# Optimal pointwise estimates for derivatives of solutions to Laplace, Lamé and Stokes equations

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**Abstract.** Various optimal estimates for solutions of the Laplace, Lamé and Stokes equations in multidimensional domains, as well as new real-part theorems for analytic functions are obtained.

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## 1 Introduction

In the present paper we extend our study of the best constants in certain inequalities for solutions of the Laplace, Lamé and Stokes equations (see Kresin and Maz'ya [10]). We also deal with optimal estimates for analytic functions in the spirit of our recent article [9]. Let us formulate some results obtained in the sequel.

By $|\cdot|$ we denote the Euclidean length of a vector or absolute value of a scalar quantity. Let $\Omega$ be a domain in $\mathbb{R}^n$. By $d_x$ we mean the distance from a point $x \in \Omega$ to $\partial\Omega$ and by $\omega_n$ we denote the area of the $(n-1)$-dimensional unit sphere.

One of the results derived in Section 2 is the following pointwise estimate of the gradient of a bounded harmonic function in the complement $\Omega$ of a convex closed domain in $\mathbb{R}^n$:

$$\big|\nabla u(x)\big| \leq \frac{C_n}{d_x} \sup_{\Omega} |u|$$

*for all* $x \in \Omega$. *Here*

$$C_n = \frac{4(n-1)^{(n-1)/2}\, \omega_{n-1}}{n^{n/2}\, \omega_n}$$

*is the best constant.*

We also state here a limit estimate with the same best constant $C_n$, valid for arbitrary domains which is established in Section 2.

*Let* $\Omega$ *be a domain in* $\mathbb{R}^n$, *and let* $\mathfrak{U}(\Omega)$ *be the set of harmonic functions* $u$ *in* $\Omega$ *with* $\sup_\Omega |u| \leq 1$. *Suppose that a point* $\xi \in \partial\Omega$ *can be touched by an interior ball* $B$. *Then*

$$\limsup_{x \to \xi} \sup_{u \in \mathfrak{U}(\Omega)} |x - \xi| \big|\nabla u(x)\big| \leq C_n ,$$

*where* $x$ *is a point of the radius of* $B$ *directed from the center to* $\xi$.

In Section 3 we obtain pointwise estimates for the directional derivative $(\boldsymbol{\ell}, \nabla)\boldsymbol{u}$, where $\boldsymbol{u}(x) = (u_1(x), \dots, u_m(x))$ is a vector field whose components are harmonic in $\Omega$. Assertions proved here are generalizations of the theorems given in Section 2. In Section 4 we present analogs of the theorems of Section 2 containing pointwise and limit estimates for $|\text{div}\, \boldsymbol{u}(x)|$, $m = n$.

By $[\mathrm{C_b}(\overline{\Omega})]^n$ we mean the space of vector-valued functions with $n$ components which are bounded and continuous on $\overline{\Omega}$. This space is endowed with the norm $||\boldsymbol{u}||_{[\mathrm{C_b}(\overline{\Omega})]^n} = \sup\{|\boldsymbol{u}(x)| : x \in \overline{\Omega})\}$. By $[\mathrm{C}^2(\Omega)]^n$ we denote the space of $n$-component vector-valued functions with continuous derivatives up to second order in $\Omega$.

Next, in Section 5 we find an optimal estimate for $|\text{div}\, \boldsymbol{u}(x)|$, where $\boldsymbol{u}$ is an elastic displacement vector in $\mathbb{R}^n_+ = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$. As a corollary, we obtain an optimal estimate for the pressure $p$ in a viscous incompressible fluid in $\mathbb{R}^n_+$. We formulate two statements following from these results.

*Let* $\Omega = \mathbb{R}^n \backslash \overline{G}$, *where* $G$ *is a convex domain in* $\mathbb{R}^n$.

(i) *Let* $\boldsymbol{u} \in [\mathrm{C}^2(\Omega)]^n \cap [\mathrm{C_b}(\overline{\Omega})]^n$ *be a solution of the Lamé system*

$$\Delta\boldsymbol{u} + (1 - 2\sigma)^{-1}\, \text{grad div}\, \boldsymbol{u} = \boldsymbol{0}$$

*in* $\Omega$, *where* $\sigma \in (-\infty, 1/2) \cup (1, +\infty)$ *is the Poisson coefficient. Then for any point* $x \in \Omega$ *the inequality*

$$|\text{div}\, \boldsymbol{u}(x)| \leq \frac{(1 - 2\sigma)E_n}{(3 - 4\sigma)d_x} \sup_\Omega |\boldsymbol{u}|$$

*holds, where*

$$E_n = \frac{4\omega_{n-1}}{\omega_n} \int_0^{\pi/2} \left[1 + n(n-2)\cos^2\vartheta\right]^{1/2} \sin^{n-2}\vartheta d\vartheta$$

*is the best constant.*

(ii) *Let* $\boldsymbol{u} \in [\mathrm{C}^2(\Omega)]^n \cap [\mathrm{C_b}(\overline{\Omega})]^n$ *be a vector component of the solution* $\{\boldsymbol{u}, p\}$ *to the Stokes system*

$$\Delta\boldsymbol{u} - \text{grad}\, p = \boldsymbol{0} , \quad \text{div}\, \boldsymbol{u} = 0 \ \textit{in} \ \Omega$$

*and let* $p(x)$ *be the pressure vanishing as* $d_x \to \infty$. *Then for any point* $x \in \Omega$ *the inequality*

$$|p(x)| \leq \frac{E_n}{d_x} \sup_\Omega |\boldsymbol{u}|$$

*holds with the same best constant* $E_n$ *as above.*

The last Section 6 is dedicated to some new real-part theorems for analytic functions (see Kresin and Maz'ya [7] and the bibliography collected there). We derive the following results.

