On Solvability of Boundary Integral Equations of Potential Theory for a Multi-Dimensional Cusp Domain

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Abstract The Dirichet and the Neumann problems for Laplace's equation on a multi-dimensional cusp domain are considered. The unique solvability of the boundary integral equation for the internal Dirichlet problem for harmonic double layer potential is established. We also prove the unique solvabilty of the boundary integral equation for the external Neumann problem for harmonic single layer potential.

1 Introduction

Boundary integral equations generated by elliptic boundary value problems have been intensively studied during more than a hundred years (for the history, see [1], [5]). By now, a comprehensive classical layer potential theory for domains with Lipschitz and piecewise smooth boundaries has been developed.

Let Γ be the common boundary of internal (Ω^+) and external (Ω^-) domains in $\mathbb{R}^n$. If Γ is sufficiently smooth, then the Fredholm theory applies. When Γ has singularities, this theory does not generally work, nevertheless, it was shown for $\Gamma \in C^{0,1}$ [2] – [6], that solutions to the Neumann and the Dirichlet problems can be written as harmonic single layer or double layer potentials with densities depending on boundary data.

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Solvability of integral equations of potential theory for a planar cusp domain was studied by Maz'ya and Soloviev [7] – [9].

In the authors work [11] a multi–dimensional cusp domain was considered whose boundary Γ contains the vertex of the cusp O. It was shown under the assumption $\Gamma \setminus \{O\} \in C^{1,\lambda}$, $\lambda \in (0,1]$, that the operator V^{-1}, inverse to the single layer potential, is an isometric isomorphism of the space $Tr(\Gamma)$ of the traces to Γ of the functions having the finite Dirichlet integral over $\mathbb{R}^n$ to the dual space $(Tr(\Gamma))^*$. One of the consequences of this result is that the solution to a two-sided Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Gamma, \quad u|_\Gamma = f$$

can be written as the single layer potential $u = V\varrho$ with density $\varrho \in Tr(\Gamma)^*$.

Results of [11] were used in [12] where the solvability of boundary integral equations for the Neumann problem was considered. For example, it turned out [12] that for domains with inward cusps the solution to the Neumann problem can be represented by the single layer potential with density in $(Tr(\Gamma))^*$. But this is generally not true for domains with outward cusps.

In the present paper we consider a bounded multi-dimensional domain with outward cusp and study the solvability of boundary integral equations for the Laplace operator. We show that these equations for the double layer potential (in case of the internal Dirichlet problem with boundary data in $Tr(\Gamma)$) and for the single layer potential (in case of the external Neumann problem with boundary data in $(Tr(\Gamma))^*$) are uniquely solvable. Incidentally we obtain that the solution to the Dirichlet problem is represented by the double layer potential and the solution to the Neumann problem is represented by the single layer potential. The last fact concerning the Neumann problem was established earlier in [12] by an alternative proof.

2 Domains and Function Spaces

We now describe a surface with cusp which we deal with in what follows. Let $\Omega \subset \mathbb{R}^n$, $n > 2$, be a bounded simply connected domain whose boundary contains the origin and some neighborhood U of the origin intersects Ω by the set

$$U \cap \Omega = \{x = (x', x_n) \in \mathbb{R}^n : \, x_n \in (0,1), x'/\varphi(x_n) \in \omega\}, \tag{1}$$

where $\varphi \in C[0,1]$ is an increasing function, $\varphi(0) = 0$, and ω is a bounded domain in $\mathbb{R}^{n-1}$. Without trying to attain a full generality, we will assume that $\partial\omega \in C^2$, $\varphi \in C^2(0,1] \cap C^1[0,1]$ $\partial\Omega \setminus \{O\} \in C^2$ and $\varphi'(0) = 0$. Furthermore, we say that O is the vertex of an outward cusp on the boundary of Ω.

