A collection of sharp dilation invariant inequalities for differentiable functions

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In memory of S. L. Sobolev

Abstract. We find best constants in several dilation invariant integral inequalities involving derivatives of functions. Some of these inequalities are new and some were known without best constants. The contents: 1. Estimate for a quadratic form of the gradient, 2. Weighted Gårding inequality for the biharmonic operator, 3. Dilation invariant Hardy's inequalities with remainder term, 4. Generalized Hardy-Sobolev inequality with sharp constant, 5. Hardy's inequality with sharp Sobolev remainder term.

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Introduction

The present article consists of five independent sections dealing with various dilation invariant integral inequalities with optimal constants. We briefly describe the contents, starting with Section 1.

Let us recall the Gagliardo-Nirenberg inequality

$$\|v\|_{L^2(\mathbb{R}^2)} \le C\,\|\nabla v\|_{L^1(\mathbb{R}^2)}, \quad v \in C_0^\infty(\mathbb{R}^2) \tag{1}$$

[G], [N]. (The best constant $C = (2\sqrt{\pi})^{-1}$ was found in [FF] and [M1], see also [M5], Sect. 1.4.2). Setting $v = |\nabla u|$, we observe that the Dirichlet integral of u admits the estimate

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx \le C\left(\int_{\mathbb{R}^2} |\nabla_2 u|\, dx\right)^2, \tag{2}$$

where

$$|\nabla_2 u|^2 = |u_{x_1x_1}|^2 + 2|u_{x_1x_2}|^2 + |u_{x_2x_2}|^2.$$

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One can see that it is impossible to improve (2), replacing $|\nabla_2 u|$ in the right-hand side by $|\Delta u|$. Indeed, it suffices to put a sequence of mollifications of the function $x \to \eta(x) \log |x|$, where $\eta \in C_0^\infty(\mathbb{R}^2)$, $\eta(0) \neq 0$, into the estimate in question in order to check its failure.

However, we show that the estimate of the same nature

$$\Big| \int_{\mathbb{R}^2} \sum_{i,j=1}^{2} a_{i,j}\, u_{x_i}\, \overline{u}_{x_j}\, dx \Big| \leq C \Big(\int_{\mathbb{R}^2} |\Delta u|\, dx \Big)^2,$$

where $a_{i,j} = const$ and u is an arbitrary complex-valued function in $C_0^\infty(\mathbb{R}^2)$, may hold if and only if $a_{11}+a_{22} = 0$. We also find the best constant C in the last inequality. This is a particular case of Theorem 1 proved in Section 1.

In Section 2 we establish a new weighted Gårding type inequality

$$\int_{\mathbb{R}^2} |\nabla_2 u|^2 \log(e^2|x|)^{-1} dx \leq \mathrm{Re} \int_{\mathbb{R}^2} \Delta^2 u \cdot \overline{u}\, \log |x|^{-1} dx \tag{3}$$

for all $u \in C_0^\infty(\mathbb{R}^2 \backslash \{0\})$. Estimates of such a kind proved to be useful in the study of boundary behavior of solutions to elliptic equations (see [M4], [MT], [M6], [M7], [E], [MM]).

Before turning to the contents of the next section, we introduce some notation. By $\mathbb{R}^n_+$ we denote the half-space $\{x = (x_1, \dots x_n) \in \mathbb{R}^n, x_n > 0\}$. Also let $\mathbb{R}^{n-1} = \partial \mathbb{R}^n_+$. As usual, $C_0^\infty(\mathbb{R}^n_+)$ and $C_0^\infty(\overline{\mathbb{R}^n_+})$ stand for the spaces of infinitely differentiable functions with compact support in $\mathbb{R}^n_+$ and $\overline{\mathbb{R}^n_+}$, respectively.

In Section 3 we are concerned with the inequality

$$\int_{\mathbb{R}^n_+} x_n\, |\nabla u|^2 dx \geq \Lambda \int_{\mathbb{R}^n_+} \frac{|u|^2}{(x_{n-1}^2 + x_n^2)^{1/2}} dx, \quad u \in C_0^\infty(\overline{\mathbb{R}^n_+}). \tag{4}$$

It was obtained in 1972 by one of the authors and proved to be useful in the study of the generic case of degeneration in the oblique derivative problem for second order elliptic differential operators [M2].

By substituting $u(x) = x_n^{-1/2} v(x)$ into (4), one deduces with the same Λ that

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{|v|^2 dx}{x_n^2} + \Lambda \int_{\mathbb{R}^n_+} \frac{|v|^2 dx}{x_n (x_{n-1}^2 + x_n^2)^{1/2}} \tag{5}$$

for all $v \in C_0^\infty(\mathbb{R}^n_+)$ (see [M5], Sect. 2.1.6).

Another inequality of a similar nature obtained in [M5] is

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{|v|^2}{x_n^2} dx + C\, \| x_n^\gamma\, v \|^2_{L_q(\mathbb{R}^n_+)}. \tag{6}$$

(This is a special case of inequality (2.1.6/3) in [M5].)

Without the second term in the right-hand sides of (5) and (6), these inequalities reduce to the classical Hardy inequality with the sharp constant 1/4 (see [Da]). An interesting feature of (5) and (6) is their dilation invariance.

Variants, extensions, and refinements of (5) and (6), usually called Hardy's inequalities with remainder term, became the theme of many subsequent studies ([ACR], [Ad], [AGS], [BCT], [BFT1], [BFT2], [BFL], [BM], [BV], [CM], [DD], [DNY], [EL],

[FMT1]–[FMT3], [FT], [FTT], [FS], [GGM], [HHL], [TT], [TZ], [Ti1], [Ti2], [VZ], [YZ] *et al*).

In Theorem 3, proved in Section 3, we find a condition on the function q which is necessary and sufficient for the inequality

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 dx - \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{|v|^2 dx}{x_n^2} \geq C \int_{\mathbb{R}^n_+} q\Big(\frac{x_n}{(x_{n-1}^2+x_n^2)^{1/2}}\Big)\frac{|v|^2 dx}{x_n\,(x_{n-1}^2+x_n^2)^{1/2}}, \tag{7}$$

where v is an arbitrary function in $C_0^\infty(\mathbb{R}^n_+)$. This condition implies, in particular, that the right-hand side of (5) can be replaced by

$$C\int_{\mathbb{R}^n_+} \frac{|v|^2 dx}{x_n^2\Big(1-\log\dfrac{x_n}{(x_{n-1}^2+x_n^2)^{1/2}}\Big)^2}.$$

The value $\Lambda = 1/16$ in (4) obtained in [M2] is not best possible. Tidblom replaced it by $1/8$ in [Ti2]. As a corollary of Theorem 3, we find an expression for the optimal value of Λ.

