and $\alpha \in \mathbb{R}$, or $q = 1$ and $\alpha \geq 0$, the space $L^A(\Omega)$ is called a Zygmund space, and is denoted by $L^q(\log L)^\alpha(\Omega)$. If

$$A(t) \text{ is equivalent to } e^{t^\beta} - 1 \text{ near infinity,}$$

for some $\beta > 0$, we denote $L^A(\Omega)$ by $\exp L^\beta(\Omega)$. Similarly, we use the notation $\exp\big(\exp L^\beta\big)(\Omega)$ for the Orlicz space associated with a Young function

$$A(t) \text{ equivalent to } e^{e^{t^\beta}} - e \text{ near infinity.}$$

2.2 Solutions

Given $p \in [1, \infty]$, we denote by $W^{1,p}(\Omega)$ the standard Sobolev space on Ω, and by $W^{1,p}_0(\Omega)$ the subspace of those functions which vanish, in the appropriate sense, on $\partial\Omega$. Their topological duals will be identified by $(W^{1,p}(\Omega))'$ and $(W^{1,p}_0(\Omega))'$, respectively. We also set

$$W^{1,p}_\perp(\Omega) = \left\{ u \in W^{1,p}(\Omega) : \int_\Omega u \, dx = 0 \right\}.$$

Assume that $f \in L^1(\Omega) \cap (W^{1,p}_0(\Omega))'$. A function $u \in W^{1,p}_0(\Omega)$ is called a weak solution to the Dirichlet problem (1.1) & (1.2) if

$$\int_\Omega a(x, \nabla u) \cdot \nabla \Phi \, dx = \int_\Omega f \Phi \, dx \qquad \text{for every } \Phi \in W^{1,p}_0(\Omega). \tag{2.10}$$

Analogously, if $f \in L^1(\Omega) \cap (W^{1,p}(\Omega))'$ and $\int_\Omega f(x)\, dx = 0$, then a function $u \in W^{1,p}(\Omega)$ is called a weak solution to the Neumann problem (1.1) & (1.3) if

$$\int_\Omega a(x, \nabla u) \cdot \nabla \Phi \, dx = \int_\Omega f \Phi \, dx \qquad \text{for every } \Phi \in W^{1,p}(\Omega). \tag{2.11}$$

Standard arguments from the direct methods of the calculus of variations, based on the strict convexity, weak lower semi-continuity, and coercivity of the functional

$$J(u) = \int_\Omega G(|\nabla u|) - f u \, dx,$$

where $G(s) = \int_0^s a(r) r \, dr$ for $s \geq 0$, yield the following existence and uniqueness result for the solution to problems (1.1) & (1.2) and (1.1) & (1.3).

Proposition 2.1 *Let Ω be a domain in $\mathbb{R}^n$.*
(i) Assume that $f \in L^1(\Omega) \cap (W^{1,p}_0(\Omega))'$. Then there exists a unique weak solution $u \in W^{1,p}_0(\Omega)$ to the Dirichlet problem (1.1) *&* (1.2)*.*
(ii) Assume that $f \in L^1(\Omega) \cap (W^{1,p}(\Omega))'$ and $\int_\Omega f(x)\, dx = 0$. Then there exists a unique weak solution $u \in W^{1,p}_\perp(\Omega)$ to the Neumann problem (1.1) *&* (1.3)*.*

When $f \notin (W^{1,p}_0(\Omega))'$, the definition of weak solution for the Dirichlet problem (1.1) & (1.2) is meaningless. The same drawback occurs for the Neumann problem (1.1) & (1.3), if $f \notin (W^{1,p}(\Omega))'$. As anticipated in Section 1, suitable generalized notions of solutions to Dirichlet and Neumann problems are based upon the use of sequences of solutions to approximating problems [DaA, DM]. Precise definitions are as follows.
Let $f \in L^1(\Omega)$. A function $u \in W^{1,p-1}_0(\Omega)$ is called an *approximable solution* to problem (1.1) & (1.2) if there exists a sequence $\{f_k\} \subset L^1(\Omega) \cap (W^{1,p}_0(\Omega))'$ such that

$$f_k \to f \qquad \text{in } L^1(\Omega),$$

and the sequence of weak solutions $\{u_k\} \subset W^{1,p}_0(\Omega)$ to problem (1.1) & (1.2), with f replaced with f_k, satisfies

$$u_k \to u \qquad \text{a.e. in } \Omega.$$

Assume now that that $f \in L^1(\Omega)$, and $\int_\Omega f(x)dx = 0$. A function $u \in W^{1,p-1}(\Omega)$ is called an *approximable solution* to problem (1.1) & (1.3) if there exists a sequence $\{f_k\} \subset L^1(\Omega) \cap (W^{1,p}(\Omega))'$ such that $\int_\Omega f_k(x)dx = 0$ for $k \in \mathbb{N}$,

$$f_k \to f \qquad \text{in } L^1(\Omega),$$

and the sequence of weak solutions $\{u_k\} \subset W^{1,p}_{\perp}(\Omega)$ to problem (1.1) & (1.3), with f replaced with f_k, satisfies

$$u_k \to u \qquad \text{a.e. in } \Omega.$$

Approximate solutions are solutions in the distributional sense. In fact, they can be regarded as distinguished members in the class of distributional solutions, which need not be unique, as shown by classical examples [Se]. An existence and uniqueness result for approximate solutions is the content of the next theorem. In particular, it turns out that the weak solution and the approximate solution agree, whenever the former is well defined.

