HIGHER REGULARITY IN THE CLASSICAL LAYER POTENTIAL THEORY FOR LIPSCHITZ DOMAINS

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Abstract. Classical boundary integral equations of the harmonic potential theory on Lipschitz surfaces are studied. We obtain higher fractional Sobolev regularity results for their solutions under weak conditions on the surface. These results are derived from a theorem on the solvability of auxiliary boundary value problems for the Laplace equation in weighted Sobolev spaces. We show that classes of domains under consideration are optimal.

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Introduction

• During more than a hundred years successful attempts to study boundary integral equations generated by elliptic boundary value problems in domains with nonsmooth boundaries were made (for the history, see [Ke2], [Ma4]). In particular, a comprehensive theory of integral equations on the boundaries of Lipschitz graph domains was developed in [JK1], [JK2], [CMM], [Ver], [Ke1], [Ca2], [Fab], [FKV], [DKV], [Cos], [MT1]-[MT5], and [MM]. All these works concern solvability and regularity properties either in $L_p(\partial\Omega)$ or in fractional Sobolev spaces $W_p^\ell(\partial\Omega)$, $0 < \ell < 1$.

Our goal is to study solutions of boundary integral equations in the fractional Sobolev space $W_p^\ell(\partial\Omega)$ for $p \in (1, \infty)$ and $\ell > 1$. Since the sole Lipschitz graph property of $\partial\Omega$ does not guarantee higher regularity of solutions, we are forced to select an appropriate subclass of Lipschitz domains depending on p and ℓ which allows to develop a solvability and regularity theory analogous to the classical one for smooth domains. This subclass of domains proves to be best possible in a certain sense.

• We consider a bounded domain $\Omega \subset \mathbb{R}^n$, and we assume that its boundary $\partial\Omega$ satisfies the Lipschitz graph property, that is $\partial\Omega \in C^{0,1}$, which means that *for every point* $O \in \partial\Omega$ *there exist a neighbourhood* U *and a Lipschitz function* f *on* $\mathbb{R}^{n-1}$ *such that*

$$U \cap \Omega = U \cap \{(x, y) : x \in \mathbb{R}^{n-1},\ y > f(x)\}. \tag{1}$$

We handle the internal and external Dirichlet problems

$$\Delta u_+ = 0 \text{ in } \Omega, \quad \operatorname{tr} u_+ = \varphi_+ \text{ on } \partial\Omega, \tag{$\mathcal{D}_+$}$$

and

$$\begin{gathered}\Delta u_- = 0 \text{ in } \mathbb{R}^n \backslash \Omega, \quad \text{tr } u_- = \varphi_- \text{ on } \partial\Omega,\\ u_-(x) = O(|x|^{2-n}) \quad \text{as } |x| \to \infty,\end{gathered} \tag{$\mathcal{D}_-$}$$

where the boundary trace is denoted by "tr", as well as the internal and external Neumann problems

$$\Delta v_+ = 0 \text{ in } \Omega, \quad \frac{\partial v_+}{\partial \nu} = \psi_+ \text{ on } \partial\Omega, \tag{$\mathcal{N}_+$}$$

and

$$\begin{gathered}\Delta v_- = 0 \text{ in } \mathbb{R}^n \backslash \Omega, \quad \frac{\partial v_-}{\partial \nu} = \psi_- \text{ on } \partial\Omega,\\ v_-(x) = O(|x|^{2-n}) \text{ as } |x| \to \infty,\end{gathered} \tag{$\mathcal{N}_-$}$$

where ν stands for the outer normal with respect to Ω.

In what follows, we exclude the case $n = 2$, which will simplify the presentation. The changes required in formulations, in comparison with dimensions $n > 2$, are the same as in the logarithmic potential theory for smooth contours. Our proofs, given for $n > 2$, apply to the two dimensional case after minor changes.

• A classical method for solving problems $(\mathcal{D}_\pm)-(\mathcal{N}_\pm)$ is representation of their solutions using the double layer potential

$$D\sigma(z) = \int\limits_{\partial\Omega} \frac{\partial}{\partial \nu_\zeta} \Gamma(\zeta - z)\, \sigma(\zeta) ds_\zeta, \quad z \in \mathbb{R}^n \backslash \partial\Omega,$$

and the single layer potential

$$S\rho(z) = \int\limits_{\partial\Omega} \Gamma(\zeta - z) \rho(\zeta) ds_\zeta, \quad z \in \mathbb{R}^n \backslash \partial\Omega,$$

where Γ is the fundamental solution of Δ with singularity at the origin. Putting $u_\pm = D\sigma_\pm$ and $v_\pm = S\rho_\pm$, one arrives at the boundary integral equations

$$\left(\pm\tfrac{1}{2}I + D\right)\sigma_\pm = \varphi_\pm, \tag{2$\pm$}$$

and

$$\left(\pm\tfrac{1}{2}I + D^*\right)\rho_\pm = \psi_\pm, \tag{3$\pm$}$$

where D^* is the adjoint of D given by

$$D^*\rho(z) = \int\limits_{\partial\Omega} \frac{\partial}{\partial\nu_z}\Gamma(\zeta - z)\,\rho(\zeta)ds_\zeta.$$

Looking for solutions of problems $(\mathcal{D}_\pm)$ and $(\mathcal{N}_\pm)$ with boundary data $\varphi_\pm = \varphi$ and $\psi_\pm = \psi$ in the form $u_\pm = S\rho$ and $v_\pm = D\sigma$, one obtains the integral equations on $\partial\Omega$

$$S\rho = \varphi, \tag{4}$$

and

$$\frac{\partial}{\partial\nu}D\sigma = \psi. \tag{5}$$

• Under the assumption that $\partial\Omega$ is sufficiently smooth, one can apply such powerful tools as pseudodifferential calculus to equations $(2_\pm)$-(5), which results in a comprehensive theory of their solvability in various spaces of differentiable functions.

We will develop a regularity theory of $(2_\pm)$-(5) with respect to the scale of the fractional Sobolev spaces $W_p^\ell(\partial\Omega)$ under weak smoothness assumptions on $\partial\Omega$, when the corresponding results in the theory of pseudodifferential operators on $\partial\Omega$ are unavailable at the present time. As a substitute, we rely upon an approach proposed in [Ma1]-[Ma4], which reduces the study of boundary integral equations to the study of the inverse operators of auxiliary boundary value problems.

In the case $p(\ell-1) > n-1$, our sole restriction on Ω is the inclusion of its boundary in the *class* W_p^ℓ which means that *every function f in the above definition of the Lipschitz graph domain belongs to* $W_p^\ell(\mathbb{R}^{n-1})$.

In the opposite case $p(\ell-1) \le n-1$, the space $MW_p^\ell(\mathbb{R}^{n-1})$ of pointwise multipliers in $W_p^\ell(\mathbb{R}^{n-1})$ is used to define an admissible class of domains. We say that $\partial\Omega$ *belongs to the class* M_p^ℓ *if every point $O \in \partial\Omega$ has a neighborhood U such that $\Omega \cap U$ is given by* (1) *with $f \in C^{0,1}(\mathbb{R}^{n-1})$ subject to*

$$\nabla f \in MW_p^{\ell-1}(\mathbb{R}^{n-1})$$

(here and elsewhere we do not differentiate between spaces of scalar and vector valued functions in our notation). Furthermore, *the surface $\partial\Omega$ is said to be in the class $M_p^\ell(\delta)$ if*

$$\|\nabla f, \mathbb{R}^{n-1}\|_{MW_p^{\ell-1}} \le \delta, \tag{6}$$

where δ is a positive number and $\|\cdot,\mathbb{R}^{n-1}\|_{MW_p^{\ell-1}}$ is the norm in the multiplier space $MW_p^\ell(\mathbb{R}^{n-1})$. Obviously, $M_p^\ell = \underset{\delta>0}{\cup} M_p^\ell(\delta)$.

Several conditions, either necessary or sufficient for $\partial\Omega \in M_p^\ell(\delta)$, will be discussed in Section 6. In particular, the inclusion $\partial\Omega \in M_p^\ell(0) := \underset{\delta>0}{\cap} M_p^\ell(\delta)$ is guaranteed by the condition

$$\int_0^1 \left(\frac{\omega_q(\nabla_{[\ell]}f,t)}{t^{\{\ell\}}}\right)^p \frac{dt}{t} < \infty,$$

where $\omega_q(\nabla_k f,t)$ is the L_q continuity modulus of the vector $\nabla_k f = \{\partial^\alpha f/\partial x_1^{\alpha_1}\ldots,\partial x_{n-1}^{\alpha_{n-1}}\}$, with $\alpha_1+\cdots+\alpha_{n-1} = |\alpha| = k$, and q is any number satisfying $(n-1)/(\ell-1) \le q \le \infty$ for $p(\ell-1) < n-1$ and $p < q \le \infty$ for $p(\ell-1) = n-1$.

Clearly, any surface in the class $C^{\ell+\epsilon}$, $\epsilon > 0$, belongs to $M_p^\ell(0)$. However, there are surfaces in C^ℓ which are not in M_p^ℓ. Note that $\partial\Omega \in M_p^\ell$ admits vertices and edges on $\partial\Omega$ in the case $p(\ell-1) < n-1$.

• We give our main result concerning the boundary integral equations $(2_\pm)$-(5). In its statement and in the sequel the notation $W_p^s(\partial\Omega)\ominus g$ with $g \in (W_p^s(\partial\Omega))^*$ stands for the subspace of functions $\psi \in W_p^s(\partial\Omega)$ such that $\int_{\partial\Omega}\psi g ds = 0$.

Theorem 1. *Let $n > 2$, $p \in (1,\infty)$, and let ℓ be a noninteger, $\ell > 1$. Suppose that $\partial\Omega$ is connected, $\partial\Omega \in W_p^\ell$ for $p(\ell-1) > n-1$ and $\partial\Omega \in M_p^\ell(\delta)$ with some $\delta = \delta(n,p,\ell) > 0$ for $p(\ell-1) \le n-1$. Then the following assertions hold.*

(i) The operator $\frac{1}{2}I + D$ is an isomorphism of $W_p^\ell(\partial\Omega)$.

(ii) The operator $\frac{1}{2}I + D^*$ is an isomorphism of $W_p^{\ell-1}(\partial\Omega)$.

(iii) The operator S maps isomorphically $W_p^{\ell-1}(\partial\Omega)$ onto $W_p^\ell(\partial\Omega)$.

(iv) The operator $(\partial/\partial\nu)D$ maps continuously $W_p^\ell(\partial\Omega)$ into $W_p^\ell(\partial\Omega)\ominus 1$. There is a continuous inverse

$$\left(\frac{\partial}{\partial\nu}D\right)^{-1} : W_p^{\ell-1}(\partial\Omega)\ominus 1 \to W_p^\ell(\partial\Omega)\ominus 1.$$

(v) There is a continuous inverse

$$\left(-\tfrac{1}{2}I + D\right)^{-1} : W_p^\ell(\partial\Omega)\ominus 1 \to W_p^\ell(\partial\Omega)\ominus \frac{\partial P}{\partial\nu},$$

where P is the Wiener capacitary potential of Ω and $\partial P/\partial\nu \in W_p^{\ell-1}(\partial\Omega)\cap(W_p^\ell(\partial\Omega))^*$. The equality $\left(-\frac{1}{2}I + D\right)1 = 0$ holds.

(vi) There is a continuous inverse

$$\left(-\tfrac{1}{2}I + D^*\right)^{-1} : W_p^{\ell-1}(\partial\Omega)\ominus 1 \to W_p^{\ell-1}(\partial\Omega)\ominus 1.$$

The equality $\left(-\frac{1}{2}I + D^*\right)\partial P/\partial\nu = 0$ holds.

Counterexamples in Section 7.3 show that Theorem 1 fails if $M_p^\ell(\delta)$ is replaced by M_p^ℓ.

• The invertibility properties of the operators $\pm\frac{1}{2}I + D$, $\pm\frac{1}{2}I + D^*$, S, and $(\partial/\partial\nu)D$ in Theorem 1 result from solvability properties of problems $(\mathcal{D}_\pm)$ and $(\mathcal{N}_\pm)$ collected in the next Theorem 2 which are of independent interest. The continuity properties of D, D^*, S, and $(\partial/\partial\nu)D$ stated in Theorem 1 are deduced from the part of Theorem 2 concerning the transmission problem

$$\begin{aligned} \Delta w_+ = 0 \text{ in } \Omega, \quad \Delta w_- = 0 \text{ in } \mathbb{R}^n\backslash\overline{\Omega}, \\ \text{tr } w_+ - \text{tr } w_- = \varphi, \quad \frac{\partial w_+}{\partial\nu} - \frac{\partial w_-}{\partial\nu} = \psi \text{ on } \partial\Omega, \\ w_-(x) = O(|x|^{2-n}) \text{ as } |x| \to \infty. \end{aligned} \tag{$\mathcal{T}$}$$

In the formulation of Theorem 2 and in the sequel, we use the weighted Sobolev space $W_p^{k,\alpha}(\Omega)$ endowed with the norm

$$\|u, \Omega\|_{W_p^{k,\alpha}} = \Big(\int\limits_\Omega (\text{dist}(z, \partial\Omega))^{p\alpha} |\nabla_k u(z)|^p dz\Big)^{1/p} + \|u, \Omega\|_{L_p}. \tag{7}$$

Besides, $W^{k,\alpha}_{p,\text{loc}}(\mathbb{R}^n\backslash\overline{\Omega})$ stands for the space of functions subject to $\|u, B\backslash\overline{\Omega}\|_{W_p^{k,\alpha}} < \infty$ for an arbitrary open ball B containing $\overline{\Omega}$.

Theorem 2. *Let $n > 2$, $p \in (1, \infty)$, $\alpha = 1 - \{\ell\} - 1/p$, where ℓ is a noninteger, $\ell > 1$. Suppose that $\partial\Omega \in W_p^\ell$ for $p(\ell - 1) > n - 1$ and $\partial\Omega \in M_p^\ell(\delta)$ with some $\delta = \delta(n, p, \ell)$, for $p(\ell - 1) \le n - 1$. Then*

(i) For every $\varphi_+ \in W_p^\ell(\partial\Omega)$ there exists a unique solution $u_+ \in W_p^{[\ell]+1,\alpha}(\Omega)$ of problem $(\mathcal{D}_+)$ and

$$\|u_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} \le c\, \|\varphi_+, \partial\Omega\|_{W_p^\ell}. \tag{8}$$

This solution is represented uniquely as $(D\sigma_+)_+$ with $\sigma_+ \in W_p^\ell(\partial\Omega)$ subject to equation (2_+). Moreover, u_+ can be represented uniquely in the form $S\rho$ with $\rho \in W_p^{\ell-1}(\partial\Omega)$ subject to equation (4).

(ii) For every $\varphi_- \in W_p^\ell(\partial\Omega)$ there exists a unique solution $u_- \in W^{[\ell]+1,\alpha}_{p,\text{loc}}(\mathbb{R}^n\backslash\overline{\Omega})$ of problem $(\mathcal{D}_-)$ and for every ball B with $B \supset \overline{\Omega}$,

$$\|u_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}} \le c(B)\, \|\varphi_-, \partial\Omega\|_{W_p^\ell}. \tag{9}$$

This solution is represented uniquely in the form

$$u_-(z) = (D\sigma_-)(z) + C\,\Gamma(z), \quad z \in \mathbb{R}^n\backslash\overline{\Omega},$$

where C is a constant, the singularity of the fundamental solution Γ is situated in Ω, and $\sigma_- \in W_p^\ell(\partial\Omega) \ominus 1$, is a solution of the equation

$$\big(-\tfrac{1}{2}I + D\big)\,\sigma_- = \varphi_- - C\Gamma \;\text{ on } \partial\Omega. \tag{10}$$

Moreover, u_- can be represented uniquely in the form $S\rho$ with $\rho \in W_p^{\ell-1}(\partial\Omega)$ subject to equation (4).

(iii) For every $\psi_+ \in W_p^{\ell-1}(\partial\Omega) \ominus 1$ there exists a unique solution $v_+ \in W_p^{[\ell]+1,\alpha}(\Omega)$ of problem $(\mathcal{N}_+)$ subject to $v_+ \perp 1$ on Ω and satisfying

$$\|v_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} \leq c\,\|\psi_+, \partial\Omega\|_{W_p^{\ell-1}}. \tag{11}$$

This solution is represented uniquely in the form

$$v_+(z) = (S\rho_+)(z) + C, \quad z \in \Omega,$$

where C is a constant, $\rho_+ \in W_p^{\ell-1}(\partial\Omega) \ominus 1$ and ρ_+ satisfies (3_+). Moreover, v_+ can be represented uniquely as

$$v_+(z) = (D\sigma)(z) + C, \quad z \in \Omega,$$

where C is a constant and $\sigma \in W_p^{\ell}(\partial\Omega) \ominus 1$ satisfies (5).