Let $\Gamma = \partial\Omega$. For all $x \in \Gamma \setminus \{O\}$ there exists a normal to Γ at x. In what follows $\nu(x)$ designates a unit outward normal at $x \in \Gamma \setminus \{O\}$. We also put

$\Omega^+ = \Omega$, $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$. The symbol $B_r(x)$ denotes an open ball in $\mathbb{R}^n$ of radius r centered at x, $B_r = B_r(0)$. Let G be a domain in $\mathbb{R}^n$. By $C_0^\infty(G)$ we mean the set of smooth functions having compact support in G. In what follows $L_2^1(G)$ is the space of functions in $L_{2,loc}(G)$ whose gradient is in $L_2(G)$. The space $\mathring{L}_2^1(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\nabla u\|_{L_2(\mathbb{R}^n)}$, where ∇u is the gradient of u. The space $\mathring{L}_2^1(\Omega^-)$ is defined as the closure in the norm $\|\nabla(\cdot)\|_{L_2(\Omega^-)}$ of the set of functions in $C^\infty(\Omega^-) \cap L_2^1(\Omega^-)$ with bounded supports in Ω^-. The space $L_2^1(\Omega^+)$ is equipped with norm

$$\|u\|_{L_2^1(\Omega^+)} = \left(\|u\|^2_{L_2(\Gamma \setminus U)} + \|\nabla u\|^2_{L_2(\Omega^+)} \right)^{1/2}.$$

where U is the same as in (1).

Let $Tr(\Gamma)$ denote the space of the traces $u|_\Gamma$ of the functions $u \in \mathring{L}_2^1(\mathbb{R}^n)$ equipped with norm

$$\|f\|_{Tr(\Gamma)} = \inf\{\|\nabla u\|_{L_2(\mathbb{R}^n)} : u \in \mathring{L}_2^1(\mathbb{R}^n), u|_\Gamma = f\}.$$

We also consider one-sided trace spaces $Tr^\pm(\Gamma)$ for Γ. The space $Tr^+(\Gamma)$ consists of the traces $u|_\Gamma$ of the elements $u \in L_2^1(\Omega^+)$. This space is endowed with norm

$$\|f\|_{Tr^+(\Gamma)} = \inf\{\|u\|_{L_2^1(\Omega^+)} : u \in L_2^1(\Omega^+), u|_\Gamma = f\}.$$

We define $Tr^-(\Gamma)$ as the trace space $\{u|_\Gamma : u \in \mathring{L}_2^1(\Omega^-)\}$ with norm

$$\|f\|_{Tr^-(\Gamma)} = \inf\{\|\nabla u\|_{L_2(\Omega^-)} : u \in \mathring{L}_2^1(\Omega^-), \quad u|_\Gamma = f\}.$$

In what follows $(Tr^\pm(\Gamma))^*$ and $(Tr(\Gamma))^*$ designate the dual spaces of the corresponding trace spaces.

Let $H(\Omega^\pm)$ be the space of harmonic functions on $\Omega^\pm$ with finite Dirichlet integral, and let $H(\mathbb{R}^n \setminus \Gamma)$ denote the space of the finctions on $\mathbb{R}^n$ with finite Dirichlet integral, which are harmonic on each of domains Ω^+ and Ω^-. Clearly $H(\Omega^+) \subset L_2^1(\Omega^+)$. According to [11], [12], $H(\Omega^-)$ and $H(\mathbb{R}^n \setminus \Gamma)$ are subspaces of $\mathring{L}_2^1(\Omega^-)$ and $\mathring{L}_2^1(\mathbb{R}^n)$, respectively, and the map $u \mapsto u|_\Gamma$ is an isometric isomorphism between the spaces $H(\Omega^\pm)$, $Tr^\pm(\Gamma)$ and between the spaces $H(\mathbb{R}^n \setminus \Gamma)$, $Tr(\Gamma)$.

3 Single and Double Layer Potentials

Let $E(\cdot,\cdot)$ be the fundamental solution of the Poisson equation:

$$E(x,y) = ((2-n)|S^{n-1}||x-y|^{n-2})^{-1}, \quad x, y \in \mathbb{R}^n,$$

where $|S^{n-1}|$ is the area of the unit sphere in $\mathbb{R}^n$. The single layer potential for Γ with density $\varrho \in L_{2,loc}(\Gamma \setminus \{O\})$ is

$$(V\varrho)(x) = \int_\Gamma \varrho(y) E(x,y) d\Gamma(y), \quad x \in \mathbb{R}^n. \tag{2}$$

The following assertion was established in [11].

Theorem 1. *The map*

$$L_2(\Gamma) \ni \varrho \mapsto V\varrho \in H(\mathbb{R}^n \setminus \Gamma)$$

is uniquely extended to an isometric isomorphism: $(Tr(\Gamma))^* \to H(\mathbb{R}^n \setminus \Gamma)$ *if* $\varrho \in L_2(\Gamma)$ *is identified with the functional* $\langle \varrho, \cdot \rangle \in (Tr(\Gamma))^*$ *defined by*

$$\langle \varrho, g \rangle = \int_\Gamma \varrho g d\Gamma, \quad g \in Tr(\Gamma).$$

Furthermore, the map $(Tr(\Gamma))^* \ni \varrho \mapsto V\varrho|_\Gamma \in Tr(\Gamma)$ *is an isometric isomorphism.*

In the following lemma we collect some known results on density of subsets in function spaces introduced above [11], [12].