Let the measure μ_b be defined by

$$\mu_b(K) = \int_K \frac{dx}{|x|^b} \tag{8}$$

for any compact set K in $\mathbb{R}^n$. In Section 4 we obtain the best constant in the inequality

$$\|u\|_{\mathcal{L}_{\tau,q}(\mu_b)} \leq C\Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p \frac{dx}{|x|^a}\Big)^{1/p},$$

where the left-hand side is the quasi-norm in the Lorentz space $\mathcal{L}_{\tau,q}(\mu_b)$, i.e.

$$\|u\|_{\mathcal{L}_{\tau,q}(\mu_b)} = \Big(\int_0^\infty \big(\mu_b\{x : |u(x)| \geq t\}\big)^{q/\tau} d(t^q)\Big)^{1/q}.$$

As a particular case of this result we obtain the best constant in the Hardy-Sobolev inequality

$$\Big(\int_{\mathbb{R}^n} |u(x)|^q \frac{dx}{|x|^b}\Big)^{1/q} \leq \mathcal{C}\Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p \frac{dx}{|x|^a}\Big)^{1/p}, \tag{9}$$

first proved by Il'in in 1961 in [Il] (Th. 1.4) without discussion of the value of $\mathcal{C}$. Our result is a direct consequence of the capacitary integral inequality from [M7] combined with an isocapacitary inequality. For particular cases the best constant $\mathcal{C}$ was found in [CW] ($p = 2$), in [M3], Sect. 2 ($p = 1$, $a = 0$), in [GMGT] ($p = 2$, $n = 3$, $a = 0$), in [L] ($p = 2$, $n \geq 3$, $a = 0$), and in [Na] ($1 < p < n$, $a = 0$), where different methods were used.

The topic of the concluding Section 5 is the best constant C in the inequality (6), where $u \in C^\infty(\overline{\mathbb{R}^n_+})$, $u = 0$ on $\mathbb{R}^{n-1}$.

Recently Tertikas and Tintarev [TT] obtained (among other results) the existence of an optimizer in (6) in the case $\gamma = 0$, $q = 2n/(n-2)$, $n \geq 4$. However, for these values of γ, q, and n the best value of C is unknown. In the case $n = 3$, $\gamma = 0$, $q = 6$ Benguria, Frank, and Loss proved the nonexistence of an optimizer and found the best value of C by an ingenious argument [BFL].

We note in Section 5 that a similar problem can be easily solved for the special case $q = 2(n+1)/(n-1)$ and $\gamma = -1/(n+1)$.

1 Estimate for a quadratic form of the gradient

Theorem 1 *Let $n \geq 2$ and let $A = \|a_{i,j}\|_{i,j=1}^n$ be an arbitrary matrix with constant complex entries. The inequality*

$$\Big| \int_{\mathbb{R}^n} \langle A\nabla u, \nabla u \rangle_{\mathbb{C}^n} \, dx \Big| \leq C \Big(\int_{\mathbb{R}^n} \big| (-\Delta)^{\frac{n+2}{4}} u \big| \, dx \Big)^2, \tag{10}$$

where C is a positive constant, holds for all complex-valued $u \in C_0^\infty(\mathbb{R}^n)$ if and only if the trace of A is equal to zero. The best value of C is given by

$$C = \frac{(4\pi)^{-n/2}}{\Gamma\big(\frac{n}{2}+1\big)} \max_{\omega \in S^{n-1}} \Big| \sum_{1 \leq i,j \leq n} a_{i,j} \, \omega_i \omega_j \Big|, \tag{11}$$

where S^{n-1} is the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$.

(The notation $(-\Delta)^s$ in (10) stands for an integer or noninteger power of $-\Delta$.)

Proof. By $\mathcal{F}$ we denote the unitary Fourier transform in $\mathbb{R}^n$ defined by

$$\mathcal{F}h(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} h(x) \, e^{-i\, x \cdot \xi} \, dx. \tag{12}$$

We set $h = (-\Delta)^{(n+2)/4} u$ and write (10) in the form

$$\Big| \int_{\mathbb{R}^n} |\mathcal{F}h(\xi)|^2 \Big\langle A\frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \Big\rangle_{\mathbb{C}^n} \frac{d\xi}{|\xi|^n} \Big| \leq C \Big(\int_{\mathbb{R}^n} |h(x)| \, dx \Big)^2. \tag{13}$$

The singular integral in the left-hand side exists in the sense of the Cauchy principal value, since

$$\int_{S^{n-1}} \langle A\omega, \omega \rangle_{\mathbb{C}^n} \, ds_\omega = n^{-1} |S^{n-1}| \operatorname{Tr} A = 0,$$

where $\operatorname{Tr} A$ is the trace of A (see, for example, [MP], Ch. 9, Sect. 1 or [SW], Theorem 4.7). Let

$$k(\xi) = |\xi|^{-n} \Big\langle A\frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \Big\rangle_{\mathbb{C}^n}.$$

The left-hand side in (13) equals

$$\Big| \int_{\mathbb{R}^n} \mathcal{F}^{-1}\Big(k(\xi) \big(\mathcal{F}h\big)(\xi) \Big)(x) \, \overline{h(x)} \, dx \Big| = (2\pi)^{-n/2} \Big| \int_{\mathbb{R}^n} \big((\mathcal{F}^{-1}k) * h \big)(x) \, \overline{h(x)} \, dx \Big|$$

with $*$ meaning the convolution. Thus, inequality (13) becomes

$$\Big| \int_{\mathbb{R}^n} \big((\mathcal{F}^{-1}k) * h \big)(x) \, \overline{h(x)} \, dx \Big| \leq (2\pi)^{n/2} \, C \Big(\int_{\mathbb{R}^n} |h(x)| \, dx \Big)^2. \tag{14}$$

We note that for $\xi \in \mathbb{R}^n$,

$$k(\xi) = |\xi|^{-n-2} \Big(\sum_{j=1}^n a_{jj} \big(\xi_j^2 - n^{-1} |\xi|^2 \big) + \sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij} \xi_i \xi_j \Big). \tag{15}$$

Hence, for $n > 2$

$$k(\xi) = \frac{1}{n\,(n-2)} \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j} |\xi|^{2-n}, \tag{16}$$

and for $n = 2$

$$k(\xi) = \frac{1}{2} \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j} \log |\xi|^{-1}. \tag{17}$$

Applying $\mathcal{F}^{-1}$ to the identity

$$-\Delta_\xi \Big(\frac{\partial^2}{\partial \xi_i \partial \xi_j} \frac{|\xi|^{2-n}}{|S^{n-1}|\,(n-2)} \Big) = \frac{\partial^2}{\partial \xi_i \partial \xi_j} \delta(\xi),$$

where $n > 2$ and δ is the Dirac function, we obtain from (16)

$$\big(\mathcal{F}^{-1} k\big)(x) = \frac{-|S^{n-1}|}{n\,(2\pi)^{n/2}} \sum_{i,j=1}^{n} a_{ij} \frac{x_i x_j}{|x|^2}. \tag{18}$$

Here $|S^{n-1}|$ stands for the $(n-1)$-dimensional measure of S^{n-1}:

$$|S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \tag{19}$$

Hence

$$\big(\mathcal{F}^{-1} k\big)(x) = \frac{-2^{-n/2}}{\Gamma(1+\frac{n}{2})} \sum_{i,j=1}^{n} a_{ij} \frac{x_i x_j}{|x|^2}. \tag{20}$$

Similarly, we deduce from (17) that (20) holds for $n = 2$ as well. Now, (10) with C given by (11) follows from (20) inserted into (14).

