On the discreteness of the spectrum of the Laplacian on noncompact Riemannian manifolds

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Abstract

Necessary and sufficient conditions for the discreteness of the Laplacian on noncompact Riemannian manifolds M are established in terms of the isocapacitary function of M. The relevant capacity takes a different form according to whether M has finite or infinite volume. Conditions involving the more classical isoperimetric function of M can also be derived, but they are only sufficient in general, as we demonstrate by concrete examples.

1 Introduction

Let M be an n-dimensional connected Riemannian manifold. We denote by Δ_M the semi-definite self-adjoint Laplace operator on the Hilbert space $L^2(M)$ associated with the closed bilinear form

$$a(u,v) = \int_M \nabla u \cdot \nabla v \, d\mathcal{H}^n, \tag{1.1}$$

defined for u and v in the Sobolev space $W^{1,2}(M)$. This definition of Δ_M encompasses various special instances. For example, if the space $C_0^\infty(M)$ of smooth compactly supported functions on M is dense in $W^{1,2}(M)$, the operator Δ_M agrees with the Friedrichs extension of the classical Laplacian, regarded as an unbounded operator on $L^2(M)$ with domain $C_0^\infty(M)$. This is certainly the case when M is complete [Ch, Ro, St], and, in particular, if M is compact. A different situation occurs when M is an open subset of $\mathbb{R}^n$, or, more generally, of a Riemannian manifold; in this case, Δ_M corresponds to the so called Neumann Laplacian on M.

We are concerned with the problem of the discreteness of the spectrum of Δ_M. This property is well known when M is compact, or when M is an open subset of $\mathbb{R}^n$ with finite measure and sufficiently regular boundary. However, the spectrum of Δ_M need not be discrete in general.

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Special situations, which are not included in this standard frameworks, have been considered in the literature. For instance, conditions for the discreteness of the spectrum of the Laplacian on noncompact complete manifolds with a peculiar structure are the object of several contributions, including [Ba, Bro, Brü, DL, Es, Kl1, Kl2]. On the other hand, a characterization, involving capacities, of open subsets of $\mathbb{R}^n$ with finite measure whose Neumann Laplacian has a discrete spectrum was established in [Ma2, Ma3].

It is the aim of the present paper to provide a necessary and sufficient condition, in a spirit similar to [Ma2, Ma3], on an arbitrary Riemannian manifold M for the spectrum of Δ_M to be discrete. Our characterization involves a function associated with M, which will be called the isocapacitary function of M. Its name stems from the fact that it is the optimal function of the measure of any subset E of M which can be estimated by a suitable capacity of E. The relevant capacity takes a different form according to whether $\mathcal{H}^n(M) < \infty$ or $\mathcal{H}^n(M) = \infty$. Here, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on M, i.e. the volume measure on M induced by its Riemannian metric, and hence $\mathcal{H}^n(M)$ is the volume of M. Interestingly, the spectrum of the operator Δ_M may actually be discrete on certain manifolds with infinite volume (as already noted in [Ba]), whereas this is never the case when M is just an open subset of $\mathbb{R}^n$ equipped with the Euclidean metric and having infinite Lebesgue measure. As a corollary of our capacitary characterization, we derive a sufficient condition for the discreteness of the spectrum of the Laplacian on M involving the isoperimetric function of M, namely the optimal function in the isoperimetric inequality on M. Our conditions depend only on the asymptotic behavior at 0, and also at infinity if $\mathcal{H}^n(M) = \infty$, of the isocapacitary or isoperimetric function of M.

Isoperimetric inequalities have a transparent geometric character, and their applications to the study of eigenvalue problems on Riemannian manifolds is quite classical – see e.g. [BGM, Cha, CF, Che, Ga, Ma5, Ya]. One aspect of our discussion that we would like to emphasize is that, although quite effective when dealing with spectral properties of manifolds with a sufficiently regular geometry, the use of isoperimetric inequalities need not yield the best possible results in general. The criterion in terms of the isoperimetric function of M that will be established is sharp, in a sense, for the discreteness of the spectrum of Δ_M. However, isocapacitary inequalities are a more appropriate tool, since they enable us to provide a full characterization of Riemannian manifolds where Δ_M is discrete. Such a characterization applies to certain manifolds with complicated geometric configurations for which the approach by the isoperimetric function fails. This will be shown by an explicit example of a family of manifolds with a sequence of clustering submanifolds.

A key step in our approach is a criterion, of independent interest, for the compactness of the embedding of the Sobolev space $W^{1,2}(M)$ into $L^2(M)$ in terms of the isocapacitary function of the manifold M.

2 Manifolds with finite volume

In this section we are concerned with the case when $\mathcal{H}^n(M) < \infty$.

2.1 Discreteness of the spectrum

Given sets $E \subset G \subset M$, the capacity $C(E,G)$ of the condenser (E,G) is defined as

$$C(E,G) = \inf \Big\{ \int_M |\nabla u|^2 \, dx : u \in W^{1,2}(M), u \geq 1 \text{ in } E \text{ and } \\ u \leq 0 \text{ in } M \setminus G \text{ (up to a set of standard capacity zero)} \Big\}. \tag{2.1}$$

Here, $W^{1,2}(M)$ denotes the Sobolev space defined as

$$W^{1,2}(M) = \{ u \in L^2(M) : \ u \text{ is weakly differentiable in } M \text{ and } |\nabla u| \in L^2(M) \ \},$$

and

$$\|u\|_{W^{1,2}(M)} = \sqrt{\|u\|^2_{L^2(M)} + \|\nabla u\|^2_{L^2(M)}}$$

for $u \in W^{1,2}(M)$. We refer to [He] for a comprehensive treatment of the theory of Sobolev spaces on Riemannian manifolds.
The isocapacitary function $\nu_M : [0, \mathcal{H}^n(M)/2] \to [0,\infty]$ is given by

$$\nu_M(s) = \inf \{ C(E,G) : E \text{ and } G \text{ are measurable subsets of } M \text{ such that } \\ E \subset G \subset M \text{ and } s \leq \mathcal{H}^n(E), \, \mathcal{H}^n(G) \leq \mathcal{H}^n(M)/2 \} \qquad \text{for } s \in [0, \mathcal{H}^n(M)/2]. \tag{2.2}$$

The function ν_M is clearly non-decreasing. The isocapacitary inequality on M is a straightforward consequence of definition (2.2), and tells us that

$$\nu_M(\mathcal{H}^n(E)) \leq C(E,G) \tag{2.3}$$

for every measurable sets $E \subset G \subset M$ with $\mathcal{H}^n(G) \leq \mathcal{H}^n(M)/2$. A version of the isocapacitary function on open subsets of $\mathbb{R}^n$ was introduced in [Ma2, Ma3], and employed to provide necessary and sufficient conditions for embeddings in the Sobolev space of functions with gradient in L^2.

Our characterization of Riemannian manifolds of finite volume with a discrete spectrum reads as follows.

Theorem 2.1 *Let M be a Riemannian manifold such that $\mathcal{H}^n(M) < \infty$. Then the spectrum of Δ_M is discrete if and only if*

$$\lim_{s \to 0} \frac{s}{\nu_M(s)} = 0 \,. \tag{2.4}$$

Incidentally, let us mention that condition (2.4) turns out also to be optimal for the existence of eigenfunction estimates in $L^q(M)$, with $q \in (2,\infty)$ [CM, Theorem 2.1]. On the other hand, eigenfunctions of the Laplacian need not be in $L^\infty(M)$ under (2.4). The boundedness of eigenfunctions is only guaranteed provided that (2.4) is strengthened to

$$\int_0 \frac{ds}{\nu_M(s)} < \infty$$

– see [CM, Theorem 2.3].