(i) *Let* $\Omega = \mathbb{C}\backslash\overline{G}$, *where* $G$ *is a convex domain in* $\mathbb{C}$, *and let* $f$ *be a holomorphic function in* $\Omega$ *with bounded real part. Then for any point* $z \in \Omega$ *the inequality*

$$\left|f^{(s)}(z)\right| \leq \frac{K_s}{d^s} \sup_{\Omega} |\Re f| \,, \qquad s = 1, 2, \dots \,,$$

*holds with* $d_z = \text{dist}\,(z, \partial\Omega)$, *where*

$$K_s = \frac{s!}{\pi} \max_{\alpha} \int_{-\pi/2}^{\pi/2} \left|\cos\big(\alpha + (s+1)\varphi\big)\right| \cos^{s-1}\varphi \, d\varphi$$

*is the best constant. In particular* $K_{2l+1} = 2[(2l+1)!!]^2[\pi(2l+1]^{-1}$.

(ii) *Let* $\Omega$ *be a domain in* $\mathbb{C}$, *and let* $\mathfrak{R}(\Omega)$ *be the set of holomorphic functions* $f$ *in* $\Omega$ *with* $\sup_\Omega |\Re f| \leq 1$. *Assume that a point* $\zeta \in \partial\Omega$ *can be touched by an interior disk* $D$. *Then*

$$\limsup_{z \to \zeta} \sup_{f \in \mathfrak{R}(\Omega)} |z - \zeta|^s \left|f^{(s)}(z)\right| \leq K_s \,, \qquad s = 1, 2, \dots \,,$$

*where* $z$ *is a point of the radius of* $D$ *directed from the center to* $\zeta$. *Here the constant* $K_s$ *is the same as above and cannot be diminished.*

More details concerning the above formulations can be found in the statements of corresponding theorems, propositions and corollaries in what follows.

## 2 Estimates for the gradient of harmonic function

We introduce some notation used henceforth. Let $\mathbb{B} = \big\{x \in \mathbb{R}^n : |x| < 1\big\}$, $\mathbb{B}_R = \big\{x \in \mathbb{R}^n : |x| < R\big\}$, and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. By $h^\infty(\Omega)$ we denote the Hardy space of bounded harmonic functions on the domain $\Omega$ with the norm $||u||_{h^\infty(\Omega)} = \sup\{|u(x)| : x \in \Omega\}$.

**Theorem 1.** *Let* $\Omega = \mathbb{R}^n\backslash\overline{G}$, *where* $G$ *is a convex domain in* $\mathbb{R}^n$, *and let* $u$ *be a bounded harmonic function in* $\Omega$. *Then for any point* $x \in \Omega$ *the inequality*

$$\left|\nabla u(x)\right| \leq \frac{C_n}{d_x} \sup_{\Omega} |u| \tag{2.1}$$

*holds, where*

$$C_n = \frac{4(n-1)^{(n-1)/2}\,\omega_{n-1}}{n^{n/2}\,\omega_n} \tag{2.2}$$

*is the best constant in the inequality*

$$|\nabla u(x)| \leq C_n\, x_n^{-1}\, ||u||_{L^\infty(\partial\mathbb{R}^n_+)}$$

*for a bounded harmonic function* $u$ *in the half-space* $\mathbb{R}^n_+$.

*In particular,*

$$C_2 = \frac{2}{\pi}\,, \qquad C_3 = \frac{4}{3\sqrt{3}}\,.$$

*Proof.* Let $\xi \in \partial\Omega$ be a point at $\partial\Omega$ nearest to $x \in \Omega$ and let $T(\xi)$ be the hyperplane containing $\xi$ and orthogonal to the line joining $x$ and $\xi$. By $\mathbb{R}^n_\xi$ we denote the open half-space with boundary $T(\xi)$ such that $\mathbb{R}^n_\xi \subset \Omega$.

Let $n \geq 3$. According to Theorem 1 [8], the inequality

$$|\nabla u(x)| \leq \frac{C_n}{d_x} \, ||u||_{h^\infty(\mathbb{R}^n_\xi)} \tag{2.3}$$

holds, where $C_n$ is given by (2.2). Using (2.3) and the obvious inequality

$$||u||_{h^\infty(\mathbb{R}^n_\xi)} \leq \sup_\Omega |u| \ ,$$

we arrive at (2.1).

