BOURGAIN-BREZIS TYPE INEQUALITY WITH EXPLICIT CONSTANTS

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This article is dedicated to Michael Cwikel

Abstract. We deduce a Bourgain-Brezis type inequality with explicit constants which implies that $\nabla u \in H^{-n/2}(\mathbb{R}^n)+L^1(\mathbb{R}^n)$ leads to $u \in H^{1-n/2}(\mathbb{R}^n)$.

Let u denote a 2π-periodic function in $\mathbb{R}^n$ with zero mean value, and let $\nabla u = \mathbf{f} + \mathbf{g}$, where $\mathbf{f}$ and $\mathbf{g}$ are 2π-periodic vector valued functions. The main result of Bourgain and Brezis in [**BB1**] can be formulated as the inequality

$$\|u\|_{L^{\frac{n}{n-1}}} \leq c\left(\|\mathbf{f}\|_{W^{-1,\frac{n}{n-1}}} + \|\mathbf{g}\|_{L^1}\right), \tag{1}$$

where $W^{-1,\frac{n}{n-1}}$ denotes the dual of the Sobolev space $W^{1,n}$ of functions with gradients in L^n.

The dual formulation of this result is the existence of a 2π-periodic $\mathbf{v} \in W^{1,n} \cap L^\infty$ satisfying

$$\operatorname{div} \mathbf{v} = h$$

for an arbitrary $h \in L^n$ with zero mean value. Several more recent related results can be found in [**BB2**], [**LS**], [**VS**].

In the present article we prove a certain inequality of the same nature as (1), whose formulation involves explicit constants.

In the sequel, we do not use different notation for spaces of scalar and vector valued distributions. Let $l \in \mathbb{R}$. The notation $H^l(\mathbb{R}^n)$ will be used for the space of distributions h with finite norm

$$\|h\|_{H^l} := \left(\int_{\mathbb{R}^n} |\hat{h}(\xi)|^2 |\xi|^{2l}\, d\xi\right)^{1/2},$$

where $\hat{h}$ stands for the Fourier transform of h (see formula (7.1.3) in [**H**]). Here and in what follows we omit $\mathbb{R}^n$ in the notations of the norms.

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THEOREM 0.1. *Let $u \in H^{1-n/2}(\mathbb{R}^n)$ be a scalar distribution defined on $\mathbb{R}^n$, $n \ge 2$, such that $\nabla u = \mathbf{f} + \mathbf{g}$, where $\mathbf{f} \in H^{-n/2}(\mathbb{R}^n)$ and $\mathbf{g} \in L^1(\mathbb{R}^n)$. Then*

$$\|u\|_{H^{1-n/2}} \le \|\mathbf{f}\|_{H^{-n/2}} + \left(\|\mathbf{f}\|^2_{H^{-n/2}} + \frac{|S^{n-1}|}{n\,(2\pi)^n}\, \|\mathbf{g}\|^2_{L^1} \right)^{1/2}. \tag{2}$$

Proof. We can assume that u, $\mathbf{f}$ and $\mathbf{g}$ are real valued. Since $\nabla u = \mathbf{f} + \mathbf{g}$ and $\Delta u = \operatorname{div} \mathbf{f} + \operatorname{div} \mathbf{g}$, we have

$$\begin{aligned}
&2(n-1)\Big((-\Delta)^{-1/2-n/4}(\Delta u - \operatorname{div} \mathbf{f})\Big)^2 \\
&= \sum_{1\le j,k\le n} \left((-\Delta)^{-1/2-n/4} \left(\frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k} \right) \right)^2 + 2(n-1)\Big((-\Delta)^{-1/2-n/4} \operatorname{div} \mathbf{g} \Big)^2 \\
&- \sum_{1\le j,k\le n} \left((-\Delta)^{-1/2-n/4} \left(\frac{\partial g_k}{\partial x_j} - \frac{\partial g_j}{\partial x_k} \right) \right)^2,
\end{aligned}$$

where $(-\Delta)^{-1/2-n/4}$ is the pseudodifferential operator with the symbol $|\xi|^{-1-n/2}$. Integrating over $\mathbb{R}^n$, we arrive at