Theorem 2.1 can be used to derive a sufficient condition for the discreteness of the spectrum of Δ_M in terms of another function, of genuinely geometric nature, associated with M. This is called the isoperimetric function of M and will be denoted by λ_M. The function $\lambda_M : [0, \mathcal{H}^n(M)/2] \to [0, \infty]$ is given by

$$\lambda_M(s) = \inf\{P(E) : s \leq \mathcal{H}^n(E) \leq \mathcal{H}^n(M)/2\}\,. \tag{2.5}$$

Here, $P(E)$ is the perimeter of E, which can be defined as

$$P(E) = \mathcal{H}^{n-1}(\partial^* E)\,,$$

where $\partial^* E$ stands for the essential boundary of E in the sense of geometric measure theory, and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure on M, namely the surface measure on M induced by its Riemannian metric. Recall that $\partial^* E$ agrees with the topological boundary ∂E of E when E is regular enough, e.g. an open subset of M with a smooth boundary.
The very definition of λ_M leads to the isoperimetric inequality on M, which reads

$$\lambda_M(\mathcal{H}^n(E)) \leq P(E) \tag{2.6}$$

for every measurable set $E \subset M$ with $\mathcal{H}^n(E) \leq \mathcal{H}^n(M)/2$.
The isoperimetric function of an open subset of $\mathbb{R}^n$ was introduced in [Ma1] (see also [Ma4]) in view of the characterization of Sobolev embeddings for functions with gradient in L^1. In more recent years, isoperimetric inequalities and corresponding isoperimetric functions have been intensively investigated on Riemannian manifolds as well – see e.g. [BC, CF, CGL, GP, Gr, MHH, Kle, MJ, Pi, Ri].

The functions ν_M and λ_M are related by the inequality

$$\nu_M(s) \geq \frac{1}{\int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2}} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2), \tag{2.7}$$

which follows along the same lines as in [Ma4, Proposition 4.3.4/1]. Since M is connected, by an analogous argument as in [Ma4, Lemma 3.2.4] one can show that $\lambda_M(s) > 0$ for $s > 0$. Owing to inequality (2.7), $\nu_M(s) > 0$ for $s > 0$ as well.

The following result is easily seen to follow from Theorem 2.1, via (2.7).

Corollary 2.2 *Let M be a Riemannian manifold such that $\mathcal{H}^n(M) < \infty$. Assume that*

$$\lim_{s \to 0} \frac{s}{\lambda_M(s)} = 0\,. \tag{2.8}$$

Then the spectrum of Δ_M is discrete.

Observe that both Theorem 2.1 and Corollary 2.2 recover, in particular, the classical result on the discreteness of the spectrum of the Laplacian on any compact Riemannian manifold M. Indeed, if M is compact, then

$$\lambda_M(s) \approx s^{\frac{n-1}{n}} \qquad \text{near } 0, \tag{2.9}$$

and

$$\nu_M(s) \approx \begin{cases} s^{\frac{n-2}{n}} & \text{if } n \geq 3, \\ \left(\log \frac{1}{s}\right)^{-1} & \text{if } n = 2, \end{cases} \tag{2.10}$$

near 0. Here, and in what follows, the notation

$$f \approx g \qquad \text{near } 0 \tag{2.11}$$

for functions $f, g : (0, \infty) \to [0, \infty)$ means that there exist positive constants c_1, c_2 and s_0 such that

$$c_1 g(c_1 s) \leq f(s) \leq c_2 g(c_2 s) \qquad \text{if } s \in (0, s_0). \tag{2.12}$$

Remark 2.3 Assumption (2.8) is essentially minimal in terms of λ_M for the spectrum of Δ_M to be discrete, in the sense that (2.8) is sharp in the class of all manifolds M with prescribed isoperimetric function λ_M. To be more specific, consider any non-decreasing function $\lambda : [0, \infty) \to [0, \infty)$, vanishing only at 0, and such that

$$\frac{\lambda(s)}{s^{\frac{n-1}{n}}} \approx \text{a non-decreasing function near } 0. \tag{2.13}$$

By [CM, Proposition 4.3], there exists an n-dimensional Riemannian manifold of revolution M fulfilling

$$\lambda_M(s) \approx \lambda(s) \qquad \text{near } 0. \tag{2.14}$$

Note that assumption (2.13) is required in the light of the fact that (2.9) holds for any compact manifold M, and that $\lambda_M(s)$ cannot decay more slowly to 0 as $s \to 0$ in the noncompact case. Now, if λ is such that $\limsup_{s\to 0} \frac{s}{\lambda(s)} > 0$, then

$$\limsup_{s\to 0} \frac{s}{\lambda_M(s)} > 0 \tag{2.15}$$

as well. Owing to [CM, Corollary], for the relevant manifold of revolution M condition (2.15) is equivalent to

$$\limsup_{s\to 0} \frac{s}{\nu_M(s)} > 0,$$

and, by Theorem 2.1, the latter implies that the spectrum of Δ_M is not discrete.

2.2 Compactness of a Sobolev embedding

We shall deduce Theorem 2.1 via Theorem 2.4 below, showing the equivalence of condition (2.4) and of the compactness of the embedding

$$W^{1,2}(M) \to L^2(M). \tag{2.16}$$

Indeed, a standard result in the theory of positive-definite self-adjoint operators in Hilbert spaces (see e.g. [BS, Chapter 10, Section 1, Theorem 5]) ensures that the discreteness of the spectrum of the operator $-\Delta_M + \mathrm{Id}$, and hence of $-\Delta_M$, on M is equivalent to the compactness of embedding (2.16).

Theorem 2.4 *Let M be a Riemannian manifold such that $\mathcal{H}^n(M) < \infty$. Then embedding* (2.16) *is compact if and only if* (2.4) *holds.*

In our proof of Theorem 2.4, we need to consider an auxiliary Sobolev type space $V^{1,2}(M)$ defined as

$$V^{1,2}(M) = \{u : \ u \text{ is weakly differentiable in } M \text{ and } |\nabla u| \in L^2(M) \ \}.$$

Given any open set $\omega \subset M$ such that $\overline{\omega}$ is compact, the expression

$$\sqrt{\|\nabla u\|^2_{L^2(M)} + \|u\|^2_{L^2(\omega)}}$$

defines a norm in $V^{1,2}(M)$. Different choices of ω result in equivalent norms in $V^{1,2}(M)$. Clearly,

$$W^{1,2}(M) = V^{1,2}(M) \cap L^2(M).$$

Note that $W^{1,2}(M)$ may be strictly contained in $V^{1,2}(M)$, due to the lack of a Poincaré type inequality between $\inf_{c\in\mathbb{R}} \|u-c\|_{L^2(M)}$ and $\|\nabla u\|_{L^2(M)}$ on noncompact manifolds with an irregular geometry.

Our first result is concerned with the equivalence of the compactness of the embeddings $W^{1,2}(M) \to L^2(M)$ and $V^{1,2}(M) \to L^2(M)$.

Lemma 2.5 *Let M be a Riemannian manifold such that $\mathcal{H}^n(M) < \infty$. Then the embedding*

$$V^{1,2}(M) \to L^2(M) \tag{2.17}$$

is compact if and only if the embedding

$$W^{1,2}(M) \to L^2(M) \tag{2.18}$$

is compact.