(iv) For every $\psi_- \in W_p^{\ell-1}(\partial\Omega)$ there exists a unique solution $v_- \in W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ of problem $(\mathcal{N}_-)$ and for every ball B with $B \supset \overline{\Omega}$,

$$\|v_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}} \leq c(B)\|\psi_-, \partial\Omega\|_{W_p^{\ell-1}}. \tag{12}$$

The solution is represented uniquely in the form $(S\rho_-)_-$ with $\rho_- \in W_p^{\ell-1}(\partial\Omega)$ subject to equation (3_-). Moreover, v_- can be represented uniquely as

$$v_-(z) = (D\sigma)(z) + C\Gamma(z), \quad z \in \mathbb{R}^n\backslash\overline{\Omega},$$

where $C = -\int_{\partial\Omega}\psi_- ds$, $\sigma \in W_p^{\ell}(\partial\Omega) \ominus 1$, and σ is subject to the equation

$$\frac{\partial}{\partial\nu}(D\sigma)_- = \psi_- \; - \; C\frac{\partial}{\partial\nu}\Gamma_-. \tag{13}$$

(v) For every $(\varphi, \psi) \in W_p^{\ell}(\partial\Omega) \times W_p^{\ell-1}(\partial\Omega)$ there exists a unique solution $(w_+, w_-) \in W_p^{[\ell]+1,\alpha}(\Omega) \times W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n, \overline{\Omega})$ of problem $(\mathcal{T})$ and for every ball B with $B \supset \overline{\Omega}$,

$$\|w_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} + \|w_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}} \leq c(B)(\|\varphi, \partial\Omega\|_{W_p^{\ell}} + \|\psi, \partial\Omega\|_{W_p^{\ell-1}}). \tag{14}$$

This solution is given explicitly by

$$w_\pm = (S\psi)_\pm + (D\varphi)_\pm \quad \text{on } \mathbb{R}^n\backslash\partial\Omega. \tag{15}$$

This theorem follows essentially from Theorem 3 in Section 3 concerning the $W_p^{[\ell]+1,\alpha}$-solvability of the Dirichlet, Neumann, and transmission problems for equations with non-zero right-hand sides. A typical statement, contained in Theorem 3, runs as follows.

Let $n \geq 2$, $1 < p < \infty$, $\ell > 1$, and $\{\ell\} > 0$. If $\partial\Omega \in W_p^\ell$ for $p(\ell - 1) > n - 1$ and $\partial\Omega \in M_p^\ell(\delta)$ with some $\delta = \delta(n, p, \ell)$, for $p(\ell - 1) \leq n - 1$, then the mapping

$$W_p^{[\ell]+1,\alpha}(\Omega) \ni u \to \{\Delta u,\ \text{tr}\ u\} \in W_p^{[\ell]-1,\alpha}(\Omega) \times W_p^\ell(\partial\Omega) \tag{16}$$

is isomorphic.

In the case $p(\ell - 1) > n - 1$ the last assertion can be inverted for a subclass of Lipschitz domains: the isomorphism property of the mapping (16) implies $\partial\Omega \in W_p^\ell$ (Theorem 4).

Note that this implication fails for the whole class of Lipschitz domains. As for the case $p(\ell - 1) \leq n - 1$, several examples in Section 6 illustrate the sharpness of the condition $\partial\Omega \in M_p^\ell(\delta)$ in formulations of Theorems 1-3. In particular, Example 8 shows that in general the condition $\partial\Omega \in M_p^\ell(\delta)$ in Theorem 3 cannot be improved by $\partial\Omega \in M_p^\ell \cap C^{[\ell]}$.

• Although we deal with the Laplace operator in this paper, it can be replaced with the operator $A_{ij}\partial^2/\partial z_i \partial z_j$ with constant matrix coefficients $A_{ij} = \|A_{ij}^{rs}\|_{r,s=1}^m$, subject to the symmetry condition $A_{ij}^{rs} = A_{ji}^{sr}$ and the Legendre-Hadamard strong ellipticity condition

$$(A_{ij}\eta, \eta)\xi_i\xi_j \geq c|\xi|^2|\eta|^2, \quad c = \text{const} > 0,$$

for all vectors $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$. The statement of the interior and exterior Dirichlet problems does not change whereas the Neumann condition is replaced by

$$\left(\nu,\ A_{ij}^{rs}\ \text{tr}\, \frac{\partial u_\pm}{\partial z_j}\right) = \psi_\pm.$$

In particular, one may include the Dirichlet and traction problems for the Lamé system of linear elastostatics

$$\mu\, \Delta u + (\lambda + \mu)\, \nabla \text{div}\ u = 0.$$

This generalization requires only obvious changes in the proofs of Theorems 2 and 3. Similarly, the proof of Theorem 1 remains intact in spite of the fact that $D\sigma$ and $D^*\rho$ become singular integrals unlike the case of the Laplacian. Our argument is independent of the existing theory of integral operators and can be carried over verbatim. In particular, we make no use of the above mentioned deep theory of the layer potentials on Lipschitz graph surfaces, because a higher regularity of the surfaces and functions allows a direct treatment of the integral operators and equations.

A straightforward modification of our arguments leads to analogous higher regularity results in the theory of hydrodynamical potentials related to the Stokes system

$$\nu\Delta u - \nabla p = 0, \quad \text{div}\, u = 0,$$

(see, for instance, [Lad] and Sect. 2.2 in [Ma4]).

Another promising extension of our results could be based upon the fact that no estimates for fundamental solutions are required and only local theory of elliptic boundary value problems is used. Hence, in principle, one can develop an analogous theory of boundary integral equations for elliptic operators with nonsmooth coefficients for domains on (p,l)-manifolds introduced in Ch. 6 [MS1].

• We outline the structure of this paper. In Section 2 we collect auxiliary information about pointwise multipliers in the fractional Sobolev spaces W_p^ℓ and weighted Sobolev spaces $W_p^{k,\alpha}$. We introduce and study a class of mappings, the so-called (p,k,α)-diffeomorphisms, preserving $W_p^{k,\alpha}$, which play a crucial role in the subsequent treatment of the boundary value problems.

Properties of problems $(\mathcal{D}_\pm)$, $(\mathcal{N}_\pm)$, and $(\mathcal{T})$ to be used in the analysis of boundary integral equations are studied in Section 3 (Proposition 6). The next section deals with continuity properties of the potentials and their normal derivatives. Here, in particular, definitions of all integral operators involved in Theorem 1 are given. Proof of Theorems 1 and 2 can be found in Section 5.

The short Section 6 is devoted to a discussion of the class $M_p^\ell(\delta)$. In Section 7 we give a number of examples of domains which demonstrate the sharpness of our solvability results for the Dirichlet and Neumann problems as well as for corresponding integral equations. Finally, the appendix contains a proof of an auxiliary result used in Section 3.

2. Tools

2.1 The spaces $MW_p^\ell(\mathbb{R}^{n-1})$ and $MW_p^{k,\alpha}(\mathbb{R}^n_+)$.

By $B_r(x)$ we mean the ball $\{\xi\in\mathbb{R}^{n-1} : |\xi - x| < r\}$ and write B_r instead of $B_r(0)$.

We shall need the spaces S_{loc} and S_{unif} of functions on $\mathbb{R}^{n-1}$ defined as follows. By S_{loc} we denote the space

$$\{u : \eta u \in S \ \text{ for all } \ \eta \in C_0^\infty(\mathbb{R}^{n-1})\}$$

and by S_{unif} we mean the space

$$\{u : \sup_\xi \|\eta_\xi u\|_S < \infty\},$$

where $\eta_\xi(x) = \eta(x-\xi)$, $\eta\in C_0^\infty(\mathbb{R}^{n-1})$, $\eta = 1$ on B_1. The space S_{unif} is endowed with the norm

$$\|u\|_{S_{\text{unif}}} = \sup_\xi \|\eta_\xi u\|_S.$$

Let $W_p^\ell(\mathbb{R}^{n-1})$ denote the fractional Sobolev space with the norm

$$\|D_{p,\ell}u, \mathbb{R}^{n-1}\|_{L_p} + \|u, \mathbb{R}^{n-1}\|_{L_p},$$

where

$$(D_{p,\ell}u)(x) = \left(\int_{\mathbb{R}^{n-1}} |\nabla_{[\ell]}u(x+h) - \nabla_{[\ell]}u(x)|^p |h|^{1-n-p\{\ell\}} dh\right)^{1/p}, \tag{17}$$

with $\nabla_{[\ell]}$ being the gradient of order $[\ell]$, i.e. $\nabla_{[\ell]} = \{\partial_{x_1}^{\tau_1}, \dots, \partial_{x_{n-1}}^{\tau_{n-1}}\}$, $\tau_1 + \cdots + \tau_{n-1} = [\ell]$.

In this section we collect some known properties of multipliers in $W_p^\ell(\mathbb{R}^{n-1})$. The equivalence $a \sim b$ means that a/b is bounded and separated from zero by positive constants depending on n, p, and ℓ.

Proposition 1. ([MS1], Ch.3). *Let ℓ be a positive noninteger and $p \in (1,\infty)$. Then*

$$\|\gamma, \mathbb{R}^{n-1}\|_{MW_p^\ell} \sim \|D_{p,\ell}\gamma, \mathbb{R}^{n-1}\|_{M(W_p^\ell \to L_p)} + \|\gamma, \mathbb{R}^{n-1}\|_{L_\infty} \tag{18}$$

which can be written as

$$\|\gamma, \mathbb{R}^{n-1}\|_{MW_p^\ell} \sim \sup_{\substack{e \subset \mathbb{R}^{n-1} \\ \operatorname{diam}(e) \le 1}} \frac{\|D_{p,\ell}\gamma, e\|_{L_p}}{(C_{\ell,p}(e))^{1/p}} + \|\gamma, \mathbb{R}^{n-1}\|_{L_\infty}, \tag{19}$$

and $C_{\ell,p}(e)$ is the $(n-1)$- dimensional (ℓ,p)-capacity of a compact set $e \subset \mathbb{R}^{n-1}$ (see [AH], Sect. 2.2).

Let $\mathbb{R}^n_+$ denote the upper half-space $\{z = (x,y) : x \in \mathbb{R}^{n-1}, y > 0\}$. We introduce the weighted Sobolev space $W_p^{k,\alpha}(\mathbb{R}^n_+)$ with the norm

$$\|(\min\{1,y\})^\alpha \nabla_k U, \mathbb{R}^n_+\|_{L_p} + \|(\min\{1,y\})^\alpha U, \mathbb{R}^n_+\|_{L_p},$$

where k is a nonnegative integer. We always assume that

$$-1 < \alpha p < p - 1.$$

It is well known that the fractional Sobolev space $W_p^{k-\alpha-1/p}(\mathbb{R}^{n-1})$ is the space of traces on $\mathbb{R}^{n-1}$ of functions in the space $W_p^{k,\alpha}(\mathbb{R}^n_+)$, where $p \in (1,\infty)$ (see [Usp]). A similar result holds for the space $MW_p^\ell(\mathbb{R}^{n-1})$.

We introduce the extension operator T for functions given on $\mathbb{R}^{n-1}$ as

$$(T\gamma)(x,y) = \int_{\mathbb{R}^{n-1}} \zeta(t)\gamma(x+yt)\,dt, \tag{20}$$

where $\zeta \in C_0^\infty(\mathbb{R}^{n-1})$, $\zeta \ge 0$, and $\int_{\mathbb{R}^{n-1}} \zeta(t)dt = 1$.

Proposition 2. ([MS1], Sect. 5.1.3). *Let k be a positive integer and $1 < p < \infty$.*

(i) If $\Gamma \in MW_p^{k,\alpha}(\mathbb{R}^n_+)$, then there exists the trace γ of Γ on $\mathbb{R}^{n-1}$, $\gamma \in MW_p^{k-\alpha-1/p}(\mathbb{R}^{n-1})$ and

$$\|\gamma, \mathbb{R}^{n-1}\|_{MW_p^{k-\alpha-1/p}} \le c\,\|\Gamma, \mathbb{R}^n_+\|_{MW_p^{k,\alpha}}.$$

(ii) Let $\nabla_s\gamma \in MW_p^{k-\alpha-1/p}(\mathbb{R}^{n-1})$, where s is a nonnegative integer and let $T\gamma$ be the extension of γ defined by (20).

Then $\nabla_{s,z}(T\gamma) \in MW_p^{k,\alpha}(\mathbb{R}^n_+)$ and

$$\|\nabla_{s,z}(T\gamma), \mathbb{R}^n_+\|_{MW_p^{k,\alpha}} \le c\,\|\nabla_{s,x}\gamma, \mathbb{R}^{n-1}\|_{MW_p^{k-\alpha-1/p}}. \tag{21}$$

Proposition 3. *Let $\Gamma \in MW_p^{k,\alpha}(\mathbb{R}^n_+)$ and let $\Gamma_0 := \|\Gamma, \mathbb{R}^n_+\|_{L_\infty}$. If $g \in C^{k-1}([-\Gamma_0, \Gamma_0])$, then $g(\Gamma) \in MW_p^{k,\alpha}(\mathbb{R}^n_+)$ and*

$$\|g(\Gamma), \mathbb{R}^n_+\|_{MW_p^{k,\alpha}} \le c \sum_{j=0}^{k} \|g^{(j)}, [-\Gamma_0, \Gamma_0]\|_{L_\infty} \|\Gamma, \mathbb{R}^n_+\|^j_{MW_p^{k,\alpha}}.$$

Proof. The assertion is obvious for $k = 1$. Let it be valid for $k-1$. For all $u \in C_0^\infty(\mathbb{R}^n_+)$ we have

$$\|ug(\Gamma), \mathbb{R}^n_+\|_{W_p^{k,\alpha}} \le \|g(\Gamma)\nabla u, \mathbb{R}^n_+\|_{W_p^{k-1,\alpha}} + \|ug'(\Gamma)\nabla\Gamma, \mathbb{R}^n_+\|_{W_p^{k-1,\alpha}} + \|ug(\Gamma), \mathbb{R}^n_+\|_{L_p}.$$

By induction assumption the first term in the right-hand side does not exceed

$$c\,\|\nabla u, \mathbb{R}^n_+\|_{W_p^{k-1,\alpha}} \sum_{j=0}^{k-1} \|g^{(j)}, [-\Gamma_0, \Gamma_0]\|_{L_\infty} \|\Gamma, \mathbb{R}^n_+\|^j_{MW_p^{k-1,\alpha}}.$$

Using the induction assumption once more, we obtain

$$\|ug'(\Gamma)\nabla\Gamma, \mathbb{R}^n_+\|_{W_p^{k-1,\alpha}}$$

$$\le c\,\|u\nabla\Gamma, \mathbb{R}^n_+\|_{W_p^{k-1,\alpha}} \sum_{j=0}^{k-1} \|g^{(j+1)}, [-\Gamma_0, \Gamma_0]\|_{L_\infty} \|\Gamma, \mathbb{R}^n_+\|^j_{MW_p^{k-1,\alpha}}. \tag{22}$$

We have

$$\|u\nabla\Gamma, \mathbb{R}^n_+\|_{W_p^{k-1,\alpha}} \le \|\nabla\Gamma, \mathbb{R}^n_+\|_{M(W_p^{k,\alpha} \to W_p^{k-1,\alpha})} \|u, \mathbb{R}^n_+\|_{W_p^{k,\alpha}}. \tag{23}$$

Unifying the inequality

$$\|\nabla\Gamma, \mathbb{R}^n_+\|_{M(W_p^{k,\alpha} \to W_p^{k-1,\alpha})} \le \varepsilon \|\Gamma, \mathbb{R}^n_+\|_{MW_p^{0,\alpha}} + c(\varepsilon)\|\Gamma, \mathbb{R}^n_+\|_{MW_p^{k,\alpha}}$$

(with an arbitrary $\varepsilon > 0$), proved in Lemma 3 [MS2], with

$$\|\Gamma, \mathbb{R}^n_+\|_{W_p^{0,\alpha}} \le c\,\|\Gamma, \mathbb{R}^n_+\|_{W_p^{k,\alpha}} \tag{24}$$

(see estimate (15) in [MS2]), we obtain

$$\|\nabla\Gamma, \mathbb{R}^n_+\|_{M(W_p^{k,\alpha} \to W_p^{k-1,\alpha})} \le c\,\|\Gamma, \mathbb{R}^n_+\|_{MW_p^{k,\alpha}}. \tag{25}$$

Interpolating between $W_p^{k,\alpha}(\mathbb{R}^n_+)$ and $W_p^{0,\alpha}(\mathbb{R}^n_+)$ we find

$$\|\Gamma, \mathbb{R}^n_+\|_{MW_p^{k-1,\alpha}} \le c\,\|\Gamma, \mathbb{R}^n_+\|^{1-1/k}_{MW_p^{k,\alpha}} \|\Gamma, \mathbb{R}^n_+\|^{1/k}_{MW_p^{0,\alpha}}$$

which by (24) implies

$$\|\Gamma, \mathbb{R}^n_+\|_{MW_p^{k-1,\alpha}} \le c\,\|\Gamma, \mathbb{R}^n_+\|_{MW_p^{k,\alpha}}.$$

Combining the last inequality and (25) with (22), (23), we complete the proof.

Corollary. *Let* $\gamma \in MW_p^\ell(\mathbb{R}^{n-1})$ *and let* $\gamma_0 := \|\gamma, \mathbb{R}^{n-1}\|_{L_\infty}$. *If* $g \in C^{[\ell],1}([-\gamma_0, \gamma_0])$, *then* $g(\gamma) \in MW_p^\ell(\mathbb{R}^{n-1})$ *and*

$$\|g(\gamma), \mathbb{R}^{n-1}\|_{MW_p^\ell} \leq c \sum_{j=0}^{[\ell]+1} \|g^{(j)}, [-\gamma_0, \gamma_0]\|_{L_\infty} \|\gamma, \mathbb{R}^{n-1}\|^j_{MW_p^\ell}.$$

Proof. The result follows from Proposition 3 by putting $\Gamma = T\gamma$, where T is defined by (20), and by referring to Proposition 2.

