On the Brezis and Mironescu conjecture concerning a Gagliardo-Nirenberg inequality for fractional Sobolev norms

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Abstract

We prove the Gagliardo-Nirenberg type inequality

$$\|u\|_{W^{\theta s,p/\theta}} \leq c(n) \left(\frac{p}{p-1}\right)^{\theta} \left(\frac{1-s}{1-\theta}\right)^{\theta/p} \|u\|^{\theta}_{W^{s,p}} \, \|u\|^{1-\theta}_{L^\infty},$$

where $0 < \theta < 1$, $0 < s < 1$, $1 < p < \infty$, and $\|u\|_{W^{s,p}}$ is the seminorm in the fractional Sobolev space $W^{s,p}(\mathbf{R}^n)$. The dependence of the constant factor in the right-hand side on each of the parameters s, θ, and p is precise in a sense.

Let $s \in (0,1)$ and let $1 < p \leq \infty$. We introduce the space $W^{s,p}(\mathbf{R}^n)$ of functions in $\mathbf{R}^n$ with the finite seminorm

$$\|u\|_{W^{s,p}} = \left(\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p}.$$

Recently Bourgain, Brezis and Mironescu [1] found the relation

$$\lim_{s \uparrow 1} (1-s)^{1/p} \|u\|_{W^{s,p}} = c \, \|u\|_{W^{1,p}}, \tag{1}$$

which subsequently motivated Brezis and Mironescu to conjecture the Gagliardo-Nirenberg type inequality

$$\|u\|_{W^{s/2,2p}} \leq c(n,p)(1-s)^{1/2p} \|u\|^{1/2}_{W^{s,p}} \|u\|^{1/2}_{L^\infty} \tag{2}$$

(see [2], Remark 5). In [2] one can also read: "It would be of interest to establish

$$\|u\|_{W^{\theta s,p/\theta}} \leq c \, \|u\|^{\theta}_{W^{s,p}} \|u\|^{1-\theta}_{L^\infty}, \quad 0 < \theta < 1, \tag{3}$$

with control of the constant c, in particular when $s \uparrow 1$".

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In the present paper we prove that (3) holds with $c = c(n,p,\theta)(1-s)^{\theta/p}$, which, obviously, contains inequality (2) predicted by Brezis and Mironescu. Our proof is straightforward and rather elementary. In concluding Remarks 1 and 2 we show that the dependence of c on each of the parameters s, θ, and p is sharp in a certain sense.

Theorem. *For all $u \in W^{s,p} \cap L^\infty$ there holds the inequality*

$$\|u\|_{W^{\theta s, p/\theta}} \le c(n)\Big(\frac{p}{p-1}\Big)^\theta \Big(\frac{1-s}{1-\theta}\Big)^{\theta/p} \|u\|^\theta_{W^{s,p}} \|u\|^{1-\theta}_{L^\infty}, \tag{4}$$

where $0 < s < 1$, $1 < p < \infty$, and $0 < \theta < 1$.

Proof. Clearly,

$$\|u\|_{W^{\theta s, p/\theta}} \le \max\{2^{\theta/p}, 2^{1-\theta}\} \|u\|^{1-\theta}_{L^\infty}\ \|u\|^\theta_{W^{s,p}}. \tag{5}$$

Hence it suffices to prove (4) only for $s \ge 1/2$.

Let $B_r(x) = \{\xi \in \mathbf{R}^n : |\xi - x| < r\}$ and $B_r(0) = B_r$. We introduce the mean value $\overline{u}_{x,y}$ of u over the ball $\mathcal{B}_{x,y} := B_{|x-y|/2}((x+y)/2)$. Since

$$|u(x) - u(y)|^{p/\theta} \le 2^{-1+p/\theta}(|u(x) - \overline{u}_{x,y}|^{p/\theta} + |\overline{u}_{x,y} - u(y)|^{p/\theta}),$$

it follows that

$$\|u\|_{W^{\theta s, p/\theta}} \le 2\Big(\int_{\mathbf{R}^n} D(x)^{p/\theta} dx\Big)^{\theta/p}, \tag{6}$$

where

$$D(x) = \Big(\int_{\mathbf{R}^n} \frac{|u(x) - \overline{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}} dy\Big)^{\theta/p}.$$

We note that

$$\int_{|x-y|>\delta} \frac{|u(x) - \overline{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}} dy \le \frac{2^{p/\theta}|\partial B_1|}{ps} \|u\|^{p/\theta}_{L^\infty}\ \delta^{-ps}, \tag{7}$$

where $|\partial B_1|$ is the area of the unit sphere.