Proof. The compactness of (2.17) trivially implies the compactness of (2.18).
Conversely, assume that (2.18) is compact. Then there exists an increasing function $\zeta : [0, \mathcal{H}^n(M)] \to [0,\infty)$ fulfilling $\lim_{s\to 0} \zeta(s) = 0$, and such that for any measurable $E \subset M$ with $|E| \leq s$

$$\int_E u^2 d\mathcal{H}^n \leq \zeta(s)\Big(\int_M |\nabla u|^2 d\mathcal{H}^n + \int_M u^2 d\mathcal{H}^n\Big) \tag{2.19}$$

for every $u \in W^{1,2}(M)$. Given any $s \in (0, \mathcal{H}^n(M))$ such that $\zeta(s) < 1$, let ω be any open set as in the definition of the norm in $V^{1,2}(M)$ such that $\mathcal{H}^n(M \setminus \omega) \leq s$. Such a set ω exists since M is, in particular, a locally compact, separable topological space with a countable basis. We deduce from (2.19) that

$$\int_{M\setminus\omega} u^2 d\mathcal{H}^n \leq \frac{\zeta(s)}{1-\zeta(s)}\Big(\int_M |\nabla u|^2 d\mathcal{H}^n + \int_\omega u^2 d\mathcal{H}^n\Big) \tag{2.20}$$

for every $u \in W^{1,2}(M)$. Inequality (2.20) continues to hold for every $u \in V^{1,2}(M)$, as it is easily seen on truncating any such function at levels t and $-t$, applying (2.20) to the resulting function (which belongs to $W^{1,2}(M)$), and then letting $t \to \infty$.
Now, let $\{u_k\}$ be any bounded sequence in $V^{1,2}(M)$. Thus, there exists $C > 0$ such that

$$\int_M |\nabla u_k|^2 \, d\mathcal{H}^n + \int_\omega u_k^2 \, d\mathcal{H}^n \leq C. \tag{2.21}$$

From (2.20) and (2.21), we obtain that

$$\int_M |\nabla u_k|^2 \, d\mathcal{H}^n + \int_M u_k^2 \, d\mathcal{H}^n \leq \frac{C}{1-\zeta(s)}.$$

By the compactness of embedding (2.18), there exists a subsequence of $\{u_k\}$ converging in $L^2(M)$. The compactness of embedding (2.17) follows. □

The next lemma shows that the embedding $V^{1,2}(M) \to L^2(M)$ is equivalent to a Poincaré type inequality. In what follows, $\mathrm{med}(u)$ denotes the median of the function u, given by

$$\mathrm{med}(u) = \sup\{t : \mathcal{H}^n\{u > t\} \geq \mathcal{H}^n(M)/2\},$$

and $\mathrm{mv}(u)$ stands for the mean value of u, defined as

$$\mathrm{mv}(u) = \frac{1}{\mathcal{H}^n(M)} \int_M u \, d\mathcal{H}^n \, . \tag{2.22}$$

Lemma 2.6 *Let M be a Riemannian manifold such that $\mathcal{H}^n(M) < \infty$. Then the embedding*

$$V^{1,2}(M) \to L^2(M) \tag{2.23}$$

holds if and only if there exists a constant C such that

$$\|u - \mathrm{med}(u)\|_{L^2(M)} \leq C\|\nabla u\|_{L^2(M)} \tag{2.24}$$

for every u in $V^{1,2}(M)$.

Proof. We claim that embedding (2.23) is equivalent to the inequality

$$\inf_{c\in\mathbb{R}} \|u - c\|_{L^2(M)} \leq C\|\nabla u\|_{L^2(M)} \tag{2.25}$$

for some constant C and for every u in $V^{1,2}(M)$. Indeed, assume that (2.23) holds. Fix any smooth open set ω such that $\overline{\omega}$ is compact. From (2.23) we obtain that

$$\|u - c\|_{L^2(M)} \leq C\big(\|\nabla u\|_{L^2(M)} + \|u - c\|_{L^2(\omega)}\big) \tag{2.26}$$

for every $u \in V^{1,2}(M)$ and $c \in \mathbb{R}$. Since (2.25) classically holds with M replaced by ω, inequality (2.25) follows via (2.26). Conversely, assume that (2.25) holds. Since any constant function $c \in V^{1,2}(M)$, we have that $V^{1,2}(M) \subset L^2(M)$. The identity map from $V^{1,2}(M)$ into $L^2(M)$ is linear and has a closed graph. By the closed graph theorem it is also continuous. Hence, embedding (2.23) holds.
Owing to the equivalence of (2.23) and (2.25) just established, inequality (2.24) implies (2.23).

In order to prove the reverse implication, it suffices exploit the equivalence of (2.23) and (2.25), to recall that

$$\|u - \mathrm{mv}(u)\|_{L^2(M)} = \inf_{c\in\mathbb{R}} \|u - c\|_{L^2(M)} \, ,$$

and to make use of the fact that

$$\|u - \mathrm{med}(u)\|_{L^2(M)} \leq \sqrt{2}\|u - \mathrm{mv}\,(u)\|_{L^2(M)} \tag{2.27}$$

for every $u \in L^2(M)$. To verify (2.27), observe that

(2.28)
$$
\begin{aligned}
\|u-\mathrm{mv}\,(u)\|^2_{L^2(M)} &= \frac{1}{\mathcal{H}^n(M)}\int_M\int_M (u(x)-u(y))^2 d\mathcal{H}^n(y)d\mathcal{H}^n(x)\\
&\geq \frac{1}{\mathcal{H}^n(M)}\int_{\{u(y)\leq \mathrm{med}(u)\}}\int_{\{u(x)\geq \mathrm{med}(u)\}} (u(x)-\mathrm{med}(u)+\mathrm{med}(u)-u(y))^2 d\mathcal{H}^n(y)d\mathcal{H}^n(x)\\
&\geq \frac{1}{\mathcal{H}^n(M)}\int_{\{u(y)\leq \mathrm{med}(u)\}}\int_{\{u(x)\geq \mathrm{med}(u)\}} (u(x)-\mathrm{med}(u))^2 d\mathcal{H}^n(y)d\mathcal{H}^n(x)\\
&\quad + \frac{1}{\mathcal{H}^n(M)}\int_{\{u(y)\leq \mathrm{med}(u)\}}\int_{\{u(x)\geq \mathrm{med}(u)\}} (\mathrm{med}(u)-u(y))^2 d\mathcal{H}^n(y)d\mathcal{H}^n(x)\\
&\geq \frac{1}{2}\int_{\{u(x)\geq \mathrm{med}(u)\}} (u(x)-\mathrm{med}(u))^2 d\mathcal{H}^n(x) + \frac{1}{2}\int_{\{u(y)\leq \mathrm{med}(u)\}} (\mathrm{med}(u)-u(y))^2 d\mathcal{H}^n(y)\\
&= \frac{1}{2}\int_M (u(x)-\mathrm{med}(u))^2 d\mathcal{H}^n(x) = \frac{1}{2}\|u-\mathrm{med}(u)\|^2_{L^2(M)}\,.
\end{aligned}
$$

