A new type of integral equations related to the co-area formula (Reduction of dimension in multi-dimensional integral equations)

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Abstract

Some classes of multi-dimensional integral equations of the second kind are introduced which admit an equivalent reduction to integral equations for functions depending on a smaller number of independent variables. This reduction is performed by factorization of the integral operator based on the co-area formulas.

1 Introduction

It is a common knowledge that the Fredholm theory of integral equations is based on the equivalence of an equation with a compact operator, on one hand, and an algebraic system, i.e. an equation in a finite-dimensional space, on the other hand. It seems interesting to ask what happens if the role of the finite-dimensional space is given to a certain space of functions of, say, one variable. Then, simply recalling the existence of detailed theories of one-dimensional integral equations of various types (see [GKre], [GKru], [KL], [P] *et al*), one can be tempted to apply these theories to a multi-dimensional case.

In the present article we introduce classes of multi-dimensional integral equations which prove to be equivalent to integral equations or systems of such equations for functions with a smaller number of independent variables.

Typical examples are provided by the equation

$$u(P) - \int_{\Omega} k(P, \varphi(Q)) u(Q)\, dQ = f(P) \;, \quad P \in \Omega \;, \tag{1}$$

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and by its adjoint. Here Ω is a domain in $\mathbb{R}^n$ and φ is a given real-valued function defined on Ω. The kernels $k(P, Q, \varphi(Q))$, weakly dependent on the second argument Q, are admissible in (1) also. We show that the equations just mentioned are equivalent to certain one-dimensional integral equations.

By the way, equation (1) arises under the linearization of the nonlinear equation

$$\int_{\Omega} F(P, v(P), v(Q))\, dQ = g(P)\ , \quad P \in \Omega\ , \tag{2}$$

in the vicinity of φ.

In the sequel, several types of integral equations which admit the diminishing of dimension are listed. The study of classes of solutions and properties of the operators involved is only slightly touched upon at the end of the article and seems to form an extensive program of future research. It is worth mentioning also the interesting open question: what kind of the singularities of multi-dimensional kernels of integral operators allow for the reduction of dimension?

We start with simple general considerations which constitute the basis for the construction of analytic examples below.

Let K denote a linear, not necessarily continuous operator acting in a linear topological space X. We are interested in solving the equation

$$(I - K)u = f\ , \tag{3}$$

where I is the identity operator in X, f is a given and u is an unknown elements of X.

By $\mathcal{X}$ we denote another linear topological space. Let us assume that there exist linear operators L and M defined on linear subsets of $\mathcal{X}$ and X and acting into X and $\mathcal{X}$, respectively. Suppose that

$$K = L\, M \tag{4}$$

and introduce the dual operator

$$\mathcal{K} = M\, L \tag{5}$$

acting in $\mathcal{X}$. All assumptions concerning the domains of K, $\mathcal{K}$, L, and M needed to legalize formulae (4) and (5) should hold. Consider the equation

$$(\mathcal{I} - \mathcal{K})U = F\ , \tag{6}$$

where $F \in \mathcal{X}$, U is an unknown element of $\mathcal{X}$ and $\mathcal{I}$ is the identity operator in $\mathcal{X}$.

If u satisfies (3), i.e. $(I - L\,M)u = f$, then

$$(\mathcal{I} - M\,L)Mu = M\,f$$

so that the existence of the inverse $(\mathcal{I} - M\,L)^{-1}$ defined on $\mathcal{X}$ implies

$$Mu = (\mathcal{I} - M\,L)^{-1}Mf$$

and

$$u = f + L(\mathcal{I} - M\,L)^{-1}Mf\ .$$

Hence, there exists the inverse $(I - K)^{-1}$ in X and

$$(I - K)^{-1} = I + L(\mathcal{I} - \mathcal{K})^{-1}M\ . \tag{7}$$

Analogously, the existence of the inverse $(I - K)^{-1}$ in X implies the existence of $(\mathcal{I} - \mathcal{K})^{-1}$ given by

$$(\mathcal{I} - \mathcal{K})^{-1} = \mathcal{I} + M(I - K)^{-1}L\ . \tag{8}$$

Assuming no existence of $(I - K)^{-1}$, let us take $u \in \ker(I - K)$. Obviously, $M\,u \in \ker(I - K)$ and since $u = L\,Mu$, it follows that

$$\ker(I - K) \subset L\ker(\mathcal{I} - \mathcal{K})\ .$$

We verify in a similar way that

$$\ker(\mathcal{I} - \mathcal{K}) \subset M\ker(I - K)\ .$$

Hence, $\dim(I - K) = \dim(\mathcal{I} - \mathcal{K})$.