Theorem 2.2 *Let Ω be a domain in $\mathbb{R}^n$.*
(i) Let $f \in L^1(\Omega)$. Then there exists an unique approximable solution $u \in W^{1,p-1}_0(\Omega)$ to (1.1) *&* (1.2)*, and*

$$\int_\Omega a(x, \nabla u) \cdot \nabla \Phi \, dx = \int_\Omega f \Phi \, dx \qquad \textit{for every } \Phi \in C^\infty_0(\Omega). \tag{2.12}$$

Moreover, if $\{u_k\}$ is a sequence of approximating solutions for u, then

$$\nabla u_k \to \nabla u \qquad \textit{a.e. in } \Omega \tag{2.13}$$

(up to subsequences).
(ii) Assume, in addition, that $\partial\Omega \in Lip$. Let $f \in L^1(\Omega)$ be such that $\int_\Omega f(x)dx = 0$. Then there exists an unique approximable solution $u \in W^{1,p-1}_{\perp}(\Omega)$ to (1.1) *&* (1.3)*, and*

$$\int_\Omega a(x, \nabla u) \cdot \nabla \Phi \, dx = \int_\Omega f \Phi \, dx \qquad \textit{for every } \Phi \in C^\infty(\Omega). \tag{2.14}$$

Moreover, if $\{u_k\}$ is a sequence of approximating solutions for u, then

$$\nabla u_k \to \nabla u \qquad \textit{a.e. in } \Omega \tag{2.15}$$

(up to subsequences).

Part (i) of the statement of Proposition 2.2 is by now standard – see e.g. [AM, DaA, DM]. Part (ii) can be established via analogous arguments; in particular, it follows as a special case of the results of [ACMM], where the case of possibly irregular domains is also analyzed.

3 The rearrangement estimate

This section is devoted to the proof of Theorem 1.1. Unless otherwise stated, in what follows the term solution either to the Dirichlet problem (1.1) & (1.2), or to the Neumann problem (1.1) & (1.3), has to be understood in the sense of approximable solution, as defined in Section 2.2.

We begin with a lemma on a strong monotonicity property of the function a appearing in (1.1).

Lemma 3.1 *Assume that $a : [0, \infty) \to [0, \infty)$ is of class $C^1(0, \infty)$, and fulfills* (1.4) *and* (1.5) *for some $p \geq 2$. Then there exists a constant C such that*

$$(a(|\xi|)\xi - a(|\eta|)\eta) \cdot (\xi - \eta) \geq C|\xi - \eta|^p \quad \textit{for } \xi, \eta \in \mathbb{R}^n. \tag{3.1}$$

Proof. Our assumptions ensure that a is non-decreasing, and satisfies the inequality $a(t) \geq Ct^{p-2}$, for some positive constant C, and for every $t > 0$. Hence,

$$C|\xi - \eta|^{p-2} \leq |\xi|^{p-2} + |\eta|^{p-2} \leq C'(a(|\xi|) + a(|\eta|)) \qquad \text{for } \xi, \eta \in \mathbb{R}^n,$$

for some constants C and C'. Thus, inequality (3.1) will follow if we show that

$$\frac{(a(|\xi|)\xi - a(|\eta|)\eta) \cdot (\xi - \eta)}{(a(|\xi|) + a(|\eta|))|\xi - \eta|^2} \geq C \qquad \text{if } \xi \neq \eta, \tag{3.2}$$

for some positive constant C. In order to prove (3.2), we may assume, without loss of generality, that $|\xi| \geq |\eta|$, and hence $a(|\xi|) \geq a(|\eta|)$. Consider first the case when $a(|\xi|) \leq 2a(|\eta|)$. Then, given any $\xi \neq \eta$,

(3.3)
$$\begin{aligned}
\frac{(a(|\xi|)\xi - a(|\eta|)\eta) \cdot (\xi - \eta)}{(a(|\xi|) + a(|\eta|))|\xi - \eta|^2} &= \frac{(\frac{a(|\xi|)}{a(|\eta|)}\xi - \eta) \cdot (\xi - \eta)}{(\frac{a(|\xi|)}{a(|\eta|)} + 1)|\xi - \eta|^2} \geq \frac{(\frac{a(|\xi|)}{a(|\eta|)}\xi - \eta) \cdot (\xi - \eta)}{3|\xi - \eta|^2} \\
&= \frac{|\xi - \eta|^2 + (\frac{a(|\xi|)}{a(|\eta|)} - 1)\xi \cdot (\xi - \eta)}{3|\xi - \eta|^2} = \frac{1}{3} + \frac{(\frac{a(|\xi|)}{a(|\eta|)} - 1)(|\xi|^2 - \xi \cdot \eta)}{3|\xi - \eta|^2} \\
&\geq \frac{1}{3} + \frac{(\frac{a(|\xi|)}{a(|\eta|)} - 1)(|\xi|^2 - |\xi||\eta|)}{3|\xi - \eta|^2} \geq \frac{1}{3}.
\end{aligned}$$