2.2. (p, k, α)**-diffeomorphisms.**

In this section U and V are open subsets of $\mathbb{R}^n_+ = \{z = (x, y) : x \in \mathbb{R}^{n-1}, y > 0\}$. By $W_p^{k,\alpha}(V)$ we denote the space of functions with the finite norm

$$\Big(\int_V (\min\{1, y\})^{p\alpha} (|\nabla_k v(x, y)|^p + |v(x, y)|^p) dz \Big)^{1/p},$$

where k is a positive integer, $-1 < p\alpha < p - 1$, and $1 \leq p \leq \infty$. A quasi-isometric homeomorphism $\varkappa : U \to V$ will be called a (p, k, α)-diffeomorphism if the elements of its Jacobi matrix $\varkappa'$ belong to the space of multipliers $MW_p^{k-1,\alpha}(U)$.

The next proposition contains basic properties of (p, k, α)-diffeomorphisms, verified in the same way as the corresponding properties of (p, k)-diffeomorphisms in Chapter 6 [MS1]. By $\|\varkappa', U\|_{MW_p^{k-1,\alpha}}$ we denote the sum of the norms of the elements of $\varkappa'$ in the space $MW_p^{k-1,\alpha}(U)$.

Proposition 4. *(i) If* $u \in W_p^{k,\alpha}(V)$ *and* $\varkappa$ *is a* (p, k, α)*-diffeomorphism:* $U \to V$, *then* $u \circ \varkappa \in W_p^{k,\alpha}(U)$ *and*

$$\|u \circ \varkappa, U\|_{W_p^{k,\alpha}} \leq c \, \|u, V\|_{W_p^{k,\alpha}}.$$

(ii) If $\varkappa$ *is a* (p, k, α)*-diffeomorphism, then* $\varkappa^{-1}$ *is also a* (p, k, α)*-diffeomorphism.*

(iii) If $\gamma \in MW_p^{k,\alpha}(V)$ *and* $\varkappa$ *is a* (p, k, α)*-diffeomorphism, then* $\gamma \circ \varkappa \in MW_p^{k,\alpha}(U)$ *and*

$$\|\gamma \circ \varkappa, U\|_{MW_p^{k,\alpha}} \leq c \, \|\gamma, V\|_{MW_p^{k,\alpha}}.$$

(iv) If $\varkappa_1 : U \to V$ *and* $\varkappa_2 : V \to W$ *are* (p, k, α)*-diffeomorphisms then their composition* $\varkappa_2 \circ \varkappa_1 : U \to W$ *is a* (p, k, α)*-diffeomorphism.*

Let T denote the extension operator defined by (20), where $\zeta(\tau) = 0$ for $|\tau| \geq 1$. Consider the Lipschitz domain

$$G = \{(x, y) : x \in \mathbb{R}^{n-1}, \ y > f(x)\}, \tag{26}$$

where f is a Lipschitz function such that $f(0)=0$ and $|\nabla f(x)| \le L$ for almost all $x \in \mathbb{R}^{n-1}$. We introduce the mapping

$$\varkappa : \ \mathbb{R}^n_+ \ni (\xi, \eta) \to (x, y) \in \ G$$

by the equalities

$$x = \xi, \ \ y = K\eta + (Tf)(\xi, \eta), \tag{27}$$

where K is a sufficiently large constant depending on L.

Proposition 5. *Let ℓ be a noninteger, $\ell > 1$, and let $p \in (1, \infty)$. If $\nabla f \in MW^{\ell-1}_p(\mathbb{R}^{n-1})$, then $\varkappa$ is a $(p, [\ell]+1, \alpha)$-diffeomorphism.*

Proof. First we show that for any $\xi \in \mathbb{R}^{n-1}$ the mapping $\lambda_\xi : \mathbb{R}^1_+ \ni \eta \to y = K\eta + (Tf)(\xi,\eta)$ is one-to-one, that the inverse mapping is Lipschitz and that

$$|(\lambda_\xi^{-1})'_y| \le (K-L)^{-1}, \tag{28}$$

$$|\lambda_{\xi_1}^{-1}(y) - \lambda_{\xi_2}^{-1}(y)| \le cL(K-cL)^{-1}\|\xi_1 - \xi_2\|_{\mathbb{R}^{n-1}}, \tag{29}$$

where c is a constant depending on n. We fix $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}^1_+$. The operator

$$\tau : \eta \to K^{-1}(y - (T\varphi)(\xi, \eta))$$

maps the segment $\{\eta : |\eta| \le |y - f(x)|\}$ into itself since

$$\begin{aligned} |\tau(\eta)| &\le K^{-1}(|y - f(x)| + |(Tf)(\xi,\eta) - (Tf)(\xi, 0)| \\ &\le K^{-1}(|y - f(x)| + L|\eta|) \le (1+L)K^{-1}|y - f(x)|. \end{aligned}$$

Besides, τ is the contraction mapping, since $|\tau(\eta_1) - \tau(\eta_2)| \le LK^{-1}|\eta_1 - \eta_2|$. Hence, there exists a unique solution η of the equation

$$K^{-1}\big(y - (Tf)(\xi,\eta)\big) = \eta,$$

or, equivalently, of the equation $\lambda_\xi(\eta) = y$.

Let y_1, y_2 be arbitrary points in $\mathbb{R}^1_+$ and let $\eta_j = \lambda_\xi^{-1}(y_j)$, $j = 1, 2$. The equality

$$\eta_1 - \eta_2 = K^{-1}\left(y_1 - y_2 - (Tf)(x, \eta_1) + (Tf)(x, \eta_2)\right)$$

implies $|\eta_1 - \eta_2| \le K^{-1}(|y_2 - y_2| + L|\eta_1 - \eta_2|)$ which proves (28) by $L < K$. Since

$$\nabla_\xi (Tf)(\xi, \eta) = \int\limits_{\mathbb{R}^{n-1}} \zeta(t) \nabla f(\xi + t\eta) dt,$$

$$\frac{\partial}{\partial \eta}(Tf)(\xi,\eta) = \int\limits_{\mathbb{R}^{n-1}} \zeta(t)\, t\, \nabla f(\xi + t\eta) dt, \tag{30}$$

it follows from the equalities

$$y = K\lambda_{\xi_j}^{-1}(y) + (Tf)(\xi_j, \lambda_{\xi_j}^{-1}(y)), \quad j = 1, 2,$$

that

$$|\lambda_{\xi_1}^{-1}(y) - \lambda_{\xi_2}^{-1}(y)| \le cLK^{-1}\big(\|\xi_1 - \xi_2\|_{\mathbb{R}^{n-1}} + |\lambda_{\xi_1}^{-1}(y) - \lambda_{\xi_2}^{-1}(y)|\big)$$

which proves (29).

Thus, we showed the existence of the Lipschitz inverse mapping $\varkappa^{-1}$ defined by $\xi = x, \eta = \lambda_x^{-1}(y)$. The Jacobi matrix of $\varkappa$ is given by

$$\varkappa' = \begin{pmatrix} I & 0 \\ \nabla_\xi(Tf) & K + \partial(Tf)/\partial\eta \end{pmatrix} \tag{31}$$

where I is the identity $(n-1)\times(n-1)$-matrix. Since $\det \varkappa' = K + \partial(Tf)/\partial\eta$ and by (30) $|\partial(Tf)/\partial\eta| \le cL$, it follows that $\det \varkappa' \ge K - cL > 0$. By Proposition 2, elements of $\varkappa'$ belong to $MW_p^{[\ell],\alpha}(\mathbb{R}^n_+)$ with $\alpha = 1 - \{\ell\} - 1/p$. The proof is complete. $\square$

We say that a function φ defined on ∂G belongs to the space $W_p^\ell(\partial G)$ if the function $\mathbb{R}^{n-1} \ni x \to \varphi(x, f(x))$ belongs to $W_p^\ell(\mathbb{R}^{n-1})$. This can be written as

$$\varphi \in W_p^\ell(\partial G) \iff \varphi \circ \varkappa|_{\mathbb{R}^{n-1}} \in W_p^\ell(\mathbb{R}^{n-1}). \tag{32}$$

By (32) and Proposition 4 (i), the inclusion $u \in W_p^{[\ell]+1,\alpha}(G)$ implies $\operatorname{tr} u \in W_p^\ell(\partial G)$ and there exists a linear extension operator: $W_p^\ell(\partial G) \to W_p^{[\ell]+1,\alpha}(G)$.

Note that (27) gives an extension of $\varkappa$ to $\mathbb{R}^n_- = \{(x,y) : x \in \mathbb{R}^{n-1},\ y < 0\}$:

$$\mathbb{R}^n_- \ni (\xi, \eta) \to (x, y) \in \mathbb{R}^n \backslash \overline{G}$$

and this extension has the same properties as the original mapping $\varkappa$. We preserve the same notation $\varkappa$ for the extended mapping so that now, $\varkappa$ is a quasiisometric mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$ and a $(p, [\ell]+1, \alpha)$-diffeomorphic mapping of $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ onto G and $\mathbb{R}^n\backslash\overline{G}$, respectively.

3. Solvability of boundary value problems in weighted Sobolev spaces

Let $W_p^{k,\alpha}(\Omega)$ be the space introduced before Theorem 2. We also need the weighted Sobolev space $W_p^{k,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ supplied with the norm

$$\|v, \mathbb{R}^n\backslash\overline{\Omega}\|_{W_p^{k,\alpha}} = \Big(\int\limits_{\mathbb{R}^n\backslash\overline{\Omega}} (\min\{\operatorname{dist}(x,\partial\Omega), 1\})^{p\alpha}(|\nabla_k v(x)|^p + |v(x)|^p)dx\Big)^{1/p}.$$

Using a partition of unity and properties of the special Lipschitz domain (26) mentioned at the end of the last section, we can introduce the space $W_p^\ell(\partial\Omega)$ and show that it is the trace space for both $W_p^{[\ell]+1,\alpha}(\Omega)$ and $W_p^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$. We also need the space $\mathring{W}_p^{1,\alpha}(\Omega)$ obtained by completion of $C_0^\infty(\Omega)$ in the norm of $W_p^{1,\alpha}(\Omega)$. By $W_p^{-1,\alpha}(\Omega)$ we denote the space of distributions $F = g_0 + \operatorname{div} \boldsymbol{g}$ with $g_0 \in W_p^{0,\alpha}(\Omega)$ and $\boldsymbol{g} \in (W_p^{0,\alpha}(\Omega))^n$. We supply $W_p^{-1,\alpha}(\Omega)$ with the norm

$$\|F,\Omega\|_{W_p^{-1,\alpha}} = \inf\big(\|g_0,\Omega\|_{W_p^{0,\alpha}} + \|\boldsymbol{g},\Omega\|_{(W_p^{0,\alpha})^n}\big),$$

where the infimum is taken over all representations $F = g_0 + \operatorname{div} \boldsymbol{g}$.

The next theorem contains all information on auxiliary boundary value problems $(\mathcal{D}_\pm)$, $(\mathcal{N}_\pm)$, and $(\mathcal{I})$ to be used in the sequel.

Theorem 3. *Let $p \in (1,\infty)$ and let $\alpha = 1 - \{\ell\} - 1/p$, where ℓ is noninteger, $\ell > 1$. Suppose that $\partial\Omega \in W_p^\ell$ for $p(\ell-1) > n-1$ and $\partial\Omega \in M_p^\ell(\delta)$ with some $\delta = \delta(n,p,\ell)$ for $p(\ell-1) \le n-1$.*

The mappings

$$W_p^{[\ell]+1,\alpha}(\Omega) \ni u \to \{\Delta u,\ \operatorname{tr} u\} \in W_p^{[\ell]-1,\alpha}(\Omega) \times W_p^\ell(\partial\Omega), \tag{33}$$

$$W_p^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega}) \ni u \to \{\Delta u - u,\ \operatorname{tr} u\} \in W_p^{[\ell]-1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega}) \times W_p^\ell(\partial\Omega), \tag{34}$$

$$W_p^{[\ell]+1,\alpha}(\Omega) \ni u \to \{\Delta u - u,\ \partial u/\partial\nu\} \in W_p^{[\ell]-1,\alpha}(\Omega) \times W_p^{\ell-1}(\partial\Omega), \tag{35}$$

$$W_p^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega}) \ni u \to \{\Delta u - u,\ \partial u/\partial\nu\} \in W_p^{[\ell]-1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega}) \times W_p^{\ell-1}(\partial\Omega), \tag{36}$$

$$\begin{aligned} &W_p^{[\ell]+1,\alpha}(\Omega) \times W_p^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega}) \ni (u_+,u_-) \\ &\to \{\Delta u_+,\ \Delta u_- - u_-,\ \operatorname{tr}(u_+ - u_-),\ \frac{\partial u_+}{\partial\nu} - \frac{\partial u_-}{\partial\nu}\} \\ &\in \{W_p^{[\ell]-1,\alpha}(\Omega) \times W_p^{[\ell]-1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega}) \times W_p^\ell(\partial\Omega) \times W_p^{\ell-1}(\partial\Omega)\} \end{aligned} \tag{37}$$

are isomorphisms.

Proof. The continuity of the mappings (33)-(37) is obvious. Dealing with their invertibility, we restrict ourselves to a detailed treatment of mapping (33), since the analysis of mappings (34)-(37) is essentially the same. Let us show that the Dirichlet problem

$$\Delta u = F \ \text{ in } \ \Omega, \quad u = \varphi \ \text{ on } \ \partial\Omega \tag{38}$$

with $F \in W_p^{[\ell]-1,\alpha}(\Omega)$ and $\varphi \in W_p^\ell(\partial\Omega)$ is uniquely solvable in $W_p^{[\ell]+1,\alpha}(\Omega)$, and that

$$\|u,\Omega\|_{W_p^{[\ell]+1,\alpha}} \le c\,(\|F,\Omega\|_{W_p^{[\ell]-1,\alpha}} + \|\varphi,\partial\Omega\|_{W_p^\ell}).$$

Our starting point is the following auxiliary assertion.

Lemma 1. *Let $p \in (1,\infty)$, and let $0 < \alpha + 1/p < 1$. Suppose that the Lipschitz constants of the functions f in* (1) *do not exceed a sufficiently small constant depending on n, p, and α. Then the mapping*

$$W_p^{1,\alpha}(\Omega) \ni u \to \{\Delta u,\ \mathrm{tr}\ u\} \in W_p^{-1,\alpha}(\Omega) \times W_p^{1-\alpha-1/p}(\partial\Omega)$$

is an isomorphism.

This lemma, which is hardly new, will be proved in Appendix. Unfortunately, we could find no direct reference to its statement and proof. We can only say that it is similar in flavor to [GG] and [Tri], Sect. 5.7.2. By Lemma 1, problem (38) has a unique solution $u \in W_p^{1,\alpha}(\Omega)$. Therefore, it suffices to prove that this solution belongs to $W_p^{[\ell]+1,\alpha}(\Omega)$.

Let U be a coordinate neighborhood of a point $O \in \partial\Omega$ and let V denote an open set such that $O \in V$ and $\overline{V} \subset U$. We take a function $\chi \in C_0^\infty(U), \chi = 1$ on V. Then

$$\Delta(\chi u) = [\Delta, \chi]u + \chi F.$$

Let $\varkappa$ be the $(p, [\ell]+1, \alpha)$-diffeomorphism defined by (27), where $K = 1$, and let σ denote its inverse. Clearly, σ maps $U \cap \partial\Omega$ onto an open subset of the hyperplane $\eta = 0$. Now, $(\chi u) \circ \varkappa$ satisfies the boundary value problem

$$\mathrm{div}\,(A\nabla((\chi u)\circ\varkappa)) = \frac{(\chi F)\circ\varkappa + ([\Delta,\chi]u)\circ\varkappa}{\det(\sigma'\circ\varkappa)} \tag{39}$$

$$(\chi u)\circ\varkappa\big|_{\mathbb{R}^{n-1}} = (\chi\varphi)\circ(\varkappa\big|_{\mathbb{R}^{n-1}}), \tag{40}$$

where

$$A = \frac{(\sigma'\circ\varkappa)^*(\sigma'\circ\varkappa)}{\det(\sigma'\circ\varkappa)}. \tag{41}$$

By Proposition 4 (i), (iii), the right-hand side of (39) belongs to $W_p^{[\ell]-1,\alpha}(\mathbb{R}^n_+)$ and the Dirichlet data (40) are in $W_p^{\ell}(\mathbb{R}^{n-1})$. These data admit an extension $\Phi \in W_p^{[\ell]+1,\alpha}(\mathbb{R}^n_+)$. Therefore, the function $v := (\chi u)\circ\varkappa - \Phi \in W_p^{1,\alpha}(\mathbb{R}^n_+)$ is a solution of the problem

$$\mathrm{div}(A\nabla v) - v = H \ \text{ on } \mathbb{R}^n_+, \quad v\big|_{\mathbb{R}^{n-1}} = 0, \tag{42}$$

where

$$H = \frac{(\chi F)\circ\varkappa + ([\Delta,\chi]u)\circ\varkappa}{\det(\sigma'\circ\varkappa)} - \mathrm{div}(A\nabla\Phi) + \Phi - (\chi u)\circ\varkappa. \tag{43}$$

We shall consider the cases $p(\ell-1) \le n-1$ and $p(\ell-1) > n-1$ separately.