The case $n = 2$ is considered analogously, the role of (2.3) being played by the estimate

$$|f'(z)| \leq \frac{2}{\pi \Im z} \sup_{\mathbb{C}_+} |\Re f| \tag{2.4}$$

(see [7], Sect. 3.7.3) by the change $f = u + iv$, $f'(z) = u'_x - iv'_y$, where $f$ is a holomorphic function in $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ with bounded real part. □

In what follows, we assume that the Cartesian coordinates with origin $\mathcal{O}$ at the center of the ball are chosen in such a way that $x = |x|\boldsymbol{e}_n$. By $\boldsymbol{\ell}$ we denote an arbitrary unit vector in $\mathbb{R}^n$ and by $\boldsymbol{\nu}_x$ we mean the unit vector of exterior normal to the sphere $|x| = r$ at a point $x$. Let $\boldsymbol{\ell}_\tau$ be the orthogonal projection of $\boldsymbol{\ell}$ on the tangent hyperplane to the sphere $|x| = r$ at $x$. If $\boldsymbol{\ell}_\tau \neq \boldsymbol{0}$, we set $\boldsymbol{\tau}_x = \boldsymbol{\ell}_\tau / |\boldsymbol{\ell}_\tau|$, otherwise $\boldsymbol{\tau}_x$ is an arbitrary unit vector tangent to the sphere $|x| = r$ at $x$. Hence

$$\boldsymbol{\ell} = \ell_\tau \boldsymbol{\tau}_x + \ell_\nu \boldsymbol{\nu}_x, \tag{2.5}$$

where $\ell_\tau = |\boldsymbol{\ell}_\tau|$ and $\ell_\nu = (\boldsymbol{\ell}, \boldsymbol{\nu}_x)$.

We premise Lemmas 1 and 2 to Theorem 2. In Lemma 1 we derive a representation for the sharp coefficient $\mathcal{K}_n(x)$ in the inequality

$$|\nabla u(x)| \leq \mathcal{K}_n(x) ||u||_{L^\infty(\partial\mathbb{B})} \ , \tag{2.6}$$

where $x \in \mathbb{B}$ and $u \in h^\infty(\mathbb{B})$. *Here and elsewhere we say that a certain coefficient is sharp if it cannot be diminished for any point $x$ in the domain under consideration.* The expression for $\mathcal{K}_n(x)$, given below, contains two factors one of which is an explicitely given function increasing to infinity as $r \to 1$ and the second factor (the double integral) is a bounded function on the interval $0 \leq r \leq 1$.

**Lemma 1.** *Let $u \in h^\infty(\mathbb{B})$, and let $x$ be an arbitrary point in $\mathbb{B}$. The sharp coefficient $\mathcal{K}_n(x)$ in inequality* (2.6) *is given by*

$$\mathcal{K}_n(x) = \frac{2^{n-2}(n-2)}{\pi(1+r)^{n-1}(1-r)} \sup_{\gamma \geq 0} \frac{1}{\sqrt{1+\gamma^2}} \int_0^\pi \sin^{n-3}\varphi \, d\varphi \int_0^{\pi/2} G_n(\vartheta, \varphi; r, \gamma) \, d\vartheta \ , \tag{2.7}$$

*where*

$$G_n(\vartheta, \varphi; r, \gamma) = \frac{\big| n\cos 2\vartheta + n\gamma \sin 2\vartheta \cos\varphi + (n-2)r \big|}{\left[ 1 + \left(\frac{1-r}{1+r}\right)^2 \tan^2\vartheta \right]^{(n-2)/2}} \sin^{n-2}\vartheta \ . \tag{2.8}$$

*Proof. 1. Representation for $\mathcal{K}_n(x)$ by an integral over $\mathbb{S}^{n-1}$.* Let $u$ stand for a harmonic function in $\mathbb{B}$ from the space $h^\infty(\mathbb{B})$. By Poisson formula we have

$$u(x) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \frac{1-r^2}{|y-x|^n} u(y) d\sigma_y \ . \tag{2.9}$$

Fix a point $x \in \mathbb{B}$. By (2.9)

$$\frac{\partial u}{\partial x_i} = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \left[ \frac{-2x_i}{|y-x|^n} + \frac{n\left(1-r^2\right)(y_i - x_i)}{|y-x|^{n+2}} \right] u(y) d\sigma_y,$$

that is

$$\nabla u(x) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \frac{n\left(1-r^2\right)(y-x) - 2|y-x|^2 x}{|y-x|^{n+2}} \ u(y) d\sigma_y.$$

Thus

$$(\nabla u(x), \boldsymbol{\ell}) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \frac{\left(n\left(1-r^2\right)(y-x) - 2|y-x|^2 x, \boldsymbol{\ell}\right)}{|y-x|^{n+2}} \ u(y) d\sigma_y \ ,$$

and therefore

$$\mathcal{K}_n(x) = \frac{1}{\omega_n} \sup_{|\boldsymbol{\ell}|=1} \int_{\mathbb{S}^{n-1}} \frac{\left|\left(n\left(1-r^2\right)(y-x) - 2|y-x|^2 x, \boldsymbol{\ell}\right)\right|}{|y-x|^{n+2}} \ d\sigma_y \ . \tag{2.10}$$