Lemma 1. (i) *The set* $C_0^\infty(\mathbb{R}^n \setminus \{O\})$ *is dense in each of the spaces* $L_2^1(\Omega^+)$, $\mathring{L}_2^1(\Omega^-)$, $\mathring{L}_2^1(\mathbb{R}^n)$.

(ii) *The set* $\widetilde{Tr}(\Gamma)$ *of the traces to* Γ *of the functions in* $C_0^\infty(\mathbb{R}^n \setminus \{O\})$ *is dense in* $Tr^\pm(\Gamma)$ *and* $Tr(\Gamma)$.

(iii) *Let* S *be a subset of* $L_2(\Gamma)$ *which is dense in* $L_2(\Gamma)$*. Then the set* $VS = \{V\varrho : \varrho \in S\}$ *is dense in* $H(\mathbb{R}^n \setminus \Gamma)$ *and the set* $VS|_\Gamma = \{V\varrho|_\Gamma : \varrho \in S\}$ *is dense in* $Tr(\Gamma)$.

(iv) *Let* S *be the same as in (iii). Then the set of functionals* $\langle s, \cdot \rangle$, $s \in S$, *defined by*

$$\langle s, g \rangle = \int_\Gamma s g d\Gamma$$

is dense in $(Tr(\Gamma))^*$.

It follows from the above assumptions on Γ that for $x \in \Gamma \setminus \{O\}$ there is a ball $B_\delta(x)$ such that

$$\left| \frac{\partial E(\xi, \eta)}{\partial \nu_\xi} \right| \leqslant c(x, \Gamma) |\xi - \eta|^{2-n}$$

for all $\xi, \eta \in B_\delta(x) \cap \Gamma$. Hence for $\varrho \in C(\Gamma)$ the integral on the right of (3)

$$\frac{\partial (\overline{V}\varrho)(x)}{\partial \nu} = \int_\Gamma \varrho(\xi) \frac{\partial E(x, \xi)}{\partial \nu_x} d\Gamma(\xi), \tag{3}$$

is a continuous function of $x \in \Gamma \setminus \{O\}$.

Let u be a function defined on $\mathbb{R}^n \setminus \Gamma$ and let $x \in \Gamma \setminus \{O\}$. The values $u^{\pm}(x)$ are defined as one-sided limits

$$u^{\pm}(x) = \lim_{t \to \mp 0} u(x + t\nu(x)).$$

It is well known that for $\varrho \in C(\Gamma)$ normal derivatves of the single layer potential have jumps on Γ given by

$$\frac{\partial(V\varrho)^{\pm}}{\partial \nu}(x) = \frac{\partial(\overline{V}\varrho)}{\partial \nu}(x) \mp \frac{\varrho(x)}{2}, \quad x \in \Gamma \setminus \{O\}. \tag{4}$$

Because of continuity of the integral in (3), one obtains that for $\varrho \in C(\Gamma)$ one-sided limit values $\partial(V\varrho)^{\pm}/\partial\nu$ are continuous on $\Gamma \setminus \{O\}$. In particular, (4) implies that

$$\left.\frac{\partial(V\varrho)^{-}}{\partial \nu}\right|_{\Gamma} - \left.\frac{\partial(V\varrho)^{+}}{\partial \nu}\right|_{\Gamma} = \varrho. \tag{5}$$

The last formula remains true for $\varrho \in (Tr(\Gamma))^{*}$. In this case (5) should be written in the form of the identity

$$\int_{\mathbb{R}^n} \nabla(V\varrho)\nabla u dx = -\langle \varrho, u \rangle,$$

which is valid for all $u \in \mathring{L}_2^1(\mathbb{R}^n)$.