Next, we show the sharpness of C given by (11). Let θ denote a point on S^{n-1} such that

$$\big|\big(\mathcal{F}^{-1} k\big)(\theta)\big| = \max_{\xi \in \mathbb{R}^n \setminus \{0\}} |\mathcal{F}^{-1} k(\xi)|. \tag{21}$$

In order to obtain the required lower estimate for C, it suffices to set

$$h(x) = \eta(|x|)\, \delta_\theta\Big(\frac{x}{|x|}\Big),$$

where $\eta \in C_0^\infty[0,\infty)$, $\eta \geq 0$, and δ_θ is the Dirac measure on S^{n-1} concentrated at θ, into the inequality (14). (The legitimacy of this choice of h can be easily checked by approximation.) Then estimate (14) becomes

$$\Big| \int_0^\infty \int_0^\infty \big(\mathcal{F}^{-1} k\big) \Big(\frac{\rho - r}{|\rho - r|} \theta \Big) \eta(r)\, r^{n-1}\, \eta(\rho)\, \rho^{n-1} dr d\rho \Big|$$
$$\leq (2\pi)^{n/2}\, C \Big(\int_0^\infty \eta(\rho)\, \rho^{n-1} d\rho \Big)^2. \tag{22}$$

In view of (18) and (20),

$$\big(\mathcal{F}^{-1} k\big)(\pm\theta) = \big(\mathcal{F}^{-1} k\big)(\theta)$$

which together with (21) enables one to write (22) in the form

$$\max_{\xi \in \mathbb{R}^n \setminus \{0\}} |\mathcal{F}^{-1} k(\xi)| \leq (2\pi)^{n/2}\, C.$$

By (18) and (20) this can be written as

$$\frac{|S^{n-1}|}{n\,(2\pi)^{n/2}} \max_{\omega \in S^{n-1}} \Big| \sum_{1 \leq i,j \leq n} a_{ij}\, \omega_i \omega_j \Big| \leq (2\pi)^{n/2}\, C.$$

The result follows from (19). $\square$

Remark 1 Let P and Q be functions, positively homogeneous of degrees $2m$ and $m+n/2$ respectively, $m > -n/2$. We assume that the restrictions of P, Q, and $P|Q|^{-2}$ to S^{n-1} belong to $L^1(S^{n-1})$, By the same argument as in Theorem 1, one concludes that the condition

$$\int_{S^{n-1}} \frac{P(\omega)}{|Q(\omega)|^2}\, ds_\omega = 0 \tag{23}$$

is necessary and sufficient for the inequality

$$\Big|\int_{\mathbb{R}^n} P(D)u \cdot \overline{u}\, dx\Big| \le C\Big(\int_{\mathbb{R}^n} \big|Q(D)\, u\big|\, dx\Big)^2 \tag{24}$$

to hold for all $u \in C_0^\infty(\mathbb{R}^n)$. Moreover, using the classical formula for the Fourier transform of a positively homogeneous function of degree $-n$ (see Theorem 4.11 in [SW][1]), one obtains that the best value of C in (24) is given by

$$\sup_{\omega\in S^{n-1}} \Big|\int_{S^{n-1}} \Big(\frac{i\pi}{2}\mathrm{sgn}(\theta\cdot\omega) + \log|\theta\cdot\omega|\Big)\frac{P(\theta)}{|Q(\theta)|^2}\, ds_\theta\Big|. \tag{25}$$

In particular, if $P(\omega)/|Q(\omega)|^2$ is a spherical harmonic, the best value of C in (24) is equal to

$$\frac{(4\pi)^{-n/2}\Gamma(m)}{\Gamma\big(\frac{n}{2}+m\big)} \max_{\omega\in S^{n-1}} \frac{|P(\omega)|}{|Q(\omega)|^2}, \tag{26}$$

which coincides with (11) for $m=1$, $P(\xi) = A\xi\cdot\xi$, and $Q(\xi) = |\xi|^{1+n/2}$.

2 Weighted Gårding inequality for the biharmonic operator

We start with an auxiliary Hardy type inequality.

Lemma 1 *Let $u \in C_0^\infty(\mathbb{R}^2)$. Then the sharp inequality*

$$\Big|\mathrm{Re}\int_{\mathbb{R}^2} (x_1 u_{x_1} + x_2 u_{x_2})\Delta\overline{u}\, \frac{dx}{|x|^2}\Big| \le \int_{\mathbb{R}^2} |\Delta u|^2 dx \tag{27}$$

holds.

Proof. Let (r,φ) denote polar coordinates in $\mathbb{R}^2$ and let

$$u(r,\varphi) = \sum_{k=-\infty}^{\infty} u_k(r)e^{ik\varphi}.$$

Then (27) is equivalent to the sequence of inequalities

$$\Big|\mathrm{Re}\int_0^\infty \Big(v'' + \frac{1}{r}v' - \frac{k^2}{r^2}v\Big)\overline{v}'\, dr\Big| \le \int_0^\infty \Big|v'' + \frac{1}{r}v' - \frac{k^2}{r^2}v\Big|^2 r\, dr, \quad k = 0,1,2,\dots \tag{28}$$

[1] Note that the definition of the Fourier transform in [SW] contains $\exp(-2\pi i\, x\cdot\xi)$ unlike (12).

where v is an arbitrary function on $C_0^\infty([0,\infty))$. Putting $t = \log r^{-1}$ and $w(t) = v(e^{-t})$, we write (28) in the form

$$\Big|\mathrm{Re}\int_{\mathbb{R}^1} \big(w'' - k^2 w\big)\, \overline{w}'\, e^{2t}\, dt\Big| \le \int_{\mathbb{R}^1} \big|w'' - k^2 w\big|^2 e^{2t}\, dt$$

which is equivalent to the inequality

$$\Big|\mathrm{Re}\int_{\mathbb{R}^1} \big(g'' - 2g' + (1-k^2)g\big)\big(\overline{g}' - \overline{g}\big)dt\Big| \le \int_{\mathbb{R}^1} \big|g'' - 2g' + (1-k^2)g\big|^2 dt, \tag{29}$$

where $g = e^t w$. Making use of the Fourier transform in t, we see that (29) holds if and only if for all $\lambda \in \mathbb{R}^1$ and $k = 0, 1, 2\ldots$

$$\Big|\mathrm{Re}\,\big(-\lambda^2 + 1 - k^2 - 2i\lambda\big)\big(1 - i\lambda\big)\Big| \le \big(\lambda^2 - 1 + k^2\big)^2 + 4\lambda^2,$$

which is the same as

$$\big|3x - 1 + k^2\big| \le x^2 + 2(k^2+1)x + (k^2-1)^2$$

with $x = \lambda^2$. This elementary inequality becomes equality if and only if $k = 0$ and $x = 0$. $\square$

Remark 2 In spite of the simplicity of its proof, inequality (27) deserves some interest. Let us denote the integral over $\mathbb{R}^2$ in the left-hand side of (27) by $Q(u,u)$ and write (27) as

$$|\,\mathrm{Re}\, Q(u,u)| \le \|\Delta u\|_{L^2(\mathbb{R}^2)}.$$

However, the absolute value of the corresponding sesquilinear form $Q(u,v)$ cannot be majorized by $C\|\Delta u\|_{L^2(\mathbb{R}^2)}\|\Delta v\|_{L^2(\mathbb{R}^2)}$. Indeed, the opposite assertion would yield an upper estimate of $\|r^{-1}\partial u/\partial r\|_{L^2(\mathbb{R}^2)}$ by the norm of Δu in $L^2(\mathbb{R}^2)$ which is wrong for a function linear near the origin. $\square$

Remark 3 Note that under the additional orthogonality assumption

$$\int_0^{2\pi} u(r,\varphi)d\varphi = 0 \quad \text{for } \ r > 0 \tag{30}$$

the above proof of Lemma 1 provides inequality (27) with the sharp constant factor 3/4 in the right-hand side. Besides, (30) implies

$$\mathrm{Re}\int_{\mathbb{R}^2} \big(x_1 u_{x_1} + x_2 u_{x_2}\big)\Delta\overline{u}\,\frac{dx}{|x|^2} \le 0. \qquad \square$$

Using (27), we establish a new weighted Gårding type inequality.