The case $p(\ell-1) \le n-1$. Let $\partial\Omega \in M_p^{\ell}(\delta)$. By (31) and Proposition 2,

$$\|I - \varkappa', \mathbb{R}^n_+\|_{MW_p^{[\ell],\alpha}} \le c\,\|\nabla f, \mathbb{R}^{n-1}\|_{MW_p^{\ell-1}}.$$

This along with (6) and Proposition 2 implies

$$\|I-A,\mathbb{R}^n_+\|_{MW_p^{[\ell],\alpha}} \le c\,\|\nabla f,\mathbb{R}^{n-1}\|_{MW_p^{\ell-1}} \le c\delta. \tag{44}$$

We can replace $[\ell]$ in the left-hand side of (44) by any $k=0,1,\dots[\ell]$ because of the imbedding $MW_p^{[\ell],\alpha}(\mathbb{R}^n_+)\subset MW_p^{k,\alpha}(\mathbb{R}^n_+)$. This imbedding follows from $MW_p^{0,\alpha}(\mathbb{R}^n_+)=L_\infty(\mathbb{R}^n_+)\supset MW_p^{k,\alpha}(\mathbb{R}^n_+)$ by interpolation between $W_p^{[\ell],\alpha}(\mathbb{R}^n_+)$ and $W_p^{0,\alpha}(\mathbb{R}^n_+)$ (see [Tri], Sect. 3.4.2).

It is standard that there exists a bounded inverse $(1-\Delta)^{-1}$ to the operator $1-\Delta$ in $\mathbb{R}^n_+$ with zero Dirichlet data on $\mathbb{R}^{n-1}$, acting from $W_p^{k,\alpha}(\mathbb{R}^n_+)$ into $W_p^{k-2,\alpha}(\mathbb{R}^n_+)$, $k=0,1,\dots$ (see [Tri]).

We write (42) in the form

$$v-(1-\Delta)^{-1}Sv=(\Delta-1)^{-1}H \tag{45}$$

with H given by (43) and

$$Sv=\operatorname{div}((A-I)\nabla v).$$

This leads to the Neumann series

$$v=\sum_{j=0}^{\infty}((1-\Delta)^{-1}S)^j(\Delta-1)^{-1}H$$

where the operator $(1-\Delta)^{-1}S$ has a small norm in $W_p^{k+1,\alpha}(\mathbb{R}^n_+)$, $k=0,1,\dots,$ owing to (44).

Since $H\in W_p^{0,\alpha}(\mathbb{R}^n_+)$ and $(\Delta-1)^{-1}H\in W_p^{2,\alpha}(\mathbb{R}^n_+)$, it follows that $v\in W_p^{2,\alpha}(\mathbb{R}^n_+)$ and therefore, $\chi u\in W_p^{2,\alpha}(\Omega)$. Using the arbitrariness of the point $O\in\partial\Omega$ we derive that $u\in W_p^{2,\alpha}(\Omega)$ and that

$$\|u,\omega\|_{W_p^{2,\alpha}}\le c(\|F,\Omega\|_{W_p^{0,\alpha}}+\|u,\Omega\|_{W_p^{1,\alpha}}).$$

Now, the result for $\ell<2$ follows by reference to Lemma 1.

Let $\ell>2$. Using Proposition 4 and $u\in W_p^{2,\alpha}(\Omega)$ we obtain $H\in W_p^{1,\alpha}(\mathbb{R}^n_+)$ which implies $v\in W_p^{3,\alpha}(\mathbb{R}^n_+)$ by (45). Repeating this argument several times if necessary, we conclude that $u\in W_p^{[\ell]+1,\alpha}(\Omega)$. This is the required result for $p(\ell-1)\le n-1$.

The case $p(\ell-1)>n-1$. We have

$$\|A,\mathbb{R}^n_+\|_{W_p^{[\ell],\alpha}}\le c\,\|f,\mathbb{R}^{n-1}\|_{W_p^{\ell}}. \tag{46}$$

Without loss of generality we may assume that $\|\nabla f, \mathbb{R}^{n-1}\|_{L_\infty} < \delta$, where δ is sufficiently small. Then

$$\|I - A, \mathbb{R}^n_+\|_{L_\infty} \le c\delta. \tag{47}$$

We introduce a cut-off function $\zeta \in C_0^\infty(B_2)$, $\zeta = 1$ on B_1 and set $\zeta_\varepsilon(\xi, \eta) = \zeta(\xi/\varepsilon, \eta/\varepsilon)$, where ε is a small positive number. By (42)

$$\operatorname{div}(A\nabla(\zeta_\varepsilon v)) - \varepsilon^{-2}\zeta_\varepsilon v = g \text{ on } \mathbb{R}^n_+, \quad \zeta_\varepsilon v\big|_{\mathbb{R}^{n-1}} = 0, \tag{48}$$

where

$$g = \zeta_\varepsilon H + \nabla\zeta_\varepsilon A\nabla v + \operatorname{div}(vA\nabla\zeta_\varepsilon) - \varepsilon^{-2}\zeta_\varepsilon v$$

with H and v defined as in the case $p(\ell - 1) \le n - 1$.

We know that $u \in W_p^{1,\alpha}(\Omega)$. Let us suppose that $u \in W_p^{k,\alpha}(\Omega)$, $1 < k \le [\ell]$. Then $v \in W_p^{k,\alpha}(\mathbb{R}^n_+)$ and $H \in W_p^{k-1,\alpha}(\mathbb{R}^n_+)$, which implies $g \in W_p^{k-1,\alpha}(\mathbb{R}^n_+)$. We introduce new coordinates $(\xi/\varepsilon, \eta/\varepsilon)$ and use the notations $\tilde{A}$, $\tilde{v}$ and $\tilde{g}$ for A, v, and g as functions of $(\xi/\varepsilon, \eta/\varepsilon)$. Written in these dilated variables, problem (48) becomes

$$(1 - \Delta)(\zeta\tilde{v}) - \operatorname{div}((\tilde{A} - I)\nabla(\zeta\tilde{v})) = \varepsilon^2\tilde{g} \text{ on } \mathbb{R}^n_+, \quad \zeta\tilde{v}\big|_{\mathbb{R}^{n-1}} = 0.$$

By (46)

$$\|\nabla_{[\ell]}\tilde{A}, \mathbb{R}^n_+\|_{W_p^{0,\alpha}} \le c\,\varepsilon^{\ell-1-(n-1)/p}\|f, \mathbb{R}^{n-1}\|_{W_p^\ell}.$$

Besides, (47) holds with A replaced by $\tilde{A}$. Therefore, $\|\tilde{A} - I, \mathbb{R}^n_+\|_{MW_p^{[\ell],\alpha}}$ is sufficiently small. This implies that the operator P given by

$$Pw = (1 - \Delta)^{-1}\operatorname{div}((\tilde{A} - I)\nabla w)$$

is contractive in $W_p^{k+1,\alpha}(\mathbb{R}^n_+)$. Hence, $\zeta\tilde{v} \in W_p^{k+1,\alpha}(\mathbb{R}^n_+)$ which implies $u \in W_p^{k+1,\alpha}(\Omega)$. This proves Theorem 3 for mapping (33).

We conclude with a few words about mappings (34)-(37). A direct analogue of Lemma 1 with $\Delta - 1$ instead of Δ and $\mathbb{R}^n\backslash\overline{\Omega}$ instead of Ω is required to prove the invertibility of (34). As the first step in the treatment of mappings (35)-(37), involving the normal derivatives, one shows that there exist the corresponding inverse mappings acting from

$$W_p^{[\ell]-1,\alpha}(\Omega) \times W_p^{\ell-1}(\partial\Omega), \quad W_p^{[\ell]-1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega}) \times W_p^{\ell-1}(\partial\Omega),$$

and

$$W_p^{[\ell]-1,\alpha}(\Omega) \times W_p^{[\ell]-1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega}) \times W_p^\ell(\partial\Omega) \times W_p^{\ell-1}(\partial\Omega)$$

into

$$W_p^{1,\alpha}(\Omega), \quad W_p^{1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega}), \quad \text{and } W_p^{1,\alpha}(\Omega) \times W_p^{1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega}),$$

respectively. When handling (35)-(37), this fact plays the same role as Lemma 1 in the above argument concerning (33). The proof of this fact is close to the proof of Lemma 1, with mapping (27) instead of (100). Next, one needs only trivial changes in comparison with the case of mapping (33) to establish the higher regularity of the solutions belonging to each of the three just mentioned weighted Sobolev spaces. □

Now we deduce certain properties of problems $(\mathcal{D}_\pm)$, $(\mathcal{N}_\pm)$ and $(\mathcal{I})$ from Theorem 3.

Proposition 6. *Let* Ω *satisfy conditions in Theorem* 3. *Then*

(i) For every $\varphi_+ \in W_p^\ell(\partial\Omega)$ *there exists a unique solution* $u_+ \in W_p^{[\ell]+1,\alpha}(\Omega)$ *of problem* $(\mathcal{D}_+)$ *subject to* (8).

(ii) For every $\varphi_- \in W_p^\ell(\partial\Omega)$ *there exists a unique solution* $u_- \in W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ *of problem* $(\mathcal{D}_-)$ *subject to* (9).

(iii) For every $\psi_+ \in W_p^{\ell-1}(\partial\Omega)\ominus 1$ *there exists a unique solution* $v_+ \in W_p^{[\ell]+1,\alpha}(\Omega)$ *of problem* $(\mathcal{N}_+)$ *subject to* $v_+ \perp 1$ *on* Ω *and* (11).

(iv) For every $\psi_- \in W_p^{\ell-1}(\partial\Omega)$ *there exists a unique solution* $v_- \in W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ *of problem* $(\mathcal{N}_-)$ *subject to* (12).

(v) For every $(\varphi,\psi) \in W_p^\ell(\partial\Omega)\times W_p^{[\ell]+1,\alpha}(\partial\Omega)$ *there exists a unique solution* $(w_+,w_-) \in W_p^{[\ell]+1,\alpha}(\Omega)\times W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ *of problem* $(\mathcal{I})$ *subject to* (14).

Proof. Assertion (i) was justified in Theorem 3.

Let us prove (ii). Since the local Lipschitz constant of $\partial\Omega$ is small, the unique solvability of problem $(\mathcal{N}_-)$ in $W_{p,\mathrm{loc}}^{1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ is standard. It suffices to prove that the solution $u \in W_{p,\mathrm{loc}}^{1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ belongs to $W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\Omega)$. Let $\chi\in C_0^\infty(\mathbb{R}^n), \chi=1$ on $\overline{\Omega}$. Clearly,

$$(1-\Delta)(\chi u) = -\chi u - [\Delta,\chi]u \text{ on } \mathbb{R}^n\backslash\overline{\Omega}, \quad \mathrm{tr}(\chi u)=0 \text{ on } \partial\Omega.$$

Since $\chi u + [\Delta,\chi]u \in W_p^{k-1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ for $u\in W_p^{k,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$, it follows from Lemma 1 with $[\ell]$ replaced by k that $u\in W_p^{k+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$. Letting $k=1,\dots,[\ell]$, we arrive at (ii).

Proofs of (iii)-(v) require only obvious changes in this argment.

4. Continuity properties of boundary integral operators

We collect basic properties of the potentials $D\sigma$ and $S\rho$ with $\sigma\in W_p^\ell(\partial\Omega)$ and $\rho\in W_p^{\ell-1}(\partial\Omega)$ where, as usual, $p\in(1,\infty)$, $\ell>1$, and $\{\ell\}>0$.

Proposition 7. *Let the notations* $D\sigma$ *and* $S\rho$ *refer to the double and single layer potentials* *defined on* $\mathbb{R}^n \backslash \partial\Omega$. *For almost all* $Q \in \partial\Omega$ *there exist the limits*

$$(D\sigma)(Q) := \lim_{\varepsilon\to 0} \frac{1}{|S^{n-1}|} \int\limits_{\partial\Omega\backslash B_\varepsilon(Q)} \frac{(\zeta - Q,\, \nu(\zeta))}{|\zeta - Q|^n}\, \sigma(\zeta) ds_\zeta,$$

$$(D^*\sigma)(Q) := \lim_{\varepsilon\to 0} \frac{1}{|S^{n-1}|} \int\limits_{\partial\Omega\backslash B_\varepsilon(Q)} \frac{(\zeta - Q,\, \nu(Q))}{|\zeta - Q|^n} \sigma(\zeta) ds_\zeta,$$

$$\lim_{\substack{z\to Q\\ z\in\Omega}} (D\sigma)(z) = \big(\tfrac{1}{2} I + D\big)\sigma(Q), \tag{49}$$

$$\lim_{\substack{z\to Q\\ z\in\mathbb{R}^n\backslash\overline{\Omega}}} (D\sigma)(z) = \big(-\tfrac{1}{2} I + D\big)\sigma(Q), \tag{50}$$

$$(S\rho)(Q) := \lim_{\substack{z\to Q\\ z\in\mathbb{R}^n\backslash\partial\Omega}} (S\rho)(z) = \frac{-1}{|S^{n-1}|(n-2)} \int\limits_{\partial\Omega} \frac{\rho(\zeta) ds_\zeta}{|\zeta - Q|^{n-2}}, \tag{51}$$

$$\frac{\partial}{\partial\nu}(S\rho)_+(Q) := \lim_{\substack{z\to Q\\ z\in\Omega}} (\nu(Q),\, (\nabla S\rho)(z)) = \big(-\tfrac{1}{2} I + D^*\big)\rho(Q), \tag{52}$$

$$\frac{\partial}{\partial\nu}(S\rho)_-(Q) := \lim_{\substack{z\to Q\\ z\in\mathbb{R}^n\backslash\overline{\Omega}}} (\nu(Q),\, (\nabla S\rho)(z)) = \big(\tfrac{1}{2} I + D^*\big)\rho(Q), \tag{53}$$

where $(S\rho)_+$ *and* $(S\rho)_-$ *are restrictions of* $S\rho$ *to* Ω *and* $\mathbb{R}^n\backslash\overline{\Omega}$.

These classical properties of the layer potentials can be found in [Ver] for σ and ρ in $L_p(\partial\Omega)$, where $z \to Q$ means a nontangential approach. As a justification, a reference is given in [Ver] to the methods developed in [CMM], [Cal1], and [FJR]. However, for our more regular σ and ρ, the above identities can be deduced directly by using the convergence of the integral

$$\int\limits_{\partial\Omega} \frac{|\sigma(\zeta) - \sigma(z)|^p + |\rho(\zeta) - \rho(z)|^p}{|\zeta - z|^{n-1+p\{\ell\}}} ds_\zeta$$

for almost every $z \in \partial\Omega$.

Proposition 8. *The operators* D, D^*, *and* S *satisfy*

$$\|D\sigma, \partial\Omega\|_{W_p^\ell} \le c\, \|\sigma, \partial\Omega\|_{W_p^\ell} \tag{54}$$

$$\|(D\sigma)_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} \le c\, \|\sigma, \partial\Omega\|_{W_p^\ell} \tag{55}$$

$$\|(D\sigma)_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}} \le c(B)\, \|\sigma, \partial\Omega\|_{W_p^\ell} \tag{56}$$

$$\|S\rho, \partial\Omega\|_{W_p^\ell} \le c\, \|\rho, \partial\Omega\|_{W_p^{\ell-1}} \tag{57}$$

$$\|(S\rho)_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} \le c\, \|\rho, \partial\Omega\|_{W_p^{\ell-1}} \tag{58}$$

$$\|(S\rho)_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}} \le c(B)\, \|\rho, \partial\Omega\|_{W_p^{\ell-1}} \tag{59}$$

$$\|D^*\rho, \partial\Omega\|_{W_p^{\ell-1}} \le c\, \|\rho, \partial\Omega\|_{W_p^{\ell-1}}, \tag{60}$$

where $(D\rho)_\pm$ and $(S\rho)_\pm$ are restrictions of $D\sigma$ and $S\rho$ to Ω and $\mathbb{R}^n\backslash\overline{\Omega}$, respectively, and B is an arbitrary ball containing $\overline{\Omega}$.

Proof. Let us prove (54)-(56). Suppose $\sigma \in W_p^\ell(\partial\Omega)$. By Proposition 6 (v) the transmission problem $(\mathcal{I})$ with $\varphi = \sigma$ and $\psi = 0$ has a unique solution $(w_+, w_-) \in W_p^{[\ell]+1,\alpha}(\Omega) \times W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ subject to

$$\|w_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} \le c\,\|\sigma, \partial\Omega\|_{W_p^\ell}, \tag{61}$$

$$\|w_-, B\backslash\Omega\|_{W_p^{[\ell]+1,\alpha}} \le c(B)\,\|\sigma, \partial\Omega\|_{W_p^\ell}. \tag{62}$$

By Green's formula, $w_\pm = D(w_+ - w_-) = D\sigma$ on $\mathbb{R}^n\backslash\partial\Omega$ which implies (55), (56), and

$$\|\mathrm{tr}\ w_+, \partial\Omega\|_{W_p^\ell} \le c\,\|\sigma, \partial\Omega\|_{W_p^\ell}.$$

Since, $D\sigma = \mathrm{tr}\, w_+ - \sigma/2$ by (49), the last inequality leads to (54).

Combining (54) with (49) and (50) we see that $\mathrm{tr}\,(D\sigma)_+$ and $\mathrm{tr}\,(D\sigma)_-$ belong to $W_p^\ell(\partial\Omega)$. This together with Theorem 2 (i), (ii) lead to (55), (56).