$$\begin{aligned}
&2(n-1)\big\|(-\Delta)^{-1/2-n/4}(\Delta u - \operatorname{div} \mathbf{f})\big\|^2_{L^2} \\
&- \sum_{1\le j,k\le n} \left\| (-\Delta)^{-1/2-n/4} \left(\frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k} \right) \right\|^2_{L^2} \\
&= 2 \sum_{1\le k\le n} \int_{\mathbb{R}^n} g_k \Big((n-1) \frac{\partial^2}{\partial x_k^2} - \sum_{\{j: j\neq k\}} \frac{\partial^2}{\partial x_j^2} \Big) (-\Delta)^{-1/2-n/4} g_k \, dx \\
&+ 2n \sum_{j\neq k} \int_{\mathbb{R}^n} g_j \frac{\partial^2}{\partial x_j \partial x_k} (-\Delta)^{-1/2-n/4} g_k \, dx.
\end{aligned} \tag{3}$$

By Parseval's formula

$$\int_{\mathbb{R}^n} \Phi \, \bar{\Psi} \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\Phi} \, \bar{\hat{\Psi}} \, d\xi$$

(see [**H**], vol. 1, p. 163), we write the right-hand side in (3) as

$$\begin{aligned}
&(2\pi)^{-n} \sum_{1\le k\le n} \int_{\mathbb{R}^n} 2\big(|\xi|^2 - n\,\xi_k^2\big)\, |\xi|^{-2-n}\, \hat{g}_k \, \bar{\hat{g}}_k \, d\xi \\
&- (2\pi)^{-n} 2n \sum_{j\neq k} \int_{\mathbb{R}^n} \xi_j \xi_k \, |\xi|^{-2-n} \hat{g}_j \, \bar{\hat{g}}_k \, d\xi.
\end{aligned} \tag{4}$$

Note that for $n > 2$

$$\begin{aligned}
\big(|\xi|^2 - n\,\xi_k^2\big)|\xi|^{-2-n} &= (2-n)^{-1} \frac{\partial^2}{\partial \xi_k^2} \frac{1}{|\xi|^{n-2}} - \frac{|S^{n-1}|}{n} \delta(\xi), \\
\xi_j \xi_k \, |\xi|^{-2-n} &= n^{-1}(n-2)^{-1} \frac{\partial^2}{\partial \xi_j \, \partial \xi_k} \frac{1}{|\xi|^{n-2}}, \qquad \text{for } j \neq k.
\end{aligned}$$

Analogously, for $n = 2$

$$\big(|\xi|^2 - 2\,\xi_k^2\big)|\xi|^{-4} = -\frac{\partial}{\partial \xi_k} \frac{\xi_k}{|\xi|^2} - \pi\, \delta(\xi), \tag{5}$$

$$\xi_j \xi_k \, |\xi|^{-4} = -\frac{1}{2}\frac{\partial}{\partial \xi_j}\frac{\xi_k}{|\xi|^2} \qquad \text{for } j \neq k. \tag{6}$$

Therefore, in the case $n > 2$, we express (4) as

$$(2\pi)^{-n} \sum_{1\leq k\leq n} \int_{\mathbb{R}^n} \left(\frac{2}{2-n}\frac{\partial^2}{\partial \xi_k^2}\frac{1}{|\xi|^{n-2}} - \frac{2|S^{n-1}|}{n}\,\delta(\xi) \right) \hat{g_k}(\xi)\,\bar{\hat{g_k}}(\xi)\,d\xi$$

$$-(2\pi)^{-n} 2n \sum_{j\neq k} \int_{\mathbb{R}^n} \frac{1}{n(n-2)} \Big(\frac{\partial^2}{\partial \xi_j \partial \xi_k}\frac{1}{|\xi|^{n-2}} \Big) \hat{g_k}(\xi)\bar{\hat{g_j}}(\xi)\,d\xi.$$