Using (2.5), we obtain

$$\mathcal{K}_n(x) = \frac{1}{\omega_n} \sup_{|\boldsymbol{\ell}|=1} \int_{\mathbb{S}^{n-1}} \frac{\left|\left(n(1-r^2)\left((y, \boldsymbol{\nu}_x) - r\right) - 2r|y-x|^2\right)\ell_\nu + n(1-r^2)(y, \boldsymbol{\tau}_x)\ell_\tau\right|}{|y-x|^{n+2}} \ d\sigma_y \ .$$

The last expression can be written as

$$\mathcal{K}_n(x) = \frac{a_n(r)}{\omega_n} \sup_{|\boldsymbol{\ell}|=1} \int_{\mathbb{S}^{n-1}} \frac{\left|b_n(r)(y, \boldsymbol{\tau}_x)\ell_\tau + \left(y_n - c_n(r)\right)\ell_\nu\right|}{\left(1 - 2ry_n + r^2\right)^{(n+2)/2}} \ d\sigma_y \ , \tag{2.11}$$

where

$$a_n(r) = n(1-r^2) + 4r^2 \ , \qquad b_n(r) = \frac{n(1-r^2)}{n(1-r^2)+4r^2} \ , \qquad c_n(r) = \frac{n(1-r^2) + 2(1+r^2)}{n(1-r^2)+4r^2} r \ . \tag{2.12}$$

*2. Representation for $\mathcal{K}_n(x)$ by a double integral.* Introducing the function

$$\mathcal{H}_n(s, t; r, \boldsymbol{\ell}) = \frac{\left|b_n(r)s\ell_\tau + \left(t - c_n(r)\right)\ell_\nu\right|}{\left(1 - 2rt + r^2\right)^{(n+2)/2}} \ , \tag{2.13}$$

we write the integral in (2.11) as the sum

$$\int_{\mathbb{S}^{n-1}_+} \mathcal{H}_n((y_\tau, \boldsymbol{\tau}_x), y_n; r, \boldsymbol{\ell}) \ d\sigma_y + \int_{\mathbb{S}^{n-1}_-} \mathcal{H}_n((y_\tau, \boldsymbol{\tau}_x), y_n; r, \boldsymbol{\ell}) \ d\sigma_y, \tag{2.14}$$

where $\mathbb{S}^{n-1}_+ = \{y \in \mathbb{S}^{n-1} : (y, \boldsymbol{e}_n) > 0\}$, $\mathbb{S}^{n-1}_- = \{y \in \mathbb{S}^{n-1} : (y, \boldsymbol{e}_n) < 0\}$.

Let $y' = (y_1, \dots, y_{n-1}) \in \mathbb{B}' = \{y' \in \mathbb{R}^{n-1} : |y'| < 1\}$. We put

$$\boldsymbol{\tau}'_x = \sum_{i=1}^{n-1} (\boldsymbol{\tau}_x, \boldsymbol{e}_i)\boldsymbol{e}_i.$$

Since $y_n = \sqrt{1-|y'|^2}$ for $y \in \mathbb{S}^{n-1}_+$ and $y_n = -\sqrt{1-|y'|^2}$ for $y \in \mathbb{S}^{n-1}_-$ and since $d\sigma_y = dy'/\sqrt{1-|y'|^2}$, it follows that each of integrals in (2.14) can be written in the form

$$\int_{\mathbb{S}^{n-1}_+} \mathcal{H}_n((y_\tau, \boldsymbol{\tau}_x), y_n; r, \boldsymbol{\ell}) d\sigma_y = \int_{\mathbb{B}'} \frac{\mathcal{H}_n\Big((y', \boldsymbol{\tau}'_x), \sqrt{1-|y'|^2}; r, \boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}} dy', \tag{2.15}$$

$$\int_{\mathbb{S}^{n-1}_-} \mathcal{H}_n((y_\tau, \boldsymbol{\tau}_x), y_n; r, \boldsymbol{\ell}) d\sigma_y = \int_{\mathbb{B}'} \frac{\mathcal{H}_n\Big((y', \boldsymbol{\tau}'_x), -\sqrt{1-|y'|^2}; r, \boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}} dy'. \tag{2.16}$$

Putting

$$\mathcal{M}_n(s, t; r, \boldsymbol{\ell}) = \mathcal{H}_n(s, t; r, \boldsymbol{\ell}) + \mathcal{H}_n(s, -t; r, \boldsymbol{\ell}), \tag{2.17}$$

and using (2.13)-(2.16), we rewrite (2.11) as

$$\mathcal{K}_n(x) = \frac{a_n(r)}{\omega_n} \sup_{|\boldsymbol{\ell}|=1} \int_{\mathbb{B}'} \frac{\mathcal{M}_n\Big((y', \boldsymbol{\tau}'_x), \sqrt{1-|y'|^2}; r, \boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}} \, dy' \,. \tag{2.18}$$

By the identity

$$\int_{\mathbb{B}^n} g\big((\boldsymbol{y}, \boldsymbol{\xi}),\ |\boldsymbol{y}|\big) dy = \omega_{n-1} \int_0^1 \rho^{n-1} d\rho \int_0^\pi g\big(|\boldsymbol{\xi}|\rho\cos\varphi,\ \rho\big) \sin^{n-2}\varphi \, d\varphi$$