Let $\sigma \in Tr(\Gamma)$. The double layer potential for Γ with density σ is

$$(W\sigma)(x) = \int_{\Gamma} \sigma(y) \frac{\partial E(x,y)}{\partial \nu_y} d\Gamma(y), \quad x \in \mathbb{R}^n \setminus \Gamma. \tag{6}$$

It is clear that $W\sigma$ is a harmonic function on each domain Ω^{+} and Ω^{-}. The assumptions on Γ, stated at the beginning of Sec. 2, along with inclusion $\sigma \in C(\Gamma)$ provide the continuity

$$(\overline{W}\sigma)(x) = \int_{\Gamma} \sigma(y) \frac{\partial E(x,y)}{\partial \nu_y} d\Gamma(y), \tag{7}$$

of the integral on the right of (7) as a function of $x \in \Gamma \setminus \{O\}$. For the same σ the following formulas for jumps of the double layer potential at Γ are well known

$$(W\sigma)^{\pm}(x) = \overline{W}\sigma(x) \pm \sigma(x)/2, \quad x \in \Gamma \setminus \{O\}. \tag{8}$$

Clearly (8) implies that

$$(W\sigma)^{+}(x) - (W\sigma)^{-}(x) = \sigma(x), \quad x \in \Gamma \setminus \{O\}. \tag{9}$$

Lemma 2 stated below plays an important role in what follows.

Lemma 2. *The maps*

$$Tr(\varGamma) \ni \sigma \mapsto (W\sigma)^{\pm}|_{\varGamma} \in L_2(\varGamma), \tag{10}$$

$$W^{\pm} : Tr(\varGamma) \to H(\varOmega^{\pm}) \tag{11}$$

are linear continuous operators.

Proof. Let $C_0(\varGamma)$ be the space of continuous functions on $\varGamma$ vanishing in the vicinity of the origin. Suppose that $\sigma \in \widetilde{Tr}(\varGamma)$ and $\varrho \in C_0(\varGamma)$. It is well known (and readily verified) that

$$\int_{\varGamma} (W\sigma)^{\pm} \varrho d\varGamma = \int_{\varGamma} \sigma \cdot \frac{\partial (V\varrho)^{\mp}}{\partial \nu} d\varGamma. \tag{12}$$

We note also that σ admits a unique extension $u \in H(\mathbb{R}^n \setminus \varGamma)$. By Green's formula the right part of (12) is

$$\mp \int_{\varOmega^{\mp}} \nabla (V\varrho) \nabla u dx,$$

and hence

$$\left| \int_{\varGamma} (W\sigma)^{\pm} \varrho d\varGamma \right| \leqslant \|\nabla u\|_{L_2(\varOmega^{\mp})} \|\nabla (V\varrho)\|_{L_2(\varOmega^{\mp})}.$$

Thus we have

$$\left| \int_{\varGamma} (W\sigma)^{\pm} \varrho d\varGamma \right| \leqslant \text{const} \cdot \|\sigma\|_{Tr^{\mp}(\varGamma)} \|\varrho\|_{L_2(\varGamma)}.$$

Since $C_0(\varGamma)$ is dense in $L_2(\varGamma)$, it follows that

$$\|(W\sigma)^{\pm}\|_{L_2(\varGamma)} \leqslant \text{const} \cdot \|\sigma\|_{Tr(\varGamma)}.$$

Because of the density of $\widetilde{Tr}(\varGamma)$ in $Tr^{\pm}(\varGamma)$ and in $Tr(\varGamma)$ the map $W^{\pm}$ can be uniquely extended to a linear continuous operator (10).

We now turn to (11). Let $\sigma \in \widetilde{Tr}(\varGamma)$ and let $u \in H(\mathbb{R}^n \setminus \varGamma)$, $u|_{\varGamma} = \sigma$. We use the following integral representation

$$u(x) = -\int_{\mathbb{R}^n} \nabla u(\xi) (\nabla_{\xi} E)(x, \xi) d\xi, \tag{13}$$

which is valid for $u \in \mathring{L}_2^1(\mathbb{R}^n)$ and almost all $x \in \mathbb{R}^n$. If $x \in \varOmega^+$, then Green's formula gives

$$\int_{\varOmega^-} \nabla u(\xi) (\nabla_{\xi} E)(x, \xi) d\xi = -\int_{\varGamma} u(\xi) \frac{\partial E(x, \xi)}{\partial \nu_{\xi}} d\varGamma(\xi). \tag{14}$$

Integrating by parts yields

$$\int_{\Omega^+} \nabla u(\xi)(\nabla_\xi E)(x,\xi)d\xi = \int_\Gamma E(x,\xi)\partial u/\partial\nu_\xi d\Gamma(\xi).$$