□

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. Assume that (2.4) holds. Fix any $s \in (0, \mathcal{H}^n(M)/2)$, and let E be any compact set in M such that $\mathcal{H}^n(M\setminus E) < s$ (such E certainly exists since M is, in particular, a locally compact, separable topological space with a countable basis). Let η be any smooth compactly supported function on M such that $0 \leq \eta \leq 1$ and $\eta = 1$ in E. Set $K = \operatorname{supp}\eta$. Given any $u \in W^{1,2}(M)$, we have that

$$
\|u\|_{L^2(M)} \leq \|(1-\eta)u\|_{L^2(M)} + \|\eta u\|_{L^2(M)}\,. \tag{2.29}
$$

Let us set
$$
v = (1-\eta)u.
$$

Clearly, $v \in W^{1,2}(M)$, and $\operatorname{supp} v \subset M\setminus E$. Thus, for every $t > 0$, $\{x \in M : |v| \geq t\} = \{x \in M\setminus E : |v| \geq t\}$, and $\mathcal{H}^n\{x \in M : |v| \geq t\}) \leq s \leq \mathcal{H}^n(M)/2$. Hence, by (2.3),

$$
\int_M v^2\, d\mathcal{H}^n = \int_0^\infty \mathcal{H}^n(\{|v|\geq t\})d(t^2) \leq \left(\sup_{r\leq s}\frac{r}{\nu_M(r)}\right)\int_0^\infty C(\{|v|\geq t\}, M\setminus E)\, d(t^2). \tag{2.30}
$$

We now make use a discretization argument related to that of [Ma4]. By the monotonicity of capacity,

$$
\int_0^\infty C(\{|v|\geq t\}, M\setminus E)\, d(t^2) \leq 3\sum_{k\in\mathbb{Z}} 2^{2k} C(\{|v|\geq 2^k\}, M\setminus E)\,. \tag{2.31}
$$

Let $\Psi : \mathbb{R} \to [0,1]$ be the function given by $\Psi(t) = 0$ if $t \leq 0$, $\Psi(t) = 1$ if $t \geq 1$, and $\Psi(t) = t$ if $t \in (0,1)$. Define $v_k : M \to [0,1]$ as

$$
v_k = \Psi(2^{1-k}|v| - 1)
$$

for $k \in \mathbb{Z}$. Note that $v_k \in W^{1,2}(M)$ for $k \in \mathbb{Z}$, since Ψ is Lipschitz continuous, and $v_k = 1$ in $\{|v| \geq 2^k\}$ and $v_k = 0$ in $\{|v| \leq 2^{k-1}\}$. In particular, $v_k = 0$ on $E = M\setminus(M\setminus E)$. Hence, by the very definition of capacity of a condenser, one has that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^{2k} C(\{|v|\geq 2^k\}, M\setminus E) &\leq \sum_{k\in\mathbb{Z}} 2^{2k}\int_M |\nabla v_k|^2 d\mathcal{H}^n \\ &= 4\sum_{k\in\mathbb{Z}}\int_{\{2^{k-1}\leq |v|<2^k\}} |\nabla v|^2 d\mathcal{H}^n = 4\int_M |\nabla v|^2 d\mathcal{H}^n\,.\end{aligned} \tag{2.32}$$

Combining inequalities (2.30)–(2.32) tells us that there exists a constant C such that

$$\int_M v^2\, d\mathcal{H}^n \leq C\sup_{r\leq s}\frac{r}{\nu_M(r)}\int_M |\nabla v|^2 d\mathcal{H}^n\,. \tag{2.33}$$

Thus,

$$\begin{aligned}\|(1-\eta)u\|_{L^2(M)} &\leq \Big(C\sup_{r\leq s}\frac{r}{\nu_M(r)}\Big)^{1/2}\|\nabla((1-\eta)u)\|_{L^2(M)} \\ &\leq \Big(C\sup_{r\leq s}\frac{r}{\nu_M(r)}\Big)^{1/2}\Big(\|\nabla u\|_{L^2(M)} + \|\nabla\eta\|_{L^\infty(M)}\|u\|_{L^2(K)}\Big)\,,\end{aligned} \tag{2.34}$$

and, trivially,

$$\|\eta u\|_{L^2(M)} \leq \|u\|_{L^2(K)}\,. \tag{2.35}$$

From (2.29), (2.34) and (2.35) we deduce that

$$\|u\|_{L^2(M)} \leq C\Big(\sup_{r\leq s}\frac{r}{\nu_M(r)}\Big)^{1/2}\|\nabla u\|_{L^2(M)} + C\|u\|_{L^2(K)}\,, \tag{2.36}$$

for a suitable constant C.

By the standard Reillich compactness embedding theorem, applied on each element of a finite covering of K with smooth bounded open sets, the embedding

$$W^{1,2}(M)\to L^2(K) \tag{2.37}$$

is compact. Let $\{u_k\}$ be a sequence in the unit ball of $W^{1,2}(M)$. By the compact embedding (2.37), we may assume (on taking a subsequence, if necessary) that $\{u_k\}$ is a Cauchy sequence in $L^2(K)$. An application of (2.36) with u replaced by $u_k - u_m$ for $k,m\in\mathbb{N}$ yields

$$\|u_k-u_m\|_{L^2(M)} \leq 2C\Big(\sup_{r\leq s}\frac{r}{\nu_M(r)}\Big)^{1/2} + C\|u_k-u_m\|_{L^2(K)}\,. \tag{2.38}$$

Since $\{u_k\}$ is a Cauchy sequence in $L^2(K)$, by (2.38)

$$\|u_k-u_m\|_{L^2(M)} \leq 3C\Big(\sup_{r\leq s}\frac{r}{\nu_M(r)}\Big)^{1/2}\,, \tag{2.39}$$

provided that k and m are sufficiently large. Owing to the arbitrariness of s and to assumption (2.4), $\{u_k\}$ is a Cauchy sequence in $L^2(M)$. The compactness of the embedding $W^{1,2}(M)\to L^2(M)$ follows.

Conversely, assume that the embedding $W^{1,2}(M)\to L^2(M)$ is compact. Then, by Lemma 2.5, embedding (2.17) is also compact. Let ω be as in the definition of the norm in $V^{1,2}(M)$, and such that $\mathcal{H}^n(\omega) \le \mathcal{H}^n(M)/2$. Thus, there exists an increasing function $\zeta : (0, \mathcal{H}^n(M)) \to [0, \infty)$ fulfilling

$$\lim_{s \to 0} \zeta(s) = 0\,, \tag{2.40}$$

and such that for any $s \in (0, \mathcal{H}^n(M)/2)$ and any measurable set $E \subset M$ with $\mathcal{H}^n(E) = s$

$$\int_E u^2 d\mathcal{H}^n \le \zeta(s) \Big(\int_M |\nabla u|^2 d\mathcal{H}^n + \int_\omega u^2 d\mathcal{H}^n \Big) \tag{2.41}$$

for every $u \in V^{1,2}(M)$. An application of (2.41) with u replaced by $u - \mathrm{med}(u)$, and Lemma 2.6 entail that

$$\begin{aligned} \int_E |u - \mathrm{med}(u)|^2 d\mathcal{H}^n &\le \zeta(s) \Big(\int_M |\nabla u|^2 d\mathcal{H}^n + \int_\omega |u - \mathrm{med}(u)|^2 d\mathcal{H}^n \Big) \\ &\le C\zeta(s) \int_M |\nabla u|^2 d\mathcal{H}^n \end{aligned} \tag{2.42}$$

for some constant C and for every $u \in V^{1,2}(M)$. Given any measurable set $G \supset E$ such that $\mathcal{H}^n(G) \le \mathcal{H}^n(M)/2$, let $u \in V^{1,2}(M)$ be any function such that $u = 1$ a.e. in E and $u = 0$ a.e. in $M \setminus G$. In particular, $\mathrm{med}(u) = 0$. We thus infer from (2.42) that

$$s = \mathcal{H}^n(E) \le C\zeta(s) \int_M |\nabla u|^2 d\mathcal{H}^n\,. \tag{2.43}$$