Using Parseval's formula once more, we write the last formula for (4) as

$$\sum_{1\leq k\leq n} \int_{\mathbb{R}^n} g_k(x)\,\mathcal{F}^{-1}_{\xi\to x} \left(\left(\frac{2}{2-n}\frac{\partial^2}{\partial \xi_k^2}\frac{1}{|\xi|^{n-2}} - \frac{2|S^{n-1}|}{n}\,\delta(\xi) \right) \hat{g_k}(\xi) \right) dx$$

$$-\sum_{j\neq k} \int_{\mathbb{R}^n} \frac{2}{n-2}\, g_j(x)\,\mathcal{F}^{-1}_{\xi\to x} \left(\left(\frac{\partial^2}{\partial \xi_j \partial \xi_k}\frac{1}{|\xi|^{n-2}} \right) \hat{g_k}(\xi) \right) dx, \tag{7}$$

where $\mathcal{F}^{-1}$ means the inverse Fourier transform (see formula (7.1.4) in [**H**]). Since $\mathcal{F}^{-1}(\hat{u}\,\hat{v}) = u * v$, where $*$ denotes the convolution, we have

$$\mathcal{F}^{-1}_{\xi\to x} \left(\left(\frac{\partial^2}{\partial \xi_j \partial \xi_k}\frac{1}{|\xi|^{n-2}} \right) \hat{h}(\xi) \right) = -\left(x_j\, x_k \Big(\mathcal{F}^{-1}_{\xi\to x} \frac{1}{|\xi|^{n-2}} \Big) \right) * h$$

$$= -(2\pi)^{-n} |S^{n-1}| \frac{x_j\, x_k}{|x|^2} * h \tag{8}$$

for $1 \leq j, k \leq n$.

Now let $n = 2$. By (5) and Parseval's formula, we write (4) in the form analogous to (7)

$$(2\pi)^{-2} \sum_{1\leq k\leq 2} \int_{\mathbb{R}^2} \Big(-2\frac{\partial}{\partial \xi_k}\frac{\xi_k}{|\xi|^2} - 2\pi\,\delta(\xi) \Big) \hat{g_k}(\xi)\,\bar{\hat{g_k}}(\xi)\,d\xi$$

$$+\, 2(2\pi)^{-2} \sum_{j\neq k} \int_{\mathbb{R}^2} \Big(\frac{\partial}{\partial \xi_j}\frac{\xi_k}{|\xi|^2} \Big) \hat{g_k}(\xi)\,\bar{\hat{g_j}}(\xi)\,d\xi$$

$$= \sum_{1\leq k\leq 2} \int_{\mathbb{R}^2} \mathcal{F}^{-1}_{\xi\to x} \left(\big(-2\frac{\partial}{\partial \xi_k}\frac{\xi_k}{|\xi|^2} - 2\pi\,\delta(\xi) \big) \hat{g_k}(\xi) \right) g_k(x)\,dx$$

$$+\, 2\sum_{j\neq k} \int_{\mathbb{R}^2} \mathcal{F}^{-1}_{\xi\to x} \left(\big(\frac{\partial}{\partial \xi_j}\frac{\xi_k}{|\xi|^2} \big) \hat{g_k}(\xi) \right) g_j(x)\,dx. \tag{9}$$

We check directly that

$$\mathcal{F}^{-1}_{\xi\to x} \left(\big(\frac{\partial}{\partial \xi_j}\frac{\xi_k}{|\xi|^2} \big) \hat{h}(\xi) \right) = i\, x_j \mathcal{F}^{-1}_{\xi\to x} \frac{\xi_k}{|\xi|^2}\hat{h}(\xi) = -(2\pi)^{-2}\frac{x_j\, x_k}{|x|^2} * h.$$

Combining this with (8), we deduce from (7) and (9) that for every $n \geq 2$ the expression (4) is equal to

$$\sum_{k}^{n} \int_{\mathbb{R}^n} \Big(\Big(\frac{2|S|^{n-1}}{(2\pi)^n} \frac{x_k^2}{|x|^2} - \frac{2|S|^{n-1}}{(2\pi)^n n} \Big) * g_k \Big) g_k \, dx + \sum_{j \neq k} \frac{2|S|^{n-1}}{(2\pi)^n} \Big(\frac{x_j \, x_k}{|x|^2} * g_k \Big) g_j \, dx$$

