# Optimal second-order regularity for the $p$-Laplace system

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**Abstract**

Second-order estimates are established for solutions to the $p$-Laplace system with right-hand side in $L^2$. The nonlinear expression of the gradient under the divergence operator is shown to belong to $W^{1,2}$, and hence to enjoy the best possible degree of regularity. Moreover, its norm in $W^{1,2}$ is proved to be equivalent to the norm of the right-hand side in $L^2$. Our global results apply to solutions to both Dirichlet and Neumann problems, and entail minimal regularity of the boundary of the domain. In particular, for convex domains our conclusions hold without any additional regularity assumption. Local estimates for local solutions are provided as well.

**Résumé**

Des estimations de second ordre sont établies pour des solutions du systéme de $p$-Laplace avec le terme à droite dans $L^2$. Nous montrons que l'expression non-linaire du gradient à qui l'operateur de divergence est appliqué, appartient à $W^{1,2}$, et donc il jouit du meilleur degré de régularité possible. Nous montrons en plus que sa norme dans $W^{1,2}$ est equivalente à la norme du terme à droite dans $L^2$. Nos résultats globaux s'appliquent à des solutions à la fois des problémes de Dirichlet et des problémes de Neumann et ils exigent une régularité minimale de la frontiére du domaine. En particulier, dans le cas des domaines convexes nos conclusions tiennent sans supposer aucune régularité supplémentaire. Des estimations locales pour des solutions locales sont également fournies.

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# 1 Introduction

We are concerned with second-order differentiability properties of solutions to the $p$-Laplace system

$$(1.1)\qquad -\mathbf{div}(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega\,.$$

Here, $\Omega$ is an open set in $\mathbb{R}^n$, $n \geq 2$, the function $\mathbf{f} : \Omega \to \mathbb{R}^N$, $N \geq 1$, is given, and $\mathbf{u} : \Omega \to \mathbb{R}^N$ is the unknown.

Local solutions to system (1.1) are included in our analysis. However, our main focus is on Dirichlet boundary value problems, obtained on coupling system (1.1) with the boundary condition

$$(1.2)\qquad \mathbf{u} = 0 \qquad \text{on } \partial\Omega,$$

and on Neumann problems, corresponding to the boundary condition

$$(1.3)\qquad \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} = 0 \qquad \text{on } \partial\Omega,$$

where $\boldsymbol{\nu}$ denotes the outward unit vector on $\partial\Omega$. Obviously, the compatibility assumption

$$(1.4)\qquad \int_\Omega \mathbf{f}\, dx = 0$$

has also to be imposed on $\mathbf{f}$ when (1.3) is in force.

It is clear that, if $\mathbf{u}$ is a solution to system (1.1) such that $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^{Nn})$, then $\mathbf{f} \in L^2(\Omega, \mathbb{R}^N)$. Our main result amounts to a converse of this fact, and furnishes a nonlinear counterpart of the classical $L^2$-coercivity theory for linear problems, that can be traced back to [Be] ($n = 2$) and [Scha] (arbitrary $n$). Namely, it asserts that if $\mathbf{f} \in L^2(\Omega, \mathbb{R}^N)$, then $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^{Nn})$, and, moreover, the norms $\|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^N)}$ and $\||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{W^{1,2}(\Omega,\mathbb{R}^{Nn})}$ are equivalent. Of course, this loose statement requires some specifications.

To begin with, the equivalence of the relevant global norms for solutions to either the Dirichlet problem (1.1)+(1.2), or the Neumann problem (1.1)+(1.3), can only hold if $\partial\Omega$ is sufficiently regular. In this connection, let us stress that we strive for weakest possible regularity assumptions on $\partial\Omega$. As will be demonstrated by apropos examples, the hypotheses to be made on $\partial\Omega$ are virtually minimal. On the other hand, local solutions obviously just admit local estimates for the $W^{1,2}$-norm of $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}$, which do not entail any assumption on the domain $\Omega$. However, besides a local $L^2$-norm of $\mathbf{f}$, they involve an additional lower-order term depending on $\nabla \mathbf{u}$.

The very definition of solution to system (1.1) calls for an elucidation. Indeed, membership of $\mathbf{f}$ in $L^2(\Omega, \mathbb{R}^N)$ does not ensure that weak solutions be well defined, if $p$ is too small – precisely, if $p < \frac{2n}{n+2}$, the Hölder conjugate of the Sobolev conjugate of the exponent 2. As a consequence, an even weaker notion of solution has to be employed. In the scalar case, namely when $N = 1$ and hence system (1.1) reduces to a single equation, diverse definitions of solutions are available in the literature that allow for right-hand sides $\mathbf{f}$ that are merely integrable in $\Omega$ – see [BBGGPV, BoGa, DaA, DuMi1, LiMu, Maz5, Mu]. Definitions of solutions involving truncations of functions, or integration over their level sets, adopted for scalar problems do not carry over to the vector-valued case. Observe that functions that solve equations in such a weak sense need not even be weakly differentiable.

A definition of solution obtained as the limit of solutions to approximating problems, whose right-hand sides are smooth, is instead suited to be extended to the case when $N > 1$. Generalized solutions of this kind are called approximable solutions hereafter, and are precisely defined in Sections 4 and 5. This approach, introduced for equations in [DaA], has been developed for systems in [DHM1, DHM2], and requires that the relevant solution be at least weakly differentiable. When $\mathbf{f} \in L^2(\Omega, \mathbb{R}^N)$, this can only be guaranteed if $p \geq \frac{3}{2} - \frac{1}{n}$, even in the case of a single equation [AFT]. Our results will be established for the slightly smaller range of exponents $p > \frac{3}{2}$. This restriction arises in a crucial differential inequality to be used in our proof.

Such a limitation on $p$ is not required when equations, instead of systems, are in question. This is shown in our earlier contribution [CiMa2], where results in the same spirit are established for the $p$-Laplace (and a somewhat more general quasilinear) equation, for every $p > 1$. We emphasize that, besides dealing with systems, another advance of the present paper consists in further relaxing the regularity of $\partial\Omega$.

Let us mention that classical results on second-order differentiability properties of solutions to the $p$-Laplacian equation or system are concerned with the expression $|\nabla u|^{\frac{p-2}{2}}\nabla u$, instead of $|\nabla u|^{p-2}\nabla u$. They can be traced back to [Uh] for $p > 2$, and [ChDi] for every $p > 1$. Recent developments are in [BeCr, Ce1, CGM]. The differentiability of the expression $|\nabla u|^{p-2}\nabla u$ has also been investigated, but under stronger regularity assumptions on the right-hand side than the natural membership in $L^2(\Omega)$. Furthermore, the available results in this direction either require smoothness of $\partial\Omega$ [DaSc], or just deal with local solutions [AKM, Lo]. Fractional-order regularity of the gradient of solutions to quasilinear equations of $p$-Laplacian type has been studied in [Si,J.], and recently in [AKM, Ce2, Mi].

## 2 Main results

Our most general global result for solutions to Dirichlet and Neumann problems entails a regularity condition on $\partial\Omega$ depending, roughly speaking, on a local isocapacitary inequality for the integral of the curvature on $\partial\Omega$.

Precisely, we assume that $\Omega$ is a bounded Lipschtz domain, and that the functions of $(n-1)$ variables that locally describe the boundary of $\Omega$ are twice weakly differentiable, briefly $\partial\Omega \in W^{2,1}$. These assumptions do not yet ensure global second-order regularity of solutions, still in the linear case – see [Maz3, Maz4]. They have to be refined as follows. Denote by $\mathcal{B}$ the weak second fundamental form on $\partial\Omega$, by $|\mathcal{B}|$ its norm, and set

$$\mathcal{K}_\Omega(r) = \sup_{\substack{E \subset \partial\Omega \cap B_r(x) \\ x \in \partial\Omega}} \frac{\int_E |\mathcal{B}| d\mathcal{H}^{n-1}}{\mathrm{cap}_{B_1(x)}(E)} \qquad \text{for } r \in (0,1)\,. \tag{2.1}$$

Here, $B_r(x)$ stands for the ball centered at $x$, with radius $r$, the notation $\mathrm{cap}_{B_1(x)}(E)$ is adopted for the capacity of the set $E$ relative to the ball $B_1(x)$, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Then we require that

$$\lim_{r\to 0^+} \mathcal{K}_\Omega(r) < c \tag{2.2}$$

for a suitable constant $c = c(n, N, p, L_\Omega, d_\Omega)$, where, $L_\Omega$ and $d_\Omega$ denote the Lipschitz constant and the diamater of $\Omega$, respectively; moreover, here and in similar occurrences in what follows, the dependence of a constant on $L_\Omega$ and $d_\Omega$ is understood just via an upper bound for them.

A definition of capacity relative to an open set is recalled in Section 3. Let us just mention here that, if $n \geq 3$, then the capacity $\mathrm{cap}_{B_1(x)}$ relative to $B_1(x)$ is equivalent to the standard capacity in the whole of $\mathbb{R}^n$, up to multiplicative constants depending on $n$.

**Theorem 2.1 [Global estimate under capacitary conditions on curvatures]** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$ such that $\partial\Omega \in W^{2,1}$. Assume that $p > \frac{3}{2}$, and $\mathbf{f} \in L^2(\Omega, \mathbb{R}^N)$. Let $\mathbf{u}$ be an approximable solution to either the Dirichlet problem* (1.1)+(1.2), *or the Neumann problem* (1.1)+(1.3). *There exists a constant $c = c(n, N, p, L_\Omega, d_\Omega)$ such that, if $\Omega$ fulfills* (2.2) *for such a constant $c$, then*

$$|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^{Nn}). \tag{2.3}$$

*Moreover,*

$$C_1\|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^N)} \leq \||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{W^{1,2}(\Omega,\mathbb{R}^{Nn})} \leq C_2\|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^N)} \tag{2.4}$$

*for some positive constants $C_1 = C_1(n, N, p)$ and $C_2 = C_2(\Omega, N, p)$.*

**Remark 2.2** The conclusions of Theorem 2.1 may fail if assumption (2.2) is just relaxed by requring that the limit in (2.2) is merely finite. Indeed, examples of domains $\Omega$ can be produced for which $\mathcal{K}_\Omega(r) < \infty$ for $r \in (0, 1)$, but the limit in in (2.2) exceeds some explicit threshold, and where the Dirichlet problem for the Laplace operator with a suitable smooth right-hand side admits a solution that does not belong to $W^{2,2}(\Omega)$ [Maz3, Maz4]. The boundary of the relevant domains is smooth, save a small portion which agrees with the graph of a function $\Theta$ of the variables $(x_1, \dots, x_{n-1})$ having the form $\Theta(x_1, \dots, x_{n-1}) = c|x_1|(\log|x_1|)^{-1}$ for a suitable constant $c$ and for small $x_1$.

Capacities can be dispensed in the description of the boundary regularity for the conclusions of Theorem 2.1 to hold, and can be replaced with a condition involving only integrability properties of $\mathcal{B}$. The relevant condition reads

$$\lim_{r\to 0^+}\Big(\sup_{x\in\partial\Omega}\|\mathcal{B}\|_{X(\partial\Omega\cap B_r(x))}\Big) < c\,, \tag{2.5}$$

for a suitable constant $c = c(n, N, p, L_\Omega, d_\Omega)$, where

$$X = \begin{cases} L^{n-1,\infty} & \text{if } n \geq 3, \\ L^{1,\infty}\log L & \text{if } n = 2. \end{cases} \tag{2.6}$$

The spaces appearing on the right-hand side of (2.6) are Marcinkiewicz type spaces, also called weak type space. Precisely, $L^{n-1,\infty}$ denotes the weak Lebesgue space $L^{n-1}$, and $L^{1,\infty}\log L$ the weak Zygmund space $L\log L$. Let us point out that the assmption $\partial\Omega \in X$ and condition (2.5) do not even entail that $\partial\Omega \in C^1$.

As observed in Remark 2.4 below, condition (2.5) is minimal as far as integrability properties of the curvatures of $\partial\Omega$ ar concerned. However, there exist open sets where the capacitary criterion of Theorem 2.1 applies, whereas the integrability condition (2.5) assumed in the following corollary does not.

**Corollary 2.3 [Global estimate under integrability conditions on curvatures]** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$ such that $\partial\Omega \in X$. Let $p$, $\mathbf{f}$ and $\mathbf{u}$ be as in Theorem 2.1. There exists a constant $c = c(n, N, p, L_\Omega, d_\Omega)$ such that, if $\Omega$ fulfills* (2.5) *for such a constant $c$, then $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^{Nn})$, and inequality* (2.4) *holds for some positive constants $C_1 = C_1(n, N, p)$ and $C_2 = C_2(\Omega, N, p)$.*

**Remark 2.4** The sharpness of assumption (2.5) in Corollary 2.3 can be demonstrated, for instance, when $n = 3$ and $p \in (\frac{3}{2}, 2]$, by an example from [KrMa]. In that paper, open sets $\Omega \subset \mathbb{R}^3$, with $\partial\Omega \in W^2L^{2,\infty}$ but whose limit in (2.5) is not small enough, are exhibited where the solution $u$ to the Dirichlet problem, with a smooth right-hand side, is such that $|\nabla u|^{p-2}\nabla u \notin W^{1,2}(\Omega)$. If $n = 2$ and $p = 2$, so that the $p$-Laplace operator reduces to the classical Laplacian, there exist open sets $\Omega$, with $\partial\Omega \in W^2L^{1,\infty}\log L$, for which the limit in (2.5) is larger than some critical value, and where the solution $u$ to the Dirichlet problem with a smooth right-hand side fails to belong to $W^{2,2}(\Omega)$ [Maz3] (see also [MazSh, Section 14.6.1], where Neumann problems are discussed).
Let us notice that the domains mentioned in Remark 2.2 are such that $\partial\Omega \notin W^2L^{2,\infty}$ if $n \geq 3$, and hence, unlike Theorem 2.1, Corollary 2.3 cannot be applied.

**Remark 2.5** Condition (2.5) holds, for instance, if $n \geq 3$ and $\partial\Omega \in W^{2,n-1}$, and if $n = 2$ and $\partial\Omega \in W^2L\log L$ (and hence, a fortiori, if $\partial\Omega \in W^{2,q}$ for some $q > 1$). In fact, in all these cases

$$\lim_{r\to 0^+} \mathcal{K}_\Omega(r) = 0\,. \tag{2.7}$$

In particular, condition (2.7), and hence (2.5), is satisfied by any domain with a smooth boundary.

In the case when $\Omega$ is convex, the conclusions of Theorem 2.1 do not require any additional assumption on $\partial\Omega$. As will be apparent from its proof, such a statement holds thanks to the fact that the curvatures (in any weak sense) on $\partial\Omega$ have a sign that enables us to disregard certain boundary integrals arising in our estimates.

**Theorem 2.6** **[Global estimate in convex domains]** *Let $\Omega$ be any bounded convex open set in $\mathbb{R}^n$, with $n \geq 2$. Let $p$, $\mathbf{f}$ and $\mathbf{u}$ be as in Theorem 2.1. Then $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^{Nn})$, and inequality* (2.4) *holds for some positive constants $C_1 = C_1(n, N, p)$ and $C_2 = C_2(\Omega, N, p)$.*

A local counterpart of the above theorems is the subject of our last main result. In the statement, $B_R$ and $B_{2R}$ denote concentric balls of radius $R$ and $2R$, respectively.

**Theorem 2.7** **[Local estimate]** *Let $\Omega$ be an open set in $\mathbb{R}^n$, with $n \geq 2$. Assume that $p > \frac{3}{2}$, and $\mathbf{f} \in L^2_{\rm loc}(\Omega, \mathbb{R}^N)$. Let $\mathbf{u}$ be an approximable local solution to system* (1.1)*. Then*

$$|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in W^{1,2}_{\rm loc}(\Omega, \mathbb{R}^{Nn}), \tag{2.8}$$

*and there exists a constant $C = C(n, N, p)$ such that*

$$\||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{W^{1,2}(B_R,\mathbb{R}^{Nn})} \leq C\Big(\|\mathbf{f}\|_{L^2(B_{2R},\mathbb{R}^N)} + (R^{-\frac{n}{2}} + R^{-\frac{n}{2}-1})\|\nabla \mathbf{u}\|^{p-1}_{L^{p-1}(B_{2R},\mathbb{R}^{Nn})}\Big) \tag{2.9}$$

*for any ball $B_{2R} \subset\subset \Omega$.*

# 3 Key inequalities

This section is devoted to the proof of local and global norm inequalities involving a smooth differential operator, modeled upon the $p$-Laplacian. Here, the global results are established in domains with a smooth boundary. They are the subject of Theorem 3.1 below, and constitute a fundamental step in our approach.

The differential operators coming into play are built upon nonnegative functions $a \in C^1([0,\infty))$, satisfying suitable assumptions, that replace the function $t^{p-2}$ appearing in the $p$-Laplace operator. For any such function $a$, we define

$$i_a = \inf_{t>0} \frac{ta'(t)}{a(t)} \qquad \text{and} \qquad s_a = \sup_{t>0} \frac{ta'(t)}{a(t)}, \tag{3.1}$$

where $a'$ stands for the derivative of $a$.

**Theorem 3.1** *Let $n \geq 2$, $N \geq 1$, and let $\Omega$ be an open set in $\mathbb{R}^n$. Assume that $a : [0,\infty) \to [0,\infty)$ is a function having the form $a(t) = \widehat{a}(t^2)$ for some function $\widehat{a} \in C^1([0,\infty))$, and such that $a(t) > 0$ for $t > 0$. Suppose that*

$$i_a > -\tfrac{1}{2}\,, \tag{3.2}$$

*and*

$$s_a < \infty\,. \tag{3.3}$$

*(i) There exists a constant $C = C(n, N, i_a, s_a)$, such that*

$$\begin{aligned} \|a(|\nabla \mathbf{u}|)\nabla \mathbf{u}\|_{W^{1,2}(B_R,\mathbb{R}^{Nn})} & \\ \leq C\big(\|\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})\|_{L^2(B_{2R},\mathbb{R}^N)} &+ (R^{-\frac{n}{2}} + R^{-\frac{n}{2}-1})\|a(|\nabla \mathbf{u}|)\nabla \mathbf{u}\|_{L^1(B_{2R},\mathbb{R}^{Nn})}\big) \end{aligned} \tag{3.4}$$

*for every function $\mathbf{u} \in C^3(\Omega, \mathbb{R}^N)$ and any ball $B_{2R} \subset\subset \Omega$.*
*(ii) Assume, in addition, that $\Omega$ is a bounded open set with $\partial\Omega \in C^2$. There exists a constant $c = c(n, N, i_a, s_a, L_\Omega, d_\Omega)$ such that, if*

$$\mathcal{K}_\Omega(r) \leq \mathcal{K}(r) \quad \textit{for } r \in (0,1), \tag{3.5}$$

*for some function $\mathcal{K} : (0,1) \to [0,\infty)$ satisfying*

$$\lim_{r\to 0^+} \mathcal{K}(r) < c\,, \tag{3.6}$$

*then*

$$\|a(|\nabla \mathbf{u}|)\nabla \mathbf{u}\|_{W^{1,2}(\Omega,\mathbb{R}^{Nn})} \leq C\big(\|\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})\|_{L^2(\Omega,\mathbb{R}^N)} + \|a(|\nabla \mathbf{u}|)\nabla \mathbf{u}\|_{L^1(\Omega,\mathbb{R}^{Nn})}\big) \tag{3.7}$$

*for some constant $C = C(n, N, i_a, s_a, L_\Omega, d_\Omega, \mathcal{K})$, and for every function $\mathbf{u} \in C^3(\Omega, \mathbb{R}^N) \cap C^2(\overline{\Omega}, \mathbb{R}^N)$ fulfilling either* (1.2) *or* (1.3)*.*
*In particular, if $\Omega$ is convex, then inequality* (3.7) *holds whatever $\mathcal{K}_\Omega$ is, and the constant $C$ in* (3.7) *only depends on $n, N, i_a, s_a, L_\Omega, d_\Omega$.*

**Remark 3.2** By Remark 2.5, equation (3.6) holds, with $\mathcal{K} = \mathcal{K}_\Omega$, for each single open set $\Omega$ with $\partial\Omega \in C^2$. The dependence of the constant $C$ in inequality (3.7) just through an upper bound $\mathcal{K}$ for $\mathcal{K}_\Omega$, satisfying (3.6), will be crucial in approximating domains $\Omega$ with minimal boundary regularity as in Theorem 2.1.