By the definition of capacity, inequality (2.43) implies that

$$s \le C\zeta(s) C(E, G)\,. \tag{2.44}$$

Since ν_M is a positive non-decreasing function in $(0, \mathcal{H}^n(M)/2)$, equation (2.4) easily follows from (2.44), (2.2) and (2.40). □

2.3 Examples

2.3.1 Manifolds of revolution

Basic instances of complete noncompact Riemannian manifolds are provided by manifolds of revolution of the form $\mathbb{R} \times \mathbb{S}^{n-1}$, endowed with the Riemannian metric

$$ds^2 = dr^2 + \varphi(r)^2 d\omega^2\,. \tag{2.45}$$

Here, $d\omega^2$ stands for the standard metric on the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$, and φ is a smooth function on $[0, \infty)$ such that $\varphi(r) > 0$ for $r > 0$, $\varphi(0) = 0$ and $\varphi'(0) = 1$. Clearly, $\mathcal{H}^n(M) < \infty$ if and only if $\int_0^\infty \varphi(r)^{n-1}\, dr < \infty$. Under the additional assumption that there exists $r_0 > 0$ such that φ is decreasing and convex in (r_0, ∞), the asymptotic behavior of λ_M and ν_M can be described [CM, Theorem 3.1]. In particular, conditions (2.4) and (2.8) are equivalent for this class of manifolds [CM, Corollary 3.2], and lead to the following characterization of the discreteness of the spectrum of Δ_M.

Proposition 2.7 *Let M be an n-dimensional Riemannian manifold of revolution as above. Then the spectrum of Δ_M (which agrees with the Friedrichs extension of the Laplacian on M) is discrete if and only if*

$$\lim_{r \to \infty} \frac{1}{\varphi(r)^{n-1}} \int_r^\infty \varphi(\rho)^{n-1} d\rho = 0. \tag{2.46}$$

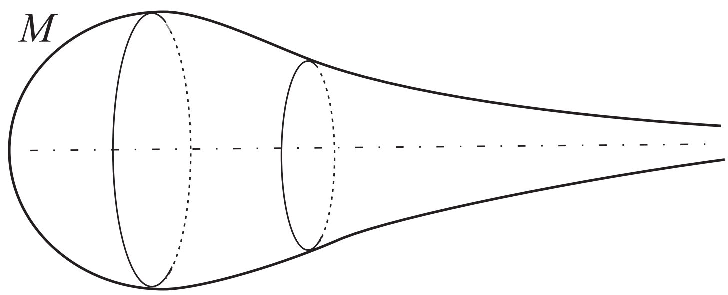

Figure 1: A manifold of revolution

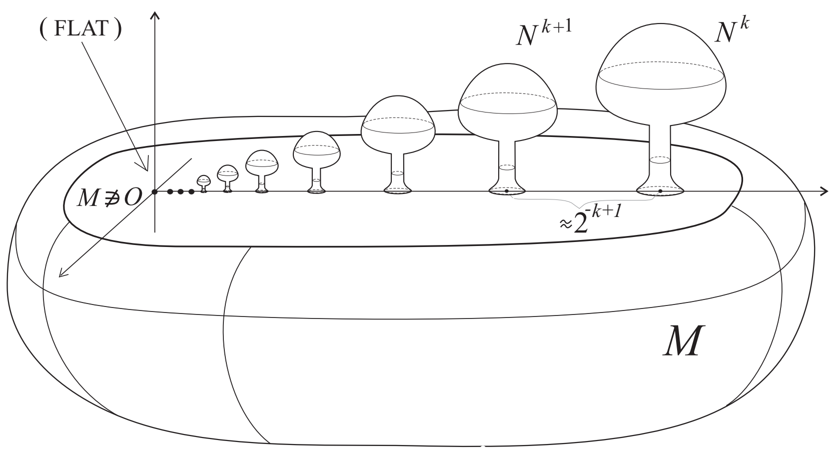

Figure 2: A manifold with a family of clustering submanifolds

Condition (2.46) suggests that profiles φ with an exponential decay at infinity are borderline for the discreteness of the spectrum of Δ_M. More precisely, consider the one-parameter family of manifolds of revolution M where $\varphi : [0, \infty) \to [0, \infty)$ is such that $\varphi(r) = e^{-r^\alpha}$ for large r, for some $\alpha > 0$. An application of Proposition 2.7 tells us that the spectrum of the Laplacian on M is discrete if and only if $\alpha > 1$. This overlaps with results of [Ba].

2.3.2 Manifolds with clustering submanifolds

We consider here a class of noncompact surfaces M embedded in $\mathbb{R}^3$, which are reminiscent of an irregular open set in $\mathbb{R}^2$ appearing in [CH]. This class of surfaces will demonstrate how Theorem 2.1, involving the isocapacitary function ν_M, can actually succeed in proving the discreteness of the spectrum of Δ_M in situations where, instead, Corollary 2.2 fails.
The main feature of the manifolds considered in this section is that they contain a sequence of mushroom-shaped submanifolds $\{N^k\}$ clustering at some point (Figure 2). The submanifolds $\{N^k\}$ are constructed in such a way that the diameter of the head and the length of the neck of

N^k decay to 0 as 2^{-k} when $k \to \infty$, whereas the width of the neck of N^k decays to 0 as $\sigma(2^{-k})$, where σ is an increasing function such that

$$\lim_{s\to 0} \frac{\sigma(s)}{s} = 0. \tag{2.47}$$

Moreover, the distance of two consecutive submanifolds $\{N^k\}$ and $\{N^{k+1}\}$ is of the order 2^{-k+1}. Loosely speaking, a faster decay to 0 of the function $\sigma(s)$ as $s \to 0$ results in a faster decay to 0 of $\lambda_M(s)$ and $\nu_M(s)$, and hence in a manifold M with a more irregular geometry. A description of the asymptotic behavior of λ_M and ν_M has been provided in Propositions 7.1 and 7.2 of [CM], to which we also refer for a rigorous definition of M. In particular, we have what follows. Assume that $\sigma \in \Delta_2$, namely a constant c exists such that $\sigma(2s) \le c\sigma(s)$ for $s > 0$. Then

$$\lambda_M(s) \le C\sigma(s^{1/2}) \qquad \text{for } s \ge 0, \tag{2.48}$$

and

$$\nu_M(s) \le C\sigma(s^{1/2})s^{-\frac{1}{2}} \qquad \text{for } s \ge 0, \tag{2.49}$$

for some positive constant C. If, in addition,

$$\frac{s^{\beta+1}}{\sigma(s)} \ \text{ is non-increasing for some } \beta > 0, \tag{2.50}$$

and

$$\frac{s^3}{\sigma(s)} \ \text{ is non-decreasing,} \tag{2.51}$$

then, in fact,

$$\nu_M(s) \approx \sigma(s^{1/2})s^{-\frac{1}{2}} \qquad \text{near } 0. \tag{2.52}$$

The manifold M is obviously not complete. This notwithstanding, the following density result holds.