A more general situation occurs if the operators L and M in (4) are the vectors $(L_1, \ldots, L_N)$ and $(M_1, \ldots, M_N)$, where $L_j : \mathcal{X} \to X$ and $M_j : X \to \mathcal{X}$. This means that

$$K = \sum_{1 \le j \le N} L_j M_j\ . \tag{9}$$

Then (6) is equivalent to the equation (7) with the matrix operator

$$\mathcal{K} = \langle M_i L_j \rangle_{i,j=1}^N : \mathcal{X}^N \to \mathcal{X}^N \ . \tag{10}$$

One can easily generalize the above consideration replacing (4) by the approximate factorization

$$K = L\,M + S \ , \tag{11}$$

where S is a an operator such that $I - S$ performs an isomorphism of X. (In particular, S can be a contraction operator in X if X is a Banach space.)

As was mentioned at the beginning of this Introduction, the classical Fredholm theory gives an example of the case just described. In fact, let K be an operator in a Banach space X whose essential norm is less than 1 (in particular, K may be a compact operator in X). Then we arrive at (11) by choosing the N-dimensional Euclidean space $\mathbb{R}^N$ with a sufficiently large integer N as the space $\mathcal{X}$. We see that (3) is equivalent to a $N \times N$ algebraic system.

An important realization of equation (3) is the integral equation

$$u(P) - \int\limits_{\Omega_n} K(P,Q) u(Q)\, dQ = f(P) \ , \quad P \in \Omega_n \ , \tag{12}$$

where Ω_n is a n-dimensional Riemannian manifold and $P \in \Omega_n$. We shall proceed formally, specifying neither X and $\mathcal{X}$ nor domains of the operators involved. Let us assume that

$$K(P,Q) = \int\limits_{\mathbb{R}^m} L(P,y) M(y,Q)\, dy \ . \tag{13}$$

Then the kernel of the integral operator $\mathcal{K}$ given by (5) has the form

$$\mathcal{K}(x,y) = \int\limits_{\mathbb{R}^m} M(x,P) L(P,y)\, dP \ , \tag{14}$$

where x and y are points in $\mathbb{R}^m$. As an example, consider the integral equation

$$u(P) - \int\limits_{\Omega_2} \frac{a(P)(p_2+q_2)b(Q)}{(p_1-q_1)^2 + (p_2+q_2)^2} u(Q)\, dQ = f(P) \ , \tag{15}$$

where $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ are points of a domain Ω_2 in the upper half-plane $\mathbb{R}^2_+$ and a and b are functions prescribed on Ω_2. By the Poisson formula for a harmonic function in $\mathbb{R}^2_+$, the kernel $K(P,Q)$ of the integral operator in (15) can be written as

$$\frac{1}{\pi}\int\limits_{\mathbb{R}} \frac{a(P)\,b(Q)\,p_2\,q_2\,dt}{((t-p_1)^2+p_2^2)((t-q_1)^2+q_2^2)}\,.$$

Hence (15) is equivalent to the integral equation on the real line

$$U(x)-\int\limits_{\mathbb{R}} \mathcal{K}(x,y)U(y)\,dy = F(x) \tag{16}$$

with the kernel

$$\mathcal{K}(x,y)=\frac{1}{\pi}\int\limits_{\Omega_2} \frac{a(P)b(P)p_2^2dP}{((x-p_1)^2+p_2^2)((y-p_1)^2+p_2^2)}\,. \qquad \square$$

In principle, the reduction of dimension of the integration domain, shown in the above example can be achieved if $K(P,Q)=a(P)b(Q)c(P,Q)$, where one of the functions $P\to c(P,Q)$ and $Q\to c(P,Q)$ satisfies a partial differential equation on a n-dimensional manifold G containing Ω_n and therefore belongs to the range of a certain co-boundary operator which maps functions given on ∂G into functions on Ω_n. In this case, equation (12) for functions of n variables is equivalent to an integral equation for functions on the $(n-1)$-dimensional manifold ∂G. Another approach to the reduction of dimension will be discussed in the next section.