(see, e.g., [13], **3.3.2(3)**), we transform the integral in (2.18):

$$\int_{\mathbb{B}'} \frac{\mathcal{M}_n\Big((y', \boldsymbol{\tau}'_x), \sqrt{1-|y'|^2}; r, \boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}} \, dy' \tag{2.19}$$

$$= \omega_{n-2} \int_0^1 \frac{\rho^{n-2}}{\sqrt{1-\rho^2}} \, d\rho \int_0^\pi \mathcal{M}_n\Big(\rho\cos\varphi,\ \sqrt{1-\rho^2}; r, \boldsymbol{\ell}\Big) \sin^{n-3}\varphi d\varphi.$$

The change $\rho = \sin\theta$ in (2.19) gives

$$\int_{\mathbb{B}'} \frac{\mathcal{M}_n\Big((y', \boldsymbol{\tau}'_x), \sqrt{1-|y'|^2}; r, \boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}} \, dy' \tag{2.20}$$

$$= \omega_{n-2} \int_0^{\pi/2} \sin^{n-2}\theta d\theta \int_0^\pi \mathcal{M}_n\Big(\sin\theta\cos\varphi,\ \cos\theta; r, \boldsymbol{\ell}\Big) \sin^{n-3}\varphi \, d\varphi \,.$$

Applying (2.13), (2.17) and introducing the notation

$$\begin{aligned} \mathcal{F}_n(\theta, \varphi; r, \boldsymbol{\ell}) &= \mathcal{H}_n\Big(\sin\theta\cos\varphi,\ \cos\theta; r, \boldsymbol{\ell}\Big) \\ &= \frac{\big|b_n(r)\ell_\tau \sin\theta\cos\varphi + \big(\cos\theta - c_n(r)\big)\ell_\nu\big|}{\big(1 - 2r\cos\theta + r^2\big)^{(n+2)/2}} \,, \end{aligned}$$

we write (2.20) as follows

$$\int_{\mathbb{B}'} \frac{\mathcal{M}_n\Big((y', \boldsymbol{\tau}'_x), \sqrt{1-|y'|^2}; r, \boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}}\, dy' \tag{2.21}$$
$$= \omega_{n-2} \int_0^{\pi/2} \sin^{n-2}\theta d\theta \int_0^{\pi}\Big(\mathcal{F}_n(\theta, \varphi; r, \boldsymbol{\ell}) + \mathcal{F}_n(\pi-\theta, \varphi; r, \boldsymbol{\ell})\Big) \sin^{n-3}\varphi\, d\varphi\ .$$

Changing the variable $\psi = \pi - \theta$, we obtain

$$\int_0^{\pi/2} \sin^{n-2}\theta\ d\theta \int_0^{\pi} \mathcal{F}_n\big(\pi-\theta, \varphi; r, \boldsymbol{\ell}\big) \sin^{n-3}\varphi\ d\varphi$$
$$= \int_{\pi/2}^{\pi} \sin^{n-2}\psi\ d\psi \int_0^{\pi} \mathcal{F}_n\big(\psi, \varphi; r, \boldsymbol{\ell}\big) \sin^{n-3}\varphi\ d\varphi,$$

which together with (2.21) leads to the representation of (2.18):

$$\mathcal{K}_n(x) = \frac{a_n(r)\omega_{n-2}}{\omega_n} \sup_{|\boldsymbol{\ell}|=1} \int_0^{\pi} \sin^{n-2}\theta\ d\theta \int_0^{\pi} \mathcal{F}_n(\theta, \varphi; r, \boldsymbol{\ell}) \sin^{n-3}\varphi\ d\varphi\ . \tag{2.22}$$

*3. Transformation of representation for $\mathcal{K}_n(x)$.* We make the change of variable

$$\theta = 2\arctan\left(\frac{1-r}{1+r}\tan\vartheta\right)$$

in (2.22). Then

$$\sin\theta = \frac{2\left(\frac{1-r}{1+r}\right)\tan\vartheta}{1+\left(\frac{1-r}{1+r}\right)^2\tan^2\vartheta}\ , \tag{2.23}$$

$$d\theta = \frac{2(1-r)}{(1+r)\cos^2\vartheta\left(1+\left(\frac{1-r}{1+r}\right)^2\tan^2\vartheta\right)}\, d\vartheta\ , \tag{2.24}$$

$$1 - 2r\cos\theta + r^2 = \frac{(1-r)^2}{\cos^2\vartheta\left(1+\left(\frac{1-r}{1+r}\right)^2\tan^2\vartheta\right)}\ , \tag{2.25}$$

$$b_n(r)\ell_\tau \sin\theta\cos\varphi + \big(\cos\theta - c_n(r)\big)\ell_\nu = \frac{(1-r)^2\big[n\ell_\tau \sin 2\vartheta\cos\varphi + \big(n\cos 2\vartheta + (n-2)r\big)\ell_\nu\big]}{\big[n(1-r^2)+4r^2\big]\cos^2\vartheta\left(1+\left(\frac{1-r}{1+r}\right)^2\tan^2\vartheta\right)}\ . \tag{2.26}$$