The proof of Theorem 3.1 relies upon the differential inequality contained in the next lemma. In what follows, we shall use the notation $\mathbf{u} = (u^1, \ldots, u^N)$ for a vector-valued function $\mathbf{u} : \Omega \to \mathbb{R}^N$.

**Lemma 3.3** *Let $n \geq 2$, $N \geq 1$, and let $\Omega$ be an open set in $\mathbb{R}^n$. Assume that $a : [0,\infty) \to [0,\infty)$ is a function having the form $a(t) = \widehat{a}(t^2)$ for some function $\widehat{a} \in C^1([0,\infty))$, and such that $a(t) > 0$ for $t > 0$. Suppose that condition (3.2) is in force. Then there exists a positive constant $C = C(n, N, i_a)$ such that*

$$
\tag{3.8}
\begin{aligned}
\big|\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})\big|^2 \geq{}& \sum_{j=1}^{n} \big(a(|\nabla \mathbf{u}|)^2 \mathbf{u}_{x_j} \cdot \Delta \mathbf{u}\big)_{x_j} \\
&- \sum_{i=1}^{n} \Big(a(|\nabla \mathbf{u}|)^2 \sum_{j=1}^{n} \mathbf{u}_{x_j} \cdot \mathbf{u}_{x_i x_j}\Big)_{x_i} + C a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2 \quad \text{in } \Omega,
\end{aligned}
$$

*for every function $\mathbf{u} \in C^3(\Omega, \mathbb{R}^N)$. Here, "$\cdot$" stands for scalar product, and $|\nabla^2 \mathbf{u}| = \big(\sum_{\alpha=1}^{N} \sum_{i,j=1}^{n} (u^\alpha_{x_i x_j})^2\big)^{\frac{1}{2}}$.*

**Proof**. Computations show that

$$
\tag{3.9}
\begin{aligned}
\big|\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})\big|^2 &= \sum_{\alpha=1}^{N} \big(\mathrm{div}(a(|\nabla \mathbf{u}|)\nabla u^\alpha)\big)^2 \\
&= \sum_{\alpha=1}^{N} \big(a(|\nabla \mathbf{u}|)\Delta u^\alpha + a'(|\nabla \mathbf{u}|)\nabla|\nabla \mathbf{u}| \cdot \nabla u^\alpha\big)^2 \\
&= a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^{N} \big((\Delta u^\alpha)^2 - |\nabla^2 u^\alpha|^2\big) + a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^{N} |\nabla^2 u^\alpha|^2 + \\
&\quad + a'(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^{N} (\nabla|\nabla \mathbf{u}| \cdot \nabla u^\alpha)^2 + 2a(|\nabla \mathbf{u}|)a'(|\nabla \mathbf{u}|) \sum_{\alpha=1}^{N} \Delta u^\alpha \nabla|\nabla \mathbf{u}| \cdot \nabla u^\alpha \\
&= a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^{N} \Big(\sum_{j=1}^{n} (u^\alpha_{x_j} \Delta u^\alpha)_{x_j} - \sum_{i,j=1}^{n} (u^\alpha_{x_j} u^\alpha_{x_i x_j})_{x_i}\Big) + a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^{N} |\nabla^2 u^\alpha|^2 \\
&\quad + a'(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^{N} (\nabla|\nabla \mathbf{u}| \cdot \nabla u^\alpha)^2 + 2a(|\nabla \mathbf{u}|)a'(|\nabla \mathbf{u}|) \sum_{\alpha=1}^{N} \Delta u^\alpha \nabla|\nabla \mathbf{u}| \cdot \nabla u^\alpha \\
&= a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^{N} \Big(\sum_{j=1}^{n} (u^\alpha_{x_j} \Delta u^\alpha)_{x_j} - \sum_{i,j=1}^{n} (u^\alpha_{x_j} u^\alpha_{x_i x_j})_{x_i}\Big) \\
&\quad + a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^{N} \sum_{i,j=1}^{n} (u^\alpha_{x_i x_j})^2 + \Big(\frac{a'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|}\Big)^2 \sum_{\alpha=1}^{N} \Big(\sum_{\beta=1}^{N} \sum_{k,j=1}^{n} u^\beta_{x_k} u^\beta_{x_k x_j} u^\alpha_{x_j}\Big)^2 \\
&\quad + 2a(|\nabla \mathbf{u}|)\frac{a'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \sum_{\alpha,\beta=1}^{N} \sum_{k,j=1}^{n} \Delta u^\alpha u^\beta_{x_k} u^\beta_{x_k x_j} u^\alpha_{x_j}\,.
\end{aligned}
$$

Here, the expression $\frac{a'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|}$ has to be interpreted as $2\,\widehat{a}'(0)$ if $\nabla \mathbf{u} = 0$. One has that

$$
\tag{3.10}
\begin{aligned}
&\sum_{\alpha=1}^{N} \sum_{j=1}^{n} (a(|\nabla \mathbf{u}|)^2 u^\alpha_{x_j} \Delta u^\alpha)_{x_j} \\
&\quad = a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^{N} \sum_{j=1}^{n} (u^\alpha_{x_j} \Delta u^\alpha)_{x_j} + 2a(|\nabla \mathbf{u}|)\frac{a'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \sum_{\alpha,\beta=1}^{N} \sum_{j,k=1}^{n} u^\beta_{x_k} u^\beta_{x_k x_j} u^\alpha_{x_j} \Delta u^\alpha\,,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.11)\qquad & \sum_{\alpha=1}^{N}\sum_{i,j=1}^{n}(a(|\nabla\mathbf{u}|)^2u^\alpha_{x_j}u^\alpha_{x_ix_j})_{x_i}\\
&=\sum_{\alpha=1}^{N}\sum_{i,j=1}^{n}(u^\alpha_{x_j}u^\alpha_{x_ix_j})_{x_i}+2a(|\nabla\mathbf{u}|)\frac{a'(|\nabla\mathbf{u}|)}{|\nabla\mathbf{u}|}\sum_{\alpha,\beta=1}^{N}\sum_{i,j,k=1}^{n}u^\beta_{x_k}u^\beta_{x_kx_i}u^\alpha_{x_j}u^\alpha_{x_ix_j}\,.
\end{aligned}
$$

On making use of equations (3.10)–(3.11), the equality between the leftmost and the rightmost sides of equation (3.9) takes the form

(3.12)

$$
\begin{aligned}
\big|\mathbf{div}(a(|\nabla\mathbf{u}|)\nabla\mathbf{u})\big|^2 &= \sum_{j=1}^{n}(a(|\nabla\mathbf{u}|)^2\mathbf{u}_{x_j}\cdot\Delta\mathbf{u})_{x_j}-\sum_{i,j=1}^{n}(a(|\nabla\mathbf{u}|)^2\mathbf{u}_{x_j}\cdot\mathbf{u}_{x_ix_j})_{x_i}\\
&+\Big(\frac{a'(|\nabla\mathbf{u}|)}{|\nabla\mathbf{u}|}\Big)^2\sum_{\alpha=1}^{N}\Big(\sum_{\beta=1}^{N}\sum_{k,j=1}^{n}u^\beta_{x_k}u^\alpha_{x_j}u^\beta_{x_kx_j}\Big)^2\\
&+2\frac{a'(|\nabla\mathbf{u}|)}{|\nabla\mathbf{u}|}\sum_{\alpha,\beta=1}^{N}\sum_{i,j,k=1}^{n}u^\beta_{x_k}u^\alpha_{x_j}u^\alpha_{x_jx_i}u^\beta_{x_kx_i}+a(|\nabla\mathbf{u}|)^2\sum_{\alpha=1}^{N}\sum_{i,j=1}^{n}(u^\alpha_{x_ix_j})^2\,.
\end{aligned}
$$

If $\nabla\mathbf{u}=0$, then inequality (3.12) tells us that inequality (3.8) holds with $C=1$. Assume next that $\nabla\mathbf{u}\neq 0$. Then, inequality (3.12) can be rewritten as

$$
\begin{aligned}
(3.13)\quad \big|\mathbf{div}(a(|\nabla\mathbf{u}|)\nabla\mathbf{u})\big|^2 &= \sum_{j=1}^{n}(a(|\nabla\mathbf{u}|)^2\mathbf{u}_{x_j}\cdot\Delta\mathbf{u})_{x_j}-\sum_{i,j=1}^{n}(a(|\nabla\mathbf{u}|)^2\mathbf{u}_{x_j}\cdot\mathbf{u}_{x_ix_j})_{x_i}\\
&+a(|\nabla\mathbf{u}|)^2\Bigg[\Big(\frac{a'(|\nabla\mathbf{u}|)|\nabla\mathbf{u}|}{a(|\nabla\mathbf{u}|)}\Big)^2\sum_{\alpha=1}^{N}\Big(\sum_{\beta=1}^{N}\sum_{k,j=1}^{n}\frac{u^\beta_{x_k}}{|\nabla\mathbf{u}|}\frac{u^\alpha_{x_j}}{|\nabla\mathbf{u}|}u^\beta_{x_kx_j}\Big)^2\\
&+2\frac{a'(|\nabla\mathbf{u}|)|\nabla\mathbf{u}|}{a(|\nabla\mathbf{u}|)}\sum_{\alpha,\beta=1}^{N}\sum_{i,j,k=1}^{n}\frac{u^\beta_{x_k}}{|\nabla\mathbf{u}|}\frac{u^\alpha_{x_j}}{|\nabla\mathbf{u}|}u^\alpha_{x_jx_i}u^\beta_{x_kx_i}+\sum_{\alpha=1}^{N}\sum_{i,j=1}^{n}(u^\alpha_{x_ix_j})^2\Bigg]\,.
\end{aligned}
$$

Now, define

$$\vartheta=\frac{|\nabla\mathbf{u}|a'(|\nabla\mathbf{u}|)}{a(|\nabla\mathbf{u}|)},$$

and

$$\boldsymbol{\omega}^\alpha=\frac{\nabla u^\alpha}{|\nabla\mathbf{u}|},\quad \mathbf{H}^\alpha=\nabla^2u^\alpha\qquad\text{for }\alpha=1,\dots,N.$$

Observe that $\boldsymbol{\omega}^\alpha\in\mathbb{R}^n$, and $\sum_{\alpha=1}^N|\boldsymbol{\omega}^\alpha|^2=\sum_{\alpha=1}^N\sum_{i=1}^n(\omega^\alpha_i)^2=1$, where we have set $\boldsymbol{\omega}^\alpha=(\omega^\alpha_1,\dots,\omega^\alpha_n)^T$, where $T$ stands for transpose. Moreover, $\mathbf{H}^\alpha$ is a symmetric matrix in $\mathbb{R}^{n^2}$, and, by assumption (3.2),

$$(3.14)\qquad \vartheta>-\tfrac{1}{2}.$$

With this notations in place, the expression in square brackets on the right-hand side of (3.13) takes the form

$$(3.15)\qquad \vartheta^2\sum_{\alpha=1}^{N}\Big(\sum_{\beta=1}^{N}\mathbf{H}^\beta\boldsymbol{\omega}^\beta\cdot\boldsymbol{\omega}^\alpha\Big)^2+2\vartheta\sum_{\alpha,\beta=1}^{N}\mathbf{H}^\alpha\boldsymbol{\omega}^\alpha\cdot\mathbf{H}^\beta\boldsymbol{\omega}^\beta+\sum_{\alpha=1}^{N}\mathrm{tr}\big((\mathbf{H}^\alpha)^2\big)\,,$$

where "tr" denotes the trace of a matrix. Since the middle term in (3.15) equals $\left\| \sum_{\alpha=1}^N \mathbf{H}^\alpha \boldsymbol{\omega}^\alpha \right\|^2$, the proof of inequality (3.8) is thus reduced to showing that

$$\vartheta^2 \sum_{\alpha=1}^N \Big( \sum_{\beta=1}^N \mathbf{H}^\beta \boldsymbol{\omega}^\beta \cdot \boldsymbol{\omega}^\alpha \Big)^2 + 2\vartheta \Big\| \sum_{\alpha=1}^N \mathbf{H}^\alpha \boldsymbol{\omega}^\alpha \Big\|^2 + \sum_{\alpha=1}^N \mathrm{tr}\big( (\mathbf{H}^\alpha)^2 \big) \geq C \sum_{\alpha=1}^N \mathrm{tr}\big( (\mathbf{H}^\alpha)^2 \big) \tag{3.16}$$

for some positive constant $C = C(n, N, i_a)$. It $\vartheta \geq 0$, inequality (3.16) holds with $C = 1$. Assume next that

$$-\tfrac{1}{2} < \vartheta < 0 . \tag{3.17}$$

For each $\alpha = 1, \dots, N$ fix an orthonormal basis in $\mathbb{R}^n$ of eigenvectors $\mathbf{e}_1^\alpha, \dots \mathbf{e}_n^\alpha$ of $\mathbf{H}^\alpha$, in which $\mathbf{H}^\alpha$ takes the diagonal form $\boldsymbol{\Lambda}^\alpha = \mathrm{diag}(\lambda_1^\alpha, \dots \lambda_n^\alpha)$, where $\lambda_1^\alpha, \dots \lambda_n^\alpha$ stand for the eigenvalues of $\mathbf{H}^\alpha$. Let $\overline{\boldsymbol{\omega}}^\alpha = (\overline{\omega}_1^\alpha, \dots, \overline{\omega}_n^\alpha)^T$ be the vector of the components of $\boldsymbol{\omega}^\alpha$ with respect to this basis. Thus, on denoting by $\mathbf{M}^\alpha = (m_{ij}^\alpha)$ the othogonal matrix whose columns are the vectors $\mathbf{e}_j$, with $j = 1, \dots, n$, one has that

$$\boldsymbol{\omega}^\alpha = \mathbf{M}^\alpha \overline{\boldsymbol{\omega}}^\alpha \qquad \text{for } \alpha = 1, \dots, N, \tag{3.18}$$

and

$$\mathbf{H}^\alpha = \mathbf{M}^\alpha \boldsymbol{\Lambda}^\alpha (\mathbf{M}^\alpha)^T \qquad \text{for } \alpha = 1, \dots, N. \tag{3.19}$$

Moreover, on setting $\mathbf{D}^\alpha = \mathbf{M}^\alpha \boldsymbol{\Lambda}^\alpha$, with $\mathbf{D}^\alpha = (d_{ij}^\alpha)$, one has that

$$d_{ij}^\alpha = m_{ij}^\alpha \lambda_j^\alpha \qquad \text{for } i, j = 1, \dots n \text{ and } \alpha = 1, \dots N. \tag{3.20}$$

Clearly,

$$\sum_{\alpha=1}^N \mathrm{tr}\big( (\mathbf{H}^\alpha)^2 \big) = \sum_{\alpha=1}^N \sum_{i=1}^n (\lambda_i^\alpha)^2 . \tag{3.21}$$

On the other hand, by (3.18) and (3.19),

$$\mathbf{H}^\alpha \boldsymbol{\omega}^\alpha = \mathbf{M}^\alpha \boldsymbol{\Lambda}^\alpha \overline{\boldsymbol{\omega}}^\alpha \qquad \text{for } \alpha = 1, \dots N.$$

Hence, the following chain holds:

(3.22)
$$\begin{aligned} \Big\| \sum_{\alpha=1}^N \mathbf{H}^\alpha \boldsymbol{\omega}^\alpha \Big\|^2 &= \Big\| \sum_{\alpha=1}^N \mathbf{M}^\alpha \boldsymbol{\Lambda}^\alpha \overline{\boldsymbol{\omega}}^\alpha \Big\|^2 = \sum_{i=1}^n \Big( \sum_{\alpha=1}^N \sum_{j=1}^n d_{ij}^\alpha \overline{\omega}_j^\alpha \Big)^2 \\ &\leq \sum_{i=1}^n \Big( \sum_{\alpha=1}^N \sum_{j=1}^n (d_{ij}^\alpha)^2 \Big) \Big( \sum_{\alpha=1}^N \sum_{j=1}^n (\overline{\omega}_j^\alpha)^2 \Big) = \sum_{\alpha=1}^N \sum_{i,j=1}^n (d_{ij}^\alpha)^2 = \sum_{\alpha=1}^N \sum_{i,j=1}^n (m_{ij}^\alpha)^2 (\lambda_j^\alpha)^2 \\ &= \sum_{\alpha=1}^N \Big( \sum_{j=1}^n \Big( (\lambda_j^\alpha)^2 \sum_{i=1}^n (m_{ij}^\alpha)^2 \Big) \Big) = \sum_{\alpha=1}^N \sum_{j=1}^n (\lambda_j^\alpha)^2 = \sum_{\alpha=1}^N \mathrm{tr}\big( (\mathbf{H}^\alpha)^2 \big) . \end{aligned}$$

Here, we have made use of the fact that, since $\mathbf{M}^\alpha$ is an orthogonal matrix for $\alpha = 1, \dots, N$,

$$\sum_{\alpha=1}^N \sum_{j=1}^n (\overline{\omega}_j^\alpha)^2 = \sum_{\alpha=1}^N \sum_{j=1}^n (\omega_j^\alpha)^2 = 1,$$

and

$$\sum_{i=1}^{n} (m_{ij}^{\alpha})^2 = 1 \qquad \text{for } j = 1, \dots, n \text{ and } \alpha = 1, \dots, N.$$

Owing to (3.17), inequality (3.22) implies that

(3.23)
$$\vartheta^2 \sum_{\alpha=1}^{N} \Big( \sum_{\beta=1}^{N} \mathbf{H}^{\beta} \boldsymbol{\omega}^{\beta} \cdot \boldsymbol{\omega}^{\alpha} \Big)^2 + 2\vartheta \Big\| \sum_{\alpha=1}^{N} \mathbf{H}^{\alpha} \boldsymbol{\omega}^{\alpha} \Big\|^2 + \sum_{\alpha=1}^{N} \mathrm{tr}\big((\mathbf{H}^{\alpha})^2\big) \geq (2\vartheta + 1) \sum_{\alpha=1}^{N} \mathrm{tr}\big((\mathbf{H}^{\alpha})^2\big),$$

whence inequality (3.16) follows with $C = 2\vartheta + 1$. Altogether, we have shown that inequality (3.8) holds with $C = \min\{2\vartheta + 1, 1\}$. □

The next lemma collects Sobolev type inequalities of a form suitable for our applications in the proof of Theorem 3.1.