Let U be an arbitrary extension of u onto $\mathbf{R}^{n+1}_+ = \{(x,z) : x \in \mathbf{R}^n, z > 0\}$ such that $\nabla U \in L^1_{loc}(\overline{\mathbf{R}^{n+1}_+})$. By $\overline{U}_{x,y}(z)$ we denote the mean value of $U(\cdot, z)$ in $\mathcal{B}_{x,y}$. Using the identity

$$|x-y|^{(1-s)p} = p(1-s)\int_0^{|x-y|} z^{-1+p(1-s)} dz,$$

we find

$$\int_{|x-y|<\delta} \frac{|u(x) - \overline{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}} dy =$$

$$p^{1/\theta}(1-s)^{1/\theta} \int_{|x-y|<\delta} \Big(\int_0^{|x-y|} z^{-1+p(1-s)} |u(x) - \overline{u}_{x,y}|^p dz\Big)^{1/\theta} \frac{dy}{|x-y|^{n+ps+(1-s)p/\theta}} \le$$

$$3^{-1+p/\theta} p^{1/\theta}(1-s)^{1/\theta}(\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3), \tag{8}$$

where

$$\mathcal{J}_1 := \int_{|x-y|<\delta} \Big(\int_0^{|x-y|} z^{-1+p(1-s)} |u(x) - U(x,z)|^p dz \Big)^{1/\theta} \frac{dy}{|x-y|^{n+ps+(1-s)p/\theta}},$$

$$\mathcal{J}_2 := \int_{|x-y|<\delta} \Big(\int_0^{|x-y|} z^{-1+p(1-s)} |\overline{U}_{x,y}(z) - \overline{u}_{x,y}|^p dz \Big)^{1/\theta} \frac{dy}{|x-y|^{n+ps+(1-s)p/\theta}},$$

and

$$\mathcal{J}_3 := \int_{|x-y|<\delta} \Big(\int_0^{|x-y|} z^{-1+p(1-s)} |U(x,z) - \overline{U}_{x,y}(z)|^p dz \Big)^{1/\theta} \frac{dy}{|x-y|^{n+ps+(1-s)p/\theta}}.$$

Clearly,

$$\mathcal{J}_1 \le \int_{|x-y|<\delta} \Big(\int_0^{|x-y|} z^{-1+p(1-s)} \Big(\int_0^z \Big| \frac{\partial U(x,t)}{\partial t} \Big| dt \Big)^p dz \Big)^{1/\theta} \frac{dy}{|x-y|^{n+ps+(1-s)p/\theta}} \le$$

$$\int_{|x-y|<\delta} \Big(\int_0^{|x-y|} z^{-1-ps} \Big(\int_0^z \Big| \frac{\partial U(x,t)}{\partial t} \Big| dt \Big)^p dz \Big)^{1/\theta} \frac{dy}{|x-y|^{n-ps(1-\theta)/\theta}}.$$

By Hardy's inequality

$$\int_0^a z^{-1-sp} \Big| \int_0^z \varphi(t) dt \Big|^p dz \le s^{-p} \int_0^a z^{-1+p(1-s)} |\varphi(z)|^p dz,$$

one has

$$\mathcal{J}_1 \le s^{-p/\theta} \int_{|x-y|<\delta} \Big(\int_0^{|x-y|} z^{-1+p(1-s)} \Big| \frac{\partial U(x,z)}{\partial z} \Big|^p dz \Big)^{1/\theta} \frac{dy}{|x-y|^{n-ps(1-\theta)/\theta}} \le$$

$$\frac{\theta |\partial B_1|}{s^{p/\theta} ps(1-\theta)} \Big(\int_0^\infty z^{-1+p(1-s)} \Big| \frac{\partial U(x,z)}{\partial z} \Big|^p dz \Big)^{1/\theta} \delta^{ps(1-\theta)/\theta}. \tag{9}$$

Duplicating the same argument, we conclude that

$$\mathcal{J}_2 \le s^{-p/\theta} \int_{|x-y|<\delta} \frac{dy}{|x-y|^{n-ps(1-\theta)/\theta}} \Big(\int_0^{|x-y|} z^{-1+p(1-s)} \Big| \frac{\partial \overline{U}_{x,y}(z)}{\partial z} \Big|^p dz \Big)^{1/\theta}. \tag{10}$$

Let $\mathcal{M}$ denote the n-dimensional Hardy-Littlewood maximal operator

$$(\mathcal{M}f)(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} |f(\xi)| d\xi.$$