Substituting (2.23)-(2.26) in (2.22), we arrive at

$$\mathcal{K}_n(x) = \frac{2^{n-2}(n-2)}{\pi(1+r)^{n-1}(1-r)} \sup_{|\boldsymbol{\ell}|=1} \int_0^{\pi} \sin^{n-3}\varphi\ d\varphi \int_0^{\pi/2} \mathcal{G}_n(\vartheta, \varphi; r, \boldsymbol{\ell})\ d\vartheta\ , \tag{2.27}$$

where

$$\mathcal{G}_n(\vartheta,\varphi;r,\boldsymbol{\ell}) = \frac{\big|n\ell_\tau \sin 2\vartheta \cos\varphi + \big(n\cos 2\vartheta + (n-2)r\big)\ell_\nu\big|}{\left[1+\left(\frac{1-r}{1+r}\right)^2 \tan^2\vartheta\right]^{(n-2)/2}} \sin^{n-2}\vartheta \,.$$

Since the integrand in (2.10) does not change when the unit vector $\boldsymbol{\ell}$ is replaced by $-\boldsymbol{\ell}$, we may assume that $\ell_\nu = (\boldsymbol{\ell}, \boldsymbol{\nu}_x) > 0$ in (2.27). Introducing the parameter $\gamma = \ell_\tau/\ell_\nu$ in (2.27) and using the equality $\ell_\tau^2 + \ell_\nu^2 = 1$, we arrive at (2.7) with $G_n(\vartheta,\varphi;r,\gamma)$ given by (2.8). $\square$

By dilation, we obtain the following result, equivalent to Lemma 1 and involving the ball $\mathbb{B}_R$ with an arbitrary $R$.

**Lemma 2.** *Let $u \in h^\infty(\mathbb{B}_R)$, and let $x$ be an arbitrary point in $\mathbb{B}_R$. The sharp coefficient $\mathcal{K}_{n,R}(x)$ in the inequality*

$$|\nabla u(x)| \le \mathcal{K}_{n,R}(x) ||u||_{L^\infty(\partial\mathbb{B}_R)}$$

*is given by*

$$\mathcal{K}_{n,R}(x) = \frac{2^{n-2}(n-2)R^{n-1}}{\pi(R+|x|)^{n-1}(R-|x|)} \sup_{\gamma\ge 0} \frac{1}{\sqrt{1+\gamma^2}} \int_0^\pi \sin^{n-3}\varphi \, d\varphi \int_0^{\pi/2} G_n\left(\vartheta,\varphi;\frac{|x|}{R},\gamma\right) d\vartheta \,,$$

*where*

$$G_n(\vartheta,\varphi;r,\gamma) = \frac{\big|n\gamma \sin 2\vartheta \cos\varphi + n\cos 2\vartheta + (n-2)r\big|}{\left[1+\left(\frac{1-r}{1+r}\right)^2 \tan^2\vartheta\right]^{(n-2)/2}} \sin^{n-2}\vartheta \,.$$

Now, we prove a limit estimate for the gradient of a bounded harmonic function.

**Theorem 2.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, and let $\mathfrak{U}(\Omega)$ be the set of harmonic functions $u$ in $\Omega$ with $\sup_\Omega |u| \le 1$. Assume that a point $\xi \in \partial\Omega$ can be touched by an interior ball $B$. Then*

$$\limsup_{x\to\xi} \sup_{u\in\mathfrak{U}(\Omega)} |x-\xi| \big|\nabla u(x)\big| \le C_n \,, \tag{2.28}$$

*where $x$ is a point at the radius of $B$ directed from the center to $\xi$. Here the constant $C_n$ is the same as in Theorem 1.*

*Proof.* Let $n \ge 3$. By Lemma 2, the relations

$$\limsup_{|x|\to R} \sup\big\{(R-|x|)|\nabla u(x)| : ||u||_{h^\infty(\mathbb{B}_R)} \le 1\big\} \le \lim_{|x|\to R} (R-|x|)\mathcal{K}_{n,R}(x) = C_n \tag{2.29}$$

hold, where

$$C_n = \frac{n-2}{2\pi} \sup_{\gamma\ge 0} \frac{1}{\sqrt{1+\gamma^2}} \int_0^\pi \sin^{n-3}\varphi \, d\varphi \int_0^{\pi/2} \big|\mathcal{P}_n(\vartheta,\varphi;\gamma)\big| \sin^{n-2}\vartheta \, d\vartheta \,, \tag{2.30}$$

with

$$\begin{aligned} \mathcal{P}_n(\vartheta,\varphi;\gamma) &= n\gamma \sin 2\vartheta \cos\varphi + n\cos 2\vartheta + (n-2) \\ &= 2\big[n\gamma \cos\vartheta \sin\vartheta \cos\varphi + (n\cos^2\vartheta - 1)\big]. \end{aligned}$$

According to Proposition 1 in [8], the sharp coefficient $\mathcal{C}_n(x)$ in the inequality

$$|\nabla u(x)| \leq \mathcal{C}_n(x)||u||_{h^\infty(\mathbb{R}^n_+)} \ , \tag{2.31}$$

where $u$ is a bounded harmonic function in the half-space $\mathbb{R}^n_+$, is equal to $\mathcal{C}_n(x) = C_n/x_n$ with the best constant $C_n$ given by (2.30). By Theorem 1 in [8], the value of $C_n$ is given by the formula

$$C_n = \frac{4(n-1)^{(n-1)/2}\ \omega_{n-1}}{n^{n/2}\ \omega_n} \ . \tag{2.32}$$