**Lemma 3.4** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Then for every $\delta > 0$, there exists a constant $C = C(n, \delta, L_\Omega, d_\Omega)$ such that*

$$\int_\Omega v^2\, dx \leq \delta \int_\Omega |\nabla v|^2\, dx + C \Big( \int_\Omega |v|\, dx \Big)^2 \tag{3.24}$$

*for every $v \in W^{1,2}(\Omega)$.*
*In particular, there exists a constant $C = C(n)$ such that, if $0 < R \leq \sigma < \tau \leq 2R$, then*

$$\int_{B_\tau \setminus B_\sigma} v^2\, dx \leq C\delta^2 R^2 \int_{B_\tau \setminus B_\sigma} |\nabla v|^2\, dx + \frac{C}{\delta^n (\tau - \sigma) R^{n-1}} \Big( \int_{B_\tau \setminus B_\sigma} |v|\, dx \Big)^2 \tag{3.25}$$

*for every $\delta \in (0, 1)$, every $v \in W^{1,2}(B_\tau \setminus B_\sigma)$, and any concentric balls $B_\sigma$ and $B_\tau$.*

Inequality (3.24) can be established as in [Maz6, Proof of Theorem 1.4.6/1]. As for inequality (3.25), see [CiMa2, Proof of Theorem 2.1].

A result of [Maz1, Maz2] (see also [Maz6, Section 2.5.2]) provides us with a necessary and sufficient capacitary condition for the validity of a Poincaré type inequality with measure. A special case of that result is enucleated in Lemma 3.5 below. Recall that, given an open set $\Omega \subset \mathbb{R}^n$, the capacity $\mathrm{cap}_\Omega(E)$ of a set $E \subset \Omega$ relative to $\Omega$ is defined as

$$\mathrm{cap}_\Omega(E) = \inf \Big\{ \int_\Omega |\nabla v|^2\, dx : v \in C_0^{0,1}(\Omega), v \geq 1 \text{ on } E \Big\}. \tag{3.26}$$

Here, $C_0^{0,1}(\Omega)$ denotes the space of Lipschitz continuous, compactly supported functions in $\Omega$.

**Lemma 3.5** *Assume that $\Omega$ is the subgraph in $\mathbb{R}^n$, $n \geq 2$, of a Lipschitz continuous function of $(n-1)$ variables. Let $\varrho$ be a nonnegative function in $L^1_{\mathrm{loc}}(\partial\Omega)$, let $x \in \partial\Omega$, $r_0 \in (0, 1)$ and $r \in (0, r_0)$. Then the following properties are equivalent:*
*(i) There exists a constant $C_1$ such that*

$$\int_{\partial\Omega \cap B_r(x)} v^2\, \varrho\, d\mathcal{H}^{n-1} \leq C_1 \int_{\Omega \cap B_r(x)} |\nabla v|^2\, dy \tag{3.27}$$

*for every* $v \in C_0^{0,1}(B_r(x))$.
*(ii) There exists a constant* $C_2$ *such that*

$$\int_E \varrho \, d\mathcal{H}^{n-1} \le C_2 \operatorname{cap}_{B_1(x)}(E), \tag{3.28}$$

*for every set* $E \subset \partial\Omega \cap B_r(x)$.
*Moreover, the best constans* $C_1$ *and* $C_2$ *in* (3.27) *and* (3.28) *are equivalent, up to multiplicative constants depending on* $n$, $r_0$ *and on an upper bound for* $L_\Omega$.
*The same statement holds if* $\Omega$ *is a bounded Lipschitz domain in* $\mathbb{R}^n$, *and* $r_0 \in (0,1)$ *is a suitable positive number depending on* $n$ *and on an upper bound for* $d_\Omega$ *and* $L_\Omega$. *In this case, the best constants* $C_1$ *and* $C_2$ *are equivalent up to multiplicative constants depending on* $n$, $r_0$, $L_\Omega$ *and* $d_\Omega$.

**Remark 3.6** A flattening argument for $\partial\Omega$, combined with an even extension of trial functions, shows that inequality (3.27), and hence also (3.28), is equivalent to

$$\int_{\partial\Omega \cap B_r(x)} v^2 \, \varrho \, d\mathcal{H}^{n-1} \le C_3 \int_{B_r(x)} |\nabla v|^2 \, dy \tag{3.29}$$

for some constant $C_3$, and every $v \in C_0^{0,1}(B_r(x))$. Furthermore, the best constant $C_3$ is equivalent (up to multiplicative constants depending on $n$, $r_0$ and $L_\Omega$) to the best constans $C_1$ and $C_2$ in (3.27) and (3.28).

As a consequence of Lemma 3.5, a capacity-free condition for the validity of a weighted trace inequality on balls centered on the boundary can be derived – see [CiMa2, Proof of Theorem 2.4]. The relevant condition can be formulated in terms of membership of the weight in the Marcinkiewicz type spaces on $\partial\Omega$, with respect to the measure $\mathcal{H}^{n-1}$, appearing in (2.6). Importantly, the constant in the Poincaré inequality turns out to be proportional to the norm of the weight in the Marcinkiewicz space.
Recall that the Marcinkiewicz space $L^{q,\infty}(\partial\Omega)$, also called weak $L^q(\partial\Omega)$ space, is the Banach function space endowed with the norm defined as

$$\|\psi\|_{L^{q,\infty}(\partial\Omega)} = \sup_{s \in (0, \mathcal{H}^{n-1}(\partial\Omega))} s^{\frac{1}{q}} \psi^{**}(s) \tag{3.30}$$

for a measurable function $\psi$ on $\partial\Omega$. Here, $\psi^{**}(s) = \int_0^s \psi^*(r)\,dr$ for $s > 0$, where $\psi^*$ denotes the decreasing rearrangement of $\psi$. The Marcinkiewicz type space $L^{1,\infty}\log L(\partial\Omega)$ is equipped with the norm given by

$$\|\psi\|_{L^{1,\infty}\log L(\partial\Omega)} = \sup_{s \in (0, \mathcal{H}^{n-1}(\partial\Omega))} s \log\left(1 + \tfrac{C}{s}\right) \psi^{**}(s), \tag{3.31}$$

for any constant $C > \mathcal{H}^{n-1}(\partial\Omega)$. Different constants $C$ result in equivalent norms in (3.31).

**Lemma 3.7** *Let* $\Omega$ *be a bounded Lipschitz domain in* $\mathbb{R}^n$, $n \ge 2$. *Let* $\varrho$ *be nonnegative function in* $X(\partial\Omega)$, *where* $X$ *is the space appearing in* (2.6). *Then, there exist constants* $C > 0$ *and* $r_0 \in (0,1)$, *depending on* $n$ *and on an upper bound for* $d_\Omega$ *and* $L_\Omega$, *such that*

$$\int_{\partial\Omega \cap B_r(x)} v^2 \, \varrho \, d\mathcal{H}^{n-1} \le C \sup_{y \in \partial\Omega} \|\varrho\|_{X(\partial\Omega \cap B_r(y))} \int_{\Omega \cap B_r(x)} |\nabla v|^2 \, dy \tag{3.32}$$

*for every* $x \in \partial\Omega$, $r \in (0, r_0)$, *and every function* $v \in C_0^{0,1}(B_r(x))$.

We are now in a position to prove Theorem 3.1.

**Proof of Theorem 3.1**. Assume that $\mathbf{u} \in C^3(\Omega, \mathbb{R}^N)$, and let $\xi \in C_0^\infty(\mathbb{R}^n)$. Multiplying through inequality (3.8) by $\xi^2$, and integrating both sides of the resulting inequality over $\Omega$ yield

$$
\begin{aligned}
(3.33)\qquad & \int_\Omega \xi^2 |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})|^2\,dx \\
&\geq \int_\Omega \xi^2 \Big[\sum_{j=1}^n \big(a(|\nabla \mathbf{u}|)^2 \mathbf{u}_{x_j}\cdot \Delta \mathbf{u}\big)_{x_j} - \sum_{i=1}^n \Big(a(|\nabla \mathbf{u}|)^2 \sum_{j=1}^n \mathbf{u}_{x_j}\cdot \mathbf{u}_{x_i x_j}\Big)_{x_i}\Big]\,dx \\
&\quad + C\int_\Omega \xi^2 a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2\,dx \\
&= \int_\Omega \xi^2 \sum_{\alpha=1}^N \Big[\sum_{j=1}^n \big(a(|\nabla \mathbf{u}|)^2 u^\alpha_{x_j} \Delta u^\alpha\big)_{x_j} - \sum_{i=1}^n \Big(a(|\nabla \mathbf{u}|)^2 \sum_{j=1}^n u^\alpha_{x_j} u^\alpha_{x_i x_j}\Big)_{x_i}\Big]\,dx \\
&\quad + C\int_\Omega \xi^2 a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2\,dx
\end{aligned}
$$

for some constant $C = C(n, N, i_a)$.

Let us first focus on Part (i) of the statement. Assume that $\xi \in C_0^\infty(\Omega)$. Then the divergence theorem yields

$$
\begin{aligned}
(3.34)\qquad & \int_\Omega \xi^2 \sum_{\alpha=1}^N \Big[\sum_{j=1}^n \big(a(|\nabla \mathbf{u}|)^2 u^\alpha_{x_j} \Delta u^\alpha\big)_{x_j} - \sum_{i=1}^n \Big(a(|\nabla \mathbf{u}|)^2 \sum_{j=1}^n u^\alpha_{x_j} u^\alpha_{x_i x_j}\Big)_{x_i}\Big]\,dx \\
&= -2\int_\Omega a(|\nabla \mathbf{u}|)^2 \xi \nabla \xi \cdot \sum_{\alpha=1}^N \Big[\Delta u^\alpha \nabla u^\alpha - \sum_{j=1}^n u^\alpha_{x_j} \nabla u^\alpha_{x_j}\Big]\,dx\,.
\end{aligned}
$$

By Young's inequality, there exists a constant $C' = C'(n, N)$ such that

$$
\begin{aligned}
(3.35)\qquad & 2\Big|\int_\Omega a(|\nabla \mathbf{u}|)^2 \xi \nabla \xi \cdot \sum_{\alpha=1}^N \Big[\Delta u^\alpha \nabla u^\alpha - \sum_{j=1}^n u^\alpha_{x_j} \nabla u^\alpha_{x_j}\Big]\,dx\Big| \\
&\leq \varepsilon C' \int_\Omega \xi^2 a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2\,dx + \frac{C'}{\varepsilon}\int_\Omega |\nabla \xi|^2 a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2\,dx
\end{aligned}
$$

for every $\varepsilon > 0$. Equations (3.33)–(3.35) ensure that

$$
\begin{aligned}
(3.36)\qquad C - \varepsilon C' \int_\Omega \xi^2 a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2\,dx &\leq \int_\Omega \xi^2 |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})|^2\,dx \\
&\quad + \frac{C'}{\varepsilon}\int_\Omega |\nabla \xi|^2 a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2\,dx\,.
\end{aligned}
$$

Let $B_{2R}$ be any ball such that $B_{2R} \subset\subset \Omega$, and let $R \leq \sigma < \tau \leq 2R$. An application of inequality (3.36), with $\varepsilon = \frac{C}{2C'}$ and any function $\xi \in C_0^\infty(B_\tau)$ such that $0 \leq \xi \leq 1$ in $B_\tau$, $\xi = 1$ in $B_\sigma$ and $|\nabla \xi| \leq C/(\tau - \sigma)$ for some constant $C = C(n)$, tells us that

$$
\begin{aligned}
(3.37)\qquad \int_{B_\sigma} a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2\,dx &\leq C\int_{B_{2R}} |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})|^2\,dx \\
&\quad + \frac{C}{(\tau-\sigma)^2}\int_{B_\tau \setminus B_\sigma} a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2\,dx
\end{aligned}
$$

for some constant $C = C(n, N, i_a)$. Inequality (3.25), applied with $\delta = (\tau - \sigma)/R$ and $v = a(|\nabla u|)u^\alpha_{x_i}$, for $i = 1 \dots, n$ and $\alpha = 1, \dots, N$, yields

$$\begin{aligned}(3.38)\quad \frac{1}{(\tau-\sigma)^2}\int_{B_\tau\setminus B_\sigma} & a(|\nabla \mathbf{u}|)^2|\nabla \mathbf{u}|^2\,dx \\ & \leq C\int_{B_\tau\setminus B_\sigma} a(|\nabla \mathbf{u}|)^2|\nabla^2 \mathbf{u}|^2\,dx + \frac{CR}{(\tau-\sigma)^{n+3}}\bigg(\int_{B_\tau\setminus B_\sigma} a(|\nabla \mathbf{u}|)|\nabla \mathbf{u}|\,dx\bigg)^2\end{aligned}$$

for some constant $C = C(n, N, s_a)$. Note that here we have made use of assumption (3.3) to infer that

$$(3.39)\qquad |\nabla(a(|\nabla \mathbf{u}|)u^\alpha_{x_i})| \leq C\, a(|\nabla \mathbf{u}|)|\nabla^2 \mathbf{u}| \qquad \text{a.e. in } \Omega,$$

for $i = 1, \dots, n$, $\alpha = 1, \dots, N$, and for some constant $C = C(n, N, s_a)$. From inequalities (3.37) and (3.38) one obtains that

$$\begin{aligned}(3.40)\quad \int_{B_\sigma} a(|\nabla \mathbf{u}|)^2|\nabla^2 \mathbf{u}|^2\,dx \leq C\int_{B_\tau\setminus B_\sigma} a(|\nabla \mathbf{u}|)^2|\nabla^2 \mathbf{u}|^2\,dx + C\int_{B_{2R}} |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})|^2\,dx \\ + \frac{CR}{(\tau-\sigma)^{n+3}}\bigg(\int_{B_{2R}} a(|\nabla \mathbf{u}|)|\nabla \mathbf{u}|\,dx\bigg)^2\end{aligned}$$

for some constant $C = C(n, N, i_a, s_a)$, whence

$$\begin{aligned}(3.41)\quad \int_{B_\sigma} a(|\nabla \mathbf{u}|)^2|\nabla^2 \mathbf{u}|^2\,dx \leq \frac{C}{1+C}\int_{B_\tau} a(|\nabla \mathbf{u}|)^2|\nabla^2 \mathbf{u}|^2\,dx + C'\int_{B_{2R}} |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})|^2\,dx \\ + \frac{C'R}{(\tau-\sigma)^{n+3}}\bigg(\int_{B_{2R}} a(|\nabla \mathbf{u}|)|\nabla \mathbf{u}|\,dx\bigg)^2\end{aligned}$$

for positive constants $C = C(n, N, i_a, s_a)$ and $C' = C'(n, N, i_a, s_a)$. Via a customary iteration argument – see e.g. [Gi, Lemma 3.1, Chapter 5] – one can deduce from (3.41) that

$$\begin{aligned}(3.42)\qquad \int_{B_R} a(|\nabla \mathbf{u}|)^2|\nabla^2 \mathbf{u}|^2\,dx \leq C\int_{B_{2R}} |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})|^2\,dx \\ + \frac{C}{R^{n+2}}\bigg(\int_{B_{2R}} a(|\nabla \mathbf{u}|)|\nabla \mathbf{u}|\,dx\bigg)^2\end{aligned}$$

for some constant $C = C(n, N, i_a, s_a)$. Moreover, inequality (3.24), applied with $\Omega = B_1$, $\delta = 1$, and $v = a(|\nabla u|)u^\alpha_{x_i}$, for $i = 1 \dots, n$ and $\alpha = 1, \dots, N$, and a scaling argument imply that there exists a constant $C = C(n, N, s_a)$ such that

$$(3.43)\qquad \int_{B_R} a(|\nabla \mathbf{u}|)^2|\nabla \mathbf{u}|^2\,dx \leq \int_{B_R} a(|\nabla \mathbf{u}|)^2|\nabla^2 \mathbf{u}|^2\,dx + \frac{C}{R^n}\bigg(\int_{B_R} a(|\nabla \mathbf{u}|)|\nabla \mathbf{u}|\,dx\bigg)^2.$$

Inequality (3.4) follows from (3.39), (3.42) and (3.43).
We now consider Part (ii). Let $\Omega$ be a bounded domain with $\partial\Omega \in C^2$. Assume that the function $\mathbf{u} \in C^3(\Omega, \mathbb{R}^N) \cap C^2(\overline{\Omega}, \mathbb{R}^N)$ fulfills either (1.2) or (1.3). Owing to [Gr, Equation (3,1,1,2)],

$$(3.44)\quad \Delta u^\alpha \frac{\partial u^\alpha}{\partial \boldsymbol{\nu}} - \sum_{i,j=1}^n u^\alpha_{x_i x_j} u^\alpha_{x_i}\nu_j$$

$$= \operatorname{div}_T\Big(\frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\nabla_T u^\alpha\Big) - \operatorname{tr}\mathcal{B}\Big(\frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\Big)^2 - \mathcal{B}(\nabla_T u^\alpha, \nabla_T u^\alpha) - 2\nabla_T u^\alpha \cdot \nabla_T \frac{\partial u^\alpha}{\partial \boldsymbol{\nu}} \qquad \text{on } \partial\Omega,$$

for $\alpha = 1, \dots, N$, where $\operatorname{tr}\mathcal{B}$ is the trace of $\mathcal{B}$, $\operatorname{div}_T$ and $\nabla_T$ denote the divergence and the gradient operator on $\partial\Omega$, respectively, and $\nu_j$ stands for the $j$-th component of $\boldsymbol{\nu}$. From the divergence theorem and equation (3.44) we deduce that

$$\begin{aligned}
(3.45)\qquad &\int_\Omega \xi^2 \sum_{\alpha=1}^N \Big[\sum_{j=1}^n \big(a(|\nabla \mathbf{u}|)^2 u^\alpha_{x_j} \Delta u^\alpha\big)_{x_j} - \sum_{i=1}^n \Big(a(|\nabla \mathbf{u}|)^2 \sum_{j=1}^n u^\alpha_{x_j} u^\alpha_{x_i x_j}\Big)_{x_i}\Big]\, dx \\
&= \int_{\partial\Omega} \xi^2 a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^N \Big[\Delta u^\alpha \frac{\partial u^\alpha}{\partial \boldsymbol{\nu}} - \sum_{i,j=1}^n u^\alpha_{x_i x_j} u^\alpha_{x_i} \nu_j\Big]\, d\mathcal{H}^{n-1} \\
&\quad - 2\int_\Omega a(|\nabla \mathbf{u}|)^2 \xi \nabla\xi \cdot \sum_{\alpha=1}^N \Big[\Delta u^\alpha \nabla u^\alpha - \sum_{j=1}^n u^\alpha_{x_j} \nabla u^\alpha_{x_j}\Big]\, dx \\
&= \int_{\partial\Omega} \xi^2 a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^N \Big[\operatorname{div}_T\Big(\frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\nabla_T u^\alpha\Big) - \operatorname{tr}\mathcal{B}\Big(\frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\Big)^2 \\
&\qquad - \mathcal{B}(\nabla_T u^\alpha, \nabla_T u^\alpha) - 2\nabla_T u^\alpha \cdot \nabla_T \frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\Big]\, d\mathcal{H}^{n-1} \\
&\quad - 2\int_\Omega a(|\nabla \mathbf{u}|)^2 \xi \nabla\xi \cdot \sum_{\alpha=1}^N \Big[\Delta u^\alpha \nabla u^\alpha - \sum_{j=1}^n u^\alpha_{x_j} \nabla u^\alpha_{x_j}\Big]\, dx\,.
\end{aligned}$$

