S.G.Mikhlin and multi-dimensional singular integrals

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Solomon G. Mikhlin (1908–1990) (S.M.) influenced essentially the development of Analysis and Mathematical Physics for almost half a century. In our article we aim at giving an outline of just one important direction of his creative heritage, namely, the theory of multi-dimensional singular integral operators. This theory, in its turn, became the base for the calculus of pseudodifferential operators, one of major means in modern mathematical analysis.

The objects of study are integral operators of the form

$$u(x) \mapsto a(x)u(x) + (\mathbf{S}u)(x), \qquad (1)$$

$$(\mathbf{S}u)(x) = \int_{\mathbb{R}^d} K(x, y-x)u(y)dy, x \in \mathbb{R}^d.$$

Here $K(x, y-x) = \frac{f(x,\theta)}{r^d}$, $\theta = \frac{y-x}{r}$, $r = |x-y|$. The singularity of the kernel is so strong here that the integral diverges if considered in the usual sense. Therefore, it should be understood in the Cauchy principal value sense. An obvious necessary condition for the latter convergence, even for nice functions u, is

$$\int_{|\theta|=1} f(x,\theta)d\theta = 0. \qquad (2)$$

In a natural way, by localization, operators (1) can be defined on closed smooth manifolds and also generalized to vector-function $u(x)$, the kernel $K(x, y-x)$ being a square matrix-function.

Integrals in (1) are usually called 'singular'. In one dimension ($d = 1$) equations with such integrals appeared at the beginning of XX century in papers by D.Hilbert and H.Poincare concerning certain boundary problems in PDE and complex analysis. Even in one dimension, the Fredholm theory fails for this kind of equations since operator (1) is not compact.

The one-dimensional case was treated mostly using tools and ideas of complex analysis. The situation was different in the multi-dimensional case where singular integrals appear naturally in the theory of elliptic boundary problems. Here, before Mikhlin's studies, very little was known (F. Tricomi, 1926, 1928, $d = 2$, and G. Giraud, 1934 - 1936, $d > 2$, solved equations of a special form). The real breakthrough was made by S.G.Mikhlin. In papers in Doklady and Mat. Sbornik, [Mikhlin, 1936A, Mikhlin, 1936B] published in 1936, he put a foundation of the general theory of singular integral equations. In the case d=2 he expanded the function f in the trigonometric series $f(x,\theta) = \sum_{n\neq 0} b_n(x)e^{in\theta}$. Modifying this series, he introduced a certain function, which he called the symbol of the operator $a(x) + \mathbf{S}$:

$$\Phi(x,\theta) = a(x) + \sum_{n\neq 0} a_n(x)e^{in\theta}, \qquad (3)$$

$$a_n(x) = \frac{2\pi i^n}{n} b_n(x).$$

In admiration by Mikhlin's paper [Mikhlin, 1936A], Giraud [Giraud, 1936] wrote: 'An ingenious procedure indicated by S.Mikhlin makes it possible to treat equations with double principal integrals of a very general type.' Giraud generalized Mikhlin's construction and published a formula for the symbol of the singular integral operator on an Euclidean space of arbitrary dimension, using spherical harmonics. Giraud has never published a proof of his formula. A proof was published by S.M. in 1955, twelve years after Giraud's death, for the first time.

The operator is recovered uniquely from its symbol, up to a compact additive term. There exists the correspondence between the sums and products of operators and their symbols, again up to a compact additive term.

Main topics of the study by S.G. Mikhlin in this field were:

– Boundedness conditions for the operator $\mathbf{S}$ in various function spaces;

– Solvability analysis for equations of the form

$$(\mathbf{A}u)(x) \equiv a(x)u(x) + (\mathbf{S}u)(x) = v(x), \qquad (4)$$

here $a(x)$, $v(x)$ are given matrix-, resp., vector-functions.

In lieu of explicit formulas for solutions of the equation (4) S.M. proposed the procedure of regularization: finding a singular integral operator

$\mathbf{R} = b(x) + \tilde{\mathbf{S}}$ such that the operators $\mathbf{AR} - \mathbf{I}$ and $\mathbf{RA} - \mathbf{I}$ are compact in proper spaces. As soon as such a regularizer is found, the singular integral equation is reduced to an equation with compact operator. The invertibility of the symbol leads to the existence of a regularizer.