We turn to the proof of (57)-(60). Let $\rho \in W_p^{\ell-1}(\partial\Omega)$. By Proposition 6 (v) the transmission problem $(\mathcal{T})$ with $\varphi = 0$ and $\psi = \rho$ has a unique solution $(w_+, w_-) \in W_p^{[\ell]+1,\alpha}(\Omega) \times W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ subject to

$$\|w_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} \le c\,\|\psi, \partial\Omega\|_{W_p^{\ell-1}},$$

$$\|w_-, B\backslash\Omega\|_{W_p^{[\ell]+1,\alpha}} \le c(B)\,\|\psi, \partial\Omega\|_{W_p^{\ell-1}}.$$

By Green's formula

$$w_\pm = S\Big(\frac{\partial w_+}{\partial\nu} - \frac{\partial w_-}{\partial\nu}\Big) = (S\psi)_\pm \tag{63}$$

which implies (58), (59), and

$$\|\mathrm{tr}\ w_-, \partial\Omega\|_{W_p^\ell} + \|\frac{\partial w_-}{\partial\nu}, \partial\Omega\|_{W_p^{\ell-1}} \le c\,\|\rho, \partial\Omega\|_{W_p^{\ell-1}}. \tag{64}$$

Since $D^*\rho = \partial w_-/\partial\nu - \frac{1}{2}\rho$ by (53), we arrive at (60). Estimate (57) follows from (63) and (64). $\Box$

We finish this section with discussion of properties of the normal derivatives of the double layer potential with density in $W_p^\ell(\partial\Omega)$. By (55), the trace of $\nabla(D\sigma)_+$ belongs to $W_p^{\ell-1}(\partial\Omega)$ and defines a continuous operator: $W_p^\ell(\partial\Omega) \to W_p^{\ell-1}(\partial\Omega)$.

Proposition 9. *Let $\sigma \in W_p^\ell(\partial\Omega)$. The operator defined by*

$$\frac{\partial}{\partial\nu}(D\sigma)_+(P) := \big(\nu(P),\, \mathrm{tr}\ \nabla(D\sigma)_+\big) \tag{65}$$

maps $W_p^\ell(\partial\Omega)$ into $W_p^{\ell-1}(\partial\Omega)\ominus 1$ continuously and

$$\frac{\partial}{\partial\nu}(D\sigma)_+ = \frac{\partial}{\partial\nu}(D\sigma)_- \ \ a.e.\ on\ \partial\Omega. \tag{66}$$

Proof. The components of ν, expressed in a local cartesian system (x, y), depend smoothly on ∇f, where f is the function in (1). Since $\nabla f \in MW_p^{\ell-1}(\mathbb{R}^{n-1})$, we conclude by Proposition 3 that

$$\nu \in MW_p^{\ell-1}(\partial\Omega). \tag{67}$$

Hence the operator

$$W_p^\ell(\partial\Omega) \ni \sigma \to \frac{\partial}{\partial\nu}(D\sigma)_+(P) \in W_p^{\ell-1}(\partial\Omega)$$

is continuous.

Let us consider the solution (w_+, w_-) of problem $(\mathcal{T})$ with boundary conditions

$$\mathrm{tr}\, w_+ - \mathrm{tr}\, w_- = \sigma \ \text{ and } \ \frac{\partial w_+}{\partial\nu} - \frac{\partial w_-}{\partial\nu} = 0 \ \text{ a.e. on } \partial\Omega. \tag{68}$$

By Green's formula,

$$w_+ = D\,\mathrm{tr}\, w_+ - S\,\frac{\partial w_+}{\partial\nu} \ \text{ and } \ S\frac{\partial w_-}{\partial\nu} = D\,\mathrm{tr}\, w_- \ \text{ on } \Omega. \tag{69}$$

Analogously,

$$w_- = S\frac{\partial w_-}{\partial\nu} - D\,\mathrm{tr}\, w_- \ \text{ and } \ S\frac{\partial w_+}{\partial\nu} = D\,\mathrm{tr}\, w_+ \ \text{ on } \mathbb{R}^n\backslash\overline{\Omega}. \tag{70}$$

Hence,

$$\begin{aligned} w_+ &= D(\mathrm{tr}\, w_+ - \mathrm{tr}\, w_-) = D\sigma \ \text{ on } \Omega, \\ w_- &= D(\mathrm{tr}\, w_+ - \mathrm{tr}\, w_-) = D\sigma \ \text{ on } \mathbb{R}^n\backslash\overline{\Omega}. \end{aligned} \tag{71}$$

Now, equality (66) is a consequence of (68) and $\partial(D\sigma)_+/\partial\nu \perp 1$ follows from (71). □

The proposition just proved enables us to introduce the operator $(\partial/\partial\nu)D$ by

$$\big(\frac{\partial}{\partial\nu}\,D\big)\sigma := \frac{\partial}{\partial\nu}(D\sigma)_\pm \tag{72}$$

and to conclude that $(\partial/\partial\nu)D$ maps $W_p^\ell(\partial\Omega)$ into $W_p^{\ell-1}(\partial\Omega)\ominus 1$.

5. Proof of Theorems 1 and 2

5.1 Proof of Theorem 1.

The continuity of the operators

$$\begin{aligned} D &: W_p^\ell(\partial\Omega) \to W_p^\ell(\partial\Omega) \\ D^* &: W_p^{\ell-1}(\partial\Omega) \to W_p^{\ell-1}(\partial\Omega) \\ S &: W_p^{\ell-1}(\partial\Omega) \to W_p^\ell(\partial\Omega) \\ \frac{\partial}{\partial\nu} D &: W_p^\ell(\partial\Omega) \to W_p^{\ell-1}(\partial\Omega) \ominus 1 \end{aligned}$$

was established in Propositions 8 and 9.

Solvability of equation (2_+). Let $u_+ \in W_p^{[\ell]+1,\alpha}(\Omega)$ satisfy problem $(\mathcal{D}_+)$ with $\varphi_+ \in W_p^\ell(\partial\Omega)$. Then $\partial u_+/\partial\nu \in W_p^{\ell-1}(\partial\Omega)$. We find a solution $v_- \in W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ of problem $(\mathcal{N}_-)$ with $\psi_- := \partial u_+/\partial\nu$. By Green's formula,

$$u_+ = D\,\mathrm{tr}\,u_+ - S\,\frac{\partial u_+}{\partial\nu} \quad\text{and}\quad S\frac{\partial v_-}{\partial\nu} = D\,\mathrm{tr}\,v_- \quad\text{on } \Omega.$$

Hence, $u_+ = D(\mathrm{tr}\,u_+ - \mathrm{tr}\,v_-)$ on Ω. This together with (49) shows that $\sigma_+ := \mathrm{tr}\,u_+ - \mathrm{tr}\,v_- \in W_p^\ell(\partial\Omega)$ is a solution of (2_+).

We have

$$\begin{aligned} \|\sigma_+, \partial\Omega\|_{W_p^\ell} &\le \|\mathrm{tr}\,u_+, \partial\Omega\|_{W_p^\ell} + \|\mathrm{tr}\,v_-, \partial\Omega\|_{W_p^\ell} \\ &\le c\big(\|u_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} + \|v_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}}\big). \end{aligned} \tag{73}$$

By Proposition 6 (iv) and (67)

$$\|v_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}} \le c\,\|\frac{\partial u_+}{\partial\nu}, \partial\Omega\|_{W_p^{\ell-1}} \le c\,\|u_+, \Omega\|_{W_p^{[\ell]+1,\alpha}}.$$

The last norm does not exceed $c\,\|\varphi_+, \partial\Omega\|_{W_p^\ell}$ by Proposition 7 (i) which together with (73) leads to the estimate

$$\|\sigma_+, \partial\Omega\|_{W_p^\ell} \le c\,\|\varphi_+, \partial\Omega\|_{W_p^\ell}. \tag{74}$$

Uniqueness for equation (2_+). Let $(\frac{1}{2}I + D)\sigma = 0$ with $\sigma \in W_p^\ell(\partial\Omega)$. By Proposition 6 (v) we can find a solution $(w_+, w_-) \in W_p^{[\ell]+1,\alpha}(\Omega) \times W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ of the transmission problem for the Laplace equation on $\mathbb{R}^n\backslash\partial\Omega$ with boundary conditions (68). By (69), $w_+ = (D\sigma)_+$. It follows from (49) and the definition of σ that $\mathrm{tr}\,w_+ = 0$. In view of

Proposition 6 (i), $w_+ = 0$ which together with (68) implies $\partial w_-/\partial\nu = 0$. Proposition 6 (iv) gives $w_- = 0$ and hence $\sigma = \operatorname{tr} w_+ - \operatorname{tr} w_- = 0$. This completes the proof of (i).

Solvability of equation (3_-). Let $v_- \in W^{[\ell]+1,\alpha}_{p,\mathrm{loc}}(\mathbb{R}^n\backslash\overline{\Omega})$ satisfy problem $(\mathcal{N}_-)$ with $\psi_- \in W^{\ell-1}_p(\partial\Omega)$. Then $\operatorname{tr} v_- \in W^{\ell}_p(\partial\Omega)$. We find a solution $u_+ \in W^{[\ell]+1,\alpha}_p(\Omega)$ of problem $(\mathcal{D}_+)$ with $\varphi_+ := \operatorname{tr} v_-$. By Green's formula, $v_- = S(\partial v_-/\partial\nu - \partial u_+/\partial\nu)$ which implies that $\rho_- = \partial v_-/\partial\nu - \partial u_+/\partial\nu \in W^{\ell-1}_p(\partial\Omega)$ satisfies (3_-).

By (67)

$$\begin{aligned}\|\rho_-,\partial\Omega\|_{W^{\ell-1}_p} &\le c\,(\|\operatorname{tr}\nabla v_-,\partial\Omega\|_{W^{\ell-1}_p} + \|\operatorname{tr}\nabla u_+,\partial\Omega\|_{W^{\ell-1}_p})\\ &\le c\,(\|v_-,B\backslash\overline{\Omega}\|_{W^{[\ell]+1,\alpha}_p} + \|u_+,\Omega\|_{W^{[\ell]+1,\alpha}_p}).\end{aligned}$$

Owing to Proposition 6 (i), the last norm does not exceed $c\,\|\operatorname{tr} v_-,\partial\Omega\|_{W^{\ell}_p}$ which is majorized by $\|v_-,B\backslash\overline{\Omega}\|_{W^{[\ell]+1,\alpha}_p}$. Hence,

$$\|\rho_-,\partial\Omega\|_{W^{\ell-1}_p} \le c\,\|v_-,B\backslash\overline{\Omega}\|_{W^{[\ell]+1,\alpha}_p}.$$

Reference to Proposition 6 (iv) results in the estimate

$$\|\rho_-,\partial\Omega\|_{W^{\ell-1}_p} \le c\,\|\psi_-,\partial\Omega\|_{W^{\ell-1}_p}.$$

Uniqueness for equation (3_-). Let $(\frac{1}{2}I + D^*)\rho_- = 0$, where $\rho_- \in W^{\ell-1}_p(\partial\Omega)$. By Proposition 6 (v) we can find a solution $(w_+,w_-) \in W^{[\ell]+1,\alpha}_p(\Omega)\times W^{[\ell]+1,\alpha}_{p,\mathrm{loc}}(\mathbb{R}^n\backslash\overline{\Omega})$ of the transmission problem for the Laplace equation on $\mathbb{R}^n\backslash\partial\Omega$ with boundary conditions

$$\operatorname{tr} w_+ - \operatorname{tr} w_- = 0 \;\text{ and }\; \frac{\partial w_-}{\partial\nu} - \frac{\partial w_+}{\partial\nu} = \rho_- \;\text{ on } \partial\Omega. \tag{75}$$

By Green's formula,

$$w_- = S\,\frac{\partial w_-}{\partial\nu} - D\,w_- \;\text{ and }\; S\frac{\partial w_+}{\partial\nu} - D\,w_+ = 0 \;\text{ on } \mathbb{R}^n\backslash\overline{\Omega}. \tag{76}$$

Hence

$$w_- = S\,\Big(\frac{\partial w_-}{\partial\nu} - \frac{\partial w_+}{\partial\nu}\Big) = S\rho_- \;\text{ on } \mathbb{R}^n\backslash\overline{\Omega}.$$

By (53)

$$\frac{\partial w_-}{\partial\nu} = \big(\tfrac{1}{2}I + D^*\big)\rho_-$$

which implies $\partial w_-/\partial\nu = 0$ on $\partial\Omega$. Using Theorem 2 (iv) we see that $w_- = 0$ on $\mathbb{R}^n\backslash\overline{\Omega}$. This and (75) gives $\operatorname{tr} w_+ = 0$. Proposition 6 (i) shows that $w_+ = 0$. Therefore, $\rho_- = 0$ by (75). This completes the proof of assertion (ii).

We turn to assertion (iii).

Solvability of equation (4). Let $u_+ \in W_p^{[\ell]+1,\alpha}(\Omega)$ be a solution of $(\mathcal{D}_+)$ with $\varphi_+ := \varphi \in W_p^\ell(\partial\Omega)$. By u_- we denote a solution of $(\mathcal{D}_-)$ with $\varphi_- := \varphi$, $u_- \in W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$. Using Green's formula we obtain $u_+ = S(\partial u_-/\partial\nu - \partial u_+/\partial\nu)$ which together with (67) implies that $\rho = \partial u_-/\partial\nu - \partial u_+/\partial\nu \in W_p^{\ell-1}(\partial\Omega)$. Hence, ρ is a solution of (4). We have

$$\|\rho, \partial\Omega\|_{W_p^{\ell-1}} \le \|\frac{\partial u_+}{\partial\nu}, \partial\Omega\|_{W_p^{\ell-1}} + \|\frac{\partial u_-}{\partial\nu}, \partial\Omega\|_{W_p^{\ell-1}}$$

$$\le c\big(\|u_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} + \|u_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}}\big),$$

and in view of Proposition 6 (i), (ii),

$$\|\rho, \partial\Omega\|_{W_p^{\ell-1}} \le c\,\|\varphi, \partial\Omega\|_{W_p^\ell}.$$

Uniqueness for equation (4). Let $\rho \in W_p^{\ell-1}(\partial\Omega)$ and $S\rho = 0$ on $\partial\Omega$. By (51), $\mathrm{tr}\,(S\rho)_\pm = 0$ which together with Proposition 6 (i), (ii) implies $(S\rho)_\pm = 0$. Since $\rho = \partial(S\rho)_-/\partial\nu - \partial(S\rho)_+/\partial\nu$ by (52) and (53), it follows that $\rho = 0$.

Our next goal is assertion (iv).

Solvability of equation (5). Let $\psi \in W_p^{\ell-1}(\partial\Omega) \ominus 1$. By Proposition 7 (iii) there exists a solution $v_+ \in W_p^{[\ell]+1,\alpha}(\Omega)$ of problem $(\mathcal{N}_+)$ with boundary data ψ, unique up to an arbitrary constant term. By v_- we denote a unique $W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$-solution of problem $(\mathcal{N}_-)$ with the same boundary data ψ which exists by Proposition 6 (iv). Let $\sigma = \mathrm{tr}\,v_+ - \mathrm{tr}\,v_-$. Then (68) holds and by (69), $v_+ = D\,\sigma$. This together with (72) gives (5). Choosing the value of an arbitrary constant term in v_+ we obtain $\sigma \perp 1$.

We have

$$\|\sigma, \partial\Omega\|_{W_p^\ell} \le \|\mathrm{tr}\,v_+ - \overline{\mathrm{tr}\,v_+}, \partial\Omega\|_{W_p^\ell} + \|\mathrm{tr}\,v_- - \overline{\mathrm{tr}\,v_-}, \partial\Omega\|_{W_p^\ell},$$

where the bar over a function stands for its mean value. Hence,

$$\|\sigma, \partial\Omega\|_{W_p^\ell} \le \|v_+ - \overline{\mathrm{tr}\,v_+}, \Omega\|_{W_p^{[\ell]+1,\alpha}} + \|v_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}},$$

where B is a ball containing $\overline{\Omega}$. Using Proposition 6 (iii), (iv), we obtain

$$\|\sigma, \partial\Omega\|_{W_p^\ell} \le c\,\|\psi, \partial\Omega\|_{W_p^{\ell-1}}.$$

Uniqueness for equation (5). Let $\sigma \in W_p^{\ell-1}(\partial\Omega)$ and let $\partial(D\sigma)/\partial\nu = 0$ on $\partial\Omega$. By (72), $\partial(D\sigma)_\pm/\partial\nu = 0$ and therefore, by Proposition 6 (ii), (iv), $(D\sigma)_+ = const$, $(D\sigma)_- = 0$. It follows from $\sigma = \mathrm{tr}\,(D\sigma)_+ - \mathrm{tr}\,(D\sigma)_-$ that $\sigma = const$.