Proposition 2.8 *Let M be the manifold described above, under assumption* (2.47). *Then*

$$\overline{C_0^\infty(M)} = W^{1,2}(M). \tag{2.53}$$

Hence, the operator Δ_M agrees with the Friedrichs extension of the Laplacian on M.

Proof. It suffices to prove (2.53). With reference to Figure 2, we denote a point in $\mathbb{R}^3$ by $x = (y, z)$, where $y \in \mathbb{R}^2$ (the horizontal plane) and $z \in \mathbb{R}$ (the vertical axis). Let $\eta \in C^\infty(\mathbb{R})$ be an increasing function such that $\eta(s) = 0$ if $s \le 0$, $\eta(s) = 1$ if $s \ge 1$. Define, for $\varepsilon > 0$, the function $\eta_\varepsilon : M \to [0, 1]$ as

$$\eta_\varepsilon(x) = \eta\left(\frac{\log(|y|/\varepsilon^2)}{\log(1/\varepsilon)}\right) \qquad \text{for } x \in M.$$

Clearly, $\eta_\varepsilon \in C_0^\infty(M)$, $\eta_\varepsilon(x) = 0$ if x belongs to the intersection of M with the half-cylinder $\{x \in \mathbb{R}^3 : |y| < \varepsilon^2, z \ge 0\}$, and $\eta(x)_\varepsilon = 1$ if x belongs to the intersection of M with $\{x \in \mathbb{R}^3 : |y| > \varepsilon, z \ge 0\}$. We have that

$$|\nabla \eta_\varepsilon(x)| \begin{cases} = \eta'\left(\frac{\log(|y|/\varepsilon^2)}{\log(1/\varepsilon)}\right) \frac{1}{|y|\log(1/\varepsilon)} \le \frac{C}{|y|\log(1/\varepsilon)} & \text{if } x \in \big(M \setminus (\cup_k N^k)\big) \cap \{z \ge 0\}, \\ \le \eta'\left(\frac{\log(|y|/\varepsilon^2)}{\log(1/\varepsilon)}\right) \frac{1}{|y|\log(1/\varepsilon)} \le \frac{C2^k}{\log(1/\varepsilon)} & \text{if } x \in N^k, \end{cases} \tag{2.54}$$

for some constant C depending on η. Here, ∇ denotes the gradient on M. Next, define $\mathcal{K} = \{k : N^k \cap \{x \in \mathbb{R}^3 : \varepsilon^2 < |y| < \varepsilon\} \neq \emptyset\}$, and note that the cardinality of $\mathcal{K}$ is of the order $\log(\varepsilon^{-1})$. Furthemore, we have that $\mathcal{H}^2(N^k) \leq C 2^{-2k}$ for some constant C. Thus,

$$
\begin{aligned}
\int_M |\nabla \eta_\varepsilon|^2 d\mathcal{H}^2 &= \int_{(M \setminus (\cup_k N^k)) \cap \{z \geq 0\}} |\nabla \eta_\varepsilon|^2 d\mathcal{H}^2 + \sum_{k \in \mathcal{K}} \int_{N^k} |\nabla \eta_\varepsilon|^2 d\mathcal{H}^2 \\
&\leq \frac{C}{\log^2(1/\varepsilon)} \int_{\{y : \varepsilon^2 \leq |y| \leq \varepsilon\}} \frac{dy}{|y|^2} + \frac{C}{\log^2(1/\varepsilon)} \sum_{k \in \mathcal{K}} 2^{2k} \\
&\leq \frac{C}{\log(1/\varepsilon)} + \frac{C'}{\log^2(1/\varepsilon)} \log(1/\varepsilon) \\
&\leq \frac{C''}{\log(1/\varepsilon)},
\end{aligned}
\tag{2.55}
$$

for some constants C, C', C''. Now, given any $u \in W^{1,2}(M)$. We have to show that u can be approximated in $W^{1,2}(M)$ by a family of functions from $C_0^\infty(M)$. Bounded functions are dense in $W^{1,2}(M)$. This can be easily seen on approximating u by its truncated at the levels t and $-t$ and letting t go to $+\infty$. Thus, we may assume, without loss of generality, that u is bounded. Moreover, smooth functions are well known to be dense in $W^{1,2}(M)$ for any Riemannian manifold M, and hence also for the present one. Thus, we may assume that $u \in C^\infty(M) \cap L^\infty(M)$. Now, set $u_\varepsilon = \eta_\varepsilon u$, and observe that $u_\varepsilon \in C_0^\infty(M)$ for $\varepsilon > 0$. We have that

$$
\|u - u_\varepsilon\|_{W^{1,2}(M)} \leq \|(1 - \eta_\varepsilon)\nabla u\|_{L^2(M)} + \|u\|_{L^\infty(M)} \|\nabla \eta_\varepsilon\|_{L^2(M)} + \|(1 - \eta_\varepsilon) u\|_{L^2(M)}. \tag{2.56}
$$

Owing to (2.55), we infer that $u_\varepsilon \to u$ in $W^{1,2}(M)$ as $\varepsilon \to 0$. Hence, assertion (2.53) follows. □

The next propostion relies upon the criterion for the discreteness of the spectrum of Δ_M in terms of the isocapacitary function of M given in Theorem 2.1, and provides us with a characterization of those manifolds M of the family considered in this section for which the spectrum of the Laplacian is discrete.

Proposition 2.9 *Assume that $\sigma \in \Delta_2$ and fulfills* (2.50)*, and that the function $\frac{s^3}{\sigma(s)}$ is monotonic. Then the spectrum of Δ_M (which, by Proposition 2.8, agrees with the Friedrichs extension of the Laplacian on M) is discrete if only if*

$$
\lim_{s \to 0} \frac{s^3}{\sigma(s)} = 0\,. \tag{2.57}
$$

Proof. By Theorem 2.4 and (2.49), condition (2.57) is necessary for the embedding $W^{1,2}(M) \to L^2(M)$ to be compact, and hence for the spectrum of Δ_M to be discrete. By Theorem 2.1, the monotonicity assumption on $\frac{s^3}{\sigma(s)}$ and (2.52), we have that condition (2.57) is also sufficient for the discreteness of the spectrum of Δ_M. □

By contrast, let us emphasize that, owing to (2.48), the use of Corollary 2.2 involving the isoperimetric function λ_M cannot yield the discreteness of the spectrum of the manifold M unless

$$
\lim_{s \to 0} \frac{s^2}{\sigma(s)} = 0\,,
$$

a condition more stringent than (2.57).

3 Manifolds with infinite volume

We assume throughout this section that $\mathcal{H}^n(M) = \infty$.

3.1 Discreteness of the spectrum

The notion of capacity $C(E)$ of a subset of M which now comes into play is:

(3.1)
$$C(E) = \inf\left\{ \int_M (|\nabla u|^2 + u^2)\,dx : u \in W^{1,2}(M), u \geq 1 \text{ in } E \text{ (up to a set of standard capacity zero)}\right\}.$$

Accordingly, the isocapacitary function $\mu_M : [0,\infty) \to [0,\infty]$ is given by

$$\mu_M(s) = \inf\{C(E) : \ E \text{ is measurable and } s \leq \mathcal{H}^n(E) < \infty\} \qquad \text{for } s \geq 0. \tag{3.2}$$

The isocapacitary inequality takes the form

$$\mu_M(\mathcal{H}^n(E)) \leq C(E) \tag{3.3}$$

for every measurable set $E \subset M$ with $\mathcal{H}^n(E) < \infty$.