To conclude Introduction I describe an example where the factorization of the type (13) has a clear physical meaning. Consider the finite right cylinder with the base Ω. It is assumed that the heat is generated at the thermally inhomogeneous bottom surface and absorbed by the thermally inhomogeneous lid surface. The emissivity coefficient $M(t,Q)$ is defined at all points Q of the bottom surface at time t, and the absorptivity coefficient $L(P,t)$ is a function of a point P on the lid surface and time t. The full heat flux emitted from the bottom at the moment t is

$$F(t)=\int\limits_{\Omega} M(t,Q)u(Q)dQ.$$

If the cylinder is sufficiently long, then the energy density becomes uniform near the lid, and therefore the temperature v at a point P of the lid over the time interval $(0, T)$ is given by

$$v(P) = \int_0^T L(P,t)F(t)dt.$$

We see that the mapping $u \to v$ is the integral operator on the domain Ω with the kernel

$$K(P,Q) = \int_0^T L(P,t)M(t,Q)dt,$$

as required.

2 Reduction of dimension by the co-area formula

We show here by formal examples that the so called co-area formula [K], [F], [MSZ] can be used to turn some multi-dimensional integral equations into equations for functions of one variable.

Example 1. Let, as before, Ω_n be a n-dimensional Riemannian manifold and let $\Omega_n \ni P \to \varphi(P)$ be a smooth function which takes all real values. Let a and b denote complex valued functions defined on Ω_n. We are interested in the integral equation

$$u(P) - \int_{\Omega_n} \frac{a(P)b(Q)}{\varphi(Q) - \varphi(P)} u(Q)\, dQ = f(P)\,, \tag{17}$$

where $P \in \Omega_n$ and the integral is understood as a principal value. Here the kernel $K(P,Q)$ can be written as

$$K(P,Q) = \int_{\mathbb{R}} \delta(\tau - \varphi(P)) \frac{a(P)b(Q)}{\varphi(Q) - \tau} d\tau$$

and δ standing for the Dirac function. This means that (13) holds with

$$L(P,\tau) = a(P)\delta(\tau - \varphi(P)) \quad \text{and} \quad M(\tau,Q) = \frac{b(Q)}{\varphi(Q) - \tau}\,. \tag{18}$$

Therefore, (14) takes the form

$$\mathcal{K}(t,\tau) = \int\limits_{\mathbb{R}} \delta(\tau - \varphi(P)) \frac{a(P)b(P)}{\varphi(P) - t} dP \ . \tag{19}$$

The co-area formula

$$\int\limits_{\Omega_n} w(P)dP = \int\limits_{\mathbb{R}} \int\limits_{\varphi^{-1}(y)} \frac{w(P)}{|\nabla\varphi(P)|} d\mathcal{H}^{n-1}(P)\, dy \ , \tag{20}$$

where $\varphi^{-1}(y)$ is a level set of φ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, shows that the right-hand side in (19) equals

$$\frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{\delta(\tau - y)}{y - t} \int\limits_{\varphi^{-1}(y)} \frac{a(P)b(P)}{|\nabla\varphi(P)|} d\mathcal{H}^{n-1}(P)\, dy = \frac{\gamma(\tau)}{\pi(\tau - t)}$$

with

$$\gamma(\tau) = \int\limits_{\varphi^{-1}(\tau)} \frac{a(P)b(P)}{|\nabla\varphi(P)|} d\mathcal{H}^{n-1}(P) \ .$$

We see that the integral equation (17) is equivalent to the classical one-dimensional singular integral equation

$$U(t) - \int\limits_{\mathbb{R}} \frac{\gamma(\tau)}{\tau - t} U(\tau)\, d\tau = F(t) \ , \quad t \in \mathbb{R} \ . \tag{21}$$

It is well known that under some requirements for γ this equation is Fredholm in various function spaces on $\mathbb{R}$ if and only if

$$\inf\nolimits_{\mathbb{R}} |1 \pm i\pi\gamma(\tau)| > 0 \ . \tag{22}$$

Hence (22) is necessary and sufficient for the Fredholm property of the multi-dimensional equation (17).

If the range of φ is an interval $[\alpha, \beta]$, then the same argument leads to the equation (21) with $[\alpha, \beta]$ instead of $\mathbb{R}$.

Example 2. Let φ and ψ be smooth real-valued functions defined on a Riemannian manifold Ω_n. Assume that all level sets $\varphi^{-1}(t)$ and $\psi^{-1}(\tau)$ are smooth

$(n-1)$-dimensional surfaces and that $\varphi^{-1}(t)$ and $\psi^{-1}(\tau)$ are transversal at the points of intersection.