By unifying the last with (13), (14), one obtains for $x \in \Omega^+$

$$W\sigma(x) = u(x) + V\varrho_+(x), \tag{15}$$

where $V\varrho_+$ is the single layer potential with density $\varrho_+ = \partial u^+/\partial\nu$. It follows from (15) that

$$\|\nabla W\sigma\|_{L_2(\Omega^+)} \leqslant \|\nabla u\|_{L_2(\Omega^+)} + \|\nabla(V\varrho_+)\|_{L_2(\mathbb{R}^n)}.$$

As it was shown in [11], the last term equals $\|\varrho_+\|_{(Tr(\Gamma))^*}$, therefore

$$\|\nabla(V\varrho_+)\|_{L_2(\mathbb{R}^n)} = \sup_v |\langle\varrho_+, v\rangle|/\|v\|_{Tr(\Gamma)} \leqslant$$

$$\leqslant \sup \left|\int_{\Omega^+} \nabla u \nabla v dx\right| \|\nabla v\|^{-1}_{L_2(\mathbb{R}^n)} \leqslant \|\nabla u\|_{L_2(\mathbb{R}^n)} = \|\sigma\|_{Tr(\Gamma)}.$$

Thus (15) implies that

$$\|\nabla W\sigma\|_{L_2(\Omega^+)} \leqslant 2\,\|\sigma\|_{Tr(\Gamma)}. \tag{16}$$

Since $\widetilde{Tr}(\Gamma)$ is dense in $Tr(\Gamma)$, (16) holds true for $\sigma \in Tr(\Gamma)$. By unifying (16) and (10), we establish he continuity of the operator

$$W^+ : Tr(\Gamma) \to H(\Omega^+). \tag{17}$$

Let $x \in \Omega^-$. Equality

$$W\sigma(x) = -u(x) + V\varrho_-(x)$$

with $\varrho_- = \partial u^-/\partial\nu$ can be deduced from (13) in the same way as (15) has been deduced above. Hence

$$\|\nabla W\sigma\|_{L_2(\Omega^-)} \leqslant 2\,\|\sigma\|_{Tr(\Gamma)},$$

which along with continuity of operator (17) gives the continuity of operator (11). This concludes the proof.

Two assertions below follow from Lemma 2.

Corollary 1. *Formula* (9) *holds true for* $\sigma \in Tr(\Gamma)$ *and almost all* $x \in \Gamma$.

Corollary 2. *Formula* (12) *is extended to the case* $\sigma \in Tr(\Gamma)$ *and* $\varrho, \partial(V\varrho)^\pm/\partial\nu \in (Tr(\Gamma))^*$. *For these parameters* (12) *should be written in the form*

$$\langle\varrho, (W\sigma)^\pm\rangle = \langle\partial(V\varrho)^\mp/\partial\nu, \sigma\rangle.$$

4 Boundary value problems

We state the Dirichlet problem, the Neumann problem and the transmission problem.

The internal (external) Dirichlet problem reads as follows. Given $f^+ \in Tr^+(\Gamma)$ $(f^- \in Tr^-(\Gamma))$, find $u^+ \in L_2^1(\Omega^+)$ $(u^- \in \mathring{L}_2^1(\Omega^-))$, such that $u^+|_\Gamma = f^+$ $(u^-|_\Gamma = f^-)$ and the identity

$$\int_{\Omega^\pm} \nabla u^\pm \nabla v dx = 0$$

holds for all $v \in C_0^\infty(\Omega^\pm)$. It is well known that the problem is uniquely solvable and its solution $u^\pm$ is in fact in $H(\Omega^\pm)$. Furthermore, the equalities

$$\|u^+\|_{L_2^1(\Omega^+)} = \|f^+\|_{Tr^+(\Gamma)}, \quad \|\nabla u^-\|_{L_2(\Omega^-)} = \|f^-\|_{Tr^-(\Gamma)}$$

are valid.

We now state the external Neumann problem

$$\Delta u = 0 \;\; \Omega^-, \quad \partial u/\partial \nu|_\Gamma = \psi^-, \tag{18}$$

where $\psi^- \in (Tr^-(\Gamma))^*$. Its solution is that function $u \in \mathring{L}_2^1(\Omega^-)$, which satisfies

$$\int_{\Omega^-} \nabla u \nabla w dx = -\langle \psi^-, w\rangle$$

for all $w \in \mathring{L}_2^1(\Omega^-)$. It is well known that problem (18) is uniquely solvable for all $\psi^- \in Tr^-(\Gamma)^*$. Moreover, $u \in H(\Omega^-)$, and the estimate

$$\|\nabla u\|_{L_2(\Omega^-)} \leqslant \text{const}\, \|\psi^-\|_{(Tr^-(\Gamma))^*}$$

holds true.