Solvability of equation (2_-). We recall that the capacitary potential P of Ω is a unique solution of problem $(\mathcal{D}_-)$ with the Dirichlet data 1 and that

$$-\int_{\partial\Omega} \frac{\partial P}{\partial \nu} ds = \operatorname{cap}\Omega > 0.$$

Suppose that $\varphi_- \in W_p^\ell(\partial\Omega) \ominus \partial P/\partial\nu$. Let $u_- \in W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ satisfy problem $(\mathcal{D}_-)$. Then $\partial u_-/\partial\nu \in W_p^{\ell-1}(\partial\Omega)$ and

$$\int_{\partial\Omega} \frac{\partial u_-}{\partial\nu} ds = \int_{\partial\Omega} \frac{\partial u_-}{\partial\nu} \operatorname{tr} P ds = \int_{\partial\Omega} \varphi_- \frac{\partial P}{\partial\nu} ds = 0.$$

By Proposition 6 (iii), there exists a solution $v_+ \in W_p^{[\ell]+1,\alpha}(\Omega)$ of problem $(\mathcal{N}_+)$ with $\psi_+ = \partial u_-/\partial\nu$ and $v_+ \perp 1$ on Ω. By Green's formula,

$$u_- = S\,\frac{\partial u_-}{\partial\nu} - D \operatorname{tr} u_- \ \text{ and } \ S\,\frac{\partial v_+}{\partial\nu} = D \operatorname{tr} v_+ \ \text{ on } \ \mathbb{R}^n\backslash\overline{\Omega}.$$

Hence, $u_- = D(\operatorname{tr} v_+ - \operatorname{tr} u_-)$. This together with (50) shows that $\operatorname{tr} v_+ - \operatorname{tr} u_- \in W_p^\ell(\partial\Omega)$ is a solution of (2_-).

From $(D1)_- = 0$ and (50) we find $(-\frac{1}{2}I + D)1 = 0$. Therefore, the function

$$\sigma_- := \operatorname{tr} v_+ - \operatorname{tr} u_- - \overline{\operatorname{tr} v_+} + \overline{\operatorname{tr} u_-}$$

satisfies (2_-). Clearly,

$$\|\sigma_-, \partial\Omega\|_{W_p^\ell} \leq c\left(\|\operatorname{tr} v_+, \partial\Omega\|_{W_p^\ell} + \|\operatorname{tr} u_-, \partial\Omega\|_{W_p^\ell}\right)$$

$$\leq c\big(\|v_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} + \|u_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}}\big). \tag{77}$$

By Proposition 6 (ii) and (67)

$$\|v_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} \leq c\,\|\frac{\partial u_-}{\partial\nu}, \partial\Omega\|_{W_p^{\ell-1}} \leq c\,\|u_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}}$$

for an arbitrary ball $B \supset \overline{\Omega}$. The last norm does not exceed $c\,\|\varphi_-, \partial\Omega\|_{W_p^\ell}$ by Proposition 6 (ii), which together with (77) leads to

$$\|\sigma_-, \partial\Omega\|_{W_p^\ell} \leq c\,\|\varphi_-, \partial\Omega\|_{W_p^\ell}.$$

Uniqueness for equation (2_-). Suppose that $\sigma \in W_p^\ell(\partial\Omega)$ and $(-\frac{1}{2}I + D)^{-1}\sigma = 0$. By Proposition 6 (v) we can find a solution $(w_+, w_-) \in W_p^{[\ell]+1,\alpha}(\Omega) \times W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ of the transmission problem for the Laplace equation on $\mathbb{R}^n\backslash\partial\Omega$ with boundary conditions (68).

In view of (70), $w_- = (D\sigma)_-$. It follows from (50) and the definition of σ that $\operatorname{tr} w_- = 0$. By Proposition 7 (ii), $w_- = 0$ which together with (68) implies $\partial w_+/\partial\nu = 0$. Proposition 6 (iii) gives $w_+ = const$ and hence $\sigma = \operatorname{tr} w_+ - \operatorname{tr} w_- = const$. The result follows by $\sigma \perp 1$.

Solvability of equation (3_+). Let $v_+ \in W_p^{[\ell]+1,\alpha}(\Omega)$ satisfy problem $(\mathcal{N}_+)$ with $\psi_+ \in W_p^{\ell-1}(\partial\Omega) \ominus 1$. We assume that $v_+ \perp 1$ on Ω. We find a solution $u_- \in W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ of problem $(\mathcal{D}_-)$ with $\varphi_- := \operatorname{tr} v_+ \in W_p^{\ell}(\partial\Omega)$. By Green's formula,

$$v_+ = D\operatorname{tr} v_+ - S\,\frac{\partial v_+}{\partial\nu} \ \text{ and } \ S\frac{\partial u_-}{\partial\nu} = D\operatorname{tr} u_- \ \text{ on } \Omega.$$

Hence,

$$v_+ = S\big(\frac{\partial u_-}{\partial\nu} - \frac{\partial v_+}{\partial\nu}\big).$$

This together with (52) shows that $\partial u_-/\partial\nu - \partial v_+/\partial\nu \in W_p^{\ell-1}(\partial\Omega)$ is a solution of (3_+).

Since $S\partial P/\partial\nu = 1$ on Ω, it follows from (52) that $(-\frac{1}{2}I + D^*)\partial P/\partial\nu = 0$. Therefore, the function

$$\rho_+ := \frac{\partial u_-}{\partial\nu} - \frac{\partial v_+}{\partial\nu} + C\,\frac{\partial P}{\partial\nu},\ \ C = const, \tag{78}$$

satisfies (3_+). The constant C can be chosen to have $\rho_+ \perp 1$ on $\partial\Omega$. By (78) and (67)

$$\|\rho_+, \partial\Omega\|_{W_p^{\ell-1}} \le c\,(\|\operatorname{tr}\nabla v_+, \partial\Omega\|_{W_p^{\ell-1}} + \|\operatorname{tr}\nabla u_-, \partial\Omega\|_{W_p^{\ell-1}})$$

$$\le c\,(\|v_+, \Omega\|_{W_p^{[\ell]+1,\alpha}} + \|u_-, B\backslash\overline{\Omega}\|_{W_p^{[\ell]+1,\alpha}}).$$

Owing to Proposition 6 (iii), the last norm does not exceed $c\,\|\operatorname{tr} v_+, \partial\Omega\|_{W_p^{\ell}}$ which is majorised by $c\,\|v_+, \Omega\|_{W_p^{[\ell]+1,\alpha}}$. Hence,

$$\|\rho_+, \partial\Omega\|_{W_p^{\ell-1}} \le c\,\|v_+, \Omega\|_{W_p^{[\ell]+1,\alpha}}.$$

Reference to Proposition 6 (iii) results in the estimate

$$\|\rho_+, \partial\Omega\|_{W_p^{\ell-1}} \le c\,\|\psi_+, \partial\Omega\|_{W_p^{\ell-1}}.$$

Uniqueness for equation (3_+). Let $(-\frac{1}{2}I + D^*)\rho_+ = 0$, where $\rho_+ \in W_p^{\ell-1}(\partial\Omega) \ominus 1$. By Proposition 6 (v) we can find a solution $(w_+, w_-) \in W_p^{[\ell]+1,\alpha}(\Omega) \times W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ of the transmission problem for the Laplace equation on $\mathbb{R}^n\backslash\partial\Omega$ with the boundary conditions

$$\operatorname{tr} w_+ - \operatorname{tr} w_- = 0 \ \text{ and } \ \frac{\partial w_-}{\partial\nu} - \frac{\partial w_+}{\partial\nu} = \rho_+ \ \text{ on } \partial\Omega. \tag{79}$$

By Green's formula,

$$w_+ = D\,\mathrm{tr}\,w_+ - S\,\frac{\partial w_+}{\partial \nu} \ \text{ and } \ S\,\frac{\partial w_-}{\partial \nu} = D\,\mathrm{tr}\,w_- \ \text{ on } \ \Omega. \tag{80}$$

Hence,

$$w_+ = S\big(\frac{\partial w_-}{\partial \nu} - \frac{\partial w_+}{\partial \nu}\big) = S\,\rho_+ \ \text{ on } \ \Omega.$$

Owing to (52)

$$\frac{\partial w_+}{\partial \nu} = (-\tfrac{1}{2}I + D^*)\rho_+$$

which implies $\partial w_+/\partial\nu = 0$ on $\partial\Omega$. Using Proposition 6 (iii), we see that $w_+ = const$ on Ω. This and (79) gives $\mathrm{tr}\,w_- = const$ which implies $w_- = constP$. Using (79) again, we obtain $\rho_+ = const\partial P/\partial\nu$. This together with $\rho_+ \perp 1$ completes the proof of assertion (v). □

Now, we are in a position to prove Theorem 2 stated in Introduction.

5.2 Proof of Theorem 2.

All assertions concerning the solvability of problems $(\mathcal{D}_\pm)$, $(\mathcal{N}_\pm)$, and $(\mathcal{T})$, as well as estimates (8)-(14) have been proved in Proposition 6. We need to justify the representations of the solutions to these problems by layer potentials.

(i) By Theorem 1 (i), there exists a unique solution $\sigma_+ \in W^{\ell}_p(\partial\Omega)$ to equation (2_+). By (49) and (55), $(D\sigma_+)_+$ is a solution of problem $(\mathcal{D}_+)$ in $W^{[\ell]+1,\alpha}_p(\Omega)$. Hence, $u_+ = (D\sigma_+)_+$ by Proposition 6 (i).

Theorem 1 (iii) implies the existence of a unique solution $\rho \in W^{\ell-1}_p(\partial\Omega)$ to equation (4). From relations (51) and (58) we obtain that $(S\rho)_+$ is a solution of problem $(\mathcal{D}_+)$ in $W^{[\ell]+1,\alpha}_p(\Omega)$. Hence, $u_+ = (S\rho)_+$ by Proposition 6 (i).

(ii) By Theorem 1 (v), equation (10) has a solution $\sigma_- \in W^{\ell}_p(\partial\Omega) \ominus 1$ if and only if

$$\int_{\partial\Omega} (\varphi_- - C\,\Gamma)\frac{\partial P}{\partial \nu} ds = 0$$

which is equivalent to

$$C = \int_{\partial\Omega} \varphi_- \frac{\partial P}{\partial \nu} ds.$$

By (50) and (56), the function $(D\sigma_-)_- + C\,\Gamma_-$ is a solution in $W^{[\ell]+1,\alpha}_{p,\mathrm{loc}}(\mathbb{R}^n\backslash\overline{\Omega})$ to problem $(\mathcal{D}_-)$. Hence, $u_- = (D\sigma_-)_- + C\,\Gamma_-$ by Proposition 6 (ii).

According to Theorem 1 (iii), there exists a unique solution $\rho \in W^{\ell-1}_p(\partial\Omega)$ to equation (4). Using (51) and (59) we find that $(S\rho)_-$ is a solution of problem $(\mathcal{D}_-)$ in $W^{[\ell]+1,\alpha}_{p,\mathrm{loc}}(\mathbb{R}^n\backslash\overline{\Omega})$. Hence, $u_- = (S\rho)_-$ by Proposition 6 (ii).

(iii) In view of Theorem 1 (vi), there exists a unique solution $\rho_+ \in W_p^{\ell-1}(\partial\Omega) \ominus 1$ to equation (3_+). From (52) and (58) we obtain that $(S\rho)_+$ is a solution of problem $(\mathcal{N}_+)$ in $W_p^{[\ell]+1,\alpha}(\Omega)$. Therefore, $v_+ = (S\rho)_+ + C$. The constant C can be chosen to ensure $v_+ \perp 1$ on Ω.

By Theorem 1 (iv), there exists a unique solution $\sigma \in W_p^{\ell}(\partial\Omega) \ominus 1$ to equation (5). From (65) and (55) we find that $(D\sigma)_+ + C$ is a solution of problem $(\mathcal{N}_+)$ in $W_p^{[\ell]+1,\alpha}(\Omega)$. Choosing C to ensure the orthogonality of $(D\sigma)_+ + C$ and 1 on Ω, we conclude that $v_+ = (D\sigma)_+ + C$.

(iv) Theorem 1 (ii) implies the existence of a unique solution $\rho_- \in W_p^{\ell-1}(\partial\Omega)$ to equation (3_-). It follows from (53) and (59) that $(S\rho_-)_-$ is a solution of problem $(\mathcal{N}_-)$ in $W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$. Hence, $v_- = (S\rho_-)_-$ by Proposition 6 (iv).

By Theorem 1 (iv), there exists a unique solution $\sigma \in W_p^{\ell}(\partial\Omega) \ominus 1$ to equation (13) provided

$$C = -\int_{\partial\Omega} \psi_- ds.$$

It follows from (72) and (56) that $(D\sigma)_- + C\Gamma_-$ is a solution of problem $(\mathcal{N}_-)$ in the space $W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$. Therefore, $v_- = (D\sigma)_- + C\Gamma_-$ by Proposition 6 (iv).

(v) We note that $(S\psi)_+ + (D\varphi)_+$ belongs to $W_p^{[\ell]+1,\alpha}(\Omega)$ and that $(S\psi)_- + (D\varphi)_-$ belongs to $W_{p,\mathrm{loc}}^{[\ell]+1,\alpha}(\mathbb{R}^n\backslash\overline{\Omega})$ by (55), (58) and (56), (59), respectively. Furthermore, $(S\psi)_\pm + (D\varphi)_\pm$ satisfies the boundary conditions of problem $(\mathcal{T})$ by (49), (50), (51), (53). The equality $w_\pm = (S\psi)_\pm + (D\varphi)_\pm$ results from Proposition 6(v). The proof of Theorem 2 is complete.

6. Surfaces of the class $M_p^{\ell}(\delta)$

Let $p(\ell-1) \le n-1$. According to Theorem 3.2.7/1 [MS1], the condition $\partial\Omega \in M_p^{\ell}(\delta)$ is equivalent to the inequality

$$\|D_{p,\ell}\, f, \mathbb{R}^{n-1}\|_{M(W_p^{\ell-1}\to L_p)} + \|\nabla f, \mathbb{R}^{n-1}\|_{L_\infty} < c\,\delta. \tag{81}$$

Using known descriptions of the space $M(W_p^{\ell-1}(\mathbb{R}^{n-1}) \to L_p(\mathbb{R}^{n-1}))$, stated in analytic terms ([MS1], [KSa], [MVe], [Ve]), one can give various equivalent formulations of (81).

The following local characterization of $M_p^{\ell}(\delta)$ is contained in Lemma 7.8.1 [MS1]. Let η be an even function in $C_0^\infty(-1,1)$, $\eta = 1$ on $(-1/2, 1/2)$. We put

$$\eta_\varepsilon(z) = \begin{cases} \eta(|z|/\varepsilon) & \text{if } p(\ell-1) < n, \\ \eta(\log\varepsilon/\log|z|) & \text{if } p(\ell-1) = n. \end{cases}$$

Inequality (81) is equivalent to

$$\limsup_{\varepsilon\to 0} \|\nabla(\eta_\varepsilon f), \mathbb{R}^{n-1}\|_{MW_p^{\ell-1}} \le c\,\delta.$$

The following local condition, equivalent to $\partial\Omega \in M_p^\ell(\delta)$, was obtained in [MS1], Sect. 7.7 and 7.8:

for every point $O \in \partial\Omega$ there exists a neighborhood U such that (1) *holds with f satisfying*

$$\lim_{\varepsilon\to 0}\left(\sup_{e\subset B_\varepsilon}\frac{\|D_{p,\ell}(f,B_\varepsilon),e\|_{L_p}}{\big(C_{\ell-1,p}(e)\big)^{1/p}}+\|\nabla f,B_\varepsilon\|_{L_\infty}\right)\le c\,\delta, \tag{82}$$

where $B_\varepsilon=\{\zeta\in\mathbb{R}^{n-1},\,|\zeta|<\varepsilon\}$,

$$D_{p,\ell}(f,B_\varepsilon)(x)=\left(\int_{B_\varepsilon}\frac{|\nabla_{[\ell]}f(x)-\nabla_{[\ell]}f(\zeta)|^p}{|x-\zeta|^{n-1+p\{\ell\}}}d\zeta\right)^{1/p}.$$

Simpler conditions sufficient for $\partial\Omega\in M_p^\ell(\delta)$ can be obtained from (82) combined with the well known inequality between the capacity and the Lebesgue measure:

if $p(\ell-1)<n-1$, *and*

$$\lim_{\varepsilon\to 0}\left(\sup_{e\subset B_\varepsilon}\frac{\|D_{p,\ell}(f,B_\varepsilon),e\|_{L_p}}{\big(\mathrm{mes}_{n-1}(e)\big)^{\frac{n-1-p(\ell-1)}{(n-1)p}}}+\|\nabla f,B_\varepsilon\|_{L_\infty}\right)\le c\,\delta,$$

and if $p(\ell-1)=n-1$ *and*

$$\lim_{\varepsilon\to 0}\left(\sup_{e\subset B_\varepsilon}|\log(\mathrm{mes}_{n-1}(e))|^{(p-1)/p}\|D_{p,\ell}(f,B_\varepsilon),e\|_{L_p}+\|\nabla f,B_\varepsilon\|_{L_\infty}\right)\le c\,\delta,$$

then $\partial\Omega\in M_p^\ell(\delta)$.

This leads to the following condition, sufficient for $\partial\Omega\in M_p^\ell(0)$:

$$\partial\Omega\in B^\ell_{q,p}\ \text{ and }\ \|\nabla f,\mathbb{R}^{n-1}\|_{L_\infty}<\ \delta,$$

which is mentioned in Introduction (see [MS1], Corollary 7.7/2). (Condition $\partial\Omega\in B^\ell_{q,p}$ can be improved for $p(\ell-1)=n-1$, if one uses the Orlicz space $L_{t^p(\log_+ t)^{p-1}}$ instead of L_q with an arbitrary q but we shall not go into this.) Note that $\partial\Omega\in B^\ell_{\infty,p}$ means that the continuity modulus $\omega_{[\ell]}$ of $\nabla_{[\ell]}f$ satisfies

$$\int_0^1\left(\frac{\omega_{[\ell]}(t)}{t^{\{\ell\}}}\right)^p\frac{dt}{t}<\infty, \tag{83}$$

which implies, in particular, that any surface $\partial\Omega$ in the class $C^{[\ell],\{\ell\}+\varepsilon}$ with an arbitrary $\varepsilon>0$ belongs to $M_p^\ell(0)$. $\Box$

The next example shows that condition (83), sufficient for $\partial\Omega\in M_p^\ell(0)$, is sharp. It demonstrates, in particular, that there exist surfaces in $C^{[\ell],\{\ell\}}$ which do not belong to M_p^ℓ.