We consider the integral equation

$$u(P) - \int_{\Omega_n} S(\varphi(P), \psi(Q))u(Q)\, dQ = f(P)\ , \tag{23}$$

where Q is a continuous function defined on $\mathbb{R}^1 \times \mathbb{R}^1$. By the co-area formula, (23) can be written in the form

$$u(P) - \int_{\mathbb{R}} S(\varphi(P), t) \int_{\psi^{-1}(t)} \frac{u(Q)}{|\nabla\psi(Q)|}\, d\mathcal{H}^{n-1}(Q)\, dt = f(P)$$

The last integral operator is the product LM, where

$$(Mu)(t) = \int_{\psi^{-1}(t)} \frac{u(Q)}{|\nabla\psi(Q)|} d\mathcal{H}^{n-1}(Q)\ ,$$

$$(Lv)(P) = \int_{\mathbb{R}} S(\varphi(P), t)v(t)\, dt\ .$$

The operator $\mathcal{K} = ML$ which acts in a space of functions of one variable has the kernel

$$\mathcal{K}(t, \tau) = \int_{\psi^{-1}(\tau)} \frac{S(\varphi(P), t)}{|\nabla\psi(P)|} d\mathcal{H}^{n-1}(P)\ .$$

By the co-area formula applied to functions defined on the $(n-1)$-dimensional manifold $\psi^{-1}(\tau)$,

$$\mathcal{K}(t, \tau) = \int_{\mathbb{R}} S(\sigma, t) \int_{\psi^{-1}(\tau)\cap\varphi^{-1}(\sigma)} \frac{d\mathcal{H}^{n-2}(P)\, d\sigma}{|\nabla\psi(P)||\nabla_\tau\varphi(P)|}\ , \tag{24}$$

where ∇_τ is the gradient of the restriction of φ to the level surface $\psi^{-1}(\tau)$. Clearly, (24) can be written in the following more symmetric form

$$\mathcal{K}(t, \tau) = \int_{\mathbb{R}} S(\sigma, t) \int_{\psi^{-1}(\tau)\cap\varphi^{-1}(\sigma)} \frac{d\mathcal{H}^{n-2}(P)\, d\sigma}{|\nabla\psi(P)||\nabla\varphi(P)||\sin\omega(P)|}$$

where $\omega(P)$ is the angle between the normal directions to $\psi^{-1}(\tau)$ and $\varphi^{-1}(\sigma)$ at the point $P \in \varphi^{-1}(\tau) \cap \varphi^{-1}(\sigma)$.

3 More general equations

Example 3. Let Ω be an open set in $\mathbb{R}^n$ and let ϕ be a Lipschitz map of Ω onto $\mathbb{R}^m$, where $1 \le m < n$. By k and A we denote the functions

$$\Omega \times \mathbb{R}^m \ni (P, y) \to k(P, y) \in \mathbb{R} \quad \text{and} \quad \Omega \ni P \to A(P) \in \mathbb{R} \,.$$

Consider the integral operator $u \to Ku$ formally given by

$$(Ku)(P) = \int\limits_{\Omega} k(P, \phi(Q))A(Q)\, u(Q) dQ \;, \quad P \in \Omega \,. \tag{25}$$

We make use of the following co-area formula, generalizing (20),

$$\int\limits_{\Omega} w(Q)\, dQ = \int\limits_{\mathbb{R}^m} \int\limits_{\phi^{-1}(y)} \frac{w(Q)}{|\phi'(Q)|}\, d\mathcal{H}^{n-m}(Q)\, dy \;, \tag{26}$$

where ϕ' is the differential of ϕ, $\mathcal{H}^{n-m}$ is the $(n-m)$-dimensional Hausdorff measure, and $|\phi'|$ is the square root of the sum of the squares of the determinants of the $m \times m$ minors of ϕ' [F], [MSZ].

Hence

$$(Ku)(P) = \int\limits_{\mathbb{R}^m} K(P, y) \int\limits_{\phi^{-1}(y)} \frac{A(Q)u(Q)\, d\mathcal{H}^{n-m}(Q)}{|\phi'(Q)|}\, dy \;,$$

i.e. $K = LM$ where

$$(Mu)(y) = \int\limits_{\phi^{-1}(y)} \frac{A(Q)}{|\phi'(Q)|} u(Q) d\mathcal{H}^{n-m}(Q) \;,$$

$$(Lv)(P) = \int\limits_{\mathbb{R}^m} K(P, y) v(y)\, dy \,.$$

Therefore the equation (12) with $\Omega_n = \Omega$ is equivalent to the integral equation (14) in $\mathbb{R}^m$ with the kernel

$$\mathcal{K}(x,y) = \int\limits_{\phi^{-1}(x)} \frac{A(Q)}{|\phi'(Q)|} K(Q,y)\, d\mathcal{H}^{n-m}(Q) \; .$$

Example 4. Consider the operator

$$(Ku)(P) = A(P) \int\limits_{\Omega} k(Q, \varphi(P)) u(Q)\, dQ \; , \tag{27}$$

which is the formal adjoint to (25).