Theorem 2 *Let $u \in C_0^\infty(\mathbb{R}^2\backslash\{0\})$. Then inequality* (3) *holds.*

Proof. Clearly, the right-hand side in (3) is equal to

$$\mathrm{Re}\int_{\mathbb{R}^2} \Delta\overline{u}\,\Delta\,(u\log|x|^{-1})\,dx$$

$$= \int_{\mathbb{R}^2} |\Delta u|^2 \log|x|^{-1} dx + 2\mathrm{Re}\int_{\mathbb{R}^2} \Delta\overline{u}\cdot\nabla u\cdot\nabla\log|x|^{-1} dx.$$

Combining this identity with (27), we arrive at the inequality

$$\int_{\mathbb{R}^2} |\Delta u|^2 \log(e^2\,|x|)^{-1} dx \le \mathrm{Re} \int_{\mathbb{R}^2} \Delta\overline{u}\,\Delta(u\,\log|x|^{-1})\,dx. \tag{31}$$

Note that

$$\int_{\mathbb{R}^2} \Delta u \cdot \Delta\overline{u} \cdot \log|x|^{-1} = -\int_{\mathbb{R}^2} \nabla u \cdot \nabla(\Delta\overline{u} \cdot \log|x|^{-1})\,dx$$

$$= \mathrm{Re}\int_{\mathbb{R}^2} \sum_{j=1}^{2} \Big(\nabla \frac{\partial u}{\partial x_j} \cdot \frac{\partial}{\partial x_j}\nabla\overline{u} \cdot \log|x|^{-1} + \nabla u \cdot \frac{\partial}{\partial x_j}\nabla\overline{u} \cdot \frac{\partial}{\partial x_j}\big(\log|x|^{-1}\big)\Big)dx,$$

which is equal to

$$\int_{\mathbb{R}^2}\left(\sum_{j,k=1}^{2}\Big|\frac{\partial^2 u}{\partial x_j \partial x_k}\Big|^2 \log|x|^{-1} + \frac{1}{2}\sum_{j=1}^{2}\frac{\partial}{\partial x_j}|\nabla u|^2 \cdot \frac{\partial}{\partial x_j}\Big(\log|x|^{-1}\Big)\right)dx.$$

Integrating by parts in the second term, we see that it vanishes. Thus, we conclude that

$$\int_{\mathbb{R}^2} |\Delta u|^2 \log|x|^{-1} dx = \int_{\mathbb{R}^2} |\nabla_2 u|^2 \log|x|^{-1} dx$$

which together with (27) and the obvious identity

$$\int_{\mathbb{R}^2} |\Delta u|^2 dx = \int_{\mathbb{R}^2} |\nabla_2 u|^2 dx$$

completes the proof of (3).

In order to see that no constant less than 1 is admissible in front of the integral in the right-hand side of (3), it suffices to put

$$u(x) = e^{i\langle x,\xi\rangle}\,\eta(x)$$

with $\eta \in C_0^\infty(\mathbb{R}^2)\backslash\{0\}$ into (3) and take the limit as $|\xi| \to \infty$. $\square$

Remark 4 If the condition $u = 0$ near the origin in Theorem 2 is removed, the above proof gives the additional term

$$\pi\Big(|\nabla u(0)|^2 - 2\,\mathrm{Re}\big(u(0)\,\Delta\overline{u}(0)\big)\Big)$$

in the right-hand side of (3). $\square$

3 Dilation invariant Hardy's inequalities with remainder term

Theorem 3 *(i) Let q denote a locally integrable nonnegative function on $(0,1)$. The best constant in the inequality*

$$\int_{\mathbb{R}^n_+} x_n\,|\nabla u|^2 dx \ge C\int_{\mathbb{R}^n_+} q\Big(\frac{x_n}{(x_{n-1}^2 + x_n^2)^{1/2}}\Big)\frac{|u|^2}{(x_{n-1}^2+x_n^2)^{1/2}}dx, \tag{32}$$

for all $u \in C_0^\infty(\overline{\mathbb{R}^n_+})$, which is equivalent to (7), is given by

$$\lambda := \inf \frac{\displaystyle\int_0^{\pi/2}\Big(\big|y'(\varphi)\big|^2 + \frac{1}{4}\big|y(\varphi)\big|^2\Big)\sin\varphi\,d\varphi}{\displaystyle\int_0^{\pi/2}\big|y(\varphi)\big|^2 q(\sin\varphi)\,d\varphi}, \tag{33}$$

where the infimum is taken over all smooth functions on $[0, \pi/2]$.

(ii) *Inequalities* (32) *and* (7) *with a positive* C *hold if and only if*

$$\sup_{t\in(0,1)} (1-\log t)\int_0^t q(\tau)\,d\tau < \infty. \tag{34}$$

Moreover,

$$\lambda \sim \Big(\sup_{t\in(0,1)} (1-\log t)\int_0^t q(\tau)\,d\tau\Big)^{-1}, \tag{35}$$

where $a \sim b$ *means that* $c_1 a \le b \le c_2 a$ *with absolute positive constants* c_1 *and* c_2.

Proof (i) Let $U \in C_0^\infty(\overline{\mathbb{R}^2_+})$, $\zeta \in C_0^\infty(\mathbb{R}^{n-2})$, $x' = (x_1, \dots x_{n-2})$, and let $N = const > 0$. Putting

$$u(x) = N^{(2-n)/2}\zeta(N^{-1}x')U(x_{n-1}, x_n)$$

into (32) and passing to the limit as $N \to \infty$, we see that (32) is equivalent to the inequality

$$\int_{\mathbb{R}^2_+} x_2\big(|U_{x_1}|^2 + |U_{x_2}|^2\big)dx_1dx_2 \ge C\int_{\mathbb{R}^2_+} q\Big(\frac{x_2}{(x_1^2+x_2^2)^{1/2}}\Big)\frac{|U|^2dx_1dx_2}{(x_1^2+x_2^2)^{1/2}}, \tag{36}$$

where $U \in C_0^\infty(\overline{\mathbb{R}^2_+})$. Let (ρ, φ) be the polar coordinates of the point $(x_1, x_2) \in \mathbb{R}^2_+$. Then (36) can be written as

$$\int_0^\infty\int_0^\pi \Big(|U_\rho|^2 + \rho^{-2}|U_\varphi|^2\Big)\sin\varphi\,d\varphi\,\rho^2 d\rho \ge C\int_0^\infty\int_0^\pi |U|^2 q(\sin\varphi)\,d\varphi d\rho.$$