Equations (3.33), (3.35) and (3.45) ensure that there exist constants $C = C(n, N, i_a)$ and $C' = C'(n, N)$ such that

(3.46)

$$\begin{aligned}
C - \varepsilon C' \int_\Omega \xi^2 a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2\, dx &\le \int_\Omega \xi^2 |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})|^2\, dx + \frac{C'}{\varepsilon}\int_\Omega |\nabla\xi|^2 a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2\, dx \\
&\quad + \Big|\int_{\partial\Omega} \xi^2 a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^N \Big[\operatorname{div}_T\Big(\frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\nabla_T u^\alpha\Big) - \operatorname{tr}\mathcal{B}\Big(\frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\Big)^2 \\
&\qquad - \mathcal{B}(\nabla_T u^\alpha, \nabla_T u^\alpha) - 2\nabla_T u^\alpha \cdot \nabla_T \frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\Big]\, d\mathcal{H}^{n-1}\Big|.
\end{aligned}$$

If $\mathbf{u}$ satisfies the boundary condition (1.2), then $\nabla_T u^\alpha = 0$ on $\partial\Omega$ for $\alpha = 1, \dots, N$, and hence

$$\begin{aligned}
(3.47)\qquad &\int_{\partial\Omega} \xi^2 a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^N \Big[\operatorname{div}_T\Big(\frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\nabla_T u^\alpha\Big) - \operatorname{tr}\mathcal{B}\Big(\frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\Big)^2 \\
&\qquad - \mathcal{B}(\nabla_T u^\alpha, \nabla_T u^\alpha) - 2\nabla_T u^\alpha \cdot \nabla_T \frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\Big]\, d\mathcal{H}^{n-1} \\
&= -\int_{\partial\Omega} \xi^2 a(|\nabla \mathbf{u}|)^2 \operatorname{tr}\mathcal{B} \sum_{\alpha=1}^N \Big(\frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\Big)^2\, d\mathcal{H}^{n-1}\,.
\end{aligned}$$

On the orther hand, if the boundary condition (1.3) is in force, then $\dfrac{\partial u^\alpha}{\partial \boldsymbol{\nu}} = 0$ on $\partial\Omega$ for $\alpha = 1, \dots, N$. Therefore,

$$(3.48)\qquad \int_{\partial\Omega} \xi^2 a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^N \Big[\operatorname{div}_T\Big(\frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\nabla_T u^\alpha\Big) - \operatorname{tr}\mathcal{B}\Big(\frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\Big)^2$$

$$- \mathcal{B}(\nabla_T u^\alpha, \nabla_T u^\alpha) - 2\nabla_T u^\alpha \cdot \nabla_T \frac{\partial u^\alpha}{\partial \boldsymbol{\nu}}\bigg] \, d\mathcal{H}^{n-1}$$

$$= - \int_{\partial\Omega} \xi^2 a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^{N} \mathcal{B}(\nabla_T u^\alpha, \nabla_T u^\alpha) \, d\mathcal{H}^{n-1} .$$

In both cases, one has that

$$\begin{aligned} \bigg| \int_{\partial\Omega} \xi^2 a(|\nabla \mathbf{u}|)^2 \sum_{\alpha=1}^{N} \bigg[ &\mathrm{div}_T\bigg( \frac{\partial u^\alpha}{\partial \boldsymbol{\nu}} \nabla_T u^\alpha \bigg) - \mathrm{tr}\mathcal{B} \bigg( \frac{\partial u^\alpha}{\partial \boldsymbol{\nu}} \bigg)^2 \\ &- \mathcal{B}(\nabla_T u^\alpha, \nabla_T u^\alpha) - 2\nabla_T u^\alpha \cdot \nabla_T \frac{\partial u^\alpha}{\partial \boldsymbol{\nu}} \bigg] \, d\mathcal{H}^{n-1} \bigg| \\ &\leq C \int_{\partial\Omega} \xi^2 a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2 |\mathcal{B}| \, d\mathcal{H}^{n-1} , \end{aligned} \tag{3.49}$$

for some constant $C(n, N)$. Now, assume that

$$\xi \in C_0^\infty(B_r(x_0)) \tag{3.50}$$

for some $x_0 \in \overline{\Omega}$ and $r \in (0, r_0)$, where $r_0 \in (0, 1)$ is as in the last assertion of the statement of Lemma 3.5.

First, suppose that $x_0 \in \partial\Omega$. Then an application of inequality (3.27), with $\varrho = |\mathcal{B}|$ and $v = \xi \, a(|\nabla \mathbf{u}|) u^\alpha_{x_i}$ , for $i = 1, \dots n$ and $\alpha = 1, \dots, N$, yields

$$\begin{aligned} &\int_{\partial\Omega} \xi^2 \, a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2 |\mathcal{B}| \, d\mathcal{H}^{n-1} \\ &\qquad \leq C \mathcal{K}_\Omega(r) \bigg( \int_\Omega \xi^2 a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2 \, dx + \int_\Omega |\nabla \xi|^2 a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2 \, dx \bigg) \end{aligned} \tag{3.51}$$

for some constant $C = C(n, N, s_a, L_\Omega, d_\Omega)$. Note that inequality (3.39) has also been exploited in deriving (3.51). Combining equations (3.46) and (3.51) tells us that

$$\begin{aligned} &\Big[ C_1 - \varepsilon C_3 - C_2 \, \mathcal{K}_\Omega(r) \Big] \int_\Omega \xi^2 a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2 \, dx \\ &\qquad \leq \int_\Omega \xi^2 |\mathbf{div}(a(|\nabla \mathbf{u}|) \nabla \mathbf{u})|^2 \, dx + \bigg[ C_2 \, \mathcal{K}_\Omega(r) + \frac{C_3}{\varepsilon} \bigg] \int_\Omega |\nabla \xi|^2 a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2 \, dx \end{aligned} \tag{3.52}$$

for some constants $C_1 = C_1(n, N, i_a)$, $C_2 = C_2(n, N, s_a, L_\Omega, d_\Omega)$ and $C_3 = C_3(n, N)$. If condition (3.6) is fulfilled with $c = \frac{C_1}{C_2}$, then there exist $\varepsilon > 0$ and $C > 0$, depending on $\Omega$ only through $L_\Omega$, $d_\Omega$, and $r' \in (0, r_0)$ depending also on the function $\mathcal{K}$, such that

$$C_1 - \varepsilon C_3 - C_2 \, \mathcal{K}_\Omega(r) \geq C_1 - \varepsilon C_3 - C_2 \, \mathcal{K}(r) \geq C \tag{3.53}$$

provided that $r \in (0, r']$. Therefore, by inequality (3.52),

$$\int_\Omega \xi^2 a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2 \, dx \leq C \int_\Omega \xi^2 |\mathbf{div}(a(|\nabla \mathbf{u}|) \nabla \mathbf{u})|^2 \, dx + C \int_\Omega |\nabla \xi|^2 a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2 \, dx \tag{3.54}$$

for some constant $C = C(n, N, i_a, s_a, L_\Omega, d_\Omega)$, if $r \in (0, r']$ in (3.50).

When $x_0 \in \Omega$ and $B_r(x_0) \subset \Omega$, inequality (3.54) just follows from (3.46), inasmuch as the integral over $\partial\Omega$ in the latter inequality vanishes. Also, the constant $C$ in (3.54) is independent

of $L_\Omega$ and $d_\Omega$.
Now, there exists $r^{''} \in (0, r')$, hence depending on $L_\Omega$, $d_\Omega$, and on the function $\mathcal{K}$, such that $\overline{\Omega}$ admits a finite covering $\{B_{r_k}\}$ by balls $B_{r_k}$, with $r^{''} \le r_k \le r'$, and having the property that either $B_{r_k}$ is centered on $\partial\Omega$, or $B_{r_k} \subset \Omega$. Note that this covering can be chosen in such a way that the multiplicity of overlapping of the balls $B_{r_k}$ only depends on $n$. Let $\{\xi_k\}$ be a family of functions such that $\xi_k \in C^\infty_0(B_{r_k})$ and $\{\xi_k^2\}$ is a partition of unity of $\overline{\Omega}$ associated with the covering $\{B_{r_k}\}$. Thus $\sum_k \xi_k^2 = 1$ in $\overline{\Omega}$. The functions $\xi_k$ can also be chosen so that $|\nabla \xi_k| \le C/r_k \le C/r^{''}$ for some absolute constant $C$. Making use of inequality (3.54) with $\xi = \xi_k$ for each $k$, and adding the resulting inequalities one obtains that

$$\int_\Omega a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2\, dx \le C \int_\Omega |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})|^2\, dx + C \int_\Omega a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2\, dx \tag{3.55}$$

for some constant $C = C(n, N, i_a, s_a, L_\Omega, d_\Omega, \mathcal{K})$. Given $\delta > 0$, an application of inequality (3.24) with $v = a(|\nabla \mathbf{u}|) u^\alpha_{x_i}$, $i = 1, \dots, n$, $\alpha = 1, \dots, N$, and equation (3.39) tell us that

$$\int_\Omega a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2\, dx \le \delta C_1 \int_\Omega a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2\, dx + C_2 \bigg( \int_\Omega a(|\nabla \mathbf{u}|) |\nabla \mathbf{u}|\, dx \bigg)^2 \tag{3.56}$$

for some constants $C_1 = C_1(n, N, s_a)$ and $C_2 = C_2(n, N, s_a, \delta, L_\Omega, d_\Omega)$. Choose $\sigma = \frac{1}{2CC_1}$ in inequality (3.56), where $C$ is the constant appearing in (3.55). Coupling the resulting inequality with inequality (3.55) enables us to conclude that

$$\int_\Omega a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2\, dx \le C \int_\Omega |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})|^2\, dx + C \bigg( \int_\Omega a(|\nabla \mathbf{u}|) |\nabla \mathbf{u}|\, dx \bigg)^2 \tag{3.57}$$

for some constant $C = C(n, N, i_a, s_a, L_\Omega, d_\Omega, \mathcal{K})$. Inequalities (3.56) and (3.57) imply (3.7), via (3.39).

Finally, under the assumption that $\Omega$ is convex, the right-hand side of identity (3.44) is nonnegative, since, in this case,

$$-\mathrm{tr}\mathcal{B} \sum_{\alpha=1}^N \left( \frac{\partial u^\alpha}{\partial \boldsymbol{\nu}} \right)^2 \ge 0 \quad \text{and} \quad -\sum_{\alpha=1}^N \mathcal{B}(\nabla_T\, u^\alpha, \nabla_T\, u^\alpha) \ge 0 \qquad \text{on } \partial\Omega\ .$$

Therefore, inequality (3.46) can be replaced with the stronger inequality

$$C - \varepsilon C' \int_\Omega \xi^2 a(|\nabla \mathbf{u}|)^2 |\nabla^2 \mathbf{u}|^2\, dx \le \int_\Omega \xi^2 |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})|^2\, dx + \frac{C'}{\varepsilon} \int_\Omega |\nabla \xi|^2 a(|\nabla \mathbf{u}|)^2 |\nabla \mathbf{u}|^2\, dx \tag{3.58}$$

for some constants $C = C(n, N, i_a)$ and $C' = C'(n, N, i_a)$. Starting from this inequality, instead of (3.46), estimate (3.7) follows analogously (and even more easily). In particular, the function $\mathcal{K}_\Omega$ does not come into play in this case. □

# 4 Local estimates

The proof of Theorem 2.7 is accomplished in this section. Approximable local solutions to equation (1.1) considered in its statement can be defined as follows.

Assume that $\mathbf{f} \in L^1_{\rm loc}(\Omega, \mathbb{R}^N) \cap (W^{1,p}_0(\Omega, \mathbb{R}^N))'$. A function $\mathbf{u} \in W^{1,p}_{\rm loc}(\Omega, \mathbb{R}^N)$ is called a local weak solution to system (1.1) if

$$\int_{\Omega'} |\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u} \cdot \nabla \boldsymbol{\varphi}\, dx = \int_{\Omega'} \mathbf{f} \cdot \boldsymbol{\varphi}\, dx \tag{4.1}$$

for every open set $\Omega' \subset\subset \Omega$, and every function $\boldsymbol{\varphi} \in W^{1,p}_0(\Omega', \mathbb{R}^N)$.
Assume that $\mathbf{f} \in L^q_{\rm loc}(\Omega, \mathbb{R}^N)$ for some $q \geq 1$. A function $\mathbf{u} \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^N)$ is called a local approximable solution to system (1.1) if $\nabla \mathbf{u} \in L^{p-1}_{\rm loc}(\Omega)$, and there exists a sequence $\{\mathbf{f}_k\} \subset C^\infty(\Omega, \mathbb{R}^N)$, with $\mathbf{f}_k \to \mathbf{f}$ in $L^q_{\rm loc}(\Omega, \mathbb{R}^N)$, such that the corresponding sequence of local weak solutions $\{\mathbf{u}_k\}$ to the system

$$-\mathbf{div}(|\nabla \mathbf{u}_k|^{p-2} \nabla \mathbf{u}_k) = \mathbf{f}_k \quad \text{in } \Omega\,, \tag{4.2}$$

satisfies

$$\mathbf{u}_k \to \mathbf{u} \quad \text{and} \quad \nabla \mathbf{u}_k \to \nabla \mathbf{u} \quad \text{a.e. in } \Omega, \tag{4.3}$$

and

$$\lim_{k\to\infty} \int_{\Omega'} |\nabla \mathbf{u}_k|^{p-1}\, dx = \int_{\Omega'} |\nabla \mathbf{u}|^{p-1}\, dx \tag{4.4}$$

for every open set $\Omega' \subset\subset \Omega$.

**Proof of Theorem 2.7**. Assume, for the time being, that

$$\mathbf{f} \in C^\infty(\Omega, \mathbb{R}^N)\,. \tag{4.5}$$

Given $\varepsilon \in (0,1)$, define the function $b_\varepsilon : \mathbb{R} \to (0, \infty)$ as

$$b_\varepsilon(t) = \sqrt{t^2 + \varepsilon^2} \quad \text{for } t \in \mathbb{R}. \tag{4.6}$$

Observe that

$$i_{b_\varepsilon^{p-2}} = \min\{p-2, 0\} \quad \text{and} \quad s_{b_\varepsilon^{p-2}} = \max\{p-2, 0\}\,, \tag{4.7}$$

where $i_{b_\varepsilon^{p-2}}$ and $s_{b_\varepsilon^{p-2}}$ are defined as in (3.1), with $a$ replaced by $b_\varepsilon^{p-2}$. Hence,

$$i_{b_\varepsilon^{p-2}} > -\tfrac{1}{2}\,, \tag{4.8}$$

inasuch as we are assuming that $p > \frac{3}{2}$.
Let $B_{2R} \subset\subset \Omega$. Given a local weak solution $\mathbf{u}$ to system (1.1), let $\mathbf{u}_\varepsilon \in \mathbf{u} + W^{1,p}_0(B_{2R}, \mathbb{R}^N)$ be the weak solution to the Dirichlet problem

$$\begin{cases} -\mathbf{div}(b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2} \nabla \mathbf{u}_\varepsilon) = \mathbf{f} & \text{in } B_{2R} \\ \mathbf{u}_\varepsilon = \mathbf{u} & \text{on } \partial B_{2R}\,. \end{cases} \tag{4.9}$$

To begin with, we notice that

$$\mathbf{u}_\varepsilon \in C^\infty(B_{2R}, \mathbb{R}^N). \tag{4.10}$$

Indeed, by [Schw, Corollary 1.26], $\nabla \mathbf{u}_\varepsilon \in L^\infty_{\rm loc}(B_{2R}, \mathbb{R}^{Nn})$. Hence, via [Schw, Corollary 1.26 and equation (1.8d) ], there exists $\alpha \in (0,1)$ such that $\nabla \mathbf{u}_\varepsilon \in C^\alpha_{\rm loc}(B_{2R}, \mathbb{R}^{Nn})$. In particular, $b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2} \in C^{1,\alpha}_{\rm loc}(B_{2R})$. The Schauder theory for linear elliptic systems then entails that $\mathbf{u}_\varepsilon \in C^{2,\alpha}_{\rm loc}(B_{2R}, \mathbb{R}^N)$. Property (4.10) then follows by iteration, via the Schauder theory again.
Next, let us observe that there exists a constant $C = C(n, N, p, R)$ such that

$$\int_{B_{2R}} |\nabla \mathbf{u}_\varepsilon|^p \, dx \le C\bigg( \int_{B_{2R}} |\mathbf{f}|^{p'} \, dx + \int_{B_{2R}} |\nabla \mathbf{u}|^p \, dx + \varepsilon^p \bigg) \tag{4.11}$$

for $\varepsilon \in (0,1)$. Here, $p' = \frac{p}{p-1}$, the Hölder conjugate. Actually, the use of $\mathbf{u}_\varepsilon - \mathbf{u} \in W^{1,p}_0(B_{2R}, \mathbb{R}^n)$ as a test function in the weak formulation of problem (4.9) yields

$$\int_{B_{2R}} b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2} \nabla \mathbf{u}_\varepsilon \cdot (\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}) \, dx = \int_{B_{2R}} \mathbf{f} \cdot (\mathbf{u}_\varepsilon - \mathbf{u}) \, dx \, . \tag{4.12}$$

One can verify that there exist nonnegative constants $c_1 = c_1(p)$ and $c_2 = c_2(p)$ such that

$$t^p - c_1 \varepsilon^p \le b_\varepsilon(t)^{p-2} t^2 \le c_2(t^p + \varepsilon^p) \quad \text{for } t \ge 0. \tag{4.13}$$