Using the obvious inequality

$$\Big| \frac{\partial \overline{U}_{x,y}(z)}{\partial z} \Big| \le \Big(\mathcal{M} \frac{\partial U}{\partial z} \Big)(x,z),$$

we find from (10)

$$\mathcal{J}_2 \le \frac{\theta |\partial B_1|}{s^{p/\theta} ps(1-\theta)} \Big(\int_0^\infty z^{-1+p(1-s)} \Big(\mathcal{M} \frac{\partial U}{\partial z} \Big)^p dz \Big)^{1/\theta} \delta^{ps(1-\theta)/\theta}. \tag{11}$$

In order to estimate $\mathcal{J}_3$ we use the Sobolev type integral representation in the form given in [3], Ch. 10, Sect. 3

$$U(x,z)-\overline{U}_{x,y}(z)=\sum_{k=1}^{n}\int_{\mathcal{B}_{x,y}}\frac{b_k(\xi,x)}{|x-\xi|^{n-1}}\frac{\partial U(\xi,z)}{\partial\xi_k}d\xi, \tag{12}$$

where $b_k(\xi,x)$ are continuous functions for $x\neq\xi$ admitting the estimate

$$|b_k(\xi,x)|\leq\frac{|x-y|^n}{n|\mathcal{B}_{x,y}|}.$$

Clearly, (12) implies the estimate

$$|U(x,z)-\overline{U}_{x,y}(z)|\leq\frac{2^n n^{1/2}}{|\partial B_1|}\int_{B_r(x)}\frac{|\nabla_\xi U(\xi,z)|}{|x-\xi|^{n-1}}d\xi,$$

where $r=|x-y|$. Integrating by parts we find

$$\int_{B_r(x)}\frac{|\nabla_\xi U(\xi,z)|}{|x-\xi|^{n-1}}d\xi=r^{1-n}\int_{B_r(x)}|\nabla_\xi U(\xi,z)|d\xi+$$

$$(n-1)\int_0^r\frac{ds}{s^n}\int_{B_s(x)}|\nabla_\xi U(\xi,z)|d\xi\leq n|x-y|(\mathcal{M}|\nabla U|)(x,z).$$

Therefore,

$$\mathcal{J}_3\leq\Big(\frac{2^n n^{3/2}}{|\partial B_1|}\Big)^{p/\theta}\int_{|x-y|<\delta}\Big(\int_0^{|x-y|}z^{-1+p(1-s)}(\mathcal{M}|\nabla U|)^p dz\Big)^{1/\theta}\frac{dy}{|x-y|^{n-ps(1-\theta)/\theta}}\leq$$

$$\frac{(2^n n^{3/2})^{p/\theta}\theta}{|\partial B_1|^{(p-\theta)/\theta}ps(1-\theta)}\Big(\int_0^\infty z^{-1+p(1-s)}(\mathcal{M}|\nabla U|)^p dz\Big)^{1/\theta}\delta^{ps(1-\theta)/\theta}. \tag{13}$$

Here and in the sequel, for the sake of brevity, by $\mathcal{M}|\nabla U|$ we mean $(\mathcal{M}|\nabla U|)(x,z)$. Putting estimates (9), (11), and (13) into (8), we arrive at

$$\int_{|x-y|<\delta}\frac{|u(x)-\overline{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}}dy\leq c(n)\frac{(1-s)^{1/\theta}}{1-\theta}\Big(\int_0^\infty z^{-1+p(1-s)}(\mathcal{M}|\nabla U|)^p dz\Big)^{1/\theta}\delta^{ps(1-\theta)/\theta}.$$

This estimate together with (7) implies that $D(x)$ is majorized by

$$c(n)\Big(\|u\|_{L^\infty}\delta^{-\theta s}+\Big(\frac{1-s}{1-\theta}\Big)^{1/p}\Big(\int_0^\infty z^{-1+p(1-s)}(\mathcal{M}|\nabla U|)^p dz\Big)^{1/p}\delta^{s(1-\theta)}\Big).$$

Minimizing the right-hand side, we conclude that

$$D(x)\leq c(n)\Big(\frac{1-s}{1-\theta}\Big)^{\theta/p}\|u\|_{L^\infty}^{1-\theta}\Big(\int_0^\infty z^{-1+p(1-s)}(\mathcal{M}|\nabla U|)^p dz\Big)^{\theta/p}.$$

Hence and by (6)

$$\|u\|_{W^{\theta s,p/\theta}}\leq c(n)\Big(\frac{1-s}{1-\theta}\Big)^{\theta/p}\|u\|_{L^\infty}^{1-\theta}\Big(\int_{\mathbf{R}^n}\int_0^\infty z^{-1+p(1-s)}(\mathcal{M}|\nabla U|)^p dzdx\Big)^{\theta/p}.$$