By the substitution

$$U(\rho,\varphi) = \rho^{-1/2}v(\rho,\varphi)$$

the left-hand side becomes

$$\int_0^\infty\int_0^\pi \Big(|\rho v_\rho|^2 + |v_\varphi|^2 + \frac14|v|^2\Big)\sin\varphi\,d\varphi\frac{d\rho}{\rho} - \mathrm{Re}\int_0^\pi\int_0^\infty \overline{v}\,v_\rho\,d\rho\,\sin\varphi\,d\varphi. \tag{37}$$

Since $v(0) = 0$, the second term in (37) vanishes. Therefore, (36) can be written in the form

$$\int_0^\infty\int_0^\pi \Big(|\rho v_\rho|^2 + |v_\varphi|^2 + \frac14|v|^2\Big)\sin\varphi\,d\varphi\frac{d\rho}{\rho} \ge C\int_0^\infty\int_0^\pi |v|^2 q(\sin\varphi)\,d\varphi\frac{d\rho}{\rho}. \tag{38}$$

Now, the definition (33) of λ shows that (38) holds with $C = \lambda$.

In order to show the optimality of this value of C, put $t = \log\rho$ and $v(\rho,\varphi) = w(t,\varphi)$. Then (38) is equivalent to

$$\int_{\mathbb{R}^1}\int_0^\pi \big(|w_t|^2 + |w_\varphi|^2 + \frac14|w|^2\big)\sin\varphi\,d\varphi\,dt \ge C\int_{\mathbb{R}^1}\int_0^\pi |w|^2 q(\sin\varphi)d\varphi\,dt. \tag{39}$$

Applying the Fourier transform $w(t,\varphi) \to \hat{w}(s,\varphi)$, we obtain

$$\int_{\mathbb{R}^1}\int_0^\pi \Big(|\hat{w}_\varphi|^2 + \Big(|s|^2 + \frac14\Big)|\hat{w}|^2\Big)\sin\varphi\,d\varphi\,ds \ge C\int_{\mathbb{R}^1}\int_0^\pi |\hat{w}|^2 q(\sin\varphi)d\varphi\,ds. \tag{40}$$

Putting here

$$\hat{w}(s,\varphi) = \varepsilon^{-1/2}\eta(s/\varepsilon)y(\varphi),$$

where $\eta \in C_0^\infty(\mathbb{R}^1)$, $\|\eta\|_{L^2(\mathbb{R}^1)} = 1$, and y is a function on $C^\infty([0,\pi])$, and passing to the limit as $\varepsilon \to 0$, we arrive at the estimate

$$\int_0^\pi \Big(|y'(\varphi)|^2 + \frac{1}{4}|y(\varphi)|^2\Big) \sin\varphi \, d\varphi \geq C \int_0^\pi |y(\varphi)|^2 q(\sin\varphi) d\varphi \tag{41}$$

where π can be changed for $\pi/2$ by symmetry. This together with (33) implies $\Lambda \leq \lambda$. The proof of (i) is complete.

(ii) Introducing the new variable $\xi = \log\cot\frac{\varphi}{2}$, we write (33) as

$$\lambda = \inf_z \frac{\displaystyle\int_0^\infty \Big(|z'(\xi)|^2 + \frac{|z(\xi)|^2}{4\,(\cosh\xi)^2}\Big) d\xi}{\displaystyle\int_0^\infty |z(\xi)|^2 \, q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}}. \tag{42}$$

Since

$$|z(0)|^2 \leq 2\int_0^1 \big(|z'(\xi)|^2 + |z(\xi)|^2\big) d\xi$$

and

$$\int_0^\infty |z(\xi)|^2 \frac{e^{2\xi}}{(1+e^{2\xi})^2} d\xi \leq 2\int_0^\infty |z(\xi) - z(0)|^2 \frac{d\xi}{\xi^2} + 2\,|z(0)|^2 \int_0^\infty \frac{e^{2\xi}}{(1+e^{2\xi})^2} d\xi$$
$$\leq 8\int_0^\infty |z'(\xi)|^2 d\xi + |z(0)|^2,$$

it follows from (42) that

$$\lambda \sim \inf_z \frac{\displaystyle\int_0^\infty |z'(\xi)|^2 d\xi + |z(0)|^2}{\displaystyle\int_0^\infty |z(\xi)|^2 \, q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}}. \tag{43}$$

Setting $z(\xi) = 1$ and $z(\xi) = \min\{\eta^{-1}\xi, 1\}$ for all positive ξ and fixed $\eta > 0$ into the ratio of quadratic forms in (43), we deduce that

$$\lambda \leq \min\Big\{\Big(\int_0^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}\Big)^{-1},\ \Big(\sup_{\eta>0} \eta \int_\eta^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}\Big)^{-1}\Big\}.$$

Hence,

$$\lambda \leq c\Big(\sup_{t\in(0,1)} (1 - \log t) \int_0^t q(\tau)\, d\tau\Big)^{-1}.$$

In order to obtain the converse estimate, note that

$$\int_0^\infty |z(\xi)|^2 \, q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}$$
$$\leq 2\,|z(0)|^2 \int_0^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi} + 2\int_0^\infty |z(\xi) - z(0)|^2 \, q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}.$$

The second term in the right-hand side is dominated by

$$8\sup_{\eta>0}\Big(\eta\int_\eta^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}\Big)\int_0^\infty |z'(\xi)|^2 d\xi$$

(see, for example, [M5], Sect. 1.3.1). Therefore,

$$\int_0^\infty |z(\xi)|^2\, q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi} \le 8\max\bigg\{\int_0^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}\,,$$

$$\sup_{\eta>0}\eta\int_\eta^\infty q\Big(\frac{1}{\cosh\sigma}\Big)\frac{d\sigma}{\cosh\sigma}\bigg\}\Big(\int_0^\infty |z'(\xi)|^2 d\xi + |z(0)|^2\Big)$$

which together with (43) leads to the lower estimate

$$\lambda \ge \min\Big\{\Big(\int_0^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}\Big)^{-1},\ \Big(\sup_{\eta>0}\eta\int_\eta^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}\Big)^{-1}\Big\}.$$

Hence,

$$\lambda \ge c\Big(\sup_{t\in(0,1)}(1-\log t)\int_0^t q(\tau)\,d\tau\Big)^{-1}.$$

The proof of (ii) is complete. $\square$

Since (34) holds for $q(t)=t^{-1}(1-\log t)^{-2}$, Theorem 3 (ii) leads to the following assertion.