Thus, equation (4.12), Young's inequality and Poincaré's inequality imply that, given $\delta \in (0,1)$,

$$\begin{aligned} \int_{B_{2R}} |\nabla \mathbf{u}_\varepsilon|^p \, dx &\le \int_{B_{2R}} |\mathbf{f}| |\mathbf{u}_\varepsilon - \mathbf{u}| \, dx + C \int_{B_{2R}} (|\nabla \mathbf{u}_\varepsilon|^{p-1} + \varepsilon^{p-1}) |\nabla \mathbf{u}| \, dx + C R^n \varepsilon^p \\ &\le C' \int_{B_{2R}} |\mathbf{f}|^{p'} \, dx + \delta \int_{B_{2R}} |\mathbf{u}_\varepsilon - \mathbf{u}|^p \, dx \\ &\quad + \delta \int_{B_{2R}} |\nabla \mathbf{u}_\varepsilon|^p \, dx + C' \int_{B_{2R}} |\nabla \mathbf{u}|^p \, dx + C'' \varepsilon^p \\ &\le C' \int_{B_{2R}} |\mathbf{f}|^{p'} \, dx + \delta C''' \int_{B_{2R}} |\nabla \mathbf{u}_\varepsilon|^p \, dx + C' \int_{B_{2R}} |\nabla \mathbf{u}|^p \, dx + C'' \varepsilon^p \end{aligned} \tag{4.14}$$

for suitable constants $C = C(p)$, $C' = C'(p, \delta)$, $C'' = C''(n, N, p, R)$ and $C''' = C'''(n, p, N, R)$, and for $\varepsilon \in (0,1)$. Choosing $\delta$ sufficiently small yields (4.11).
Now, we claim that

$$\nabla \mathbf{u}_\varepsilon \to \nabla \mathbf{u} \quad \text{in } L^p(B_{2R}, \mathbb{R}^{Nn}) \tag{4.15}$$

as $\varepsilon \to 0^+$. This claim can be verified via the following argument from [AKM, Proof of Theorem 4.1]. Making use of $\mathbf{u}_\varepsilon - \mathbf{u}$ as a test function in the weak formulation of systems (1.1) and (4.9), subtracting the resulting equalities, and applying Hölder's inequality yield

$$\begin{aligned} &\int_{B_{2R}} \big( b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2} \nabla \mathbf{u}_\varepsilon - b_\varepsilon(|\nabla \mathbf{u}|)^{p-2} \nabla \mathbf{u} \big) \cdot (\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}) \, dx \\ &\quad = \int_{B_{2R}} \big( |\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u} - b_\varepsilon(|\nabla \mathbf{u}|)^{p-2} \nabla \mathbf{u} \big) \cdot (\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}) \, dx \\ &\quad \le C \bigg( \int_{B_{2R}} |\nabla \mathbf{u}_\varepsilon|^p + |\nabla \mathbf{u}|^p \, dx \bigg)^{\frac{1}{p}} \bigg( \int_{B_{2R}} \big| |\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u} - b_\varepsilon(|\nabla \mathbf{u}|)^{p-2} \nabla \mathbf{u} \big|^{p'} \bigg)^{\frac{1}{p'}} \end{aligned} \tag{4.16}$$

for some constant $C = C(p)$. On the other hand,

(4.17)

$$C(|\boldsymbol{\xi}|^2 + |\boldsymbol{\eta}|^2 + \varepsilon^2)^{\frac{p-2}{2}} |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \le \big| b_\varepsilon(|\boldsymbol{\xi}|)^{\frac{p-2}{2}} \boldsymbol{\xi} - b_\varepsilon(|\boldsymbol{\eta}|)^{\frac{p-2}{2}} \boldsymbol{\eta} \big|^2$$

$$\leq C'\big(b_\varepsilon(|\boldsymbol{\xi}|)^{p-2}\boldsymbol{\xi} - b_\varepsilon(|\boldsymbol{\eta}|)^{p-2}\boldsymbol{\eta}\big)\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) \qquad \text{for } \boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^{Nn},$$

for suitable positive constants $C = C(p)$ and $C' = C'(p)$. From (4.16) and (4.17) we deduce that there exists a constant $C = C(p)$ such that

$$\begin{aligned}(4.18)\qquad &\int_{B_{2R}}(|\nabla\mathbf{u}_\varepsilon|^2+|\nabla\mathbf{u}|^2+\varepsilon^2)^{\frac{p-2}{2}}|\nabla\mathbf{u}_\varepsilon-\nabla\mathbf{u}|^2\,dx\\ &\qquad\leq C\bigg(\int_{B_{2R}}|\nabla\mathbf{u}_\varepsilon|^p+|\nabla\mathbf{u}|^p\,dx\bigg)^{\frac1p}\bigg(\int_{B_{2R}}\big||\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}-b_\varepsilon(|\nabla\mathbf{u}|)^{p-2}\nabla\mathbf{u}\big|^{p'}\bigg)^{\frac{1}{p'}}.\end{aligned}$$

By inequality (4.11), the first integral on the right-hand side of inequality (4.18) is uniformly bounded for $\varepsilon\in(0,1)$. Moreover, the second integral converges to 0 as $\varepsilon\to 0$, by dominated convergence. Thus,

$$(4.19)\qquad \lim_{\varepsilon\to0^+}\int_{B_{2R}}(|\nabla\mathbf{u}_\varepsilon|^2+|\nabla\mathbf{u}|^2+\varepsilon^2)^{\frac{p-2}{2}}|\nabla\mathbf{u}_\varepsilon-\nabla\mathbf{u}|^2\,dx=0\,.$$

If $p\geq 2$, equation (4.15) follows from (4.19). If $1<p<2$, conclusion (4.15) still holds, since, by Hölder's inequality,

$$\begin{aligned}(4.20)\qquad &\int_{B_{2R}}|\nabla\mathbf{u}_\varepsilon-\nabla\mathbf{u}|^p\,dx\\ &\leq\bigg(\int_{B_{2R}}(|\nabla\mathbf{u}_\varepsilon|^2+|\nabla\mathbf{u}|^2+\varepsilon^2)^{\frac{p-2}{2}}|\nabla\mathbf{u}_\varepsilon-\nabla\mathbf{u}|^2\,dx\bigg)^{\frac p2}\bigg(\int_{B_{2R}}(|\nabla\mathbf{u}_\varepsilon|^2+|\nabla\mathbf{u}|^2+\varepsilon^2)^{\frac p2}\,dx\bigg)^{\frac{2-p}{2}}.\end{aligned}$$

From property (4.15) one can deduce that

$$(4.21)\qquad \|b_\varepsilon(|\nabla\mathbf{u}_\varepsilon|)^{p-2}\nabla\mathbf{u}_\varepsilon\|_{L^1(B_{2R},\mathbb{R}^{Nn})}\leq C$$

for some constant $C$ independent of $\varepsilon\in(0,1)$. Owing to (4.8), the function $b_\varepsilon^{p-2}$ satisfies the hypotheses on the function $a$ in Theorem 3.1. An application of inequality (3.4) of this theorem to the function $\mathbf{u}_\varepsilon$, and the equation in (4.9), tell us that

$$\begin{aligned}(4.22)\qquad &\|b_\varepsilon(|\nabla\mathbf{u}_\varepsilon|)^{p-2}\nabla\mathbf{u}_\varepsilon\|_{W^{1,2}(B_R,\mathbb{R}^{Nn})}\\ &\qquad\leq C\big(\|\mathbf{f}\|_{L^2(B_{2R},\mathbb{R}^N)}+(R^{-\frac n2}+R^{-\frac n2-1})\|b_\varepsilon(|\nabla\mathbf{u}_\varepsilon|)^{p-2}\nabla\mathbf{u}_\varepsilon\|_{L^1(B_{2R},\mathbb{R}^{Nn})}\big)\,,\end{aligned}$$

where $C = C(n,N,p)$, and, in particular, is indepedent of $\varepsilon$. Inequalities (4.21) and (4.22) ensure that the sequence $\{b_\varepsilon(|\nabla\mathbf{u}_\varepsilon|)^{p-2}\nabla\mathbf{u}_\varepsilon\}$ is bounded in $W^{1,2}(B_R,\mathbb{R}^{Nn})$, and hence there exists a function $\mathbf{U}\in W^{1,2}(B_R,\mathbb{R}^{Nn})$, and a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k\to0^+$,

$$(4.23)\qquad b_{\varepsilon_k}(|\nabla\mathbf{u}_{\varepsilon_k}|)^{p-2}\nabla\mathbf{u}_{\varepsilon_k}\to\mathbf{U}\quad\text{in } L^2(B_R,\mathbb{R}^{Nn})\quad\text{and}\quad b_{\varepsilon_k}(|\nabla\mathbf{u}_{\varepsilon_k}|)^{p-2}\nabla\mathbf{u}_{\varepsilon_k}\rightharpoonup\mathbf{U}\quad\text{in } W^{1,2}(B_R,\mathbb{R}^{Nn})$$

as $k\to\infty$. Since $\lim_{k\to\infty}b_{\varepsilon_k}(t)^{p-2}=t^{p-2}$ for $t>0$, thanks to (4.23) and (4.15),

$$(4.24)\qquad |\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}=\mathbf{U}\in W^{1,2}(B_R,\mathbb{R}^{Nn}).$$

Moreover, equations (4.22) and (4.23) entail that

$$(4.25)\qquad \||\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}\|_{W^{1,2}(B_R,\mathbb{R}^{Nn})}\leq C\big(\|\mathbf{f}\|_{L^2(B_{2R},\mathbb{R}^N)}+(R^{-\frac n2}+R^{-\frac n2-1})\|\nabla\mathbf{u}\|^{p-1}_{L^{p-1}(B_{2R},\mathbb{R}^{Nn})}\big)\,.$$

It remains to remove assumption (4.5). Suppose that $\mathbf{f} \in L^2_{\rm loc}(\Omega, \mathbb{R}^N)$, let $\mathbf{u}$ be an approximable local solution to equation (1.1), and let $\mathbf{f}_k$ and $\mathbf{u}_k$ be as in the definition of this kind of solution given at the begining of the present section. An application of inequality (4.25) to $\mathbf{u}_k$ tells us that $|\nabla \mathbf{u}_k|^{p-2}\nabla \mathbf{u}_k \in W^{1,2}(B_R, \mathbb{R}^{Nn})$, and

(4.26)
$$\||\nabla \mathbf{u}_k|^{p-2}\nabla \mathbf{u}_k\|_{W^{1,2}(B_R,\mathbb{R}^{Nn})} \leq C\big(\|\mathbf{f}_k\|_{L^2(B_{2R},\mathbb{R}^N)} + (R^{-\frac{n}{2}} + R^{-\frac{n}{2}-1})\|\nabla \mathbf{u}_k\|^{p-1}_{L^{p-1}(B_{2R},\mathbb{R}^{Nn})}\big)\,,$$

where the constant $C$ is independent of $k$. Therefore, owing to equation (4.4), the sequence $\{|\nabla \mathbf{u}_k|^{p-2}\nabla \mathbf{u}_k\}$ is bounded in $W^{1,2}(B_R, \mathbb{R}^{Nn})$, and hence there exists a function $\mathbf{U} \in W^{1,2}(B_R, \mathbb{R}^{Nn})$, and a subsequence, still indexed by $k$, such that

$$|\nabla \mathbf{u}_k|^{p-2}\nabla \mathbf{u}_k \to \mathbf{U} \quad \text{in } L^2(B_R, \mathbb{R}^{Nn}) \quad \text{and} \quad |\nabla \mathbf{u}_k|^{p-2}\nabla \mathbf{u}_k \rightharpoonup \mathbf{U} \quad \text{in } W^{1,2}(B_R, \mathbb{R}^{Nn}). \tag{4.27}$$

By assumption (4.3), $\nabla \mathbf{u}_k \to \nabla \mathbf{u}$ a.e. in $\Omega$. Hence, owing to (4.27),

$$|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} = \mathbf{U} \in W^{1,2}(B_R, \mathbb{R}^{Nn})\,. \tag{4.28}$$

Inequality (2.9) follows from (4.26), via (4.4), (4.27) and (4.28). $\square$

## 5 Global estimates

This section is devoted to the proofs of Theorems 2.1 and 2.6 and Corollary 2.3. As a preliminary, we recall some relevant notions of solutions to Dirichlet and Neumann boundary value problems.

We begin with Dirichlet problems. Let $\Omega$ be an open set with finite Lebesgue measure $|\Omega|$. Assume that $\mathbf{f} \in L^1(\Omega, \mathbb{R}^N) \cap (W^{1,p}_0(\Omega, \mathbb{R}^N))'$. A function $\mathbf{u} \in W^{1,p}_0(\Omega, \mathbb{R}^N)$ is called a weak solution to the Dirichlet problem (1.1)+(1.2) if

$$\int_\Omega |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \cdot \nabla \boldsymbol{\varphi}\, dx = \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\, dx \tag{5.1}$$

for every $\boldsymbol{\varphi} \in W^{1,p}_0(\Omega, \mathbb{R}^N)$. Classically, a unique weak solution to (1.1)+(1.2) exists under the present assumptions on $\Omega$ and $\mathbf{f}$.
Assume next that $\mathbf{f} \in L^q(\Omega, \mathbb{R}^N)$ for some $q \geq 1$. A function $\mathbf{u} \in W^{1,1}_0(\Omega, \mathbb{R}^N)$ is called an approximable solution to the Dirichlet problem (1.1)+(1.2) if there exists a sequence $\{\mathbf{f}_k\} \subset C^\infty_0(\Omega, \mathbb{R}^N)$ such that $\mathbf{f}_k \to \mathbf{f}$ in $L^q(\Omega, \mathbb{R}^N)$, and the sequence $\{\mathbf{u}_k\}$ of weak solutions to the Dirichlet problems

$$\begin{cases} -\mathbf{div}(|\nabla \mathbf{u}_k|^{p-2}\nabla \mathbf{u}_k) = \mathbf{f}_k & \text{in } \Omega \\ \mathbf{u}_k = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2}$$

satisfies

$$\mathbf{u}_k \to \mathbf{u} \quad \text{and} \quad \nabla \mathbf{u}_k \to \nabla \mathbf{u} \quad \text{a.e. in } \Omega. \tag{5.3}$$

Parallel definitions can be given for Neumann problems. Let $\Omega$ be a bounded Lipschitz domain. Assume that $\mathbf{f} \in L^1(\Omega, \mathbb{R}^N) \cap (W^{1,p}(\Omega, \mathbb{R}^N))'$ and satisfies the compatibility condition (1.4). A function $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^N)$ is called a weak solution to the Neumann problem (1.1)+(1.3) if

$$\int_\Omega |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \cdot \nabla \boldsymbol{\varphi}\, dx = \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\, dx \tag{5.4}$$

for every $\boldsymbol{\varphi} \in W^{1,p}(\Omega, \mathbb{R}^N)$. A weak solution to (1.1)+(1.3) exists if $\Omega$ and $\mathbf{f}$ iare as above, and is unique, up to additive constant vectors in $\mathbb{R}^N$.
Assume now that $\mathbf{f} \in L^q(\Omega, \mathbb{R}^N)$ for some $q \geq 1$. A function $\mathbf{u} \in W^{1,1}(\Omega, \mathbb{R}^N)$ is called an approximable solution to the Neumann problem (1.1)+(1.3) if there exists a sequence $\{\mathbf{f}_k\} \subset C_0^\infty(\Omega, \mathbb{R}^N)$ such that $\int_\Omega \mathbf{f}_k \, dx = 0$ for $k \in \mathbb{N}$, $\mathbf{f}_k \to \mathbf{f}$ in $L^q(\Omega, \mathbb{R}^N)$, and a sequence $\{\mathbf{u}_k\}$ of weak solutions to the Neumann problems

$$\begin{cases} -\mathbf{div}(|\nabla \mathbf{u}_k|^{p-2} \nabla \mathbf{u}_k) = \mathbf{f}_k & \text{in } \Omega \\ \dfrac{\partial \mathbf{u}_k}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega \end{cases} \tag{5.5}$$

satisfies

$$\mathbf{u}_k \to \mathbf{u} \quad \text{and} \quad \nabla \mathbf{u}_k \to \nabla \mathbf{u} \quad \text{a.e. in } \Omega. \tag{5.6}$$

The following a priori estimate for the gradient of weak solutions to Dirichlet and Neumann problems will be of use in the proof of our global main results.

**Proposition 5.1** *Assume that $n \geq 2$, $N \geq 1$ and $p > 1$. Let $\Omega$ be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$.*
(i) *Assume that $\mathbf{f} \in L^q(\Omega, \mathbb{R}^N) \cap (W_0^{1,p}(\Omega, \mathbb{R}^N))'$ for some $q > 1$. Let $\mathbf{u}$ be the weak solution to the Dirichlet problem* (1.1)+(1.2)*. Then, there exists a constant $C = C(n, N, p, q, |\Omega|)$ such that*

$$\|\nabla \mathbf{u}\|_{L^{p-1}(\Omega, \mathbb{R}^{Nn})} \leq C \|\mathbf{f}\|_{L^q(\Omega, \mathbb{R}^N)}^{p-1} . \tag{5.7}$$

(ii) *Suppose, in addition, that $\Omega$ is a bounded Lipschitz domain. Assume that $\mathbf{f} \in L^q(\Omega, \mathbb{R}^N) \cap (W^{1,p}(\Omega, \mathbb{R}^N))'$ and satifies condition* (1.4)*. Let $\mathbf{u}$ be a weak solution to the Neumann problem* (1.1)+(1.3)*. Then there exists a constant $C = C(n, N, p, q, L_\Omega, d_\Omega)$ such that*

$$\|\nabla \mathbf{u}\|_{L^{p-1}(\Omega, \mathbb{R}^{Nn})} \leq C \|\mathbf{f}\|_{L^q(\Omega, \mathbb{R}^N)}^{p-1} . \tag{5.8}$$

**Proof**. Part (i). Given $\delta > 0$ and $\gamma \in (0,1)$, choose the test function $(|\mathbf{u}|+\delta)^{-\gamma}\mathbf{u}$ in the definition of weak solution to problem (1.1)+(1.2). Since, by the chain rule for Sobolev functions,

$$((|\mathbf{u}| + \delta)^{-\gamma} \mathbf{u})_{x_i} = (|\mathbf{u}| + \delta)^{-\gamma} \mathbf{u}_{x_i} - \gamma (|\mathbf{u}| + \delta)^{-\gamma - 1} \frac{\mathbf{u}}{|\mathbf{u}|} \mathbf{u} \cdot \mathbf{u}_{x_i} \quad \text{a.e. in } \Omega, \tag{5.9}$$

for $i = 1, \dots, n$, we obtain that

$$\begin{aligned} &\int_\Omega |\nabla \mathbf{u}|^{p-2} \sum_{i=1}^n \left[ (|\mathbf{u}| + \delta)^{-\gamma} \mathbf{u}_{x_i} - \gamma (|\mathbf{u}| + \delta)^{-\gamma-1} \frac{\mathbf{u}}{|\mathbf{u}|} \mathbf{u} \cdot \mathbf{u}_{x_i} \right] \cdot \mathbf{u}_{x_i} \, dx \\ &\quad = \int_\Omega |\nabla \mathbf{u}|^{p-2} \sum_{i=1}^n \left[ (|\mathbf{u}| + \delta)^{-\gamma} |\mathbf{u}_{x_i}|^2 - \gamma (|\mathbf{u}| + \delta)^{-\gamma-1} \frac{(\mathbf{u} \cdot \mathbf{u}_{x_i})^2}{|\mathbf{u}|} \right] dx \\ &\quad = \int_\Omega (|\mathbf{u}| + \delta)^{-\gamma} \mathbf{f} \cdot \mathbf{u} \, dx . \end{aligned} \tag{5.10}$$