Here we can put

$$\begin{aligned}(Mu)(x) &= \int\limits_{\Omega} k(Q,x) u(Q) dQ \; , \\ (Lv)(P) &= A(P) \int\limits_{\mathbb{R}^m} \delta(x - \varphi(P)) v(x)\, dx \; ,\end{aligned}$$

and therefore the dual operator (5) is given by

$$(\mathcal{K}v)(x) = (MLv)(x) = \int\limits_{\Omega} k(P,x) A(P) \int\limits_{\mathbb{R}^m} (\delta - \varphi(P)) v(y)\, dy\, dP \; .$$

We see that the kernel of the operator $\mathcal{K}$ is

$$\mathcal{K}(x,y) = \int\limits_{\Omega} \delta(y, \varphi(P)) k(P,x) A(P)\, dP \; .$$

By the co-area formula (26)

$$\begin{aligned}\mathcal{K}(x,y) &= \int\limits_{\mathbb{R}^m} \delta(y - z) \int\limits_{\varphi^{-1}(z)} \frac{k(P,x)A(P)}{|\phi'(P)|}\, d\mathcal{H}^{n-m}(P)\, dz \\ &= \int\limits_{\varphi^{-1}(y)} \frac{k(P,x)A(P)}{|\phi'(P)|}\, d\mathcal{H}^{n-m}(P) \; .\end{aligned}$$

A more general situation is described in the following

Example 5. Let the kernel of the multi-dimensional operator K have the form

$$K(P,Q) = \sum_{1\le l\le N^+} k_l^+(P,\varphi_l^+(Q))A_l(Q) + \sum_{1\le m\le N^-} A_m^-(P)k_m^-(\varphi_m^-(P),Q) \ .$$

Then Examples 3 and 4 enable one to write this kernel as

$$K(P,Q) = \sum_{1\le l\le N^+} L_l^+ M_l^+ + \sum_{1\le m\le N^-} L_m^- M_m^- \ ,$$

where the operators M_l^+, M_m^- transform functions of n-variables to functions of one variable and L_l^+, L_m^- act in the opposite sense. Therefore the equation (3) is equivalent to a system of $N^- + N^+$ one-dimensional integral equations (compare with (9) and (10))

4 Solvability in weighted Lebesgue spaces

In order to emphasise the algorithmic side and avoid technicalities we treated all previous examples formally without specifying function spaces and the properties of the functions involved. Here we violate this rule for a subclass of integral operators (25).

As in section 3, ϕ will denote a Lipschitz mapping of Ω onto $\mathbb{R}^m$, $m < n$.

We introduce the measurable functions a and b and define for almost all y:

$$\alpha(y) = \left(\int\limits_{\varphi^{-1}(y)} |a(Q)|^r \frac{d\mathcal{H}^{n-m}(Q)}{|\phi'(Q)|} \right)^{1/r} \ ,$$

$$\beta(y) = \left(\int\limits_{\phi^{-1}(y)} |b(Q)|^{r'} \frac{d\mathcal{H}^{n-m}(Q)}{|\phi'(Q)|} \right)^{1/r'} \ ,$$

where $r \in [1,\infty]$ and $r + r' = rr'$.

Let ϱ denote a measurable nonnegative function given in $\mathbb{R}^m$. We introduce the weighted Lebesgue spaces $\mathcal{L}^r(\Omega,\varrho)$ and $L^r(\mathbb{R}^m,\varrho)$ supplied with the norms

$$\|u\|_{\mathcal{L}^r(\Omega,\varrho)} = \left(\int\limits_{\Omega} |u(Q)\varrho(\phi(Q))|^r \, dQ \right)^{1/r} \ ,$$

$$\|v\|_{L^r(\mathbb{R}^m,\varrho)} = \left(\int\limits_{\mathbb{R}^m} |v(y)\varrho(y)|^r \, dy \right)^{1/r} .$$

Lemma 1 *Suppose that there exists a constant C such that*

$$\alpha(y)\beta(y) \leq C \, , \tag{28}$$

for almost all $y \in \mathbb{R}^m$. Then the operator

$$v(y) \to (Lv)(P) = a(P) \int\limits_{\mathbb{R}^m} \delta(y - \varphi(P)) v(y) \, dy \, , \tag{29}$$

where δ is Dirac's function in $\mathbb{R}^m$, maps $L^r(\mathbb{R}^m, \varrho)$ continuously into $\mathcal{L}^r(\Omega, \beta\varrho)$.