Assume now that $a(|\xi|) \geq 2a(|\eta|)$. Then, given any $\xi \neq \eta$,

(3.4)
$$\begin{aligned}
\frac{(a(|\xi|)\xi - a(|\eta|)\eta) \cdot (\xi - \eta)}{(a(|\xi|) + a(|\eta|))|\xi - \eta|^2} &= \frac{(\frac{a(|\xi|)}{a(|\eta|)}\xi - \eta) \cdot (\xi - \eta)}{(\frac{a(|\xi|)}{a(|\eta|)} + 1)|\xi - \eta|^2} \geq \inf_{s \geq 2} \frac{(s\xi - \eta) \cdot (\xi - \eta)}{(s+1)|\xi - \eta|^2} = \frac{(2\xi - \eta) \cdot (\xi - \eta)}{3|\xi - \eta|^2} \\
&= \frac{2|\xi|^2 - 3\xi \cdot \eta + |\eta|^2}{3|\xi - \eta|^2} = \frac{|\xi - \eta|^2 + |\xi|^2 - \xi \cdot \eta}{3|\xi - \eta|^2} \\
&\geq \frac{1}{3} + \frac{|\xi|^2 - |\xi||\eta|}{3|\xi - \eta|^2} \geq \frac{1}{3}.
\end{aligned}$$

Inequality (3.2) is fully proved. □

We now recall a differential inequality for the distribution function of Sobolev functions first established in [Ma3]. In the statement, u_+ and u_- denote the positive and the negative part of a function u, respectively, namely $u_+ = \frac{1}{2}(|u| + u)$ and $u_- = \frac{1}{2}(|u| - u)$. Moreover,

$$\text{med}(u) = \sup\{t \in \mathbb{R} : |\{u \geq t\}| \geq |\Omega|/2\},$$

the median of u.

Lemma 3.2 [Ma3] *Let Ω be a domain in $\mathbb{R}^n$, and let $p \in (1, \infty)$.*
(i)

$$1 \leq \frac{1}{n\omega_n^{\frac{1}{n}}}(-\mu_u'(t))^{1/p'}\mu_u(t)^{-1/n'}\left(-\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|^p dx\right)^{1/p} \textit{for a.e. } t \geq 0, \tag{3.5}$$

for every $u \in W_0^{1,p}(\Omega)$. Here $\omega_n = \pi^{\frac{n}{2}}/\Gamma(1 + \frac{n}{2})$, the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

(ii) Assume, in addition, that $\partial\Omega \in \mathrm{Lip}$*. Then there exists a constant* $C = C(\Omega, p)$ *such that*

$$1 \le C(-\mu'_{u_\pm}(t))^{1/p'}\mu_{u_\pm}(t)^{-1/n'}\left(-\frac{d}{dt}\int_{\{u_\pm>t\}}|\nabla u|^p dx\right)^{1/p} \text{for a.e. } t \ge 0, \tag{3.6}$$

for every $u \in W^{1,p}(\Omega)$ *such that* $\mathrm{med}(u) = 0$.

Our first step in the proof of Theorem 1.1 is a stability result for the gradient of solutions to the boundary value problems under consideration, with right-hand side in $L^1(\Omega)$. This is the content of Proposition 3.4 below. Its proof in turn makes use of a parallel stability result for the solutions themselves, which is established in the following proposition.

Proposition 3.3 *Let* Ω *be a domain in* $\mathbb{R}^n$*,* $n \ge 3$*, such that* $\partial\Omega \in \mathrm{Lip}$*, and let* $p \in [2, n)$*. Assume that* $f, g \in L^1(\Omega)$*.*
(i) Let u *be the solution to* (1.1) *&* (1.2)*, and let* v *be the solution to the same problem with* f *replaced with* g*. Then*

$$\|u - v\|_{L^{\frac{n(p-1)}{n-p},\infty}(\Omega)} \le C\|f - g\|_{L^1(\Omega)}^{\frac{1}{p-1}} \tag{3.7}$$

for some constant $C = C(n, p, |\Omega|)$*.*
(ii) Assume, in addition, that $\int_\Omega f(x)dx = 0$ *and* $\int_\Omega g(x)dx = 0$*. Let* u *be the solution to* (1.1) *&* (1.3)*, and let* v *be the solution to the same problem with* f *replaced with* g*. Then*

$$\|(u - v) - \mathrm{med}(u - v)\|_{L^{\frac{n(p-1)}{n-p},\infty}(\Omega)} \le C\|f - g\|_{L^1(\Omega)}^{\frac{1}{p-1}} \tag{3.8}$$

for some constant $C = C(p, \Omega)$*.*

Proof. Let us prove part (ii). By the definition of approximable solution and by Fatou's Lemma, it suffices to prove inequality (3.8) under the additional assumption that $f, g \in L^1(\Omega) \cap (W^{1,p}(\Omega))'$. Then u is the weak solution to (1.1) & (1.3), and v is the weak solution to (1.1) & (1.3) with f replaced by g. Define

$$w = (u - v) - \mathrm{med}(u - v), \tag{3.9}$$

and, given any $t, h > 0$, make use of the test function

$$\Phi = \begin{cases} 0 & \text{if } w \le t, \\ (w - t) & \text{if } t < w < t + h, \\ h & \text{if } t + h \le w, \end{cases}$$

in the definitions of weak solution for u and v. Subtracting the resulting integral equalities and dividing through by h yields

$$\begin{aligned} &\frac{1}{h}\int_{\{t<w<t+h\}}(a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v) \cdot (\nabla u - \nabla v)\, dx \\ &\qquad = \int_{\{w>t+h\}}(f - g)dx + \frac{1}{h}\int_{\{t<w<t+h\}}(f - g)(w - t)dx. \end{aligned} \tag{3.10}$$