Corollary 1 *There exists an absolute constant $C>0$ such that the inequality*

$$\int_{\mathbb{R}^n_+}|\nabla v|^2 dx - \frac{1}{4}\int_{\mathbb{R}^n_+}\frac{|v|^2dx}{x_n^2} \ge C\int_{\mathbb{R}^n_+}\frac{|v|^2dx}{x_n^2\Big(1-\log\dfrac{x_n}{(x_{n-1}^2+x_n^2)^{1/2}}\Big)^2} \tag{44}$$

holds for all $v\in C_0^\infty(\mathbb{R}^n_+)$. The best value of C is equal to

$$\lambda := \inf\frac{\displaystyle\int_0^\pi\Big[\big|y'(\varphi)\big|^2+\frac{1}{4}\big|y(\varphi)\big|^2\Big]\sin\varphi\,d\varphi}{\displaystyle\int_0^\pi\big|y(\varphi)\big|^2(\sin\varphi)^{-1}\big(1-\log\sin\varphi\big)^{-2}d\varphi}, \tag{45}$$

where the infimum is taken over all smooth functions on $[0,\pi/2]$. By numerical approximation, $\lambda=0.16\ldots$

A particular case of Theorem 3 corresponding to $q=1$ is the following assertion.

Corollary 2 *The sharp value of Λ in (4) and (5) is equal to*

$$\lambda := \inf\frac{\displaystyle\int_0^\pi\Big[\big|y'(\varphi)\big|^2+\frac{1}{4}\big|y(\varphi)\big|^2\Big]\sin\varphi\,d\varphi}{\displaystyle\int_0^\pi\big|y(\varphi)\big|^2d\varphi}\,, \tag{46}$$

where the infimum is taken over all smooth functions on $[0,\pi]$. By numerical approximation, $\lambda=0.1564\ldots$

Remark 5 Let us consider the Friedrichs extension $\tilde{\mathcal{L}}$ of the operator

$$\mathcal{L}: z\to -\big((\sin\varphi)z'\big)' + \frac{\sin\varphi}{4}z \tag{47}$$

defined on smooth functions on $[0, \pi]$. It is a simple exercise to show that the energy space of $\tilde{\mathcal{L}}$ is compactly imbedded into $L^2(0, \pi)$. Hence, the spectrum of $\tilde{\mathcal{L}}$ is discrete and λ defined by (46) is the smallest eigenvalue of $\tilde{\mathcal{L}}$.

Remark 6 The argument used in the proof of Theorem 3 (i) with obvious changes enables one to obtain the following more general fact. Let P and Q be measurable nonnegative functions in $\mathbb{R}^n$, positive homogeneous of degrees 2μ and $2\mu - 2$, respectively. The sharp value of C in

$$\int_{\mathbb{R}^n} P(x)|\nabla u|^2 dx \geq C \int_{\mathbb{R}^n} Q(x)|u|^2 dx, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{48}$$

is equal to

$$\lambda := \inf \frac{\displaystyle\int_{S^{n-1}} P(\omega)\Big(|\nabla_\omega Y|^2 + \big(\mu - 1 + \frac{n}{2}\big)^2 |Y|^2\Big) ds_\omega}{\displaystyle\int_{S^{n-1}} Q(\omega)|Y|^2 ds_\omega},$$

where the infimum is taken over all smooth functions on the unit sphere S^{n-1}. $\square$

A direct consequence of this assertion is the following particular case of (48).

Remark 7 Let p and q stand for locally integrable nonnegative functions on $(0, 1]$ and let $\mu \in \mathbb{R}^1$. If $n > 2$, the best value of C in

$$\int_{\mathbb{R}^n} |x|^{2\mu} p\big(\frac{x_n}{|x|}\big)|\nabla u|^2 dx \geq C \int_{\mathbb{R}^n} |x|^{2\mu-2} q\big(\frac{x_n}{|x|}\big)|u|^2 dx, \tag{49}$$

where $u \in C_0^\infty(\mathbb{R}^n)$, is equal to

$$\inf \frac{\displaystyle\int_0^\pi \Big(|y'(\theta)|^2 + \big(\mu - 1 + \frac{n}{2}\big)^2 |y(\theta)|^2\Big) p(\cos\theta)(\sin\theta)^{n-2} d\theta}{\displaystyle\int_0^\pi |y(\theta)|^2 q(\cos\theta)(\sin\theta)^{n-2} d\theta}, \tag{50}$$

with the infimum taken over all smooth functions on the interval $[0, \pi]$.

Formula (50) enables one to obtain a necessary and sufficient condition for the existence of a positive C in (49). Let us assume that the function

$$\theta \to \frac{(\sin\theta)^{2-n}}{p(\cos\theta)}$$

is locally integrable on $(0, \pi)$. We make the change of variable $\xi = \xi(\theta)$, where

$$\xi(\theta) = \int_{\pi/2}^\theta \frac{(\sin\tau)^{2-n}}{p(\cos\tau)} d\tau,$$

and suppose that $\xi(0) = -\infty$ and $\xi(\pi) = \infty$. Then (50) can be written in the form

$$\lambda = \inf_z \frac{\displaystyle\int_{\mathbb{R}^1} |z'(\xi)|^2 d\xi + \big(\mu - 1 + \frac{n}{2}\big)^2 \int_{\mathbb{R}^1} |z(\xi)|^2 \big(p(\cos\theta(\xi))\,(\sin\theta(\xi))^{n-2}\big)^2 d\xi}{\displaystyle\int_{\mathbb{R}^1} |z(\xi)|^2\, p(\cos\theta(\xi))\, q(\cos\theta(\xi))(\sin\theta(\xi))^{2(n-2)}\, d\xi},$$

where $\theta(\xi)$ is the inverse function of $\xi(\theta)$. By Theorem 1 in [M10],

$$\lambda \sim \inf_{\substack{\xi \in \mathbb{R}^1 \\ d>0, \delta>0}} \frac{\displaystyle\frac{1}{\delta} + \int_{\theta(\xi-d-\delta)}^{\theta(\xi+d+\delta)} p(\cos\theta)\,(\sin\theta)^{n-2}\, d\theta}{\displaystyle\int_{\theta(\xi-d)}^{\theta(\xi+d)} q(\cos\theta)\,(\sin\theta)^{n-2}\, d\theta}. \tag{51}$$

Here the equivalence $a \sim b$ means that $c_1 b \le a \le c_2 b$, where c_1 and c_2 are positive constants depending only on μ and n. Hence (49) holds with a positive Λ if and only if the infimum (51) is positive. $\square$

Remark 8 In the case $n = 2$ the best constant in (49) is equal to

$$\lambda := \inf_y \frac{\displaystyle\int_0^{2\pi} \Big(|y'(\varphi)|^2 + \mu^2 |y(\varphi)|^2\Big) p(\sin\varphi)\, d\varphi}{\displaystyle\int_0^{2\pi} |y(\varphi)|^2 q(\sin\varphi)\, d\varphi}, \tag{52}$$

where the infimum is taken over all smooth functions on the interval $[0, 2\pi]$. Note that (33) is a particular case of (52) with $\mu = 1/2$ and $p(t) = t$. As another application of (52) we obtain the following special case of inequality (49) with $n = 2$.

For all $u \in C_0^\infty(\mathbb{R}^2)$ the inequality

$$\int_{\mathbb{R}^2} \frac{x_2^2\, |u(x)|^2\, dx}{(x_1^2 + x_2^2)\Big(\dfrac{\pi^2}{4} - \Big(\arcsin\dfrac{x_1}{(x_1^2+x_2^2)^{1/2}}\Big)^2\Big)} \le \frac{1}{2} \int_{\mathbb{R}^2} x_2^2\, |\nabla u(x)|^2 dx \tag{53}$$

holds, where $1/2$ is the best constant.