Proof. By definition of L

$$\begin{aligned} \|Lv\|_{\mathcal{L}(\Omega,\beta\varrho)} &= \left(\int\limits_{\Omega} \left| \int\limits_{\mathbb{R}^m} a(P)\delta(y - \varphi(P))v(y) \, dy \right|^r |\beta(\varphi(P))\varrho(\phi(P))|^r \, dP \right)^{1/r} \\ &= \left(\int\limits_{\Omega} |a(P)\beta(\phi(P))\varrho(\phi(P))|^r \, dP \right)^{1/r} . \end{aligned}$$

By the co-area formula (26) the last norm can be written in the form

$$\left(\int\limits_{\mathbb{R}^m} |\varrho(y)\alpha(y)\beta(y)v(y)|^r \, dy \right)^{1/r}$$

and the result follows from (28).

□

Lemma 2 *Suppose that a convolution operator $v \to k * v$ acts continuously in $L^r(\mathbb{R}^m, \varrho)$. Then the operator*

$$u(Q) \to (Mu)(y) = \int\limits_{\Omega} k(\varphi(Q) - y) b(Q) u(Q) \, dQ$$

is a continuous mapping: $\mathcal{L}^r(\Omega, \beta\varrho) \to L^r(\mathbb{R}^m, \varrho)$.

Proof. By (26)

$$(Mu)(y) = \int\limits_{\mathbb{R}^m} k(x-y) \int\limits_{\phi^{-1}(x)} \frac{b(Q)u(Q)}{|\phi'(Q)|}\, d\mathcal{H}^{n-m}(Q)\ ,$$

and therefore

$$\|\varrho M u\|_{L^r(\mathbb{R}^m)} \le c \left(\int\limits_{\mathbb{R}^m} \left| \varrho(x) \int\limits_{\phi^{-1}(x)} \frac{b(Q)u(Q)}{|\phi'(Q)|}\, d\mathcal{H}^{n-m}(Q) \right|^r dx \right)^{1/r} .$$

Using Hölder's inequality and (28) we obtain

$$\|\varrho M u\|_{L^r(\mathbb{R}^m)} \le cC \left(\int\limits_{\mathbb{R}^m} \beta(x)^r \varrho(x)^r \int\limits_{\phi^{-1}(x)} \frac{|u(Q)|^r}{|\phi'(Q)|}\, d\mathcal{H}^{n-m}(Q)\, dx \right)^{1/r} ,$$

which together with (26) gives

$$\|Mu\|_{L^r(\mathbb{R}^m,\varrho)} \le cC \|u\|_{\mathcal{L}^r(\Omega,\beta\varrho)}\ .$$

The result follows. □

Let us consider the operator $K = LM$, which can be written as

$$u(Q) \to (Ku)(P) = a(P) \int\limits_{\Omega} k(\varphi(Q) - \varphi(P)) b(Q) u(Q)\, dQ\ .$$

Lemmas 1 and 2 show that K is a continuous operator in $\mathcal{L}^r(\Omega, \beta\varrho)$. The same Lemmas guarantee that the dual operator $\mathcal{K} = ML$ is well-defined and is a continuous operator in $\mathcal{L}^r(\mathbb{R}^m, \varrho)$. The operator $\mathcal{K}$ can be written in the form

$$v \to (\mathcal{K}v)(x) = \int\limits_{\mathbb{R}^m} k(y-x)\gamma(y)v(y)\, dy\ ,$$

where

$$\gamma(y) = \int\limits_{\phi^{-1}(y)} \frac{a(Q)b(Q)}{|\phi'(Q)|}\, d\mathcal{H}^{n-m}(Q)\ .$$

Making use of the formulae (7), (8) we obtain the following result.

Theorem 3 *The operator $I - K$ is invertible in $\mathcal{L}^r(\Omega, \beta\varrho)$ if and only if the operator $\mathcal{I} - \mathcal{K}$ is invertible in $L^r(\mathbb{R}^m, \varrho)$.*

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