A solution of the internal Neumann problem

$$\Delta u = 0 \text{ in } \Omega^+, \quad \partial u/\partial \nu|_\Gamma = \psi^+, \quad \psi^+ \in (Tr^+(\Gamma))^*, \tag{19}$$

is that function $u \in L_2^1(\Omega^+)$, for which

$$\int_{\Omega^+} \nabla u \nabla w dx = \langle \psi^+, w\rangle$$

with an arbitrary $w \in L_2^1(\Omega^+)$. Let $(Tr^+(\Gamma))^* \ominus 1$ denote the subspace of the functionals in $(Tr^+(\Gamma))^*$, orthogonal to 1 in $L_2(\Gamma)$. It is known that for any $\psi^+ \in (Tr^+(\Gamma))^* \ominus 1$ problem (19) has a solution, uniquely determined up to a constant summand, and the solution is in fact in $H(\Omega^+)$. Furthermore, the following estimate holds

$$\|\nabla u\|_{L_2(\Omega^+)} \leqslant \text{const}\,\|\psi^+\|_{(Tr^+(\Gamma))^*}.$$

Let $(f, \varrho) \in Tr(\Gamma) \times (Tr(\Gamma))^*$ be given. The transmission problem is to find a pair of functions $(w^+, w^-) \in H(\Omega^+) \times H(\Omega^-)$ such that

$$w^+|_\Gamma - w^-|_\Gamma = f, \tag{20}$$

$$\left.\frac{\partial w^-}{\partial \nu}\right|_\Gamma - \left.\frac{\partial w^+}{\partial \nu}\right|_\Gamma = \varrho. \tag{21}$$

Lemma 3. *For arbitrary $(f, \varrho) \in Tr(\Gamma) \times (Tr(\Gamma))^*$ the transmission problem with boundary conditions* (20), (21) *has a unique solution (w^+, w^-), that can be written explicitly in the form*

$$w^+ = (Wf)^+ + (V\varrho)^+, \quad w^- = (Wf)^- + (V\varrho)^-,$$

where V and W are the single and the double layer potentials defined by (2) *and* (6). *The following estimate holds*

$$\|\nabla w^+\|_{L_2(\Omega^+)} + \|\nabla w^-\|_{L_2(\Omega^-)} \leqslant \text{const}(\|f\|_{Tr(\Gamma)} + \|\varrho\|_{(Tr(\Gamma))^*}). \tag{22}$$

Proof. Uniqueness. One should check that for $f = \varrho = 0$ the corresponding transmission problem has only one solution $w^+ = 0$, $w^- = 0$.

Let (w^+, w^-) be a solution of the homogeneous transmission problem. Define $w|_{\Omega^+} = w^+$ and $w|_{\Omega^-} = w^-$. Because of the homogeneity of condition (20) we have $w \in H(\mathbb{R}^n \setminus \Gamma)$. Hence $w = V\delta$ for some $\delta \in (Tr(\Gamma))^*$ by Theorem 1. An application of (5) gives

$$\delta = \left.\frac{\partial w^-}{\partial \nu}\right|_\Gamma - \left.\frac{\partial w^+}{\partial \nu}\right|_\Gamma,$$

so that $\delta = 0$ because of the homogeneity of condition (21). Thus $w = V\delta = 0$ and $w^+ = w^- = 0$.

Solvability. Let $w^+ = (Wf)^+ + (V\varrho)^+$, $w^- = (Wf)^- + (V\varrho)^-$. It follows from Theorem 1 and Lemma 2 that $(w^+, w^-) \in H(\Omega^+) \times H(\Omega^-)$. We now check (20), (21). Since $(V\varrho)^+|_\Gamma = (V\varrho)^-|_\Gamma$, we have

$$w^+|_\Gamma - w^-|_\Gamma = (Wf)^+|_\Gamma - (Wf)^-|_\Gamma = f.$$

Next, with the aid of (5), we obtain that

$$\left.\frac{\partial w^-}{\partial \nu}\right|_\Gamma - \left.\frac{\partial w^+}{\partial \nu}\right|_\Gamma = \left.\frac{\partial (Wf)^+}{\partial \nu}\right|_\Gamma - \left.\frac{\partial (Wf)^-}{\partial \nu}\right|_\Gamma + \varrho.$$

It remains to verify the equality

$$\partial(Wf)^+/\partial\nu = \partial(Wf)^-/\partial\nu, \quad f \in Tr(\Gamma). \tag{23}$$

If the derivatives of the double layer potential are continuous, the required equality is a statement of the Liapunov theorem [13], ch. XV, § 5. In the general case (23) is equivalent to the identity

$$\int_{\Omega^+} \nabla (Wf)^+ \nabla v dx + \int_{\Omega^-} \nabla (Wf)^- \nabla v dx = 0, \tag{24}$$

which is valid for all $v \in \mathring{L}_2^1(\mathbb{R}^n)$.