Owing to (3.1), the left-hand side of (3.10) is bounded from below by

$$\frac{C}{h}\int_{\{t<w<t+h\}} |\nabla w|^p\,dx\,. \tag{3.11}$$

Making use of this bound in (3.10), and passing to the limit as $h \to 0^+$ yield

$$-\frac{d}{dt}\int_{\{w>t\}} |\nabla w|^p\,dx \le \int_{\{w>t\}} (f-g)dx \le \|f-g\|_{L^1(\Omega)} \quad \text{for a.e. } t>0. \tag{3.12}$$

From (3.12) and Lemma 3.2 (ii), we deduce that there exists a constant $C = C(\Omega, p)$ such that

$$1 \le C(-\mu'_{w_+}(t))\mu_{w_+}(t)^{-\frac{p'}{n'}}\|f-g\|_{L^1(\Omega)}^{\frac{1}{p-1}} \quad \text{for a.e. } t \ge 0.$$

Integration in this equation tells us that

$$t \le C\|f-g\|_{L^1(\Omega)}^{\frac{1}{p-1}}\int_0^t (-\mu'_{w_+}(\tau))\mu_{w_+}(\tau)^{-\frac{p'}{n'}}\,d\tau \le C'\|f-g\|_{L^1(\Omega)}^{\frac{1}{p-1}}\mu_{w_+}(t)^{\frac{p-n}{n(p-1)}} \quad \text{for } t \ge 0,$$

for some constants $C = C(\Omega, p)$ and $C' = C'(\Omega, p)$. From this inequality, and the very definition of decreasing rearrangement, one can show that

$$w_+^*(s) \le C' s^{\frac{p-n}{n(p-1)}}\|f-g\|_{L^1(\Omega)}^{\frac{1}{p-1}} \quad \text{for } s \in (0, |\Omega|/2),$$

and hence

$$\|w_+\|_{L^{\frac{n(p-1)}{n-p},\infty}(\Omega)} \le C\|f-g\|_{L^1(\Omega)}^{\frac{1}{p-1}},$$

for some constant $C = C(\Omega, p)$. An analogous argument yields the same estimate with w_+ replaced with w_-. Inequality (3.8) follows.
The proof of part (i) is analogous, and even somewhat simpler. One has just to define $w = u - v$, and to make use of part (i) of Lemma 3.2 instead of part (ii). We omit the details for brevity. □

Proposition 3.4 *Under the same assumptions and with the same notations as in Proposition 3.3,*

$$\|\nabla u - \nabla v\|_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \le C\|f-g\|_{L^1(\Omega)}^{\frac{1}{p-1}} \tag{3.13}$$

for some constant $C = C(n, p, |\Omega|)$, or $C = C(p, \Omega)$, according to whether (1.2) *or* (1.3) *is in force.*

Proof. As in the proof of Proposition 3.3, we limit ourselves to proving the statement for weak solutions to (1.1) & (1.3). In the sprit of an argument of [ACMM, AM], we begin by constructing a family of test functions as follows. Let w be defined as in (3.9). Given any integrable function $\zeta : (0, |\Omega|/2) \to [0, \infty)$, define $\Lambda : [0, |\Omega|/2] \to [0, \infty)$ as

$$\Lambda(r) = \int_0^r \zeta(\rho)\,d\rho \qquad \text{for } r \in [0, |\Omega|/2]. \tag{3.14}$$

Moreover, given any $s \in [0, |\Omega|/2]$, define $I : [0, |\Omega|/2] \to [0, \infty)$ as

$$I(r) = \begin{cases} \Lambda(r) & \text{if } \quad 0 \le r \le s, \\ \Lambda(s) & \text{if } \quad s < r \le |\Omega|/2, \end{cases} \tag{3.15}$$

and $\Phi : \Omega \to [0,\infty)$ as

$$\Phi(x) = \int_0^{w_+(x)} I(\mu_{w_+}(t))\, dt \qquad \text{for } x \in \Omega. \tag{3.16}$$

Since $I \circ \mu_{w_+}$ is a bounded function, the chain rule for derivatives in Sobolev spaces tells us that $\Phi \in W^{1,p}(\Omega)$, and

$$\nabla\Phi = \chi_{\{u-v>0\}} I(\mu_{w_+}(w))(\nabla u - \nabla v) \qquad \text{a.e. in } \Omega. \tag{3.17}$$

Choosing Φ as test function in the definitions of weak solution for u and v and subtracting the resulting equations yields

$$\int_{\{u-v>0\}} I(\mu_{w_+}(w_+(x)))(a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v)) \cdot (\nabla u - \nabla v)\, dx = \int_\Omega (f-g)\Phi\, dx. \tag{3.18}$$

Observe that

$$\begin{aligned}
\|\Phi\|_{L^\infty(\Omega)} &\le \int_0^\infty I(\mu_{w_+}(t))\, dt \\
&= \int_{(w_+)^*(s)}^\infty \Lambda(\mu_{w_+}(t))\, dt + \int_0^{(w_+)^*(s)} \Lambda(s)\, dt \\
&= \int_{(w_+)^*(s)}^\infty \int_0^{\mu_{w_+}(t)} \zeta(\rho) d\rho\, dt + \Lambda(s)(w_+)^*(s) \\
&= \int_0^s ((w_+)^*(\rho) - (w_+)^*(s))\zeta(\rho)\, d\rho + \Lambda(s)(w_+)^*(s) \\
&= \int_0^s (w_+)^*(\rho)\zeta(\rho)\, d\rho\,.
\end{aligned} \tag{3.19}$$