Let $R$ denote the radius of the ball $B \subset \Omega$ tangent to $\partial\Omega$ at the point $\xi$. We put the origin $\mathcal{O}$ at the center of $B$. Let the point $x$ belong to the interval joining $\mathcal{O}$ and $\xi$. Then $R - |x| = |x - \xi|$. By (2.29) with $C_n$ from (2.32) on the right-hand side we conclude the proof in the case $n \geq 3$ by reference to the inequality

$$||u||_{h^\infty(B)} \leq \sup_\Omega |u| \ . \tag{2.33}$$

The proof of Theorem 2 in the case $n = 2$ is analogous, estimate (2.29) follows from D. Khavinson's [6] inequality

$$|f'(z)| \leq \frac{4R}{\pi(R^2 - |z|^2)} \sup_{|\zeta|<R} |\Re f(\zeta)| \tag{2.34}$$

by the change $f = u + iv$, $f'(z) = u'_x - iv'_y$, where $f$ is holomorphic in $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$. The estimate (2.31) results from (2.4) by the change $f = u + iv$, $f'(z) = u'_x - iv'_y$, where $f$ is holomorphic in $\mathbb{C}_+$. □

**Remark 1**. The following inequality for the modulus of the gradient of a harmonic function is known (see [12], Ch. 2, Sect. 13)

$$|\nabla u(x)| \leq \frac{A_n}{d_x} \operatorname{osc}_\Omega(u) \ ,$$

where

$$A_n = \frac{n\omega_{n-1}}{(n-1)\omega_n} \ .$$

It is equivalent to the estimate

$$|\nabla u(x)| \leq \frac{2A_n}{d_x} \sup_\Omega |u| \ , \tag{2.35}$$

where $u$ is a bounded harmonic function in $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and $\operatorname{osc}_\Omega(u)$ is the oscillation of $u$ on $\Omega$.

The coefficient on the right-hand side of (2.35) is less than that in the well known gradient estimate (see, e.g., [2], Sect. 2.7)

$$|\nabla u(x)| \leq \frac{n}{d_x} \sup_\Omega |u| \ .$$

By

$$\frac{C_n}{2A_n} = \frac{2}{\sqrt{n}}\left(1 - \frac{1}{n}\right)^{(n+1)/2} < 1 \ ,$$

inequality (2.1) with $C_n$ from (2.2) improves (2.35) for domains complementary to convex closed domains.

Sharp estimates of derivatives of harmonic functions can be found in the books [7], [10]. We also mention the articles [1], [3], [5] dealing with estimates of harmonic functions.

## 3 Estimates for the maximum value of the modulus of directional derivative of a vector field with harmonic components

Let in the domain $\Omega \subset \mathbb{R}^n$, there is a $m$-component vector field $\boldsymbol{a}(x) = (a_1(x), \dots, a_m(x))$, $m \geq 1$. Let, further $\boldsymbol{\ell} = (\ell_1, \dots, \ell_n)$ be a unit $n$-dimensional vector. The derivative of the field $\boldsymbol{a}(x)$ in the direction $\boldsymbol{\ell}$ is defined by

$$\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\ell}} = \lim_{t \to 0} \frac{\boldsymbol{a}(x + t\boldsymbol{\ell}) - \boldsymbol{a}(x)}{t} \ ,$$

that is

$$\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\ell}} = (\boldsymbol{\ell}, \nabla)\boldsymbol{a} \ . \tag{3.1}$$

Let us introduce some notation used in the sequel. By $||\boldsymbol{u}||_{[L^\infty(\partial\Omega)]^m} = \text{ess } \sup\{|\boldsymbol{u}(x)| : x \in \partial\Omega\}$ we denote the norm in the space $[L^\infty(\partial\Omega)]^m$ of vector-valued functions $\boldsymbol{u}$ on $\partial\Omega$ with $m$ components from $L^\infty(\partial\Omega)$. By $[h^\infty(\Omega)]^m$ we mean the Hardy space of vector-valued functions $\boldsymbol{u}(x) = (u_1(x), \dots, u_m(x))$ with bounded harmonic components on $\Omega$ endowed with the norm $||\boldsymbol{u}||_{[h^\infty(\Omega)]^m} = \sup\{|\boldsymbol{u}(x)| : x \in \Omega\}$.

It is known that any element of $[h^\infty(\mathbb{R}^n_+)]^m$ can be represented by the Poisson integral

$$\boldsymbol{u}(x) = \frac{2}{\omega_n} \int_{\partial\mathbb{R}^n_+} \frac{x_n}{|y - x|^n} \, \boldsymbol{u}(y) dy' \tag{3.2}$$

with boundary values in $[L^\infty(\partial\mathbb{R}^n_+)]^m$, where $y = (y', 0)$, $y' \in \mathbb{R}^{n-1}$.