First we establish (24) for $v = V\varrho$, $\varrho \in C_0(\Gamma)$ and $f \in \widetilde{Tr}(\Gamma)$. By Green's formula the left part of (24) is

$$\int_\Gamma \Big((Wf)^+ \partial (V\varrho)^+ / \partial \nu - (Wf)^- \partial (V\varrho)^- / \partial \nu \Big) d\Gamma.$$

The last integral equals zero because of (12) (see also Corollary 2). So (24) holds with v replaced by $V\varrho$. By using Lemma 1 (iii), we obtain (24) for all $v \in \mathring{L}_2^1(\mathbb{R}^n)$.

Estimate (22) follows from Theorem 1 and Lemma 2. This completes the proof.

5 Solvability of Boundary Integral Equations

Here we establish the following assertion.

Theorem 2. (i) *Let $\overline{W}$ be given by* (7). *The operator*

$$\overline{W} + \frac{1}{2} I : Tr(\Gamma) \to Tr(\Gamma), \tag{25}$$

is continuous and has a bounded inverse.
(ii) *The operator*

$$\frac{\partial \overline{V}}{\partial \nu} + \frac{1}{2} I : (Tr(\Gamma))^* \to (Tr(\Gamma))^*, \tag{26}$$

where $\partial \overline{V}/\partial \nu$ is given by (3), *is continuous and has a bounded inverse.*

Proof. Continuity of operator (25) follows from Lemma 2. We shall check that the operator is surjective and one to one. The desired result then follows from the Banach theorem.

Suppose that $\overline{W}\sigma + \sigma/2 = 0$ for some $\sigma \in Tr(\Gamma)$. There is a unique extension of σ to a function in $H(\mathbb{R}^n \setminus \Gamma)$ (we relabel this extension again as σ). Let $\varrho \in C_0(\Gamma)$. By (12) and Green's formula, we have

$$0 = \int_\Gamma (W\sigma)^+ \varrho d\Gamma = \int_\Gamma \sigma \frac{\partial (V\varrho)^-}{\partial \nu} d\Gamma = - \int_{\Omega^-} \nabla \sigma \nabla (V\varrho) dx.$$

By Lemma 1 (iii) σ can be aproximated by the elements $V\varrho$ in $H(\mathbb{R}^n \setminus \Gamma)$, hence $\sigma|_{\Omega^-} = \text{const}$. Since $\sigma(x) \to 0$ for $|x| \to \infty$, it follows that $\sigma = 0$ almost everywhere in Ω^- and thus $\sigma = 0$ in $\mathbb{R}^n$ because $\sigma \in H(\mathbb{R}^n \setminus \Gamma)$. So operator (25) is one to one.

We now turn to the solvability of equation

$$\overline{W}\sigma + \sigma/2 = f^+ \tag{27}$$

in the space $Tr(\Gamma)$. As it was mentioned above, the internal Dirichlet problem has a unique solution u^+ for any boundary data $f^+ \in Tr(\Gamma)$. Along with this problem we state the external Neumann problem: to find $u^- \in H(\Omega^-)$ such that

$$\left.\frac{\partial u^-}{\partial \nu}\right|_\Gamma = \left.\frac{\partial u^+}{\partial \nu}\right|_\Gamma.$$

Since the space $(Tr^+(\Gamma))^*$ is continuously imbedded in $(Tr^-(\Gamma))^*$, [10], ch. 7, the Neumann problem is stated correctly and has a unique solution. Let $f^- = u^-|_\Gamma$. Then $f^- \in Tr(\Gamma)$ since $Tr^-(\Gamma) = Tr(\Gamma)$ [12]. Thus the pair (u^+, u^-) is a solution of the transmission problem with boundary conditions

$$u^+|_\Gamma - u^-|_\Gamma = f^+ - f^- \in Tr(\Gamma),$$

$$\left.\frac{\partial u^-}{\partial \nu}\right|_\Gamma - \left.\frac{\partial u^+}{\partial \nu}\right|_\Gamma = 0.$$

Now Lemma 3 implies that $u^+ = (W\sigma)^+$ for $\sigma = f^+ - f^-$. For the same density σ (27) holds.