$$= \frac{2|S|^{n-1}}{(2\pi)^n} \Bigg(\int_{\mathbb{R}^n} \sum_{1 \le j,k \le n} \Big(\frac{x_j \, x_k}{|x|^2} * g_j \Big) g_k \, dx - \frac{1}{n} \sum_{k=1}^{n} \Big(\int_{\mathbb{R}^n} g_k \, dx \Big)^2 \Bigg)$$

$$= \frac{2|S|^{n-1}}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{M}\Big(\frac{x-y}{|x-y|} \Big) g(x) \, g(y) \, dx \, dy,$$

where $\mathcal{M}(\omega)$ is the $(n \times n)$-matrix with the elements $\omega_j \, \omega_k - n^{-1} \delta_j^k$. Since the norm of $\mathcal{M}(\omega)$ does not exceed $(n-1) \, n^{-1}$, it follows that the last integral is not greater than

$$\frac{2|S|^{n-1}(n-1)}{(2\pi)^n \, n} \Big(\int_{\mathbb{R}^n} |g(x)| \, dx \Big)^2. \tag{10}$$

Hence (10) is a majorant for the left-hand side of (3). This fact implies the estimate

$$\|(\Delta)^{-1/2-n/4} (\Delta \, u - \operatorname{div} \mathbf{f})\|^2_{L^2}$$

$$\le \frac{1}{n-1} \sum_{1 \le k < j \le n} \Big\| (\Delta)^{-1/2-n/4} \Big(\frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k} \Big) \Big\|^2_{L^2} + \frac{|S^{n-1}|}{n \, (2\pi)^n} \, \|\mathbf{g}\|^2_{L^1}$$

which is the same as

$$\|\Delta \, u - \operatorname{div} \mathbf{f}\|^2_{H^{-1-n/2}} \le \frac{1}{n-1} \sum_{1 \le k < j \le n} \Big\| \frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k} \Big\|^2_{H^{-1-n/2}} + \frac{|S^{n-1}|}{n \, (2\pi)^n} \, \|\mathbf{g}\|^2_{L^1}.$$

This implies

$$\|u\|_{H^{1-n/2}} \le \|\operatorname{div} \mathbf{f}\|_{H^{-1-n/2}}$$

$$+ \Big(\frac{1}{n-1} \sum_{1 \le k < j \le n} \Big\| \frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k} \Big\|^2_{H^{-1-n/2}} + \frac{|S^{n-1}|}{n \, (2\pi)^n} \, \|\mathbf{g}\|^2_{L^1} \Big)^{1/2}. \tag{11}$$

Since, obviously,

$$\sum_{1 \le k < j \le n} (\xi_k \, \eta_j - \xi_j \, \eta_k)^2 \le (n-1) \, |\xi|^2 \, |\eta|^2$$

and $(\xi, \eta)^2 \le |\xi|^2 \, |\eta|^2$ for all ξ and η in $\mathbb{R}^n$, we have

$$\sum_{1 \le k < j \le n} \Big\| \frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k} \Big\|^2_{H^{-1-n/2}} \le (n-1) \, \|\mathbf{f}\|^2_{H^{-n/2}}$$

and

$$\|\operatorname{div} \mathbf{f}\|_{H^{-1-n/2}} \le \|\mathbf{f}\|_{H^{-n/2}}.$$

These inequalities combined with (11) lead to (2).

The theorem just proved implies the following result by duality .

Corollary 0.2. *For any* $h \in H^{(n-2)/2}(\mathbb{R}^n)$ *there exists a solution*

$$\mathbf{v} \in H^{n/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

of the equation $\operatorname{div} \mathbf{v} = h$ *in* $\mathbb{R}^n$.

For $n = 2$ the above assertion is essentially contained in [**BB1**].