Our characterization of Riemannian manifolds of infinite volume such that Δ_M has a discrete spectrum reads as follows.

Theorem 3.1 *Let M be a Riemannian manifold such that $\mathcal{H}^n(M) = \infty$. Then the spectrum of Δ_M is discrete if and only if*

$$\lim_{s\to 0} \frac{s}{\mu_M(s)} = 0 \qquad \text{and} \qquad \lim_{s\to\infty} \frac{s}{\mu_M(s)} = 0\,. \tag{3.4}$$

Defining the isoperimetric function $\sigma_M : [0,\infty) \to [0,\infty)$ of M as

$$\varrho_M(s) = \inf\{P(E) : s \leq \mathcal{H}^n(E) < \infty\}\,, \tag{3.5}$$

results in the isoperimetric inequality on M

$$\varrho_M(\mathcal{H}^n(E)) \leq P(E) \tag{3.6}$$

for every measurable set $E \subset M$ with $\mathcal{H}^n(E) < \infty$. The counterpart of (2.7) is the inequality

$$\mu_M(s) \geq \frac{1}{\int_s^\infty \frac{dr}{\varrho_M(r)^2}} + s \qquad \text{for } s \in (0,\infty), \tag{3.7}$$

whose proof follows via an easy modification of that of (2.7).
An analogue of Corollary 2.2 in the present framework follows from Theorem 3.1 and inequality (3.7).

Corollary 3.2 *Let M be a Riemannian manifold such that $\mathcal{H}^n(M) = \infty$. Assume that*

$$\lim_{s\to 0} \frac{s}{\varrho_M(s)} = 0 \qquad \text{and} \qquad \lim_{s\to\infty} \frac{s}{\varrho_M(s)} = 0\,. \tag{3.8}$$

Then the spectrum of Δ_M is discrete.

3.2 Compactness of a Sobolev embedding

Analogously to the case of manifolds of finite volume, Theorem 3.1 is a consequence of a compactness result, which is the content of the next theorem.

Theorem 3.3 *Let M be a Riemannian manifold such that $\mathcal{H}^n(M)=\infty$. Then the embedding*

$$W^{1,2}(M) \to L^2(M)$$

is compact if and only if (3.4) *holds.*

Proof. Assume that (3.4) holds. Let $u \in W^{1,2}(M)$. Since the set of continuous functions is dense in $W^{1,2}(M)$, we may assume, without loss of generality, that u is continuous. Given $\delta > 0$, set

$$M_\delta = \{|u| > \delta\}.$$

Owing to our assumptions, M_δ is an open set, and $\mathcal{H}^n(M_\delta) < \infty$. We have that

$$\|u\|_{L^2(M)} \leq \|u\|_{L^2(M_\delta)} + \|u\|_{L^2(M\setminus M_\delta)}. \tag{3.9}$$

The first term on the right hand side of (3.9) can be estimated via an argument similar to that of the proof of Theorem 2.4. Specifically, fix any $s > 0$, and let E be any compact subset of M_δ such that $\mathcal{H}^n(M_\delta \setminus E) < s$. Let η be any smooth compactly supported function on M_δ such that $0 \leq \eta \leq 1$ and $\eta = 1$ in E. Set $K = \operatorname{supp} \eta$. We have that

$$\|u\|_{L^2(M_\delta)} \leq \|(1-\eta)u\|_{L^2(M_\delta)} + \|\eta u\|_{L^2(M_\delta)}\,. \tag{3.10}$$

Let us set

$$v = (1-\eta)u.$$

Then $v \in W^{1,2}(M)$, and $v = 0$ on E. Thus, for every $t > 0$, $\{x \in M_\delta : |v| \geq t\} = \{x \in M_\delta \setminus E : |v| \geq t\}$, and $\mathcal{H}^n\{x \in M_\delta : |v| \geq t\}) \leq s$. Hence, by (3.3),

$$\begin{aligned} \int_{M_\delta} v^2\, d\mathcal{H}^n &= \int_0^\infty \mathcal{H}^n(\{x \in M_\delta : |v| \geq t\}) d(t^2) \\ &\leq \left(\sup_{r\leq s} \frac{r}{\mu_M(r)}\right) \int_0^\infty C(\{x \in M_\delta : |v| \geq t\})\, d(t^2). \end{aligned} \tag{3.11}$$

By the monotonicity of capacity,

$$\int_0^\infty C(\{x \in M_\delta : |v| \geq t\})\, d(t^2) \leq \int_0^\infty C(\{|v| \geq t\})\, d(t^2) \leq 3 \sum_{k\in\mathbb{Z}} 2^{2k} C(\{|v| \geq 2^k)\,. \tag{3.12}$$

Let Ψ be the function defined in the proof of Theorem 2.4. Define $v_k : M \to [0,1]$ as

$$v_k = \Psi(2^{1-k}|v| - 1)$$

for $k \in \mathbb{Z}$. A similar chain as in (2.32) now yields

$$\begin{aligned} \sum_{k\in\mathbb{Z}} 2^{2k} C(\{|v| \geq 2^k\}) &\leq \sum_{k\in\mathbb{Z}} 2^{2k} \int_M \left(|\nabla v_k|^2 + v_k^2\right) d\mathcal{H}^n \\ &\leq \sum_{k\in\mathbb{Z}} \int_{\{2^{k-1}\leq|v|<2^k\}} \left(4|\nabla v|^2 + 2^{2k}\right) d\mathcal{H}^n \\ &\leq \sum_{k\in\mathbb{Z}} \int_{\{2^{k-1}\leq|v|<2^k\}} \left(4|\nabla v|^2 + 4v^2\right) d\mathcal{H}^n \\ &= 4\int_M \left(|\nabla v|^2 + v^2\right) d\mathcal{H}^n\,. \end{aligned} \tag{3.13}$$

From (3.11) – (3.13) we infer that there exists a constant C such that

$$\int_{M_\delta} v^2\, d\mathcal{H}^n \le C\Big(\sup_{r\le s}\frac{r}{\mu_M(r)}\Big)\int_M \big(|\nabla v|^2+v^2\big)\, d\mathcal{H}^n\,. \tag{3.14}$$

Thus,

(3.15)

$$\begin{aligned}\|(1-\eta)u\|_{L^2(M_\delta)} &\le \Big(C\sup_{r\le s}\frac{r}{\mu_M(r)}\Big)^{1/2}\Big(\|\nabla((1-\eta)u)\|_{L^2(M)}+\|(1-\eta)u\|_{L^2(M)}\Big)\\ &\le \Big(C\sup_{r\le s}\frac{r}{\mu_M(r)}\Big)^{1/2}\Big(\|\nabla u\|_{L^2(M)}+\|\nabla\eta\|_{L^\infty(M)}\|u\|_{L^2(K)}+\|u\|_{L^2(M)}\Big)\,,\end{aligned}$$

and, trivially,

$$\|\eta u\|_{L^2(M_\delta)} \le \|u\|_{L^2(K)}\,. \tag{3.16}$$

From (3.10), (3.15) and (3.16) we deduce that

$$\|u\|_{L^2(M_\delta)} \le C\Big(\sup_{r\le s}\frac{r}{\mu_M(r)}\Big)^{1/2}\|u\|_{W^{1,2}(M)}+\|u\|_{L^2(K)}\,, \tag{3.17}$$

for a suitable constant C.
Let us now focus on the second term on the right-hand side of (3.9). We have that

$$\begin{aligned}\|u\|^2_{L^2(M\setminus M_\delta)} &= \int_{\{|u|\le\delta\}} u^2 d\mathcal{H}^n = \sum_{k=0}^\infty \int_{\{\delta 2^{-k-1}<|u|\le\delta 2^{-k}\}} u^2 d\mathcal{H}^n\\ &\le \sum_{k=0}^\infty \delta^2 4^{-k}\mathcal{H}^n(\{|u|>\delta 2^{-k-1}\}).\end{aligned} \tag{3.18}$$