Example 1. Let T denote a domain in $\mathbb{R}^2$ with compact closure and the boundary ∂T. By $B_r^{(2)}$ we denote the open disk of a sufficiently small radius r centered at an arbitrary point $O \in \partial T$. We assume that $B_r^{(2)} \cap T = \{(x_1, y) \in B_r^{(2)} : x_1 \in \mathbb{R}^1,\, y > F(x_1)\}$. Let $B_\rho^{(n-2)} = \{x' \in \mathbb{R}^{n-2} : |x'| < \rho\}$, where $x' = (x_2, \dots, x_n)$ and let $\eta \in C_0^\infty(B_2^{(n-2)})$, $\eta = 1$ on $B_1^{(n-2)}$. Also let $f(x) = F(x_1)\,\eta(x')$ and $U = B_r^{(2)} \times B_2^{(n-2)}$. We construct a bounded domain $\Omega \subset \mathbb{R}^n$ satisfying (1) whose boundary is smooth outside U. According to [MS1], Sect. 3.3.2, for any increasing function $\omega \in C[0,1]$ satisfying the inequality

$$\delta \int_\delta^1 \omega(t) \frac{dt}{t^2} + \int_0^\delta \omega(t) \frac{dt}{t} \le c\,\omega(\delta),$$

as well as the condition

$$\int_0^1 \left(\frac{\omega(t)}{t^{\{\ell\}}} \right)^p \frac{dt}{t} = \infty, \tag{84}$$

one can construct a function f of the above form such that the continuity modulus of $\nabla_{[\ell]} f$ does not exceed $c\,\omega$ with $c = const$, and

$$f \notin W_p^\ell(\mathbb{R}^{n-1}). \tag{85}$$

Therefore, $\partial\Omega \notin M_p^\ell$. In the case $\partial\Omega \in C^{[\ell],\{\ell\}}$ we have $\omega(t) = t^{\{\ell\}}$ which implies (84). Hence, the last inclusion is not sufficient for $\partial\Omega$ to be in M_p^ℓ. $\square$

Now we shall see that surfaces in the class $M_p^\ell(\delta)$ with $p(\ell - 1) < n - 1$ may have conic vertices and s-dimensional edges if $s < n - 1 - p(\ell - 1)$.

Example 2. Let s be an integer, $1 \le s \le n-1$, and let $x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. We use the notations $\xi = (x_1, \dots, x_s)$ and $\eta = (x_{s+1}, \dots, x_{n-1})$. Consider the domain $G = K_{n-s} \times \mathbb{R}^s$, where K_{n-s} is the $(n-s)$-dimensional cone

$$\{(\eta, y) : y > -A\,|\eta|\}, \quad A = const > 0. \tag{86}$$

The well known equivalence relation

$$\|v, \mathbb{R}^{n-1}\|_{W_p^{\ell-1}} \sim \left(\int_{\mathbb{R}^s} \|v(\xi, \cdot), \mathbb{R}^{n-1-s}\|_{W_p^\ell}^p \, d\xi \right)^{1/p}$$

$$+ \left(\int_{\mathbb{R}^{n-1-s}} \|v(\cdot, \eta), \mathbb{R}^s\|_{W_p^\ell}^p d\eta \right)^{1/p} \tag{87}$$

implies that the Hardy type inequality

$$\int_{\mathbb{R}^{n-1}} \frac{|v|^p dx}{|\eta|^{p(\ell-1)}} \le c\,\|v, \mathbb{R}^{n-1}\|_{W_p^{\ell-1}}^p \tag{88}$$

holds for all $v \in C_0^\infty(\mathbb{R}^{n-1})$ if and only if

$$\int_{\mathbb{R}^{n-1-s}} \frac{|w|^p d\eta}{|\eta|^{p(\ell-1)}} \leq c\, \|w, \mathbb{R}^{n-1-s}\|^p_{W_p^{\ell-1}}$$

holds for all $w \in C_0^\infty(\mathbb{R}^{n-1-s})$. It is standard that the last inequality is valid if and only if $p(\ell-1) < n-1-s$. One can easily check that $D_{p,\ell}|\eta| = c\,|\eta|^{1-\ell}$. Hence, (88) is equivalent to $D_{p,\ell}|\eta| \in M(W_p^{\ell-1}(\mathbb{R}^{n-1}) \to L_p(\mathbb{R}^{n-1}))$. By (18) and (19), the last inclusion can be written as $\nabla|\eta| \in MW_p^{\ell-1}(\mathbb{R}^{n-1})$. Thus, the domain G belongs to $M_p^\ell \cap C^{0,1}$ if and only if $s < n-1-p(\ell-1)$. Under this restriction on the dimension of the edge, $\partial G \in M_p^\ell(c\,A)$. $\square$

Remark 1. Suppose that for any point $O \in \partial\Omega$ there exists a neighborhood U such that $U \cap \Omega$ is C^∞-diffeomorphic to the domain $\mathbb{R}^s \times \{(x,y) : y > f(x_{s+1}, \ldots, x_{n-1})\}$, $0 \leq s \leq n-2$, i.e. the dimensions of boundary singularities are at most $n-1-s$. Then relation (87) shows that (6) is equivalent to

$$\|\nabla f, \mathbb{R}^{n-1-s}\|_{MW_p^{\ell-1}} \leq c\,\delta$$

and, in particular, it takes the form

$$\|\nabla f, \mathbb{R}^{n-1-s}\|_{W^{\ell-1}_{p,\mathrm{unif}}} \leq c\,\delta$$

if $n-1-s < p(\ell-1) \leq n-1$. In other words, $\partial\Omega \in M_p^\ell(\delta)$ if and only if $(n-1-s)$-dimensional domain $\{(x,y) : y > f(x_{s+1}, \ldots, x_{n-1})\}$ belongs to $M_p^\ell(c\,\delta)$.

7. Sharpness of conditions imposed on $\partial\Omega$

7.1. On the necessity of inclusion $\partial\Omega \in W_p^\ell$ in Theorem 3.

We start out with showing that the condition $\partial\Omega \in W_p^\ell$ is necessary for the solvability in $W_p^{[\ell]+1,\alpha}(\Omega)$ of the Dirichlet problem

$$\Delta\, u = g \in W_p^{[\ell]-1,\alpha}(\Omega), \quad u|_{\partial\Omega} = \varphi \in W_p^\ell(\partial\Omega) \tag{89}$$

provided Ω is subject to some regularity assumptions. It is worth noting that certain additional conditions on $\partial\Omega$ should be imposed to guarantee the above statement. For example, it is well known that the problem

$$\Delta\, u = g \in L_2(\Omega), \quad u|_{\partial\Omega} = 0$$

is uniquely solvable in $W_2^2(\Omega)$ for any convex domain which is not necessarily in $W_2^{3/2}$ ($p = 2$, $\alpha = 0$, $\ell = 3/2$).

Theorem 4. *Let one of the following conditions hold:*

either $\ell \in (1,2)$, $\partial\Omega \in C^1$, *and the continuity modulus* ω *of the normal to* $\partial\Omega$ *satisfies the Dini condition*

$$\int_0^1 \omega(t)\frac{dt}{t} < \infty, \tag{90}$$

or $\ell > 2$ *and* $\partial\Omega \in C^{[\ell]-1,1}$.

If for every $\varphi \in W_p^\ell(\partial\Omega)$ *problem* (89) *has a solution* $u \in W_p^{[\ell]+1,\alpha}(\Omega)$, $\alpha = 1-\{\ell\}-1/p$, *then* $\partial\Omega \in W_p^\ell$.

Proof. Let φ be a nonnegative function vanishing on $U \cap \partial\Omega$, where U is an arbitrary coordinate neighborhood. It is well known that condition (90) guarantees that $u \in C^1(\overline{\Omega})$ and the outer normal derivative at any point of $U \cap \partial\Omega$ is positive. Let us use the mapping $\lambda = \varkappa^{-1}$ with $\varkappa$ introduced in Section 2.2. Since $f \in C^{[\ell]-1,1}(\mathbb{R}^{n-1})$ and $\nabla_x u$, $u_y \in W^{[\ell],\alpha}(V \cap \Omega)$ for any set V with $\overline{V} \subset U$, it follows that $\nabla_x u \circ \lambda$ and $u_y \circ \lambda$ belong to $W^{[\ell],\alpha}(\varkappa(V \cap \Omega))$. Therefore, $\operatorname{tr}(\nabla_x u \circ \lambda)$ and $\operatorname{tr}(u_y \circ \lambda)$ are in $W_p^\ell(\varkappa(V \cap \partial\Omega))$. Observing that $(W_p^\ell \cap L_\infty)(\varkappa(V \cap \partial\Omega))$ is a multiplication algebra, we conclude that

$$\nabla f = -\operatorname{tr}(\nabla_x u \circ \lambda)/\operatorname{tr}(u_y \circ \lambda) \in W_p^\ell(\varkappa(V \cap \partial\Omega)).$$

The result follows from the arbitrariness of V and U.

Remark 2. By the above proof we have shown that the inclusion $\partial\Omega \in W_p^\ell$ is necessary for the solvability of problem $(\mathcal{D}_+)$ in $W_p^{[\ell]+1,\alpha}(\Omega)$ for all $\varphi \in W_p^\ell(\partial\Omega)$ under the conditions imposed on $\partial\Omega$ in Theorem 4. Note that $\partial\Omega \in W_p^\ell$ is also sufficient in the case $p(\ell-1) > n-1$ by Theorem 2.

7.2. Sharpness of inclusion $\partial\Omega \in B_{\infty,p}^\ell$.

Using Remark 2 we show in the following example that no condition on $\partial\Omega$ weaker than $\partial\Omega \in B_{\infty,p}^\ell$ (condition (83)) can give the solvability of problem $(\mathcal{D}_+)$ in $W_p^{[\ell]+1,\alpha}(\Omega)$ for all $\varphi_+ \in W_p^\ell(\partial\Omega)$. We recall that $\partial\Omega \in B_{\infty,p}^\ell$ is sufficient for $\partial\Omega \in M_p^\ell(\delta)$ and hence for this solvability in the case $p(\ell-1) \le n-1$ (see Theorem 2 and Sect. 5).

Example 3. Let Ω be the domain described in Example 1. By (85) and Theorem 4, problem $(\mathcal{D}_+)$ for Ω is not generally solvable in $W_p^{[\ell]+1,\alpha}(\Omega)$ if $\varphi_+ \in W_p^\ell(\partial\Omega)$.

The next example of the same nature demonstrates the sharpness of the conditions $\partial\Omega \in W_p^\ell$ and $\partial\Omega \in B_{\infty,p}^\ell$ for the solvability of the Neumann problem.

Example 4. We use the domains T and Ω from Example 1. Let ∂T be a simple contour and let (α,β) denote the arc $B_r^{(2)} \cap \partial T$. We choose an arbitrary point $\tau \in \partial T \backslash (\alpha,\beta)$ and introduce a function $\gamma \in W_p^{\ell-1}(\partial T)$ equal to zero on (α,β) and at the point τ, negative on (τ,α) and positive on (β,τ). We require also that γ is orthogonal to one on ∂T. Since $\partial T \in C^{[\ell]-1,1}$, the problem

$$\Delta h = 0 \text{ in } \mathbb{R}^2 \backslash \overline{T}, \quad \partial h/\partial\nu = \gamma \text{ on } \partial T$$

has a solution $h \in (W^{[\ell]}_{p,\mathrm{loc}} \cap L_\infty)(\mathbb{R}^2\backslash\overline{T})$.

Let $\zeta \in C_0^\infty(\mathbb{R}^2)$, $\zeta = 1$ on $\overline{T}$ and let η be the cut-off function from Example 1. The function $v(x,y) = h(x_1,y)\,\zeta(x_1,y)\,\eta(x')$ satisfies the Neumann problem

$$\Delta v - v = g \text{ in } \mathbb{R}^n\backslash\overline{\Omega}, \quad \partial v/\partial\nu = \psi \text{ on } \partial\Omega \tag{91}$$

with

$$g = h\Delta\eta + 2\eta\nabla h\nabla\zeta \in W^{[\ell]}_p(\mathbb{R}^n\backslash\overline{\Omega}) \subset W^{[\ell]-1,\alpha}_p(\mathbb{R}^n\backslash\overline{\Omega})$$

and

$$\psi = \eta\partial h/\partial\nu + h\partial\eta/\partial x \in W^{\ell-1}_p(\partial\Omega).$$

If the problem (91) is solvable in $W^{[\ell]+1,\alpha}_p(\mathbb{R}^n\backslash\overline{\Omega})$ for all $g \in W^{[\ell]-1,\alpha}_p(\mathbb{R}^n\backslash\overline{\Omega})$ and $\psi \in W^{\ell-1}_p(\partial\Omega)$, then $v \in W^{[\ell]+1,\alpha}_p(\mathbb{R}^n\backslash\overline{\Omega})$ and hence $h \in W^{[\ell]+1,\alpha}_{p,\mathrm{loc}}(\mathbb{R}^2\backslash\overline{T})$. By χ we denote a conjugate harmonic function of h such that $h(\alpha) = 0$. Clearly, $\chi \in W^{[\ell]+1,\alpha}_p(\mathbb{R}^2\backslash\overline{T})$. Since the first derivative of $\chi|_{\partial T}$ is equal to γ, it follows that $\chi = 0$ on $B^{(2)}_r \cap \partial T$ and $\chi \geq 0$ on ∂T. Repeating the proof of Theorem 4 with $\mathbb{R}^2\backslash\overline{T}$ and $\chi|_{\partial T}$ instead of Ω and φ, respectively, we obtain that $\partial T \in W^\ell_p$ which implies $\partial\Omega \in W^\ell_p$. However, this is not true in view of (85), and therefore, problem (91) is not solvable in $W^{[\ell]+1,\alpha}_p(\mathbb{R}^n\backslash\overline{\Omega})$, in general. One sees that no matter how weak the violation of the inclusion $\partial\Omega \in B^\ell_{\infty,p}$ be, it may lead to the breakdown of the solvability in $W^{[\ell]+1,\alpha}_p(\Omega)$, $p(\ell-1) \leq n$, for the Neumann problem (91).

7.3 Sharpness of the condition $\partial\Omega \in M^\ell_p(\delta)$ in Theorem 3.

It was mentioned before Theorem 4, that the inclusion $\partial\Omega \in W^\ell_p$ is not necessary for the solvability of the Dirichlet problem in $W^{[\ell]+1,\alpha}_p(\Omega)$. Hence, there is no necessity of the condition $M^\ell_p(\delta)$. However, we show in this section that inclusion $\partial\Omega \in M^\ell_p(\delta)$ is best possible in a certain sense. In fact, the following two examples demonstrate that inequality (6), where $p(\ell-1) < n-1$ and δ is not small, does not suffice, in general, for the $W^{[\ell]+1,\alpha}_p$-solvability of the Dirichlet and Neumann problems.

Example 5. Let a domain Ω coincide with the domain G in Example 2 in a neighborhood of the origin. We adopt the same notations as in Example 2. Let u be a positive harmonic function in Ω, satisfying $\mathrm{tr}\, u = \varphi_+ \in W^l_p(\partial\Omega)$ with φ_+ vanishing on $U \cap \partial\Omega$. It is well known that for small $r = (|\eta|^2 + y^2)^{1/2}$

$$u(x) = C(\xi)r^\lambda\Phi(\omega) + O(r^{\lambda_1}), \tag{92}$$

where $1 > \lambda_1 > \lambda > 0$, $\omega = (\eta/r, y/r)$, Φ is smooth on $\{(\eta,y) \in K_{n-s} : r = 1\}$, and C is smooth and positive near the origin of $\mathbb{R}^s$. Moreover, the asymptotic relation (92) is infinitely differentiable and therefore $u \in W^{[\ell]+1,\alpha}_p(\Omega)$ if and only if $n-1-s > p(\ell-\lambda)$. If

$s < n-2$, one can make λ arbitrarily small by choosing sufficiently large A in (86). In the case $s = n-2$, we have $\lambda > 1/2$ and $\lambda - 1/2$ can be made arbitrarily small by increasing the value of A.

According to Example 2, $\partial\Omega \in M_p^\ell$ if and only if $n-1-s > p(\ell-1)$. At the same time, one can choose A to have $u \notin W_p^{[\ell]+1,\alpha}(\Omega)$ if and only if $n-1-s < p\ell$ for $s < n-2$ and $1 < p(\ell - 1/2)$ for $s = n-2$. Thus, the inclusion $\partial\Omega \in M_p^\ell \cap C^{0,1}$ does not imply the solvability of $(\mathcal{D}_+)$ in $W_p^{[\ell]+1,\alpha}(\Omega)$ for all $\varphi_+ \in W_p^\ell(\partial\Omega)$ if $p\ell > n-1-s > p(\ell-1)$ for $s < n-2$ and $p(\ell-1/2) > 1 > p(\ell-1)$ for $s = n-2$. $\square$

In the next example we shall see that the inclusion $\partial\Omega \in M_p^\ell(\delta)$ in Theorem 3 cannot be replaced by $\partial\Omega \in M_p^\ell \cap C^{[\ell]}$, for a particular choice of p and ℓ.