In order to prove (53), we choose $p(t) = t^2$ and $\mu = 1$ in (52), which becomes

$$\lambda = \inf_y \frac{\displaystyle\int_0^{2\pi} \Big(|y'(\varphi)|^2 + |y(\varphi)|^2\Big) (\sin\varphi)^2 d\varphi}{\displaystyle\int_0^{2\pi} |y(\varphi)|^2 q(\sin\varphi)\, d\varphi}, \tag{54}$$

where y is an arbitrary smooth 2π-periodic function. Putting $\eta(\varphi) = y(\varphi)\sin\varphi$, we write (54) in the form

$$\lambda = \inf_\eta \frac{\displaystyle\int_0^{2\pi} |\eta'(\varphi)|^2 d\varphi}{\displaystyle\int_0^{2\pi} |\eta(\varphi)|^2 q(\sin\varphi)(\sin\varphi)^{-2} d\varphi} \tag{55}$$

with the infimum taken over all 2π-periodic functions satisfying $\eta(0) = \eta(\pi) = 0$. Let

$$q(\sin\varphi) = \frac{(\sin\varphi)^2}{\varphi(\pi - \varphi)}.$$

In view of the well-known sharp inequality

$$\int_0^1 \frac{|z(t)|^2}{t(1-t)} dt \le \frac{1}{2} \int_0^1 |z'(t)|^2 dt$$

(see [HLP], Th. 262), we have $\lambda = 2$ in (55). Therefore, (49) becomes (53). $\square$

4 Generalized Hardy-Sobolev inequality with sharp constant

Let Ω denote an open set in $\mathbb{R}^n$ and let $p \in [1, \infty)$. By (p, a)-capacity of a compact set $K \subset \Omega$ we mean the set function

$$\mathrm{cap}_{p,a}(K, \Omega) = \inf\Big\{\int_\Omega |\nabla u|^p |x|^{-a}\, dx : \ u \in C_0^\infty(\Omega),\ u \ge 1 \text{ on } K\Big\}.$$

In the case $a = 0$, $\Omega = \mathbb{R}^n$ we write simply $\mathrm{cap}_p(K)$.

The following inequality is a particular case of a more general one obtained in [M8], where Ω is an open subset of an arbitrary Riemannian manifold and $|\Phi(x, \nabla u(x)|$ plays the role of $|\nabla u(x)|\,|x|^{-a/p}$.

Theorem 4 (see [M3] for $q = p$ and [M8] for $q \geq p$)
(i) Let $q \geq p \geq 1$ and let Ω be an open set in $\mathbb{R}^n$. Then for an arbitrary $u \in C_0^\infty(\Omega)$,

$$\Big(\int_0^\infty \big(\mathrm{cap}_{p,a}(M_t, \Omega)\big)^{q/p} d(t^q)\Big)^{1/q} \leq \mathcal{A}_{p,q} \Big(\int_\Omega |\nabla u(x)|^p |x|^{-a} dx\Big)^{1/p}, \qquad (56)$$

where $M_t = \{x \in \Omega : \ |u(x)| \geq t\}$ and

$$\mathcal{A}_{p,q} = \Big(\frac{\Gamma\big(\frac{pq}{q-p}\big)}{\Gamma\big(\frac{q}{q-p}\big)\Gamma\big(p\frac{q-1}{q-p}\big)}\Big)^{1/p-1/q} \qquad (57)$$

for $q > p$, and

$$\mathcal{A}_{p,p} = p(p-1)^{(1-p)/p}. \qquad (58)$$

(ii) The sharpness of this constant is checked by a sequence of radial functions in $C_0^\infty(\mathbb{R}^n)$. Moreover, there exists a radial optimizer vanishing at infinity.

Being combined with the isocapacitary inequality

$$\mu(K)^\gamma \leq \Lambda_{p,\gamma}\, \mathrm{cap}_{p,a}(K, \Omega) \qquad (59)$$

where μ is a Radon measure in Ω, (56) implies the estimate

$$\Big(\int_0^\infty \big(\mu(M_t)\big)^{\gamma q/p} d(t^q)\Big)^{1/q} \leq \mathcal{A}_{p,q} \Lambda_{p,\gamma}^{1/p} \Big(\int_\Omega |\nabla u(x)|^p |x|^{-a} dx\Big)^{1/p} \qquad (60)$$

for all $u \in C_0^\infty(\Omega)$.

This estimate of u in the Lorentz space $\mathcal{L}_{p/\gamma,q}(\mu)$ becomes the estimate in $L_q(\mu)$ for $\gamma = p/q$:

$$\|u\|_{L_q(\mu)} \leq \mathcal{A}_{p,q} \Lambda_{p,\gamma}^{1/p} \Big(\int_\Omega |\nabla u(x)|^p |x|^{-a} dx\Big)^{1/p}.$$

In the next assertion we find the best value of $\Lambda_{p,\gamma}$ in (59) for the measure $\mu = \mu_b$ defined by (8).

Lemma 2 *Let*

$$1 \leq p < n, \quad 0 \leq a < n-p, \quad \text{and} \quad a + p \geq b \geq \frac{an}{n-p}. \qquad (61)$$

Then

$$\Big(\int_{\mathbb{R}^n} \frac{dx}{|x|^b}\Big)^{\frac{n-p-a}{n-b}} \leq \Big(\frac{p-1}{n-p-a}\Big)^{p-1} \frac{|S^{n-1}|^{\frac{b-p-a}{n-b}}}{(n-b)^{\frac{n-p-a}{n-b}}}\, \mathrm{cap}_{p,a}(K). \qquad (62)$$

The value of the constant factor in front of the capacity is sharp and the equality in (62) is attained at any ball centered at the origin.

Proof. Introducing spherical coordinates (r,ω) with $r>0$ and $\omega \in S^{n-1}$, we have

$$\mathrm{cap}_{p,a}(K) = \inf_{u\big|_K \ge 1} \int_{S^{n-1}} \int_0^\infty \Big(\Big|\frac{\partial u}{\partial r}\Big|^2 + \frac{1}{r^2} |\nabla_\omega u|^2 \Big)^{\frac{p}{2}} r^{n-1-a} dr ds_\omega. \tag{63}$$

Let us put here $r = \rho^{1/\varkappa}$, where

$$\varkappa = \frac{n-p-a}{n-p}$$

and $y = (\rho, \omega)$. The mapping $(r,\omega) \to (\rho,\omega)$ will be denoted by σ. Then (63) takes the form

$$\mathrm{cap}_{p,a}(K) = \varkappa^{p-1} \inf_v \int_{\mathbb{R}^n} \Big(\Big|\frac{\partial u}{\partial \rho}\Big|^2 + (\varkappa\rho)^{-2} |\nabla_\omega u|^2 \Big)^{\frac{p}{2}} dy, \tag{64}$$

where the infimum is taken over all $v = u \circ \sigma^{-1}$. Since $0 \le \varkappa \le 1$ owing to the conditions $p<n$, $0<a<n-p$, and $a \ge 0$, inequality (64) implies

$$\mathrm{cap}_{p,a}(K) \ge \varkappa^{p-1} \inf_v \int_{\mathbb{R}^n} |\nabla u|^p dy \ge \varkappa^{p-1} \mathrm{cap}_p(\sigma(K)) \tag{65}$$

which together with the isocapacitary property of cap_p (see Corollary 2.2.3/2 [M5]) leads to the estimate

$$\mathrm{cap}_p(\sigma(K)) \ge \Big(\frac{n-p}{p-1}\Big)^{p-1} |S^{n-1}|^{\frac{p}{n}} n^{\frac{n-p}{n}} \big(\mathrm{mes}_n(\sigma(K)\big)^{\frac{n-p}{n}}. \tag{66}$$