The notion of symbol enables one to treat many problems concerning singular integrals. For instance, it was established by S.M. that boundedness of the symbol, together with certain smoothness, guarantees boundedness of the operator in $L_2(\mathbb{R}^d)$.

For 15 years S.M. was the only researcher who worked in the theory of multi-dimensional singular integral operators. It is in 1952 that the fundamental paper by A.P. Calderón and A.Zygmund [Calderón-Zygmund, 1952] appeared. They extended M.Riesz' theorem on the L_p–boundedness of the Hilbert transform

$$(Hu)(x) = (\pi i)^{-1} \int_{-\infty}^{\infty} \frac{u(y)}{y-x} dy.$$

These authors showed that the multi-dimensional singular operator of convolution type $u \mapsto \frac{f(x/|x|)}{|x|^d} * u$ is bounded in $L_p(\mathbb{R}^d)$, $1 < p < \infty$, provided f satisfies (2) and

$$\int_{|\theta|=1} (1 + \log^+ |f(\theta)|)|f(\theta)|) d\theta < \infty. \quad (5)$$

Further on, Calderón and Zygmund succeeded in extending their results to a wide class of singular integrals of nonconvolutional type. Namely, with a kernel $Q(x, y)$ satisfying $|Q(x, y)| \leq C|x - y|^{-d}$ and

$$|Q(x, y + h) - Q(x, y)| + |Q(x + h, y) - Q(x, y)| \leq \quad (6)$$
$$Ch^{\alpha}|x - y|^{-d-\alpha}, \alpha \in (0, 1),\ 2h \leq |x - y|,$$

one associates the operator $\mathbf{T}_Q$ defined by the bilinear form

$$\langle \mathbf{T}_Q \varphi, \psi \rangle = \int\int \psi(x) Q(x, y) \varphi(y) dx dy, \quad (7)$$

for functions $\varphi, \psi \in C_0^{\infty}(\mathbb{R}^d)$ with disjoint supports. If the operator $\mathbf{T}_Q$ extends to a bounded operator in L_2, i.e., for all $g, h \in C_0^{\infty}$, the estimate $|\langle T_Q g, h \rangle| \leq C \|g\|_{L_2} \|h\|_{L_2}$ holds, then the operator is bounded in all L_p, $1 < p < \infty$. Thus the boundedness problem is reduced to the single case of $p = 2$. An essential progress in this latter question was made by E.Stein, A.McIntosh, M.Christ and others. In particular, due to the impressive $T\mathbf{1}$ theorem, the operator with Calderón-Zygmund kernel Q is bounded in L_2 iff both T_Q and its transposed T'_Q map just one single function, that equals identically one, into the space BMO (see, e.g., [Hofmann-McIntosh, 2011] for details and references).

The notion of *index* of an operator, the difference of dimensions of the null spaces of the operator and its adjoint, introduced in F.Noether's works of 1920-s for the one-dimensional case, was investigated by S.M. for the multi-dimensional case. The symbol of a singular operator, as S.M. introduced it, became central in this theory and had numerous applications. S.M. proved that a scalar elliptic singular integral operator in the multi-dimensional case has index zero. The matrix case turned out to be much more involved. In 1960-s the general index problem was settled by M.Atiyah and I.Singer, with far-reaching consequences for Analysis, Algebraic Topology, Noncommutative Geometry, with further expansions into Theoretical Physics. With his ideas and results in singular integral operators, S.M. became a forerunner of the revolutionary progress in analysis in 60-s, the theory of pseudodifferential operators, enveloping both singular integral, as well as differential operators and their resolvents. R.T.Seeley, one of pioneers in the field, acknowledges the contribution by Mikhlin to the topic, see [Seeley, 1963]: 'It will be clear that the author is indebted to the work of Mikhlin, which introduces many of the concepts and questions considered here.'

The results by S.G.Mikhlin as well as the development of the theory are presented in the books [Mikhlin, 1962], [Mikhlin-Prössdorff, 1986].

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