Since

$$\|\mathcal{M}u\|_{L^p} \le \frac{n8^n p}{|\partial B_1|(p-1)}\|u\|_{L^p}$$

(see [4], Sect. 2.5), we have

$$\|u\|_{W^{\theta s,p/\theta}} \le c(n)\Big(\frac{p}{p-1}\Big)^{\theta}\Big(\frac{1-s}{1-\theta}\int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)}|\nabla U(x,z)|^p dxdz\Big)^{\theta/p}\|u\|_{L^\infty}^{1-\theta}. \quad (14)$$

Now we define U by the formula

$$U(x,z) := \int_{\mathbf{R}^n} \psi(h)u(x+zh)dh, \quad (15)$$

where

$$\psi(h) = |\partial B_1|n(n+1)(1-|h|)_+$$

with plus standing for the nonnegative part of a real valued function. It follows directly from (15) that

$$|\nabla U(x,z)| \le \frac{n(n+1)(n+2)}{z\,|\partial B_1|}\int_{|h|<1} |u(x+zh)-u(x)|dh.$$

Hence and by Hölder's inequality

$$\int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)}|\nabla U(x,z)|^p dxdz \le$$

$$\frac{n}{|\partial B_1|}(n+1)^p(n+2)^p\int_0^\infty z^{-1-ps}\int_{|h|<1}\int_{\mathbf{R}^n}|u(x+zh)-u(x)|^p dxdhdz. \quad (16)$$

We have

$$\int_0^\infty z^{-1-ps}\int_{|h|<1}|u(x+zh)-u(x)|^p dhdz =$$

$$\int_0^\infty z^{-1-ps-n}\int_0^z \rho^{n-1}d\rho\int_{\partial B_1}|u(x+\rho\theta)-u(x)|^p d\theta =$$

$$(ps+n)^{-1}\int_0^\infty \rho^{-ps-1}d\rho\int_{\partial B_1}|u(x+\rho\theta)-u(x)|^p d\theta.$$

Thus,

$$\int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)}|\nabla U(x,z)|^p dxdz \le \frac{n(n+1)^p(n+2)^p}{|\partial B_1|(ps+n)}\|u\|_{W^{s,p}}^p. \quad (17)$$

Combining (17) with (14) we complete the proof.

Remark 1. Let

$$\|u\|_{W^{1,p}} = \Big(\int_{\mathbf{R}^n}|\nabla u(x)|^p dx\Big)^{1/p}.$$

As a particular case of a more general inequality, Brezis and Mironescu [2] obtained (3) for $s=1$. They commented on this in the following way: "We do not know any elementary (i.e., without the Littlewood-Paley machinery) proof of (3) when $s=1$".

Obviously, the above proof of (4), complemented by the reference to formula (1), provides an elementary proof of the inequality

$$(1-\theta)^{\theta/p}\|u\|_{W^{\theta,p/\theta}} \le c(n,p)\|u\|^{\theta}_{W^{1,p}}\|u\|^{1-\theta}_{L^\infty}.$$

The factor $(1-\theta)^{\theta/p}$ controls the blow up of the norm in $W^{\theta,p/\theta}$ as $\theta \uparrow 1$.

Remark 2. Note that passing to the limit as $p \to \infty$ in both sides of (4) one obtains inequality (3) with $p = \infty$ and with a certain finite constant c. Let us consider the case $p \to 1$ when the constant factor in (4) tends to infinity. It follows from (4) that the best value of $c(n,p,\theta)$ in the inequality

$$\|u\|_{W^{\theta s,p/\theta}} \le c(n,p,\theta)(1-s)^{\theta/p}\|u\|^{\theta}_{W^{s,p}}\|u\|^{1-\theta}_{L^\infty} \tag{18}$$

admits the upper estimate

$$\limsup_{p\downarrow 1}\,(p-1)^{\theta}c(n,p,\theta) \le c(n)(1-\theta)^{-\theta}. \tag{19}$$

Now we obtain the analogous lower estimate

$$\liminf_{p\downarrow 1}\,(p-1)^{\theta}c(n,p,\theta) \ge 1. \tag{20}$$

In fact, the characteristic function χ of the ball B_1 belongs to $W^{s,p}$ and $W^{\theta s,p/\theta}$ if and only if $sp < 1$, and there holds

$$\|\chi\|_{W^{\theta s,p/\theta}} = \|\chi\|^{\theta}_{W^{s,p}}.$$

Putting $u = \chi$ into (18), where $s = p^{-1} - \varepsilon$ with an arbitrarily small $\varepsilon > 0$, we obtain

$$1 \le c(n,p,\theta)((p-1)/p)^{\theta/p},$$

which implies (20). Thus, the growth $O((p-1)^{-\theta})$ of the constant in (4) as $p \downarrow 1$ is best possible.

References

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