Hence,

$$\int_\Omega |\nabla \mathbf{u}|^p \big[ (|\mathbf{u}| + \delta)^{-\gamma} - \gamma (|\mathbf{u}| + \delta)^{-\gamma-1} |\mathbf{u}| \big] \, dx \leq \int_\Omega (|\mathbf{u}| + \delta)^{-\gamma} |\mathbf{f}| |\mathbf{u}| \, dx . \tag{5.11}$$

Passing to the limit as $\delta \to 0^+$ in (5.11) yields, via monotone convergence,

$$(1-\gamma)\int_\Omega |\nabla \mathbf{u}|^p |\mathbf{u}|^{-\gamma}\, dx \le \int_\Omega |\mathbf{f}||\mathbf{u}|^{1-\gamma}\, dx\,. \tag{5.12}$$

Notice that the right-hand side of inequality (5.12) is finite, if $\gamma$ is sufficiently close to 1, by Hölder's inequality and the Sobolev embedding theorem, since $\mathbf{f} \in L^q(\Omega,\mathbb{R}^N)$ and $\mathbf{u} \in W^{1,p}_0(\Omega,\mathbb{R}^N)$. As a consequence of (5.12), one can thus show that $|\mathbf{u}|^{-\frac{\gamma}{p}}\mathbf{u} \in W^{1,p}_0(\Omega,\mathbb{R}^N)$. To verify this assertion, observe that

$$\big|\nabla\big((|\mathbf{u}|+\delta)^{-\frac{\gamma}{p}}\mathbf{u}\big)\big|^p \le C|\nabla \mathbf{u}|^p(|\mathbf{u}|+\delta)^{-\gamma} \le |\nabla \mathbf{u}|^p|\mathbf{u}|^{-\gamma} \quad \text{a.e. in } \Omega, \tag{5.13}$$

for every $\delta > 0$, and for some constant $C = C(n,N,p,\gamma)$. Therefore, by (5.12) the family of functions $\{(|\mathbf{u}|+\delta)^{-\frac{\gamma}{p}}\mathbf{u}\}$ is bounded in $W^{1,p}_0(\Omega,\mathbb{R}^N)$ as $\delta \to 0^+$. Hence, by weak compactness, its limit $|\mathbf{u}|^{-\frac{\gamma}{p}}\mathbf{u} \in W^{1,p}_0(\Omega,\mathbb{R}^N)$.
Let $r \in (p, \frac{np}{n-p}]$ if $p < n$, or $r > p$ if $p \ge n$. The Poincaré-Sobolev inequality applied to the function $|\mathbf{u}|^{-\frac{\gamma}{p}}\mathbf{u}$ tells us that

$$\begin{aligned}\Big(\int_\Omega |\mathbf{u}|^{(1-\frac{\gamma}{p})r}\, dx\Big)^{\frac{1}{r}} = \Big(\int_\Omega \big||\mathbf{u}|^{-\frac{\gamma}{p}}\mathbf{u}|^r\, dx\Big)^{\frac{1}{r}} &\le C\Big(\int_\Omega \big|\nabla\big(|\mathbf{u}|^{-\frac{\gamma}{p}}\mathbf{u}\big)\big|^p\, dx\Big)^{\frac{1}{p}}\\ &\le C'\Big(\int_\Omega |\nabla \mathbf{u}|^p|\mathbf{u}|^{-\gamma}\, dx\Big)^{\frac{1}{p}}\end{aligned} \tag{5.14}$$

for some constants $C = C(p,n,r,|\Omega|)$ and $C' = C'(p,n,r,\gamma,|\Omega|)$. If

$$q'(1-\gamma) \le (1-\tfrac{\gamma}{p})r\,, \tag{5.15}$$

then from inequality (5.12), Hölder's inequality and inequality (5.14) one can infer that

$$\begin{aligned}(1-\gamma)\int_\Omega |\nabla \mathbf{u}|^p|\mathbf{u}|^{-\gamma}\, dx &\le \Big(\int_\Omega |\mathbf{f}|^q\, dx\Big)^{\frac{1}{q}}\Big(\int_\Omega |\mathbf{u}|^{q'(1-\gamma)}\Big)^{\frac{1}{q'}}\\ &\le C\Big(\int_\Omega |\mathbf{f}|^q\, dx\Big)^{\frac{1}{q}}\Big(\int_\Omega |\mathbf{u}|^{(1-\frac{\gamma}{p})r}\Big)^{\frac{1-\gamma}{(1-\frac{\gamma}{p})r}}\\ &\le C'\Big(\int_\Omega |\mathbf{f}|^q\, dx\Big)^{\frac{1}{q}}\Big(\int_\Omega |\nabla \mathbf{u}|^p|\mathbf{u}|^{-\gamma}\, dx\Big)^{\frac{1-\gamma}{p-\gamma}},\end{aligned} \tag{5.16}$$

for some constants $C = C(n,N,p,q,r,\gamma,|\Omega|)$ and $C' = C'(n,N,p,q,r,\gamma,|\Omega|)$. Hence,

$$\Big(\int_\Omega |\nabla \mathbf{u}|^p|\mathbf{u}|^{-\gamma}\, dx\Big)^{\frac{p-1}{p-\gamma}} \le C\Big(\int_\Omega |\mathbf{f}|^q\, dx\Big)^{\frac{1}{q}} \tag{5.17}$$

for some constant $C = C(n,N,p,q,r,\gamma,|\Omega|)$. On the other hand, if

$$\gamma(p-1) \le (1-\tfrac{\gamma}{p})r\,, \tag{5.18}$$

then the use of Hölder's inequality (twice), and of inequality (5.14) yields

$$\int_\Omega |\nabla \mathbf{u}|^{p-1}\, dx \le \Big(\int_\Omega |\nabla \mathbf{u}|^p|\mathbf{u}|^{-\gamma}\, dx\Big)^{\frac{1}{p'}}\Big(\int_\Omega |\mathbf{u}|^{\gamma(p-1)}\, dx\Big)^{\frac{1}{p}} \tag{5.19}$$

$$\leq C\Big(\int_\Omega |\nabla \mathbf{u}|^p|\mathbf{u}|^{-\gamma}\,dx\Big)^{\frac{1}{p'}}\Big(\int_\Omega |\mathbf{u}|^{(1-\frac{\gamma}{p})r}\,dx\Big)^{\frac{\gamma(p-1)}{(p-\gamma)r}}$$
$$\leq C'\Big(\int_\Omega |\nabla \mathbf{u}|^p|\mathbf{u}|^{-\gamma}\,dx\Big)^{\frac{p-1}{p-\gamma}}$$

for some constants $C = C(n,N,p,r,\gamma,|\Omega|)$ and $C' = C'(n,N,p,r,\gamma,|\Omega|)$. Notice that both conditions (5.15) and (5.18) are fulfilled owing to our choice of $r$, provided that $\gamma$ is sufficiently close to 1. Inequality (5.7) follows from (5.17) and (5.19).

Part (ii). The proof follows along the same lines as that of Part (i). One has just to replace (if necessary) the original solution $\mathbf{u}$ to the Neumann problem (1.1)+(1.3), by the solution $\mathbf{u}-\boldsymbol{\zeta}_0$, where $\boldsymbol{\zeta}_0 \in \mathbb{R}^N$ is chosen in such a way that

$$\int_\Omega |\mathbf{u}-\boldsymbol{\zeta}_0|^{-\frac{\gamma}{p}}(\mathbf{u}-\boldsymbol{\zeta}_0)\,dx = 0\,. \tag{5.20}$$

To verify that this choice is indeed possible, consider the function $\Phi : \mathbb{R}^N \to [0,\infty)$, given by

$$\Phi(\boldsymbol{\zeta}) = \int_\Omega |\mathbf{u}-\boldsymbol{\zeta}|^{2-\frac{\gamma}{p}}\,dx \quad \text{for } \boldsymbol{\zeta}\in\mathbb{R}^N. \tag{5.21}$$

This function is actually finite-valued if $\gamma$ is sufficiently close to 1, since $\mathbf{u}\in W^{1,p}(\Omega,\mathbb{R}^N)$, and hence $\mathbf{u}\in L^{2-\frac{\gamma}{p}}(\Omega,\mathbb{R}^N)$ by the Sobolev embedding theorem. Moreover, $\Phi$ is (strictly) convex, continuosly differentiable, and $\lim_{|\boldsymbol{\zeta}|\to\infty}\Phi(\boldsymbol{\zeta}) = \infty$. Thus, there exists a (unique) minimum point $\boldsymbol{\zeta}_0$ of $\Phi$, and

$$\big(2-\tfrac{\gamma}{p}\big)\int_\Omega |\mathbf{u}-\boldsymbol{\zeta}_0|^{-\frac{\gamma}{p}}(\mathbf{u}-\boldsymbol{\zeta}_0)\,dx = \nabla\Phi(\boldsymbol{\zeta}_0) = 0\,. \tag{5.22}$$

On denoting the solution $\mathbf{u}-\boldsymbol{\zeta}_0$ simply by $\mathbf{u}$, condition (5.20) reads

$$\int_\Omega |\mathbf{u}|^{-\frac{\gamma}{p}}\mathbf{u}\,dx = 0\,. \tag{5.23}$$

Under condition (5.23), inequality (5.14) continues to hold, by the Poincaré-Sobolev inequality for functions with vanishing mean value, for some constants $C$ and $C'$. The dependence of these constants on $\Omega$ is as described in the last part of the statement. Having disposed of inequality (5.14), inequality (5.8) can be derived via the same argument as in the proof of (5.7). $\square$

Our proof of Theorem 2.1 entails an approximation of the domain $\Omega$ by a sequence of domains $\Omega_m$ with a smooth boundary. Such an approximation requires, in particular, that the quantities $L_{\Omega_m}$ and $d_{\Omega_m}$, and the functions $\mathcal{K}_{\Omega_m}$ be uniformly bounded (up to multiplicative constants) by $L_\Omega$ and $d_\Omega$, and $\mathcal{K}_\Omega$. This is the subject of the next lemma. The main steps of its proof are described in the Appendix.

**Lemma 5.2** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n\geq 2$ such that $\partial\Omega\in W^{2,1}$. Assume that the function $\mathcal{K}_\Omega(r)$, defined as in (2.1), is finite-valued for $r\in(0,1)$. Then there exist positive constants $r_0$ and $C$ and a sequence of bounded open sets $\{\Omega_m\}$, such that $\partial\Omega_m\in C^\infty$, $\Omega\subset\Omega_m$, $\lim_{k\to\infty}|\Omega_m\setminus\Omega| = 0$, the Hausdorff distance between $\Omega_m$ and $\Omega$ tends to 0 as $m\to\infty$,*

$$L_{\Omega_m}\leq CL_\Omega\,,\quad d_{\Omega_m}\leq Cd_\Omega \tag{5.24}$$

*and*

$$\mathcal{K}_{\Omega_m}(r)\leq C\mathcal{K}_\Omega(r) \tag{5.25}$$

*for $r\in(0,r_0)$ and $m\in\mathbb{N}$.*

**Proof of Theorem 2.1**. We begin by dealing with the case when $\mathbf{u}$ is a solution to the Dirichlet problem (1.1)+(1.2). The proof relies on a combination of Theorem 3.1, Part (ii), with approximation arguments for the differential operator, the domain and the right-hand side. The relevant approximations are accomplished in separate steps.
*Step 1.* Here, we assume the additional conditions

$$\mathbf{f} \in C_0^\infty(\Omega, \mathbb{R}^N)\,, \tag{5.26}$$

and

$$\partial\Omega \in C^\infty\,. \tag{5.27}$$

Given $\varepsilon \in (0,1)$, denote by $\mathbf{u}_\varepsilon$ the weak solution to the system

$$\begin{cases} -\mathbf{div}(b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2}\nabla \mathbf{u}_\varepsilon) = \mathbf{f} & \text{in } \Omega \\ \mathbf{u}_\varepsilon = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{5.28}$$

with $b_\varepsilon$ defined as in (4.6). By [CiMa1, Theorem 2.1], there exists a constant $C$, independent of $\varepsilon$, such that

$$\|\nabla \mathbf{u}_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \leq C. \tag{5.29}$$

Hence, for each $\varepsilon \in (0,1)$, there exist constants $c_2 > c_1 > 0$ such that

$$c_1 \leq b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|) \leq c_2 \qquad \text{in } \Omega. \tag{5.30}$$

Properties (5.26), (5.27) and (5.30) permit an application of a result by Elcrat and Meyers [BF, Theorem 8.2], ensuring that $\mathbf{u}_\varepsilon \in W^{2,2}(\Omega, \mathbb{R}^N)$. Thus, $\mathbf{u}_\varepsilon \in W_0^{1,2}(\Omega, \mathbb{R}^n) \cap W^{1,\infty}(\Omega, \mathbb{R}^N) \cap W^{2,2}(\Omega, \mathbb{R}^N)$. By standard approximation [Bu, Chapter 2, Corollary 3], there exists a sequence $\{\mathbf{u}_k\} \subset C^\infty(\Omega, \mathbb{R}^N) \cap C^2(\overline{\Omega}, \mathbb{R}^N)$ such that $\mathbf{u}_k = 0$ on $\partial\Omega$ for $k \in \mathbb{N}$, and

$$\mathbf{u}_k \to \mathbf{u}_\varepsilon \quad \text{in } W_0^{1,2}(\Omega, \mathbb{R}^N), \quad \mathbf{u}_k \to \mathbf{u}_\varepsilon \quad \text{in } W^{2,2}(\Omega, \mathbb{R}^N), \quad \nabla \mathbf{u}_k \to \nabla \mathbf{u}_\varepsilon \quad \text{a.e. in } \Omega, \tag{5.31}$$

as $k \to \infty$. Furthermore,

$$\|\nabla \mathbf{u}_k\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \leq C \|\nabla \mathbf{u}_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \tag{5.32}$$

for some constant $C$ independent of $k$, and, by the chain rule for vector-valued Sobolev functions [MaMi, Theorem 2.1], $|\nabla|\nabla \mathbf{u}_k|| \leq |\nabla^2 \mathbf{u}_k|$ a.e. in $\Omega$.
Finally, one can show that

$$-\mathbf{div}(b_\varepsilon(|\nabla \mathbf{u_k}|)^{p-2}\nabla \mathbf{u_k}) \to \mathbf{f} \quad \text{in } L^2(\Omega, \mathbb{R}^N), \tag{5.33}$$

as $k \to \infty$, see [CiMa1, Equation (6.12)]. Owing to assumption (2.2), inequality (3.7), applied with $a$ replaced by $b_\varepsilon^{p-2}$ and $\mathbf{u}$ replaced by $\mathbf{u}_k$, yields

$$\begin{aligned} &\|b_\varepsilon(|\nabla \mathbf{u}_k|)^{p-2}\nabla \mathbf{u}_k\|_{W^{1,2}(\Omega,\mathbb{R}^{Nn})} \\ &\qquad \leq C\Big(\|\mathbf{div}(b_\varepsilon(|\nabla \mathbf{u}_k|)^{p-2}\nabla \mathbf{u}_k)\|_{L^2(\Omega,\mathbb{R}^N)} + \|b_\varepsilon(|\nabla \mathbf{u}_k|)^{p-2}\nabla \mathbf{u}_k\|_{L^1(\Omega,\mathbb{R}^{Nn})}\Big) \end{aligned} \tag{5.34}$$

for $k \in \mathbb{N}$, and for some constant $C = C(n, N, p, L_\Omega, d_\Omega, \mathcal{K}_\Omega)$. Observe that this constant is independent of $\varepsilon$, thanks to (4.7). Owing to equations (5.32)–(5.34), the sequence $\{b_\varepsilon(|\nabla \mathbf{u}_k|)^{p-2}\nabla \mathbf{u}_k\}$

is bounded in $W^{1,2}(\Omega,\mathbb{R}^{Nn})$. Hence, there exists a subsequence of $\{\mathbf{u}_k\}$, still denoted by $\{\mathbf{u}_k\}$, and a function $\mathbf{U}_\varepsilon \in W^{1,2}(\Omega,\mathbb{R}^{Nn})$ such that

$$b_\varepsilon(|\nabla \mathbf{u}_k|)^{p-2}\nabla \mathbf{u}_k \to \mathbf{U}_\varepsilon \quad \text{in } L^2(\Omega,\mathbb{R}^{Nn}), \quad b_\varepsilon(|\nabla \mathbf{u}_k|)^{p-2}\nabla \mathbf{u}_k \rightharpoonup \mathbf{U}_\varepsilon \quad \text{in } W^{1,2}(\Omega,\mathbb{R}^{Nn}). \tag{5.35}$$

Since, by (5.31), $\nabla \mathbf{u}_k \to \nabla \mathbf{u}_\varepsilon$ a.e. in $\Omega$, one has that

$$b_\varepsilon(|\nabla \mathbf{u}_k|)^{p-2}\nabla \mathbf{u}_k \to b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2}\nabla \mathbf{u}_\varepsilon \quad \text{a.e. in } \Omega. \tag{5.36}$$

Coupling (5.36) with (5.35) ensures that

$$b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2}\nabla \mathbf{u}_\varepsilon = \mathbf{U}_\varepsilon \in W^{1,2}(\Omega,\mathbb{R}^{Nn})\,, \tag{5.37}$$

and passing to the limit as $k \to \infty$ in (5.34) yields

$$\|b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2}\nabla \mathbf{u}_\varepsilon\|_{W^{1,2}(\Omega,\mathbb{R}^{Nn})} \le C\big(\|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^N)} + \|b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2}\nabla \mathbf{u}_\varepsilon\|_{L^1(\Omega,\mathbb{R}^{Nn})}\big)\,. \tag{5.38}$$

Note that, in deriving inequality (5.38), we have made use of (5.35) and (5.37) on the left-hand side, and of (5.32) and (5.33) on the right-hand side. From inequalities (5.38) and (5.29), we infer that there exists a constant $C$, independent of $\varepsilon$, such that

$$\|b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2}\nabla \mathbf{u}_\varepsilon\|_{W^{1,2}(\Omega,\mathbb{R}^{Nn})} \le C(1+\varepsilon^{p-1})\,. \tag{5.39}$$

Owing to inequality (5.39), the family of functions $b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2}\nabla \mathbf{u}_\varepsilon$ is uniformly bounded in $W^{1,2}(\Omega,\mathbb{R}^{Nn})$ for $\varepsilon \in (0,1)$, and hence there exists a sequence $\{\varepsilon_m\}$ and a function $\mathbf{U} \in W^{1,2}(\Omega,\mathbb{R}^{Nn})$ such that $\varepsilon_m \to 0$, and

$$b_{\varepsilon_m}(|\nabla \mathbf{u}_{\varepsilon_m}|)^{p-2}\nabla \mathbf{u}_{\varepsilon_m} \to \mathbf{U} \text{ in } L^2(\Omega,\mathbb{R}^{Nn}), \quad b_{\varepsilon_m}(|\nabla \mathbf{u}_{\varepsilon_m}|)^{p-2}\nabla \mathbf{u}_{\varepsilon_m} \rightharpoonup \mathbf{U} \text{ in } W^{1,2}(\Omega,\mathbb{R}^{Nn}). \tag{5.40}$$

Now, an analogous (and even simpler) argument as in the proof of (4.15) tells us that

$$\nabla \mathbf{u}_{\varepsilon_m} \to \nabla \mathbf{u} \qquad \text{in } L^p(\Omega,\mathbb{R}^{Nn}). \tag{5.41}$$

In particular, notice that, in this argument, inequality (4.11) has to be replaced by

$$\int_\Omega |\nabla \mathbf{u}_{\varepsilon_m}|^p\,dx \le C\bigg(\int_\Omega |\mathbf{f}|^{p'}\,dx + \varepsilon_m^p\bigg)\,,$$

an easy consequence of the use of $\mathbf{u}_{\varepsilon_m}$ as a test function in the definition of weak solution to problem (5.28), with $\varepsilon = \varepsilon_m$. From equations (5.40) and (5.41) one infers that

$$|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} = \mathbf{U} \in W^{1,2}(\Omega,\mathbb{R}^{Nn})\,. \tag{5.42}$$

Also, equation (5.41), inequality (5.29), the dominated convergence theorem for Lebesgue integrals and inequality (5.7) ensure that

$$\lim_{m\to\infty} \|b_{\varepsilon_m}(|\nabla \mathbf{u}_{\varepsilon_m}|)^{p-2}\nabla \mathbf{u}_{\varepsilon_m}\|_{L^1(\Omega,\mathbb{R}^{Nn})} = \|\nabla \mathbf{u}\|_{L^{p-1}(\Omega,\mathbb{R}^{Nn})} \le C\|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^N)} \tag{5.43}$$

for some constant $C = C(n,N,p,|\Omega|)$. From (5.38), we obtain via (5.40), (5.42) and (5.43) that

$$\||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{W^{1,2}(\Omega,\mathbb{R}^{Nn})} \le C\|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^N)} \tag{5.44}$$

for some constant $C = C(n,N,p,L_\Omega,d_\Omega,\mathcal{K}_\Omega)$.