By inequality (3.8), there exists a constant $C = C(\Omega, p)$ such that

$$\int_0^s (w_+)^*(\rho)\zeta(\rho)\, d\rho \le C\|f-g\|_{L^1(\Omega)}^{\frac{1}{p-1}} \int_0^s \zeta(\rho)\rho^{\frac{p-n}{n(p-1)}}\, dr. \tag{3.20}$$

On the other hand, by (3.1), there exists a constant $C = C(a)$ such that

$$\begin{aligned}
\int_{\{u-v>0\}} & I(\mu_{w_+}(w_+(x)))(a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v)) \cdot (\nabla u - \nabla v)\, dx \\
&\ge C \int_{\{u-v>0\}} |\nabla w|^p I(\mu_{w_+}(w_+(x)))\, dx \\
&\ge C \int_0^{\frac{|\Omega|}{2}} |\nabla w_+|^*(r)^p \big(I \circ \mu_{w_+} \circ (w_+)^*\big)_*(r)\, dr \\
&\ge C \int_0^{\frac{|\Omega|}{2}} |\nabla w_+|^*(r)^p I(r)\, dr \\
&\ge C \int_0^s |\nabla w_+|^*(r)^p I(r)\, dr \\
&\ge C|\nabla w_+|^*(s)^p \int_0^s \int_0^r \zeta(\rho)\ \ d\rho\, dr \\
&\ge C|\nabla w_+|^*(s)^p \int_0^s \zeta(\rho)(s-\rho)\, d\rho.
\end{aligned} \tag{3.21}$$

Note, in particular, that the second inequality holds owing to the first inequality in (2.3). Owing to the arbitrariness of ζ, we obtain from (3.18)–(3.21) that

$$C|\nabla w_+|^*(s)^p \sup_{\zeta} \frac{\int_0^s \zeta(\rho)(s-\rho)\ d\rho}{\int_0^s \rho^{\frac{p-n}{n(p-1)}}\zeta(\rho)\,d\rho} \leq \|f-g\|_{L^1(\Omega)}^{p'} \quad \text{for } s \in (0, |\Omega|/2), \tag{3.22}$$

for some constant $C = C(\Omega, a)$. Clearly,

$$\sup_{\zeta} \frac{\int_0^s \zeta(\rho)(s-\rho)\ d\rho}{\int_0^s \rho^{\frac{p-n}{n(p-1)}}\zeta(\rho)\,d\rho} \geq C s^{\frac{p(n-1)}{n(p-1)}} \quad \text{for } s \in (0, |\Omega|/2), \tag{3.23}$$

for some constant $C = C(p, n)$. Thus, inequality (3.13) follows from (3.22)–(3.23), and from analogous inequalities which can be deduced with w_+ replaced with w_-. □

Another main ingredient in our proof of Theorem 1.1 is an L^∞ estimate for $|\nabla u|$, which easily follows from recent results of [CM2].

Theorem 3.5 *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$. Assume that either $\partial\Omega \in W^2L^{n-1,1}$, or Ω is convex. Let $f \in L^{n,1}(\Omega)$, and let u be a solution either to* (1.1) *&* (1.2) *or, under the additional assumption $\int_\Omega f(x)dx = 0$, to* (1.1) *&* (1.3). *Then*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^{n,1}(\Omega)}^{\frac{1}{p-1}} \tag{3.24}$$

for some constant $C = C(p, \Omega)$.

We are now in a position to accomplish the proof of Theorem 1.1.

Proof of Theorem 1.1. Our proof rests upon a nonlinear interpolation argument, exploiting the endpoint pieces of information contained in Proposition 3.4 and Theorem 3.5. Consider the Neumann problem (1.1) & (1.3). For any $f \in L^1(\Omega)$, let us set

$$f_\Omega = \frac{1}{|\Omega|}\int_\Omega f(x)\,dx,$$

the mean value of f over Ω. Let

$$T : L^1(\Omega) \to \big(L^{\frac{n(p-1}{n-1},\infty}(\Omega)\big)^n$$

be the operator defined as

$$Tf = \nabla u,$$

where u is the solution to

$$\begin{cases} -\mathrm{div}(a(|\nabla u|)\nabla u) = f(x) - f_\Omega & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.25}$$

Assume now that $f \in L^1(\Omega)$, and $f_\Omega = 0$. Any decomposition

$$f = f^0 + f^1,$$

with $f^0 \in L^{n,1}(\Omega)$ and $f^1 \in L^1(\Omega)$, induces a decomposition

$$Tf = Tf^0 + (Tf - Tf^0).$$

By the definition of K-functional, Proposition 3.4 and Theorem 3.5, there exists a constant $C = C(p,\Omega)$ such that

$$\begin{aligned}(3.26)\quad K\big(Tf, s; \big(L^{\frac{n(p-1)}{n-1},\infty}(\Omega)\big)^n, \big(L^\infty(\Omega)\big)^n\big) &\le \|Tf - Tf^0\|_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} + s\|Tf^0\|_{L^\infty(\Omega)} \\ &\le C\|f - (f^0 - f^0_\Omega)\|_{L^1(\Omega)}^{\frac{1}{p-1}} + sC\|f^0 - f^0_\Omega\|_{L^{n,1}(\Omega)}^{\frac{1}{p-1}}\end{aligned}$$

for $s \in (0, |\Omega|)$. Since $f_\Omega = 0$, we have that $f^0_\Omega = -f^1_\Omega$. Thus,