Remark 1. Let $n = 2$. Then (11) becomes

$$\|u\|_{L^2(\mathbb{R}^2)} \le \|\operatorname{div} \mathbf{f}\|_{H^{-2}(\mathbb{R}^2)} + \Big(\|\operatorname{curl} \mathbf{f}\|^2_{H^{-2}(\mathbb{R}^2)} + \frac{1}{4\pi} \|\mathbf{g}\|^2_{L^1(\mathbb{R}^2)}\Big)^{1/2}. \tag{12}$$

According to [**FF**] and [**M1**], the best constant in the Gagliardo-Nirenberg inequality

$$\|u\|_{L^{n/(n-1)}} \le c_n \|\nabla u\|_{L^1}$$

is equal to the best constant in the isoperimetric inequality

$$(\operatorname{mes}_n E)^{(n-1)/n} \le c_n \mathcal{H}_{n-1}(\partial E),$$

where $E \subset \mathbb{R}^n$ and $\mathcal{H}_{n-1}(\partial E)$ is the $(n-1)$-dimensional Hausdorff measure, that is,

$$c_n = \big(n^{n-1} |S^{n-1}|\big)^{-1/n}$$

and, in particular, $c_2 = (4\pi)^{-1/2}$. Setting $\mathbf{f} = 0$, we see that *the constant* $(4\pi)^{-1}$ *in* (12) *is sharp.*

Clearly, the inequality (12) becomes equality if $\mathbf{g} = \mathbf{0}$, i.e. $\nabla u = \mathbf{f}$.

Remark 2. Let χ_Ω be the characteristic function of a domain Ω in $\mathbb{R}^2$ whose boundary has a finite perimeter in the sense of De Giorgi (see, for instance, [**M2**], Ch. 6 for the definitions and properties of the perimeter $P(\Omega)$, the reduced boundary $\partial^*\Omega$ and Federer's normal ν at the points of $\partial^*\Omega$.) By u we denote an arbitrary Lipschitz function with compact support in $\mathbb{R}^2$. Then

$$\nabla(\chi_\Omega u) = \chi_\Omega \nabla u + u \nabla \chi_\Omega.$$

Now we see that the same proof allows us to replace an integrable $\mathbf{g}$ in (11) by a vector valued charge $\mathbf{g}$ with the norm

$$\|\mathbf{g}\|_{C^*(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\mathbf{g}(dx)|.$$

Therefore, we can choose $\mathbf{f} = \chi_\Omega \nabla u$ and $\mathbf{g} = u \nabla \chi_\Omega$ and obtain the inequality

$$\|u\|_{L^2(\Omega)} \le \|\chi_\Omega \nabla u\|_{H^{-1}(\mathbb{R}^2)} + \Big(\|\chi_\Omega \nabla u\|^2_{H^{-1}(\mathbb{R}^2)} + \frac{1}{4\pi}\Big(\int_{\partial^*\Omega} |u|\, \mathcal{H}_1(dx)\Big)^2\Big)^{1/2}$$

One can verify that

$$\|\chi_\Omega \nabla u\|_{H^{-1}(\mathbb{R}^2)} = (2\pi)^{-1} \Big\| \int_\Omega \frac{\nabla u(y)\, dy}{|\cdot - y|} \Big\|_{L^2(\mathbb{R}^2)}$$

The alternate choice $\mathbf{f} = u \nabla \chi_\Omega$ and $\mathbf{g} = \chi_\Omega \nabla u$ leads to the inequality

$$\|u\|_{L^2(\Omega)} \le \|u \nabla \chi_\Omega\|_{H^{-1}(\mathbb{R}^2)} + \Big(\|u \nabla \chi_\Omega\|^2_{H^{-1}(\mathbb{R}^2)} + \frac{1}{4\pi}\Big(\int_\Omega |\nabla u(x)|\, dx\Big)^2\Big)^{1/2}.$$

Note that

$$\|u\nabla\,\chi_\Omega\|_{H^{-1}(\mathbb{R}^2)} = (2\pi)^{-1}\Big\|\int_{\partial^*\Omega}\frac{u(y)\nu(y)}{|\cdot - y|}\mathcal{H}_1(dy)\Big\|_{L^2(\mathbb{R}^2)},$$

where ν is the unit normal vector in the sense of Federer. A somewhat similar sharp inequality

$$\|u\|_{L^2(\Omega)} \le (4\pi)^{-1/2}\Big(\int_\Omega |\nabla\,u|\,dx + \int_{\partial\Omega}|u|\,\mathcal{H}_1(dx)\Big)$$

can be found in [**M1**] and in [**M2**], Corollary 3.6.3.

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