By the second limit in (3.4), for every $\varepsilon>0$, there exists $s_\varepsilon>0$ such that if $s>s_\varepsilon$, then $s\le\varepsilon\mu_M(s)$. Thus, if δ is so small that $\mathcal{H}^n(\{|u|>\delta\})>s_\varepsilon$, then

$$\begin{aligned}\sum_{k=0}^\infty \delta^2 4^{-k}\mathcal{H}^n(\{|u|>\delta 2^{-k-1}\}) &\le \varepsilon\sum_{k=0}^\infty \delta^2 4^{-k}\mu_M\big(\mathcal{H}^n(\{|u|>\delta 2^{-k-1}\})\big)\\ &\le \varepsilon\sum_{k=0}^\infty \delta^2 4^{-k} C(\{|u|\ge\delta 2^{-k-1}\}).\end{aligned} \tag{3.19}$$

Now, let Ψ be as above, and let

$$u_k = \Psi(2^{k+2}\delta^{-1}|u|-1).$$

We have that $u_k=1$ if $|u|\ge\delta 2^{-k-1}$, $u_k=0$ if $|u|\le\delta 2^{-k-2}$ and $0\le u_k\le 1$ on M. Moreover, $u_k\in W^{1,2}(M)$. Thus,

$$C(\{|u|\ge\delta 2^{-k-1}\}) \le \int_M \big(|\nabla u_k|^2+u_k^2\big)\, d\mathcal{H}^n, \tag{3.20}$$

and hence

$$\varepsilon \sum_{k=0}^{\infty} \delta^2 4^{-k} C(\{|u| \geq \delta 2^{-k-1}\}) \leq \varepsilon \sum_{k=0}^{\infty} \delta^2 4^{-k} \int_M \left(|\nabla u_k|^2 + u_k^2\right) d\mathcal{H}^n$$
$$\leq \varepsilon \sum_{k=0}^{\infty} \delta^2 4^{-k} \int_{\{\delta 2^{-k-2} < |u| \leq \delta 2^{-k-1}\}} \left(2^{2k+4} \delta^{-2} |\nabla u|^2 + 1\right) d\mathcal{H}^n$$
$$\leq 16\varepsilon \sum_{k=0}^{\infty} \int_{\{\delta 2^{-k-2} < |u| \leq \delta 2^{-k-1}\}} \left(|\nabla u|^2 + u^2\right) d\mathcal{H}^n$$
$$\leq 16\varepsilon \int_{\{|u| \leq \delta\}} \left(|\nabla u|^2 + u^2\right) d\mathcal{H}^n. \tag{3.21}$$

Combining (3.18), (3.19) and (3.21) yields

$$\|u\|^2_{L^2(M \setminus M_\delta)} \leq 16\varepsilon \|u\|^2_{W^{1,2}(M \setminus M_\delta)}\,. \tag{3.22}$$

From (3.9), (3.17) and (3.22) we infer that there exists a constant C such that

$$\|u\|_{L^2(M)} \leq C\left(\sup_{r \leq s} \frac{r}{\mu_M(r)}\right)^{1/2} \|u\|_{W^{1,2}(M)} + C\varepsilon^{1/2} \|u\|_{W^{1,2}(M \setminus M_\delta)} + \|u\|_{L^2(K)}. \tag{3.23}$$

By the first limit in (3.4), $\sup_{r \leq s} \frac{r}{\mu_M(r)} \leq \varepsilon$ provided that s is sufficiently small. Hence, for such a choice of s,

$$\|u\|_{L^2(M)} \leq C\varepsilon^{1/2} \|u\|_{W^{1,2}(M)} + \|u\|_{L^2(K)}. \tag{3.24}$$

Starting from (3.24) instead of (2.36), we conclude as in proof of Theorem 2.4 that the embedding $W^{1,2}(M) \to L^2(M)$ is compact.

Assume now that the embedding $W^{1,2}(M) \to L^2(M)$ is compact. Then there exits an increasing function $\zeta : (0, \infty) \to [0, \infty)$ fulfilling

$$\lim_{s \to 0} \zeta(s) = 0\,, \tag{3.25}$$

and such that for any $s > 0$ and any measurable set $E \subset M$ with $\mathcal{H}^n(E) = s$

$$\int_E u^2 d\mathcal{H}^n \leq \zeta(s) \int_M \left(|\nabla u|^2 + u^2\right) d\mathcal{H}^n \tag{3.26}$$

for every $u \in W^{1,2}(M)$. Moreover, given any sequence $\{G_k\}_{k \in \mathbb{N}}$ of compact sets G_k such that $G_k \subset G_{k+1}$ for $k \in \mathbb{N}$ and $\cup_k G_k = M$, for every $\varepsilon > 0$ there exists k such that

$$\int_{M \setminus G_k} u^2 d\mathcal{H}^n \leq \varepsilon \int_M \left(|\nabla u|^2 + u^2\right) d\mathcal{H}^n \tag{3.27}$$

for every $u \in W^{1,2}(M)$. By the definition of capacity, inequality (3.26) implies that

$$s \leq C\zeta(s) C(E)\,. \tag{3.28}$$

Since ν_M is a positive non-decreasing function, the first limit in (3.4) follows from (3.28) and (3.25).

To prove the second limit in (3.4), let us begin by observing that

$$\lim_{s\to\infty} \mu_M(s) = \infty. \tag{3.29}$$

This follows from the fact that for every measurable set $E \subset M$ and every function $u \in W^{1,2}(M)$ such that $u \geq 1$ q.e. on E we have

$$\int_M \left(|\nabla u|^2 + u^2\right) d\mathcal{H}^n \geq \mathcal{H}^n(E).$$

Now, given $\varepsilon > 0$, let k be such that (3.27) holds. For any measurable set E, one has that

$$\frac{\mathcal{H}^n(E)}{C(E)} \leq \frac{\mathcal{H}^n(G_k)}{C(E)} + \frac{\mathcal{H}^n(E \setminus G_k)}{C(E)}. \tag{3.30}$$

Inequality (3.27) applied with any function $u \in W^{1,2}(M)$ such that $u \geq 1$ a.e on E tells us that

$$\mathcal{H}^n(E \setminus G_k) \leq \varepsilon C(E). \tag{3.31}$$

On the other hand, by (3.29), if $\mathcal{H}^n(E)$ is sufficiently large, then

$$\frac{\mathcal{H}^n(G_k)}{C(E)} < \varepsilon. \tag{3.32}$$

Combining (3.30)–(3.32) entails that

$$\frac{\mathcal{H}^n(E)}{C(E)} \leq 2\varepsilon, \tag{3.33}$$

provided that $\mathcal{H}^n(E)$ is sufficiently large. Hence, the second limit in (3.4) follows. □

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