$$\begin{aligned}(3.27)\quad \|f - (f^0 - f^0_\Omega)\|_{L^1(\Omega)}^{\frac{1}{p-1}} &+ s\|f^0 - f^0_\Omega\|_{L^{n,1}(\Omega)}^{\frac{1}{p-1}} \\ &\le \big(\|f^1\|_{L^1(\Omega)} + \|f^1_\Omega\|_{L^1(\Omega)}\big)^{\frac{1}{p-1}} + s\big(\|f^0\|_{L^{n,1}(\Omega)} + \|f^0_\Omega\|_{L^{n,1}(\Omega)}\big)^{\frac{1}{p-1}} \\ &\le \big(2\|f^1\|_{L^1(\Omega)}\big)^{\frac{1}{p-1}} + s\big(2\|f^0\|_{L^{n,1}(\Omega)}\big)^{\frac{1}{p-1}} \\ &\le C'\big(\|f^1\|_{L^1(\Omega)} + s^{p-1}\|f^0\|_{L^{n,1}(\Omega)}\big)^{\frac{1}{p-1}} \quad \text{for } s \in (0,|\Omega|),\end{aligned}$$

for some constant $C' = C'(p)$. Coupling (3.26) with (3.27) yields, owing to the arbitrariness of the decomposition of f,

(3.28)

$$K\big(Tf, s; \big(L^{\frac{n(p-1)}{n-1},\infty}(\Omega)\big)^n, \big(L^\infty(\Omega)\big)^n\big) \le CK(f, s^{p-1}; L^1(\Omega), L^{n,1}(\Omega))^{\frac{1}{p-1}} \quad \text{for } s \in (0,|\Omega|).$$

By [Ho, Equation (4.8)] and equation (2.6),

(3.29)

$$K\big(Tf, s; \big(L^{\frac{n(p-1)}{n-1},\infty}(\Omega)\big)^n, \big(L^\infty(\Omega)\big)^n\big) \approx \|r^{\frac{n-1}{n(p-1)}}(Tf)^*(r)\|_{L^\infty(0,s^{\frac{n(p-1)}{n-1}})} \quad \text{for } s \in (0,|\Omega|),$$

and, by [Ho, Theorem 4.2],

$$(3.30)\qquad K(f, s; L^1(\Omega), L^{n,1}(\Omega)) \approx \int_0^{s^{n'}} f^*(r)dr + s\int_{s^{n'}}^{|\Omega|} f^*(r)r^{-\frac{1}{n'}}dr \quad \text{for } s \in (0,|\Omega|),$$

with equivalence constants depending on p and n. From (3.28)–(3.30) we deduce that

(3.31)

$$s^{\frac{n-1}{n(p-1)}}|\nabla u|^*(s) = s^{\frac{n-1}{n(p-1)}}(Tf)^*(s) \le C\bigg(\int_0^s f^*(r)dr + s^{\frac{1}{n'}}\int_s^{|\Omega|} f^*(r)r^{-\frac{1}{n'}}dr\bigg)^{\frac{1}{p-1}} \quad \text{for } s \in (0,|\Omega|),$$

for some constant $C = C(p,\Omega)$. Hence, inequality (1.7) easily follows.

The proof of (1.7) for solutions to the Dirichlet problem (1.1) & (1.2) is completely analogous, and even simpler, since f does not have to be normalized by subtracting f_Ω. □

4 Applications

We are concerned here with gradient norm estimates in some customary function spaces which can be deduced via our main results.

We begin with a few comments on possible alternative formulations or simplifications of Corollaries 1.2 and 1.4 of use in our applications. The couple of conditions (1.8)–(1.9) is equivalent to

$$(4.1)\qquad \bigg\|\int_s^{|\Omega|} r^{-1-\frac{1}{n'}}\int_0^r \varphi(\rho)\,d\rho\,dr\bigg\|_{\overline{Y}(0,|\Omega|)} \le C\|\varphi\|_{\overline{X}(0,|\Omega|)}$$

for every non-decreasing function $\varphi \in \overline{X}(0,|\Omega|)$. It is in fact (4.1) the condition which immediately follows from (1.7); the equivalence of (4.1) to (1.8)–(1.9) is a consequence of an application of Funbini's theorem in the integral appearing on the right-hand side of (1.7).
Inequality (4.1) is stronger, in general, than just (1.9), since, if $\varphi : (0,|\Omega|) \to [0,\infty)$ is non-increasing, then

$$\varphi(s) \leq \frac{1}{s}\int_0^s \varphi(r)dr \qquad \text{for } s>0. \tag{4.2}$$

However, inequalities (4.1) and (1.9) are equivalent in the case when the quasi-norm in $X(\Omega)$ fulfils

$$\left\| \frac{1}{s}\int_0^s \varphi(r)dr \right\|_{\overline{X}(0,|\Omega|)} \leq C\|\varphi\|_{\overline{X}(0,|\Omega|)} \tag{4.3}$$

for some constant C and for every $\varphi \in \overline{X}(0,|\Omega|)$. Thus, if $X(\Omega)$ satisfies (4.3), then (1.9) implies the gradient estimate (1.10). The r.i. Banach function spaces $X(\Omega)$ making inequality (4.3) true can be characterized in terms of their upper Boyd index $I(X)$. Indeed, inequality (4.3) holds if and only if $I(X) < 1$ [BS, Theorem 5.15].

Our first result concerns gradient estimates in classical Lebesgue spaces. In the statements below, C denotes a constant independent of u and f.