Clearly,

$$\mu_b(K) = \frac{1}{\varkappa} \int_{\sigma(K)} \frac{dy}{|y|^\alpha}$$

with

$$\alpha = n - \frac{n-b}{\varkappa} = \frac{b(n-p)-an}{n-p-a} \ge 0. \tag{67}$$

Furthermore, one can easily check that

$$\mu_b(K) \le \frac{n^{1-\frac{\alpha}{n}}}{n-b} |S^{n-1}|^{\frac{\alpha}{n}} \big(\mathrm{mes}_n(\sigma(K)\big)^{1-\frac{\alpha}{n}} \tag{68}$$

(see, for instance, [M8], Example 2.2). Combining (68) with (66), we find

$$\big(\mu_b(K)\big)^{\frac{n-p-a}{n-b}} \le \Big(\frac{p-1}{n-p}\Big)^{p-1} \frac{|S^{n-1}|^{\frac{b-p-a}{n-b}}}{(n-b)^{\frac{n-p-a}{n-b}}} \mathrm{cap}_p(\sigma(K)) \tag{69}$$

which together with (65) completes the proof of (62). $\square$

The main result of this section is as follows.

Theorem 5 *Let conditions* (61) *hold and let* $q \ge p$. *Then for all* $u \in C_0^\infty(\mathbb{R}^n)$

$$\Big(\int_0^\infty \big(\mu_b(M_t)\big)^{\frac{(n-p-a)q}{(n-b)p}} d(t^q) \Big)^{\frac{1}{q}} \le \mathcal{C}_{p,q,a,b} \Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p \frac{dx}{|x|^a} \Big)^{\frac{1}{p}}, \tag{70}$$

where

$$\mathcal{C}_{p,q,a,b} = \Big(\frac{\Gamma\big(\frac{pq}{q-p}\big)}{\Gamma\big(\frac{q}{q-p}\big)\Gamma\big(p\frac{q-1}{q-p}\big)} \Big)^{\frac{1}{p}-\frac{1}{q}} \Big(\frac{p-1}{n-p-a} \Big)^{1-\frac{1}{p}} \Big(\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} (n-b)^{\frac{n-p-a}{p+a-b}}} \Big)^{\frac{p+a-b}{(n-b)p}}. \tag{71}$$

The constant (71) *is best possible which can be shown by constructing a radial optimizing sequence in* $C_0^\infty(\mathbb{R}^n)$.

Proof. Inequality (70) is obtained by substitution of (57) and (62) into (60). The sharpness of (71) follows from the part (*ii*) of Theorem 4 and from the fact that the isocapacitary inequality (62) becomes equality for balls. $\square$

The last theorem contains the best constant in the Il'in inequality (9) as a particular case $q = (n-b)p/(n-p-a)$. We formulate this as the following assertion.

Corollary 3 *Let conditions* (61) *hold. Then for all* $u \in C_0^\infty(\mathbb{R}^n)$

$$\Big(\int_{\mathbb{R}^n} |u(x)|^{\frac{(n-b)p}{(n-p-a)}} \frac{dx}{|x|^b}\Big)^{\frac{n-p-a}{n-b}} \le \mathcal{C}_{p,a,b} \Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p \frac{dx}{|x|^a}\Big)^{\frac{1}{p}}, \tag{72}$$

where

$$\mathcal{C}_{p,a,b} = \Big(\frac{p-1}{n-p-a}\Big)^{1-\frac{1}{p}} \Big(\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}(n-b)^{\frac{n-p-a}{p+a-b}}}\Big)^{\frac{p+a-b}{(n-b)p}} \Big(\frac{\Gamma\big(\frac{p(n-b)}{p-b}\big)}{\Gamma\big(\frac{n-b}{p-b}\big)\Gamma\big(1+\frac{(n-b)(p-1)}{p-b}\big)}\Big)^{\frac{p-b}{p(n-b)}}.$$

This constant is best possible which can be shown by constructing a radial optimizing sequence in $C_0^\infty(\mathbb{R}^n)$.

5 Hardy's inequality with sharp Sobolev remainder term

Theorem 6 *For all* $u \in C^\infty(\overline{\mathbb{R}^n_+})$, $u = 0$ *on* $\mathbb{R}^{n-1}$, *the sharp inequality*

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx \ge \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{|u|^2}{x_n^2} dx + \frac{\pi^{n/(n+1)}(n^2-1)}{4\big(\Gamma\big(\frac{n}{2}+1\big)\big)^{2/(n+1)}} \| x_n^{-1/(n+1)} u \|^2_{L_{\frac{2(n+1)}{n-1}}(\mathbb{R}^n_+)} \tag{73}$$

holds.

Proof. We start with the Sobolev inequality

$$\int_{\mathbb{R}^{n+1}} |\nabla w|^2 dz \ge \mathcal{S}_{n+1} \|w\|^2_{L_{\frac{2(n+1)}{n-1}}(\mathbb{R}^{n+1})} \tag{74}$$

with the best constant

$$\mathcal{S}_{n+1} = \frac{\pi^{(n+2)/(n+1)}(n^2-1)}{4^{n/(n+1)}\big(\Gamma\big(\frac{n}{2}+1\big)\big)^{2/(n+1)}} \tag{75}$$

(see Rosen [R], Aubin [Au], and Talenti [Ta]).

Let us introduce the cylindrical coordinates (r, φ, x'), where $r \ge 0$, $\varphi \in [0, 2\pi)$, and $x' \in \mathbb{R}^{n-1}$. Assuming that w does not depend on φ, we write (74) in the form

$$2\pi \int_{\mathbb{R}^{n-1}} \int_0^\infty \Big(\Big|\frac{\partial w}{\partial r}\Big|^2 + |\nabla_{x'} w|^2\Big) r\, dr dx'$$
$$\ge (2\pi)^{(n-1)/(n+1)} \mathcal{S}_{n+1} \Big(\int_{\mathbb{R}^{n-1}} \int_0^\infty |w|^{2(n+1)/(n-1)} r\, dr dx'\Big)^{(n-1)/(n+1)}.$$

Replacing r by x_n, we obtain

$$\int_{\mathbb{R}^n_+} |\nabla w|^2 x_n\, dx \ge (2\pi)^{-2/(n+1)} \mathcal{S}_{n+1} \Big(\int_{\mathbb{R}^n_+} |w|^{2(n+1)/(n-1)} x_n\, dx\Big)^{(n-1)/(n+1)}.$$

It remains to set here $w = x_n^{1/2} v$ and to use (75). $\square$

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