Example 6. Let the domain Ω be described in a neighborhood of O by the inequality $y > f(x)$, where

$$f(x) = C\,\eta(x,0)|x_1|/\log(|x_1|)$$

with $C \geq \pi/4$ and $\eta \in C_0^\infty(B_{1/2})$, $\eta = 1$ on $B_{1/4}$. By $\zeta(t)$ we denote the conformal mapping of the domain

$$\{t = x_1 + ix_2 :\ |t| < 1/2,\ x_2 > C|x_1|/\log(|x_1|)\}$$

into the half-disk $\{\zeta : \operatorname{Im}\zeta > 0, |\zeta| < 1\}$, $\zeta(0) = 0$. By virtue of Sect. 7.6.1 [MS1], the function $u(z) = \eta(2z)\operatorname{Im}\zeta(x_1 + ix_2)$ does not belong to $W_2^2(\Omega)$ and satisfies the Dirichlet problem

$$\Delta\, u = f \in L_2(\Omega), \quad \operatorname{tr} u = \varphi \in W_2^{3/2}(\partial\Omega). \tag{93}$$

Replacing Ω by $\mathbb{R}^n\backslash\overline{\Omega}$ and using the function $v(z) = \eta(2z)\operatorname{Re}\zeta(x_1 + ix_2)$, we arrive at a solution of the Neumann problem

$$\Delta\, v - v = f \in L_2(\Omega), \quad \partial v/\partial\nu = \psi \in W_2^{1/2}(\partial\Omega), \tag{94}$$

which does not belong to $W_2^2(\Omega)$. Thus, there is no solvability in $W_2^2(\Omega)$ and in $W_2^2(\mathbb{R}^n\backslash\overline{\Omega})$ of problems (93) and (94) in spite of the inclusion $\partial\Omega \in M_2^{3/2} \cap C^1$.

The same result can be obtained for problem $(\mathcal{N}_-)$ by making small changes in the above argument. We require that the domain Ω is such that $\mathbb{R}^n\backslash\overline{\Omega}$ coincides with the domain G from Example 2 near the origin O. We note that there exists a harmonic function in $\mathbb{R}^n\backslash\overline{\Omega}$ satisfying $\partial u/\partial\nu = \psi_- \in W_p^{\ell-1}(\partial\Omega)$ with $\psi_- = 0$ in a neighborhood of O such that the asymptotic representation (92) holds with $C(0) \neq 0$. The rest of the argument is literally the same as in the case of problem $(\mathcal{D}_+)$.

7.4 Sharpness of the condition $\partial\Omega \in M_p^\ell(\delta)$ in Theorem 1.

Here we give counterexamples concerning the solutions σ and ρ of integral equations (2_+) and (3_-). First we show that solvability properties of (2_+) and (3_-) proved in Theorem 1 may fail if $\partial\Omega \in M_p^\ell \cap C^{0,1}$ and $\partial\Omega \notin M_p^\ell(\delta)$.

Example 7. Let us consider the domain Ω at the beginning of Example 2 with $n = 3$ and $s = 0$. Now we deal with the three dimensional conic singularity $\{z = (r, \theta, \omega) : r > 0,\ 0 \le \theta < \pi - \varepsilon,\ 0 \le \omega < 2\pi\}$, where $\varepsilon > 0$ and θ is the angle between y-axis and z.

We asume that the functions φ_+ and ψ_- in (2_+) and (3_-) vanish near the vertex of this cone. It was proved in [LeM] that solutions of (2_+) and (3_-) may have the asymptotic representations

$$\sigma_+(z) = \sigma(0) + c_1\, |z|^\lambda + O(|z|^{1+\varepsilon}),$$
$$\rho_-(z) = c_2\, |z|^{\lambda-1}(1 + O(|z|^\mu))$$

with $\mu > 0$, $0 < \lambda < 1$, and nonzero c_1 and c_2. The exponent λ can be made arbitrarily small by diminishing the value of ε. Also note that these asymptotic formulae can be differentiated. According to Example 2, $\partial\Omega \in M_p^\ell$ if and only if $p(\ell - 1) < 2$. However, for $p\ell > 2$, one can choose A in the cone (70) so large that $\sigma_+ \notin W_p^\ell(\partial\Omega)$ and $\rho_- \notin W_p^{\ell-1}(\partial\Omega)$.

Now, suppose that $\rho_- \in W_p^{\ell-1}(\partial\Omega)$ is a solution of (4) with $\varphi = 1$ near O, $0 \le \varphi \le 1$ on $\partial\Omega$. Let us denote the solutions of the interior and exterior Dirichlet problems for the Laplace equation with the same boundary data φ by u_+ and u_-. It is well known that

$$\nabla_k u_-(z) = o(|z|^{k-1}) \quad \text{as } |z| \to 0 \text{ for } \ k = 1, 2$$

and

$$u_+(z) = c\, |z|^\lambda\, \alpha(\theta)\, (1 + o(|z|^\mu)) \quad \text{as } |z| \to 0,$$

where $\mu > 0$, α is smooth, $\alpha'(\pi - \varepsilon) \ne 0$, and $\lambda > 0$ can be made arbitrarily small by choosing a sufficiently small $\varepsilon > 0$. The above asymptotics of u_+ can be differentiated. Hence, $\rho = \partial u_-/\partial\nu - \partial u_+/\partial\nu$ admits the differentiable representation

$$\rho(z) = c_3\, |z|^{\lambda-1}(1 + o(|z|^\mu))$$

which contradicts the inclusion $\rho \in W_p^{\ell-1}(\partial\Omega)$. $\square$

We finish with an example demonstrating that, in general, the condition $\partial\Omega \in M_p^\ell(\delta)$ in Theorem 1 (iii) cannot be improved by $\partial\Omega \in M_p^\ell \cap C^{[\ell]}$.

Example 8. Consider the same domain Ω as in Example 6. Let $\rho \in W_2^{1/2}(\partial\Omega)$ be a solution of (4) with $\varphi = 1$ near O and $0 \le \varphi \le 1$ on $\partial\Omega$. By u_+ and u_- we mean the solution of interior and exterior Dirichlet problems for the Laplace equation with $\operatorname{tr} u_\pm = \varphi$. Using the conformal mapping $t \to \zeta(t)$ one can show that u_+ admits the differentiable asymptotic representation

$$u_+(z) = H(\xi)\,\mathrm{Im}\,\zeta(t)\,(1 + |\log|t||^{-1}) \quad \text{as } |t| \to 0,$$

where $\xi = (x_3, \dots, x_{n-1}, y)$ and H is a smooth function, $H(0) \neq 0$. We also have $\nabla_k u_-(z) = o(|z|^{k-1})$ as $|z| \to 0$ for $k = 1, 2$. Hence $\rho = \partial u_-/\partial\nu - \partial u_+/\partial\nu$ admits the differentiable representation

$$\rho(z) = c\,H(\xi)\,|\log|t||^{2C/\pi}\,(1 + |\log|t||^{-1})$$

for sufficiently small $|\xi|$ and $|t| \to 0$. One can check directly that the function in the right-hand side does not belong to $W_2^{1/2}$ in any neighborhood of O for $C \geq \pi/4$. If the condition $\partial\Omega \in W_2^{3/2}(\delta)$ in Theorem 1 (iii) could be replaced by $\partial\Omega \in W_2^{3/2} \cap C^1$, one would have a contradiction.

7.5 Appendix: proof of Lemma 1.

Since $\partial\Omega \in C^{0,1}$, it follows from the extension theorem from [Usp] mentioned between Propositions 1 and 2 that problem (38) can be reduced to the case $\varphi = 0$.

Let $\mathring{W}_q^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in the norm of $W_q^1(\Omega)$ and let $W_{q'}^{-1}(\Omega)$ stand for the dual of $W_q^1(\Omega)$. We choose $s = s(p, \alpha)$ so that the imbeddings

$$\mathring{W}_p^{1,\alpha}(\Omega) \subset \mathring{W}_s^1(\Omega), \tag{95}$$

$$\mathring{W}_{s'}^1(\Omega) \subset \mathring{W}_p^{1,\alpha}(\Omega), \tag{96}$$

$$W_p^{-1,\alpha}(\Omega) \subset W_s^{-1}(\Omega) \tag{97}$$

hold. By Hölder's inequality these imbeddings follow from

$$\begin{aligned} &s' \geq p, \quad s < p/(1+\alpha p) && \text{for } \alpha > 0,\\ &s' \geq p, \quad s \leq p && \text{for } \alpha = 0,\\ &s' > p/(1+\alpha p), \quad s \leq p && \text{for } \alpha < 0. \end{aligned}$$

We can put, for example,

$$s = \tfrac{1}{2}\big(1 + \min\{p/(p-1-\alpha p),\ \ p\}\big) \quad \text{for } \alpha \leq 0$$

and

$$s = \tfrac{1}{2}\big(1 + \min\{p/(1+\alpha p),\ \ p\}\big) \quad \text{for } \alpha > 0.$$

Since $s' > 2$, the operator

$$\mathring{W}_{s'}^1(\Omega) \ni u \to \Delta u \in W_{s'}^{-1}(\Omega) \tag{98}$$

is a monomorphism. We show the existence of a bounded inverse to (98) defined on $W^{-1}_{s'}(\Omega)$.

Let $F \in W^{-1}_{s'}(\Omega)$ and let $u \in W^1_2(\Omega)$ be a solution to problem (38) with $\varphi = 0$. We denote by U a small coordinate neighborhood of a point $O \in \partial\Omega$ and by V an open set such that $O \in V$ and $\overline{V} \subset U$. We take a function $\chi \in C^\infty_0(U), \chi = 1$ on V. Then

$$\Delta(\chi u) = [\Delta, \chi]u + \chi F. \tag{99}$$

Let $\varkappa$ be the bi-Lipschitz diffeomorphism: $\mathbb{R}^n_+ \ni (\xi, \eta) \to (x, y) \in G$ defined by

$$x = \xi, \quad y = \eta + f(\xi), \tag{100}$$

and let σ denote its inverse. Clearly, σ maps $U \cap \partial\Omega$ onto an open subset of the hyperplane $\eta = 0$. Now, $(\chi u) \circ \varkappa$ satisfies the boundary value problem

$$\operatorname{div}(A\nabla((\chi u) \circ \varkappa)) = (\chi F) \circ \varkappa + ([\Delta, \chi]u) \circ \varkappa \text{ on } \mathbb{R}^n_+ \tag{101}$$

$$(\chi u) \circ \varkappa\big|_{\mathbb{R}^{n-1}} = 0, \tag{102}$$

where

$$A = (\sigma' \circ \varkappa)^*(\sigma' \circ \varkappa). \tag{103}$$

Obviously, the right-hand side of (39) belongs to $W^{-1}_{s'}(\Omega)$. Therefore, the function $v := (\chi u) \circ \varkappa \in W^1_{s'}(\mathbb{R}^n_+)$ is a solution of the problem

$$\operatorname{div}(A\nabla v) - v = H \text{ on } \mathbb{R}^n_+, \quad v\big|_{\mathbb{R}^{n-1}} = 0, \tag{104}$$

where

$$H = (\chi F) \circ \varkappa + ([\Delta, \chi]u) \circ \varkappa - (\chi u) \circ \varkappa. \tag{105}$$

Clearly,

$$\|I - \varkappa', \mathbb{R}^n_+\|_{L_\infty} \leq c\, \|\nabla f, \mathbb{R}^{n-1}\|_{L_\infty}$$

which implies

$$\|I - A, \mathbb{R}^n_+\|_{L_\infty} < \varepsilon, \tag{106}$$

where ε is sufficiently small.

It is a classical fact that the Dirichlet problem

$$-\Delta w + w = g_0 + \operatorname{div} \boldsymbol{g} \quad \text{on } \mathbb{R}^n_+, \quad w\big|_{\mathbb{R}^{n-1}} = 0 \tag{107}$$

with $g_0 \in L_q(\mathbb{R}^n_+)$ and $\boldsymbol{g} \in (L_q(\mathbb{R}^n_+))^n$, $1 < q < \infty$, is uniquely solvable in $\mathring{W}^1_q(\mathbb{R}^n_+)$. (This follows from the representation of w and the continuity of a singular integral operator in

$L_q(\mathbb{R}^n)$.) Let $(1-\Delta)^{-1}$ stand for the inverse operator of problem (107). We write (104) in the form

$$v-(1-\Delta)^{-1}Sv=(\Delta-1)^{-1}H \tag{108}$$

with H given by (105) and

$$Sv=\operatorname{div}((A-I)\nabla v). \tag{109}$$

This leads to the Neumann series

$$v=\sum_{j=0}^{\infty}((1-\Delta)^{-1}S)^j(\Delta-1)^{-1}H, \tag{110}$$

where the operator $(1-\Delta)^{-1}S$ has a small norm in $W^1_q(\mathbb{R}^n_+)$ for every $q\in(2,s']$ owing to (106). Hence

$$\|v,\mathbb{R}^n_+\|_{W^1_q}\le c\,\|(\Delta-1)^{-1}H,\mathbb{R}^n_+\|_{W^1_q}.$$

Using (105) and the arbitrariness of the point $O\in\partial\Omega$, we obtain

$$\|u,\Omega\|_{W^1_q}\le c\,(\|F,\Omega\|_{W^{-1}_q}+\|u,\Omega\|_{L_q}). \tag{111}$$

By Sobolev's imbedding theorem, $u\in L_{2n/(n-2)}(\Omega)$ if $n>2$. Thus, $u\in W^1_{2n/(n-2)}(\Omega)$ by (111). Using Sobolev's theorem again, we see that $u\in L_{2n/(n-4)}(\Omega)$ if $n>4$ and $u\in L_{s'}(\Omega)$ if $n\le 4$. Therefore, by (111), $u\in W^1_{2n/(n-4)}(\Omega)$ if $n>4$ and $u\in W^1_{s'}(\Omega)$ if $n\le 4$. Repeating this argument m times, $m>n(s'-2)/2s'$, we conclude that $u\in W^1_{s'}(\Omega)$ and arrive at the estimate

$$\|u,\Omega\|_{W^1_{s'}}\le c\,(\|F,\Omega\|_{W^{-1}_{s'}}+\|u,\Omega\|_{L_{2n/(n-2)}}).$$

This implies

$$\begin{aligned}\|u,\Omega\|_{W^1_{s'}}&\le c\,(\|F,\Omega\|_{W^{-1}_{s'}}+\|u,\Omega\|_{W^1_2})\\ &\le c\,(\|F,\Omega\|_{W^{-1}_{s'}}+\|F,\Omega\|_{W^{-1}_2})\le c\,\|F,\Omega\|_{W^{-1}_{s'}}.\end{aligned}$$

Hence, operator (98) is isomorphic. By duality, the operator

$$\mathring{W}^1_s(\Omega)\ni u\to\Delta u\in W^{-1}_s(\Omega) \tag{112}$$

is isomorphic as well. This fact, combined with (95), shows that the operator

$$\Delta:\mathring{W}^{1,\alpha}_p(\Omega)\to W^{-1,\alpha}_p(\Omega)$$

is a monomorphism.

Let $F \in W_{s'}^{-1}(\Omega)$ and let $u \in W_{s'}^1(\Omega)$ be a solution of (38) with $\varphi = 0$. In view of (96), $u \in \overset{\circ}{W}{}_p^{1,\alpha}(\Omega)$. It remains to prove the estimate

$$\|u, \Omega\|_{W_p^{1,\alpha}} \leq c\, \|F, \Omega\|_{W_p^{-1,\alpha}}. \tag{113}$$

It is well known that there exists a bounded inverse $(1-\Delta)^{-1}$ to the operator $1-\Delta$ in $\mathbb{R}_+^n$ with zero Dirichlet data on $\mathbb{R}^{n-1}$ acting from $W_p^{k-2,\alpha}(\mathbb{R}_+^n)$ into $W_p^{k,\alpha}(\mathbb{R}_+^n)$, $k = 1, 2, \ldots$ (cfr. [GG], [Tri]). Using this inverse, we write (104) in the form (108) and arrive at the Neumann series (110), where the operator $(1-\Delta)^{-1}S$ has a small norm in $W_p^{1,\alpha}(\mathbb{R}_+^n)$ owing to (106). Hence,

$$\|v, \mathbb{R}_+^n\|_{W_p^{1,\alpha}} \leq c\, \|(\Delta - 1)^{-1} H, \mathbb{R}_+^n\|_{W_p^{1,\alpha}}.$$

Using the arbitrariness of the point $O \in \partial\Omega$ and (105), we obtain

$$\|u, \Omega\|_{W_p^{1,\alpha}} \leq c\, (\|F, \Omega\|_{W_p^{-1,\alpha}} + \|u, \Omega\|_{W_p^{0,\alpha}}). \tag{114}$$

It follows from the one-dimensional Hardy inequality that for an arbitrary small $\varepsilon_0 > 0$

$$\|u, \Omega\|_{W_p^{0,\alpha}} \leq \varepsilon_0\, \|u, \Omega\|_{W_p^{1,\alpha}} + C(\varepsilon_0)\, \|u, \Omega\|_{L_1}. \tag{115}$$

Since the operator (112) is isomorphic and the imbedding (97) holds, we have

$$\|u, \Omega\|_{L_1} \leq c_1\, \|u, \Omega\|_{W_s^1} \leq c_2\, \|F, \Omega\|_{W_s^{-1}} \leq c_3\, \|F, \Omega\|_{W_p^{-1,\alpha}},$$

which, together with (114) and (115) completes the proof.

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