Theorem 4.1 *Let Ω, p and u be as in Theorem 1.1. Assume that $f \in L^q(\Omega)$.*
(i) If $q = 1$, then for every $\sigma < \frac{n(p-1)}{n-1}$

$$\|\nabla u\|_{L^\sigma(\Omega)} \leq C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}. \tag{4.4}$$

(ii) If $1 < q < n$, then

$$\|\nabla u\|_{L^{\frac{qn(p-1)}{n-q}}(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}. \tag{4.5}$$

(iii) If $q = n$, then for every $\sigma < \infty$,

$$\|\nabla u\|_{L^\sigma(\Omega)} \leq C\|f\|_{L^n}^{\frac{1}{p-1}}. \tag{4.6}$$

(v) If $q > n$, then

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}. \tag{4.7}$$

Theorem 4.1 overlaps with various contributions, including [ACMM, AM, BBGGPV, BG, DMOP, Di, Ma1, Ma3, Li3, Ta1, Ta2].
A proof of Theorem 4.1 makes use of Corollary 1.2 and of one-dimensional Hardy type inequalities in Lebesgue spaces [Ma8, Section 1.3.2]. Theorem 4.1 can also be derived from the more general, sharper, estimates in Lorentz and Lorentz-Zygmund spaces which are the object of the next result.

Theorem 4.2 *Let Ω, p and u be as in Theorem 1.1. Assume that $f \in L^{q,k}(\Omega)$.*
(i) If $q = 1$ and $0 < k \leq 1$, then

$$\|\nabla u\|_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \leq C\|f\|_{L^{1,k}(\Omega)}^{\frac{1}{p-1}}.$$

(ii) If $1 < q < n$ and $0 < k \leq \infty$, then

$$\|\nabla u\|_{L^{\frac{qn(p-1)}{n-q},k(p-1)}(\Omega)} \leq C\|f\|_{L^{q,k}(\Omega)}^{\frac{1}{p-1}}.$$

(iii) If $q = n$ and $k > 1$, then

$$\|\nabla u\|_{L^{\infty,k(p-1)}(\log L)^{-\frac{1}{p-1}}(\Omega)} \leq C\|f\|_{L^{n,k}(\Omega)}^{\frac{1}{p-1}}.$$

(iv) If either $q = n$ and $k \leq 1$, or $q > n$ and $0 < k \leq \infty$, then

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^{q,k}(\Omega)}^{\frac{1}{p-1}}.$$

Various cases of Theorem 4.2 are known, possibly under stronger assumption on Ω – see e.g. [ACMM, AFT, AM, BBGGPV, CM2]. Local gradient estimates in Lorentz spaces are established [DM4, DM3, DM5, Mi1].

Proof of Theorem 4.2 By Theorem 1.1,

$$\begin{aligned}\|\nabla u\|^{p-1}_{L^{\frac{n(p-1)}{n-1},\infty}(|\Omega|)} &\leq C \sup_{s>0} s^{\frac{n-1}{n}} \int_s^{\Omega} f^{**}(r) r^{-\frac{1}{n'}}\, dr \\ &\leq \|f\|_{L^1(\Omega)} \sup_{s>0} s^{\frac{1}{n'}} \int_s^\infty r^{-1-\frac{1}{n'}}\, dr = C'\|f\|_{L^1(\Omega)}\end{aligned} \tag{4.8}$$

for some constants $C = C(\Omega)$ and $C' = C'(\Omega)$. Hence, assertion (i) follows.
By Corollary 1.2 and (2.8), part (ii) is easily reduced to the inequality

$$\left\| s^{\frac{n-q}{nq}-\frac{1}{k}} \int_s^{|\Omega|} r^{-\frac{1}{n'}} f^{**}(r) dr \right\|_{L^k(0,|\Omega|)} \leq C \left\| s^{\frac{1}{q}-\frac{1}{k}} f^{**}(s) \right\|_{L^k(0,|\Omega|)} \tag{4.9}$$

for some constant $C = C(n, q, k)$, and for every $f \in L^{q,k}(\Omega)$. Inequality (4.9) follows via a classical weighted Hardy type inequality in Lebesgue spaces if $k \geq 1$ [Ma8, Section 1.3.2], and via a weighted Hardy type inequality in Lebesgue spaces for non-increasing functions if $0 < k < 1$ [CS].
Similarly, by Corollary 1.2 and (2.8), case (iii) is a consequence of the inequality

$$\left\| s^{-\frac{1}{k}} \big(1 + \log(|\Omega|/s)\big)^{-1} \int_s^{|\Omega|} f^{**}(r) r^{-\frac{1}{n'}}\, dr \right\|_{L^k(0,|\Omega|)} \leq C \left\| s^{\frac{1}{n}-\frac{1}{k}} f^{**}(s) \right\|_{L^k(0,|\Omega|)}, \tag{4.10}$$

for some constant $C = C(n, k, |\Omega|)$, and for every $f \in L^{n,k}(\Omega)$, which holds by standard criteria for one-dimensional Hardy inequalities [Ma8, Section 1.3.2].
Finally, by Theorem 1.1,

$$\begin{aligned}\|\nabla u\|^{p-1}_{L^\infty(\Omega)} &\leq C \int_0^{|\Omega|} f^{**}(r) r^{-\frac{1}{n'}}\, dr \\ &\leq C \int_0^{|\Omega|} f^*(\rho) \int_\rho^\infty r^{-1-\frac{1}{n'}}\, dr\, d\rho = Cn'\|f\|_{L^{n,1}(\mathbb{R}^n)},\end{aligned} \tag{4.11}$$

where C is the constant appearing in (1.7). Hence, part (iv) follows, owing to the inclusion relations between Lorentz spaces recalled in Subsection 2.5. □