*Step 2.* Here, we remove assumption (5.27). Let $\{\Omega_m\}$ be a sequence of open sets approximating $\Omega$ in the sense of Lemma 5.2. Consider, for each $m \in \mathbb{N}$, the weak solution $\mathbf{u}_m$ to the Dirichlet problem

$$\begin{cases} -\mathbf{div}(|\nabla \mathbf{u}_m|^{p-2}\nabla \mathbf{u}_m) = \mathbf{f} & \text{in } \Omega_m \\ \mathbf{u}_m = 0 & \text{on } \partial\Omega_m\,, \end{cases} \tag{5.45}$$

where $\mathbf{f}$ still fulfils (5.26), and is extended by 0 outside $\Omega$. By inequality (5.44) of Step 2, applied to $\mathbf{u}_m$,

$$\begin{aligned} \| |\nabla \mathbf{u}_m|^{p-2}\nabla \mathbf{u}_m\|_{W^{1,2}(\Omega,\mathbb{R}^{Nn})} &\le \| |\nabla \mathbf{u}_m|^{p-2}\nabla \mathbf{u}_m\|_{W^{1,2}(\Omega_m,\mathbb{R}^{Nn})} \\ &\le C\|\mathbf{f}\|_{L^2(\Omega_m,\mathbb{R}^N)} = C\|\mathbf{f}\|_{L^2(\Omega,\mathbb{R}^N)}, \end{aligned} \tag{5.46}$$

for some constant $C(n, N, p, L_\Omega, d_\Omega, \mathcal{K}_\Omega)$. Note that this dependence of the constant $C$ is guaranteed by properties (5.24) and (5.25) of the sequence $\{\Omega_m\}$.
Thanks to (5.46), the sequence $\{|\nabla \mathbf{u}_m|^{p-2}\nabla \mathbf{u}_m\}$ is bounded in $W^{1,2}(\Omega,\mathbb{R}^{Nn})$, and hence there exists a subsequence, still denoted by $\{\mathbf{u}_m\}$ and a function $\mathbf{U} \in W^{1,2}(\Omega,\mathbb{R}^{Nn})$, such that

$$|\nabla \mathbf{u}_m|^{p-2}\nabla \mathbf{u}_m \to \mathbf{U} \quad \text{in } L^2(\Omega,\mathbb{R}^{Nn}), \qquad |\nabla \mathbf{u}_m|^{p-2}\nabla \mathbf{u}_m \rightharpoonup \mathbf{U} \quad \text{in } W^{1,2}(\Omega,\mathbb{R}^{Nn}). \tag{5.47}$$

Next, we claim that there exists $\alpha \in (0,1)$ such that $\mathbf{u}_m \in C^{1,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, and for every open set $\Omega' \subset\subset \Omega$ there exists a constant $C$, independent of $m$, such that

$$\|\mathbf{u}_m\|_{C^{1,\alpha}(\Omega',\mathbb{R}^N)} \le C\,. \tag{5.48}$$

Actually, it follows from a special case of [BCDKS, Corollary 5.6] that, for each open set $\Omega'$ as above, there exists a constant $C$, independent of $m$, such that

$$\|\nabla \mathbf{u}_m\|_{L^\infty(\Omega',\mathbb{R}^{Nn})} \le C. \tag{5.49}$$

Thanks to [Schw, Corollary 1.26 and equation (1.8d)] and inequality (5.49),

$$\|\nabla \mathbf{u}_m\|_{C^\alpha(\Omega',\mathbb{R}^{Nn})} \le C \tag{5.50}$$

for some constant $C$ independent of $m$. Since assumtpion (5.26) is still in force, a basic energy estimate yields

$$\|\nabla \mathbf{u}_m\|_{L^p(\Omega_m,\mathbb{R}^{Nn})} \le C \tag{5.51}$$

for some constant $C$ independent of $m$. Hence, by the Poincaré inequality,

$$\|\mathbf{u}_m\|_{L^p(\Omega_m,\mathbb{R}^N)} \le C\,, \tag{5.52}$$

with $C$ indepedent of $m$, inasmuch as $\mathbf{u}_m \in W^{1,p}_0(\Omega_m,\mathbb{R}^N)$, and $\Omega_m$ satisfies (5.24). Owing to (5.49) and (5.52), a Sobolev type inequality yields

$$\|\mathbf{u}_m\|_{L^\infty(\Omega',\mathbb{R}^N)} \le C \tag{5.53}$$

for some constant $C$ independent of $m$. In this connection, notice that, without loss of generality, $\partial\Omega'$ can be assumed to be smooth. Inequality (5.48) follows from (5.50) and (5.53).
Thus, on taking, if necessary, a further subsequence,

$$\mathbf{u}_m \to \mathbf{v} \quad \text{and} \quad \nabla \mathbf{u}_m \to \nabla \mathbf{v} \quad \text{in } \Omega, \tag{5.54}$$

for some function $\mathbf{v} \in C^1(\Omega, \mathbb{R}^N)$. In particular,

$$|\nabla \mathbf{u}_m|^{p-2} \nabla \mathbf{u}_m \to |\nabla \mathbf{v}|^{p-2} \nabla \mathbf{v} \quad \text{in } \Omega. \tag{5.55}$$

By (5.55) and (5.47),

$$|\nabla \mathbf{v}|^{p-2} \nabla \mathbf{v} = \mathbf{U} \in W^{1,2}(\Omega, \mathbb{R}^{Nn}). \tag{5.56}$$

Now, let us make use of a test function $\boldsymbol{\varphi} \in C_0^\infty(\Omega, \mathbb{R}^N)$ (extended by 0 outside $\Omega$) in the definition of weak solution to problem (5.45), and pass to the limit as $m \to \infty$ in the resulting equation, namely in the equation

$$\int_{\Omega_m} |\nabla \mathbf{u}_m|^{p-2} \nabla \mathbf{u}_m \cdot \nabla \boldsymbol{\varphi} \, dx = \int_{\Omega_m} \mathbf{f} \cdot \boldsymbol{\varphi} \, dx. \tag{5.57}$$

So doing, we infer from (5.47) and (5.56) that

$$\int_{\Omega} |\nabla \mathbf{v}|^{p-2} \nabla \mathbf{v} \cdot \nabla \boldsymbol{\varphi} \, dx = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} \, dx. \tag{5.58}$$

Inequality (5.51) ensures that $\|\nabla \mathbf{u}_m\|_{L^p(\Omega, \mathbb{R}^{Nn})} \leq C$ for some constant $C$ independent of $m$. The same inequality thus holds with $\mathbf{u}_m$ replaced with $\mathbf{v}$, whence, in particular, $|\nabla \mathbf{v}|^{p-2} \nabla \mathbf{v} \in L^{p'}(\Omega, \mathbb{R}^{Nn})$. Therefore, by a density argument, equation (5.58) also holds for every $\boldsymbol{\varphi} \in W_0^{1,p}(\Omega, \mathbb{R}^N)$. This means that $\mathbf{v}$ is a weak solution to the Dirichlet problem (1.1)+(1.2). Its uniqueness ensures that $\mathbf{v} = \mathbf{u}$. Furthermore, owing to (5.46) and (5.47),

$$\||\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u}\|_{W^{1,2}(\Omega, \mathbb{R}^{Nn})} \leq C \|\mathbf{f}\|_{L^2(\Omega, \mathbb{R}^N)} \tag{5.59}$$

for some constant $C = C(n, N, p, L_\Omega, d_\Omega, \mathcal{K}_\Omega)$.

*Step 3.* We conclude by removing the remaining additional assumption (5.26). Let $\mathbf{f} \in L^2(\Omega, \mathbb{R}^N)$. By the definition of approximable solution, there exists a sequence $\{\mathbf{f}_k\} \subset C_0^\infty(\Omega, \mathbb{R}^N)$, with $\mathbf{f}_k \to \mathbf{f}$ in $L^2(\Omega, \mathbb{R}^N)$, such that the sequence of weak solutions $\{\mathbf{u}_k\} \subset W_0^{1,p}(\Omega, \mathbb{R}^N)$ to problems (5.2), satisfies $\mathbf{u}_k \to \mathbf{u}$ and $\nabla \mathbf{u}_k \to \nabla \mathbf{u}$ a.e. in $\Omega$. By inequality (5.59) of the previous step, applied with $\mathbf{u}$ and $\mathbf{f}$ replaced by $\mathbf{u}_k$ and $\mathbf{f}_k$, we have that $|\nabla \mathbf{u}_k|^{p-2} \nabla \mathbf{u}_k \in W^{1,2}(\Omega, \mathbb{R}^{Nn})$, and there exist constants $C_1$ and $C_2$, depending on $N$, $p$ and $\Omega$, such that

$$\||\nabla \mathbf{u}_k|^{p-2} \nabla \mathbf{u}_k\|_{W^{1,2}(\Omega, \mathbb{R}^{Nn})} \leq C_1 \|\mathbf{f}_k\|_{L^2(\Omega, \mathbb{R}^N)} \leq C_2 \|\mathbf{f}\|_{L^2(\Omega, \mathbb{R}^N)}. \tag{5.60}$$

Hence, the sequence $\{|\nabla \mathbf{u}_k|^{p-2} \nabla \mathbf{u}_k\}$ is uniformly bounded in $W^{1,2}(\Omega, \mathbb{R}^{Nn})$, and there exists a subsequence, still indexed by $k$, and a function $\mathbf{U} \in W^{1,2}(\Omega, \mathbb{R}^{Nn})$ fulfilling

$$|\nabla \mathbf{u}_k|^{p-2} \nabla \mathbf{u}_k \to \mathbf{U} \quad \text{in } L^2(\Omega, \mathbb{R}^{Nn}), \quad |\nabla \mathbf{u}_k|^{p-2} \nabla \mathbf{u}_k \rightharpoonup \mathbf{U} \quad \text{in } W^{1,2}(\Omega, \mathbb{R}^{Nn}). \tag{5.61}$$

Since $\nabla \mathbf{u}_k \to \nabla \mathbf{u}$ a.e. in $\Omega$, we thus infer that $|\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u} = \mathbf{U} \in W^{1,2}(\Omega, \mathbb{R}^{Nn})$, and the second inequality in (2.4) follows via (5.60) and (5.61). The first inequality in (2.4) holds trivially. The statement concerning the solution to the Dirichlet problem (1.1)+(1.2) is thus fully proved.

The outline of the proof for the solution to the Neumann problem (1.1)+(1.3) is the same as that for the Dirichlet problem. We point out hereafter just the variants required in the various steps.

*Step 1.* The solution $\mathbf{u}_\varepsilon$ to the Dirichlet problem (5.28) must be replaced, of course, by a solution $\mathbf{u}_\varepsilon$ to the approximating Neumann problem

$$\begin{cases} -\mathbf{div}(b_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)^{p-2}\nabla \mathbf{u}_\varepsilon) = \mathbf{f} & \text{in } \Omega \\ \dfrac{\partial \mathbf{u}_\varepsilon}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{5.62}$$

Such a solution is only unique up to additive constant vectors. Estimate (5.29) is a consequence of [CiMa1, Theorem 2.4]. As shown in the proof of that theorem, $\mathbf{u}_\varepsilon \in W^{1,\infty}(\Omega,\mathbb{R}^N)\cap W^{2,2}(\Omega,\mathbb{R}^N)$, and there exists a sequence $\{\mathbf{u}_k\} \subset C^\infty(\Omega,\mathbb{R}^N)\cap C^2(\overline{\Omega},\mathbb{R}^N)$ enjoying the following properties:

$$\frac{\partial \mathbf{u}_k}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \partial\Omega\,,$$

$$\mathbf{u}_k \to \mathbf{u}_\varepsilon \quad \text{in } W^{2,2}(\Omega,\mathbb{R}^N), \quad \nabla \mathbf{u}_k \to \nabla \mathbf{u}_\varepsilon \quad \text{a.e. in } \Omega, \tag{5.63}$$

$$\|\nabla \mathbf{u}_k\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \leq C\|\nabla \mathbf{u}_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \tag{5.64}$$

for some constant $C$ independent of $k$, $|\nabla|\nabla \mathbf{u}_k|| \leq |\nabla^2 \mathbf{u}_k|$ a.e. in $\Omega$, and

$$-\mathbf{div}(b_\varepsilon(|\nabla \mathbf{u_k}|)^{p-2}\nabla \mathbf{u_k}) \to \mathbf{f} \quad \text{in } L^2(\Omega,\mathbb{R}^N), \tag{5.65}$$

as $k \to \infty$. Furthermore, equation (5.43) now holds owing to (5.8), with a constant $C = C(\Omega, N, p)$ depending on $\Omega$ via an upper bound for $d_\Omega$ and $L_\Omega$. With these variants in place, inequality (5.44) follows via the same argument as in the case of the Dirichlet problem.

*Step 2.* The Dirichlet problem (5.45) has to be replaced with the Neumann problem

$$\begin{cases} -\mathbf{div}(|\nabla \mathbf{u}_m|^{p-2}\nabla \mathbf{u}_m) = \mathbf{f} & \text{in } \Omega_m \\ \dfrac{\partial \mathbf{u}_m}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega_m\,, \end{cases} \tag{5.66}$$

and the corresponding sequence of solutions $\{\mathbf{u}_m\}$ has to be normalized by a suitable sequence of additive constant vectors $\boldsymbol{\xi}_m \in \mathbb{R}^N$ in such a way that inequality (5.52) still holds. For instance, one can choose $\boldsymbol{\xi}_m = -\fint_\Omega \mathbf{u}_m\, dx$. After this normalization, the sequence $\{\mathbf{u}_m\}$ admits a subsequence, still denoted by $\{\mathbf{u}_m\}$, converging to a function $\mathbf{v}$ with the same properties as in the case of the Dirichlet problem.

Next, any test function $\boldsymbol{\varphi} \in W^{1,\infty}(\Omega,\mathbb{R}^N)$ can be exended to a function in $W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^N)$, still denoted by $\boldsymbol{\varphi}$. The use of this test function in the weak formulation of problem (5.66) yields

$$\int_{\Omega_m} |\nabla \mathbf{u}_m|^{p-2}\nabla \mathbf{u}_m \cdot \nabla \boldsymbol{\varphi}\, dx = \int_{\Omega_m} \mathbf{f}\cdot \boldsymbol{\varphi}\, dx\,. \tag{5.67}$$

Passage to the limit as $m \to \infty$ on the left-hand side of equation (5.67), to obtain

$$\int_\Omega |\nabla \mathbf{v}|^{p-2}\nabla \mathbf{v} \cdot \nabla \boldsymbol{\varphi}\, dx = \int_\Omega \mathbf{f}\cdot \boldsymbol{\varphi}\, dx\,, \tag{5.68}$$

can be justified as follows. The left-hand side of equation (5.67) can be split as

$$\int_{\Omega_m} |\nabla \mathbf{u}_m|^{p-2}\nabla \mathbf{u}_m \cdot \nabla \boldsymbol{\varphi}\, dx = \int_\Omega |\nabla \mathbf{u}_m|^{p-2}\nabla \mathbf{u}_m \cdot \nabla \boldsymbol{\varphi}\, dx + \int_{\Omega_m\setminus\Omega} |\nabla \mathbf{u}_m|^{p-2}\nabla \mathbf{u}_m \cdot \nabla \boldsymbol{\varphi}\, dx\,. \tag{5.69}$$

The first integral on the right-hand side of (5.69) converges to the left-hand side of (5.68) as $m \to \infty$, owing to (5.47) and (5.56). The second integral tends to 0, by (5.46) and the fact that $|\Omega_m \setminus \Omega| \to 0$.
Since $\Omega$ is a bounded Lipschitz domain, the density of $W^{1,\infty}(\Omega, \mathbb{R}^N)$ in $W^{1,p}(\Omega, \mathbb{R}^N)$ entails that equation (5.68) continues to hold for any test function $\boldsymbol{\varphi}$ in the latter space, and hence $\mathbf{v}$ is the (unique, up to additive constant vectors) weak solution to the Neumann problem (1.1)+(1.3).