(ii) Let $\sigma \in \widetilde{Tr}(\Gamma)$ and $\varrho \in C_0(\Gamma)$. Let σ be extended to a function in $H(\mathbb{R}^n \setminus \Gamma)$. Green's formula gives

$$\langle \partial\overline{V}\varrho/\partial\nu + \varrho/2, \sigma\rangle = \langle \partial(V\varrho)^-/\partial\nu, \sigma\rangle = -\int_{\Omega^-} \nabla(V\varrho)\nabla\sigma dx,$$

whence

$$\begin{aligned} |\langle \partial(V\varrho)^-/\partial\nu, \sigma\rangle| &\leqslant \|\nabla(V\varrho)\|_{L_2(\mathbb{R}^n)} \|\nabla\sigma\|_{L_2(\mathbb{R}^n)} \\ &= \|\varrho\|_{(Tr(\Gamma))^*} \|\sigma\|_{Tr(\Gamma)}. \end{aligned}$$

Here we have used that $\|\nabla(V\varrho)\|_{L_2(\mathbb{R}^n)} = \|\varrho\|_{(Tr(\Gamma))^*}$ (see[11]). So, by density of $\widetilde{Tr}(\Gamma)$ in $Tr(\Gamma)$, we arrive at the estimate

$$\|\partial(V\varrho)^-/\partial\nu\|_{(Tr(\Gamma))^*} \leqslant \|\varrho\|_{(Tr(\Gamma))^*}.$$

By Lemma 1 (iv) this estimate is valid for all $\varrho \in (Tr(\Gamma))^*$ and thus (26) is a continuous operator.

Suppose that $\partial(V\varrho)^-/\partial\nu = 0$ for some $\varrho \in (Tr(\Gamma))^*$. By using (12) (see also Corollary 2), we obtain

$$\int_\Gamma \varrho \cdot (W\sigma)^+ d\Gamma = 0$$

for all $\sigma \in Tr(\Gamma)$. Since equation (27) is solvable for all $f^+ \in Tr(\Gamma)$, it follws that $\{(W\sigma)^+ : \sigma \in Tr(\Gamma)\} = Tr(\Gamma)$, hence $\varrho = 0$ and operator (26) is one to one.

It remains to check the solvability of the equation

$$\partial \overline{V} \varrho / \partial \nu + \varrho/2 = \psi^- \tag{28}$$

for all $\psi^- \in (Tr^-(\Gamma))^*$. It will suffice to verify that solution of the Neumann problem

$$u^- \in H(\Omega^-), \quad \partial u^- / \partial \nu|_\Gamma = \psi^-$$

(which is uniquely solvable as it has been mentioned above) can be represented in the form of the single layer potential with density in $(Tr(\Gamma))^*$. To this end we consider the internal Dirichlet problem: to find $u^+ \in H(\Omega^+)$ such that $u^+|_\Gamma = u^-|_\Gamma$. This Dirichlet problem is again uniquely solvable. Let $\psi^+ = \partial u^+ / \partial \nu$. Clearly $\psi^+ \in (Tr^+(\Gamma))^*$ and because of the imbedding $Tr(\Gamma) \subset Tr^+\Gamma)$ one has $\psi^+ \in (Tr(\Gamma))^*$. We observe that the pair $(u^+, u^-) \in H(\Omega^+) \times H(\Omega^-)$ is a solution of the transmission problem

$$u^+|_\Gamma - u^-|_\Gamma = 0, \quad \left.\frac{\partial u^-}{\partial \nu}\right|_\Gamma - \left.\frac{\partial u^+}{\partial \nu}\right|_\Gamma = \psi^- - \psi^+ \in (Tr(\Gamma))^*.$$

According to Lemma 3, $u^- = (V\varrho)^-$ with $\varrho = \psi^- - \psi^+$ thus concluding the proof of the theorem.

Incidentally, we have proved the following assertion.

Corollary 3. *The solution to the internal Dirichlet problem with boundary data in $Tr(\Gamma)$ is the double layer potential with uniquely determined density in $Tr(\Gamma)$, and the solution to the external Neumann problem with boundary data in $(Tr(\Gamma))^*$ is the single layer potential with uniquely determined density in $(Tr(\Gamma))^*$.*[1]

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[1] the second statement of Corollary 3 is also obtained in [12] using another proof.

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