Let us next derive gradient estimates in Orlicz spaces. Their formulation requires the notions of Young conjugate and Sobolev conjugate of a Young function A.
The Young conjugate of A is the Young function $\widetilde{A}$ given by

$$\widetilde{A}(t) = \sup\{st - A(s) : s \geq 0\} \qquad \text{for } t \geq 0.$$

The Sobolev conjugate, introduced in [Ci2, Ci3], of a Young function A such that

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt < \infty\,, \tag{4.12}$$

is the Young function A_n defined as

$$A_n(t) = A\big(H^{-1}(t)\big) \qquad \text{for } t \geq 0, \tag{4.13}$$

where $H : [0, \infty) \to [0, \infty)$ is given by

$$H(s) = \left(\int_0^s \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt\right)^{1/n'} \qquad \text{for } s \geq 0, \tag{4.14}$$

and H^{-1} is the left-continuous generalized inverse of H. Accordingly, given a Young function B such that

$$\int_0 \left(\frac{t}{\widetilde{B}(t)}\right)^{\frac{1}{n-1}} dt < \infty\,, \tag{4.15}$$

we denote by $\big(\widetilde{B}\big)_n$ the Sobolev conjugate of $\widetilde{B}$, obtained as in (4.13)–(4.14), on replacing A with $\widetilde{B}$.

Theorem 4.3 *Let Ω, p and u be as in Theorem 1.1. Let A and B be Young functions fulfilling* (4.12) *and* (4.15)*, respectively. Assume that $f \in L^A(\Omega)$, and that there exist $c > 0$ and $t_0 > 0$ such that*

$$B(t) \leq A_n(ct) \quad \text{and} \quad \widetilde{A}(t) \leq \big(\widetilde{B}\big)_n(ct) \quad \text{for } t \geq t_0. \tag{4.16}$$

Let E be the Young function given by

$$E(t) = B(t^{p-1}) \quad \text{for } t \geq 0.$$

Then

$$\|\nabla u\|_{L^E(\Omega)} \leq C\|f\|_{L^A(\Omega)}^{\frac{1}{p-1}}. \tag{4.17}$$

We emphasize that assumptions (4.12) and (4.15) are, in fact, immaterial. Indeed, the functions A and B can be replaced, if necessary, by Young functions equivalent near infinity, which fulfil (4.12) and (4.15). Such a replacement leaves the spaces $L^A(\Omega)$ and $L^B(\Omega)$ unchanged, up to equivalent norms.

Proof of Theorem 4.3. The first inequality in (4.16) ensures that

$$\left\| \int_s^{|\Omega|} \varphi(r) r^{-\frac{1}{n'}} dr \right\|_{L^B(0,|\Omega|)} \leq C\|\varphi\|_{L^A(0,|\Omega|)} \tag{4.18}$$

and the second inequality in (4.16) ensures that

$$\left\| s^{-\frac{1}{n'}} \int_0^s \varphi(r) dr \right\|_{L^B(0,|\Omega|)} \leq C\|\varphi\|_{L^A(0,|\Omega|)} \tag{4.19}$$

for every $\varphi \in L^A(0, |\Omega|)$. These are are consequences of [Ci2, Lemma 1], and of [Ci4, Lemma 2]. Hence, (4.17) follows via Corollary 1.2. □

Theorem 4.3 can be easily specialized to the case when $L^A(\Omega)$ is a Zygmund space. This is the content of our last result.

Theorem 4.4 *Let Ω, p and u be as in Theorem 1.1. Let $f \in L^q(\log L)^\alpha(\Omega)$.*
(i) If $q = 1$ and $\alpha \geq 0$, then

$$\|\nabla u\|_{L^{\frac{n(p-1)}{n-1}}(\log L)^{\frac{n\alpha}{n-1}-1}(\Omega)} \leq C\|f\|^{\frac{1}{p-1}}_{L(\log L)^\alpha(\Omega)}. \tag{4.20}$$

(ii) If $1 < q < n$ and $\alpha \in \mathbb{R}$, then

$$\|\nabla u\|_{L^{\frac{nq(p-1)}{n-q}}(\log L)^{\frac{n\alpha}{n-q}}(\Omega)} \leq C\|f\|^{\frac{1}{p-1}}_{L^q(\log L)^\alpha(\Omega)}. \tag{4.21}$$

(iii) If $q = n$ and $\alpha < n - 1$, then

$$\|\nabla u\|_{\exp L^{\frac{n(p-1)}{n-1-\alpha}}(\Omega)} \leq C\|f\|^{\frac{1}{p-1}}_{L^n(\log L)^\alpha(\Omega)}. \tag{4.22}$$

(iv) If $q = n$ and $\alpha = n - 1$, then

$$\|\nabla u\|_{\exp\left(\exp L^{\frac{n(p-1)}{n-1}}\right)(\Omega)} \leq C\|f\|^{\frac{1}{p-1}}_{L^n(\log L)^{n-1}(\Omega)}. \tag{4.23}$$

(v) If either $q = n$ and $\alpha > n - 1$, or $q > n$ and $\alpha \in \mathbb{R}$, then

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\|f\|^{\frac{1}{p-1}}_{L^q(\log L)^\alpha(\Omega)}. \tag{4.24}$$

Special cases of Theorem 4.4 are known. In particular, some instances of case (i) can be found in [BBGGPV, De].

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