*Step 3.* This step is completely analogous, provided that the sequences $\{\mathbf{f}_k\}$ and $\{\mathbf{u}_k\}$ are taken as in the definition of approximable solution $\mathbf{u}$ to the Neumann problem (1.1)+(1.3) . □

**Proof of Corollary 2.3**. Lemmas 3.5 and 3.7 ensure that

$$\mathcal{K}_\Omega(r) \leq C \sup_{x \in \partial\Omega} \|\mathcal{B}\|_{X(\partial\Omega \cap B_r(x))} \qquad \text{for } r \in (0, r_0), \tag{5.70}$$

for suitable constants $r_0$ and $C$ depending on $n$, $L_\Omega$ and $d_\Omega$. The conclusion then follows from Theorem 2.1, via inequality (5.70). □

**Proof of Theorem 2.6**. The proof is analogous to that of Theorem 2.1. The only difference is that, in Step 2, one has to choose a sequence $\{\Omega_m\}$ of bounded convex open sets, with $\partial\Omega_m \in C^\infty$, approximating $\Omega$ from outside with respect to the Hausdorff distance. Conditions (5.24) are authomatically fulfilled in this case. On the other hand, condition (5.25) is irrelevant in the present situation, thanks to the fact that the constant $C$ in inequality (3.7) is independent of the function $\mathcal{K}_\Omega$ in the case of convex domains $\Omega$. □

## Appendix.

Here, we present an outline of the proof of Lemma 5.2.
Let $\{\sigma_m\}$ be a sequence of radially symmetric mollifiers in $\mathbb{R}^{n-1}$, namely, $\sigma_m \in C_0^\infty(\mathbb{R}^{n-1})$, $\operatorname{supp} \sigma_m \subset B_{1/m}^{n-1}(0)$, $\sigma_m \geq 0$ and $\int_{\mathbb{R}^{n-1}} \sigma_m \, dx' = 1$ for $m \in \mathbb{N}$. Here, the index $n-1$ attached to the notation of a ball denotes a ball in $\mathbb{R}^{n-1}$. Given $g \in L^1_{\rm loc}(\mathbb{R}^{n-1}, \mathbb{R}^d)$, with $d \in \mathbb{N}$, we denote by $M_m(g)$ the convolution of $g$ with $\sigma_m$, namely the function $M_m(g) : \mathbb{R}^n \to \mathbb{R}^d$ defined as

$$M_m(g)(x') = \int_{\mathbb{R}^{n-1}} g(y')\sigma_m(x' - y') \, dy' \quad \text{for } x' \in \mathbb{R}^{n-1}.$$

Moreover, given a function $h \in L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^d)$, define $\widetilde{M}_m(h) : \mathbb{R}^n \to \mathbb{R}^d$ as the convolution of $h$ with $\sigma_m$ with respect to the first $n-1$ variables, i.e.

$$\widetilde{M}_m(h)(x', x_n) = \int_{\mathbb{R}^{n-1}} h(y', x_n)\sigma_m(x' - y') \, dy' \quad \text{for } (x', x_n) \in \mathbb{R}^n.$$

Note that, if $\operatorname{supp} h \subset\subset B_r$ for some ball $B_r \subset \mathbb{R}^n$, then $\operatorname{supp}\widetilde{M}_m(h) \subset\subset B_r$ as well, if $m$ is sufficiently large.
Assume, for the time being, that $\Omega \subset \mathbb{R}^n$ globally agrees with the subgraph of a Lipschitz continuous function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, let $\varrho$ be a nonnegative function in $L^1_{\rm loc}(\partial\Omega)$, and let $r_0 \in (0, 1)$. Define, for $x \in \partial\Omega$ and $r \in (0, r_0)$,

$$R_{B_r(x)}(\varrho, \Omega) = \sup_{v \in C_0^{0,1}(B_r(x))} \frac{\int_{\partial\Omega \cap B_r(x)} v^2 \, \varrho d\mathcal{H}^{n-1}}{\int_{B_r(x)} |\nabla v|^2 \, dx}, \tag{5.71}$$

and

$$Q_{B_r(x)}(\varrho,\Omega)=\sup_{E\subset\partial\Omega\cap B_r(x)}\frac{\int_E \varrho\, d\mathcal{H}^{n-1}}{\mathrm{cap}_{B_1(x)}(E)}\,. \tag{5.72}$$

Then, by Lemma 3.5 and Remark 3.6,

$$R_{B_r(x)}(\varrho,\Omega)\approx Q_{B_r(x)}(\varrho,\Omega)\,, \tag{5.73}$$

up to positive multilicative constants depending only on $n$, $r_0$ and on an upper bound for the Lipschitz constant $L$ of $\phi$.
Assume, in addition, that $\phi\in W^{2,1}(\mathbb{R}^{n-1})$, and let $\mathcal{B}_\phi$ denote the (weak) second fundamental form of the graph of $\phi$. Then,

$$\mathcal{B}_\phi=\frac{\nabla^2\phi}{\sqrt{1+|\nabla\phi|^2}}\,. \tag{5.74}$$

Hence,

$$\frac{|\nabla^2\phi(x')|}{\sqrt{1+L^2}}\leq|\mathcal{B}_\phi(x')|\leq|\nabla^2\phi(x')|\quad\text{for a.e. } x'\in\mathbb{R}^{n-1}, \tag{5.75}$$

and

$$|\mathcal{B}_{M_m(\phi)}(x')|\leq|\nabla^2 M_m(\phi)(x')|=|M_m(\nabla^2\phi)(x')|\leq M_m(|\nabla^2\phi|)(x')\quad\text{for a.e. } x'\in\mathbb{R}^{n-1}. \tag{5.76}$$

Given $v\in C^{0,1}_0(B_r(0))$ for some $r\in(0,r_0)$, define the function $w_m:\mathbb{R}^n\to\mathbb{R}$ as

$$w_m=\sqrt{\widetilde{M}_m(v^2)}\,.$$

As noticed above,

$$\operatorname{supp} w_m\subset\subset B_r(0) \tag{5.77}$$

if $m$ is sufficiently large, owing to our choice of $v$. Moreover, on denoting by $\nabla_{x'}$ the gradient operator with respect to the sole variables $x'\in\mathbb{R}^{n-1}$, one has that

$$\big|\nabla_{x'}\big(\widetilde{M}_m(v^2)\big)\big|=\big|\widetilde{M}_m\big(\nabla_{x'}(v^2)\big)\big|=2\big|(\widetilde{M}_m\big(v\nabla_{x'}v)\big)\big|\leq 2\sqrt{\widetilde{M}_m(v^2)}\,\sqrt{\widetilde{M}_m\big(|\nabla_{x'}v|^2\big)}\,. \tag{5.78}$$

Also,

$$\big|\big(\widetilde{M}_m(v^2)\big)_{x_n}\big|=\big|\widetilde{M}_m\big((v^2)_{x_n}\big)\big|=2\big|(\widetilde{M}_m(vv_{x_n}))\big|\leq 2\sqrt{\widetilde{M}_m(v^2)}\,\sqrt{\widetilde{M}_m\big((v_{x_n})^2\big)}. \tag{5.79}$$

Thus, $w_m$ is Lipschitz continuous, and
(5.80)

$$|\nabla w_m|\leq C\frac{\big|\nabla_{x'}\big(\widetilde{M}_m(v^2)\big)\big|+\big|\big(\widetilde{M}_m(v^2)\big)_{x_n}\big|}{\sqrt{\widetilde{M}_m(v^2)}}\leq C'\sqrt{\widetilde{M}_m\big(|\nabla_{x'}v|^2\big)+\widetilde{M}_m\big((v_{x_n})^2\big)}\quad\text{a.e. in } \mathbb{R}^{n-1}$$

for some absolute constants $C$ and $C'$. Hence, via an application of Fubini's theorem, we deduce that

$$\int_{\mathbb{R}^n}|\nabla w_m|^2\,dx\leq C'\int_{\mathbb{R}^n}\widetilde{M}_m\big(|\nabla_{x'}v|^2\big)+\widetilde{M}_m\big((v_{x_n})^2\big)\,dx \tag{5.81}$$

$$= C' \int_{\mathbb{R}^n} |\nabla_{x'} v|^2 + (v_{x_n})^2 \, dx = C' \int_{\mathbb{R}^n} |\nabla v|^2 \, dx \,.$$

Owing to equations (5.75) and (5.76),

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} v^2(x',0) \, |\mathcal{B}_{M_m(\phi)}(x')| \, dx' &\le \int_{\mathbb{R}^{n-1}} v^2(x',0) \, M_m\big(|\nabla^2 \phi(x')|\big) \, dx' \\
&= \int_{\mathbb{R}^{n-1}} \widetilde{M}_m(v^2(x',0)) \, |\nabla^2 \phi(x')| \, dx' \\
&\le \sqrt{1+L^2} \int_{\mathbb{R}^{n-1}} \widetilde{M}_m(v^2(x',0)) |\mathcal{B}_\phi(x')| \, dx \\
&= \sqrt{1+L^2} \int_{\mathbb{R}^{n-1}} w_m(x',0)^2 \, |\mathcal{B}_\phi(x')| \, dx' \,.
\end{aligned} \tag{5.82}$$

By definition (5.71) and equation (5.77),

$$\frac{\int_{\mathbb{R}^{n-1}} w_m(x',0)^2 \, |\mathcal{B}_\phi(x')| \, dx'}{\int_{\mathbb{R}^n} |\nabla w_m|^2 \, dx} \le R_{B_r(0)}(|\mathcal{B}_\phi|, \mathbb{R}^n_-) \,, \tag{5.83}$$

provided that $m$ is sufficiently large, where we have set $\mathbb{R}^n_- = \{(x', x_n) : x_n < 0\}$. Inequalities (5.81)–(5.83) imply that

$$R_{B_r(0)}(|\mathcal{B}_{M_m(\phi)}|, \mathbb{R}^n_-) = \sup_{v \in C^{0,1}_0(B_r(0))} \frac{\int_{\mathbb{R}^{n-1}} v^2(x',0) \, |\mathcal{B}_{M_m(\phi)}(x')| \, dx'}{\int_{\mathbb{R}^n} |\nabla v|^2 \, dx} \le C R_{B_r(0)}(|\mathcal{B}_\phi|, \mathbb{R}^n_-) \,, \tag{5.84}$$

for some constant $C = C(n, L, r_0)$.
Now, denote by $G_\phi$ the graph of $\phi$ and by $S_\phi$ the subgraph of $\phi$, and define $G_{M_m(\phi)}$ and $S_{M_m(\phi)}$ analogously. Set $x^0 = (0, \phi(0)) \in \mathbb{R}^n$ and $x^0_m = (0, M_m(\phi)(0)) \in \mathbb{R}^n$. Given $v \in C^{0,1}_0(B_r(x^0_m))$, we have that

$$\begin{aligned}
\int_{G_{M_m(\phi)}} v^2 |\mathcal{B}_{M_m(\phi)}| d\mathcal{H}^{n-1} &= \int_{\mathbb{R}^{n-1}} v^2(x', M_m(\phi)(x')) \, |\mathcal{B}_{M_m(\phi)}(x')| \sqrt{1 + |\nabla M_m(\phi)|^2} \, dx' \\
&\le \sqrt{1+L^2} \int_{\mathbb{R}^{n-1}} v^2(x', M_m(\phi)(x')) \, |\mathcal{B}_{M_m(\phi)}(x')| \, dx' \,.
\end{aligned} \tag{5.85}$$

Define the function $w_m : \mathbb{R}^n \to \mathbb{R}$ as

$$w_m(x', x_n) = v(x', x_n + M_m(\phi)(x')) \quad \text{for } (x', x_n) \in \mathbb{R}^n. \tag{5.86}$$

Then, $\operatorname{supp} w_m \subset\subset B_r(0)$ if $m$ is sufficiently large. Furthermore,

$$\int_{\mathbb{R}^{n-1}} v^2(x', M_m(\phi)(x')) \, |\mathcal{B}_{M_m(\phi)}(x')| \, dx' = \int_{\mathbb{R}^{n-1}} w^2_m(x',0) \, |\mathcal{B}_{M_m(\phi)}(x')| \, dx' \,. \tag{5.87}$$

By (5.84) (with $v$ replaced by $w_m$) and (5.81),

$$\int_{\mathbb{R}^{n-1}} w^2_m(x',0) \, |\mathcal{B}_{M_m(\phi)}(x')| \, dx' \le C R_{B_r(0)}(|\mathcal{B}_\phi|, \mathbb{R}^n_-) \int_{\mathbb{R}^n} |\nabla w_m|^2 \, dx \,. \tag{5.88}$$

On the other hand, since

$$|\nabla M_m(\phi)| = |M_m(\nabla \phi)| \le L \quad \text{a.e. in } \mathbb{R}^{n-1} \,,$$

for $m \in \mathbb{N}$, there exists a constant $C = C(n, L)$ such that

$$|\nabla w_m(x', x_n)| \leq C |\nabla v(x', x_n + M_m(\phi)(x'))| \quad \text{for a.e. } (x', x_n) \in \mathbb{R}^n. \tag{5.89}$$

Thereby,

$$\int_{\mathbb{R}^n} |\nabla w_m|^2\, dx \leq C^2 \int_{\mathbb{R}^n} |\nabla v(x', x_n + M_m(\phi)(x'))|^2\, dx = C^2 \int_{\mathbb{R}^n} |\nabla v|^2\, dx\,. \tag{5.90}$$

Combining equations (5.85), (5.87), (5.88) and (5.90) yields

$$\int_{G_{M_m(\phi)}} v^2 |\mathcal{B}_{M_m(\phi)}| d\mathcal{H}^{n-1} \leq C R_{B_r(0)}(|\mathcal{B}_\phi|, \mathbb{R}^n_-) \int_{\mathbb{R}^n} |\nabla v|^2\, dx \tag{5.91}$$

for some constant $C = C(n, L, r_0)$ and every function $v \in C^{0,1}_0(B_r(x^0_m))$. Therefore,

$$R_{B_r(x^0_m)}(|\mathcal{B}_{M_m(\phi)}|, S_{M_m(\phi)}) \leq C R_{B_r(0)}(|\mathcal{B}_\phi|, \mathbb{R}^n_-) \tag{5.92}$$

for some constant $C = C(n, L, r_0)$. On the other hand, an analogous change of variable as in (5.86), with $M_m(\phi)$ replaced by $-\phi$, tells us that

$$R_{B_r(0)}(|\mathcal{B}_\phi|, \mathbb{R}^N_-) \leq C R_{B_r(x^0)}(|\mathcal{B}_\phi|, S_\phi) \tag{5.93}$$

for some constant $C = C(n, L, r_0)$. Inequalities (5.92) and (5.93) yield

$$R_{B_r(x^0_m)}(|\mathcal{B}_{M_m(\phi)}|, S_{M_m(\phi)}) \leq C R_{B_r(x^0)}(|\mathcal{B}_\phi|, S_\phi) \tag{5.94}$$

for some constant $C = C(n, L, r_0)$, for every sufficiently large $m \in \mathbb{N}$. Of course, inequality (5.94) continues to hold if $x^0 = (x', \phi(x'))$ and $x^0_m = (x', M_m(\phi)(x'))$ for any $x' \in \mathbb{R}^{n-1}$.
Assume now that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$. Let $\mathcal{U} = B^{n-1}_\rho \times (-\delta, \delta)$ for some $\rho > 0$ and $\delta > 0$, and let $\phi \in W^{2,1}(\mathbb{R}^{n-1})$ is a Lipschitz continuous function, with $\operatorname{supp} \phi \subset B^{n-1}_\rho$, such that (on translating and rotating $\Omega$, if necessary) $\Omega \cap \mathcal{U} = \{(x', x_n) : x' \in B^{n-1}_\rho, -\delta < x_n < \phi(x')\}$ and $\partial\Omega \cap \mathcal{U} = \{(x', x_n) : x' \in B^{n-1}_\rho, x_n = \phi(x')\}$. Let $M_m(\phi)$ be defined as above for $m \in \mathbb{N}$. Denote by $\Omega_m$ the open subset which agrees with $\Omega$ outside $\mathcal{U}$, and satisfying $\Omega_m \cap \mathcal{U} = \{(x', x_n) : x' \in B^{n-1}_\rho, -\delta < x_n < M_m(\phi)(x')\}$ and $\partial\Omega_m \cap \mathcal{U} = \{(x', x_n) : x' \in B^{n-1}_\rho, x_n = M_m(\phi)(x')\}$. From inequality (5.94) one can deduce that

$$R_{B_r(x^0_m)}(|\mathcal{B}_{M_m(\phi)}|, \Omega_m) \leq C R_{B_r(x^0)}(|\mathcal{B}_\phi|, \Omega), \tag{5.95}$$

for every ball $B_r(x^0_m) \subset \mathcal{U}$. Since $M_m(\phi) \to \phi$ as $m \to \infty$, inequality (5.95) can be used, via (5.73), to prove inequality (5.25) when $\{\Omega_m\}$ is a sequence of sets obtained from a local smoothening of $\partial\Omega$.
The construction of a sequence $\{\Omega_m\}$ of sets as in the statement can be accomplished on combining this local construction with a partition of unit argument. For brevity, we limit ourselves to sketching this argument. To begin with, note that the sequence of functions $\{M_m(\phi)\}$ above can be modified, if necessary, by an additive constant in such a way the modified sequence satisfies $\phi \leq M_m(\phi)$ in $B^{n-1}_r$, still converges to $\phi$ and fulfills inequality (5.95). Next, since $\Omega$ is a bounded Lipschitz domain with $\partial\Omega \in W^{2,1}$, there exists a finite family of open sets $\{\mathcal{U}_i\}$, and a corresponding family $\{\phi_i\}$ of Lipschitz continuous functions with $\phi_i \in W^{2,1}(\mathcal{U}_i)$, such that

$$\partial\Omega = \bigcup_i \mathcal{V}_i\,, \tag{5.96}$$

where $\mathcal{V}_i$ are open subsets in $\partial\Omega$ (with respect to the topology induced by $\mathbb{R}^n$), such that

$$\mathcal{V}_i = \Phi_i(\mathcal{W}_i)\,, \tag{5.97}$$

$\mathcal{W}_i$ is an open subset of $\mathbb{R}^{n-1}$, and

$$\Phi_i : \mathcal{W}_i \to \mathcal{V}_i \tag{5.98}$$

is a homeomerphism obtained by composing the function

$$\mathcal{U}_i \ni x' \mapsto (x', \phi_i(x')) \in \mathbb{R}^n$$

with an isometry. Let $M_m(\phi_i)$ be the function defined as above, with $\phi$ replaced by $\phi_i$, and let $\Phi_i^m : \mathcal{W}_i \to \mathbb{R}^n$ be the function obtained on replacing $\phi_i$ with $M_m(\phi_i)$ in definition (5.98) of $\Phi_i$. Let $\{\xi_i\}$ be a partition of unity associated with the covering $\{\mathcal{V}_i\}$ of $\partial\Omega$. Define, for $m \in \mathbb{R}$, the function $\Psi_m : \partial\Omega \to \mathbb{R}^n$ as

$$\Psi_m = \sum_i \big(\Phi_i^m \circ \Phi_i^{-1}\big)\xi_i\,.$$

Then $\Omega_m$ can be defined as the open set in $\mathbb{R}^n$ such that $\partial\Omega_m = \Psi_m(\partial\Omega)$. The fact that the sequence $\{\Omega_m\}$ fulfills property (5.25) can be deduced as a consequence of the local argument described above. The remaining properties described in the statement are easier to verify.

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