Now, we find a representation for the sharp coefficient $\mathcal{C}_{m,n}(x)$ in the inequality

$$\max_{|\boldsymbol{\ell}|=1} \big|(\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x)\big| \leq \mathcal{C}_{m,n}(x) ||\boldsymbol{u}||_{[L^\infty(\partial\mathbb{R}^n_+)]^m} \ , \tag{3.3}$$

where $\boldsymbol{u} \in [h^\infty(\mathbb{R}^n_+)]^m$ and $x \in \mathbb{R}^n_+$.

**Lemma 3.** *Let $\boldsymbol{u} \in [h^\infty(\mathbb{R}^n_+)]^m$, and let $x$ be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_{m,n}(x)$ in* (3.3) *is given by*

$$\mathcal{C}_{m,n}(x) = C_{m,n} x_n^{-1} \ , \tag{3.4}$$

*where*

$$C_{m,n} = \frac{1}{\omega_n} \max_{|\boldsymbol{\ell}|=1} \int_{\mathbb{S}^{n-1}} \big|\big(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma, \ \boldsymbol{\ell}\big)\big| \, d\sigma \ , \tag{3.5}$$

*and $\boldsymbol{e}_\sigma$ stands for the $n$-dimensional unit vector joining the origin to a point $\sigma$ on the sphere $\mathbb{S}^{n-1}$.*

*Proof.* Let $x = (x', x_n)$ be a fixed point in $\mathbb{R}^n_+$. The representation (3.2) implies

$$\frac{\partial \boldsymbol{u}}{\partial x_j} = \frac{2}{\omega_n} \int_{\partial\mathbb{R}^n_+} \left[ \frac{\delta_{nj}}{|y - x|^n} + \frac{n x_n (y_j - x_j)}{|y - x|^{n+2}} \right] \boldsymbol{u}(y) dy',$$

that is, by (3.1),

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\ell}} &= \frac{2}{\omega_n} \sum_{j=1}^{n} \ell_j \int_{\partial\mathbb{R}^n_+} \left[ \frac{\delta_{nj}}{|y - x|^n} + \frac{n x_n (y_j - x_j)}{|y - x|^{n+2}} \right] \boldsymbol{u}(y) dy' \\ &= \frac{2}{\omega_n} \int_{\partial\mathbb{R}^n_+} \frac{(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy}, \ \boldsymbol{\ell})}{|y - x|^n} \, \boldsymbol{u}(y) dy, \end{aligned}$$

where $\boldsymbol{e}_{xy} = (y-x)|y-x|^{-1}$. For any $\boldsymbol{z} \in \mathbb{S}^{m-1}$,

$$\big((\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x), \boldsymbol{z}\big) = \frac{2}{\omega_n} \int_{\partial \mathbb{R}^n_+} \frac{(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{\ell})}{|y-x|^n}\ (\boldsymbol{u}(y), \boldsymbol{z}) dy'.$$

Hence,

$$\begin{aligned}\mathcal{C}_{m,n}(x) &= \frac{2}{\omega_n} \max_{|\boldsymbol{\ell}|=1} \int_{\partial \mathbb{R}^n_+} \frac{\big|(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy}, \boldsymbol{\ell})\big|}{|y-x|^n}\ dy' \\ &= \frac{1}{\omega_n x_n} \max_{|\boldsymbol{\ell}|=1} \int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{\ell})\big|\ d\sigma\ .\end{aligned}$$

The last equality proves (3.4) and (3.5). □

By Lemma 3, the sharp coefficient $\mathcal{C}_{m,n}(x)$ in inequality (3.3) does not depend on $m$. Thus, $\mathcal{C}_{m,n}(x) = \mathcal{C}_{1,n}(x) = \mathcal{C}_n(x)$, where $\mathcal{C}_n(x) = C_n x_n^{-1}$ is the sharp coefficient in (2.31). Thus, we arrive at the following generalization of Theorem 1 in our paper [8], where the case $m = 1$ is treated.

**Proposition 1.** *Let $\boldsymbol{u} \in [h^\infty(\mathbb{R}^n_+)]^m$ and let $x$ be an arbitrary point in $\mathbb{R}^n_+$. The inequality*

$$\max_{|\boldsymbol{\ell}|=1} \big|(\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x)\big| \leq C_n x_n^{-1} ||\boldsymbol{u}||_{[L^\infty(\partial \mathbb{R}^n_+)]^m} \tag{3.6}$$

*holds, where the best constant $C_n$ is the same as in Theorem 1.*

The assertion below is an extension of Theorem 1.

**Proposition 2.** *Let $\Omega = \mathbb{R}^n \backslash \overline{G}$, where $G$ is a convex subdomain of $\mathbb{R}^n$, and let $\boldsymbol{u}$ be a vector-valued function with $m$ bounded harmonic components in $\Omega$. Then for any point $x \in \Omega$ the inequality*

$$\max_{|\boldsymbol{\ell}|=1} \big|(\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x)\big| \leq \frac{C_n}{d_x} \sup_\Omega |\boldsymbol{u}| \tag{3.7}$$

*holds, where the constant $C_n$ is the same as in Theorem 1.*

*Proof.* Let $\xi \in \partial\Omega$ be the point at $\partial\Omega$ nearest to $x \in \Omega$. Let the notation $\mathbb{R}^n_\xi$ be the same as in the proof of Theorem 1. By Proposition 1,

$$\max_{|\boldsymbol{\ell}|=1} \big|(\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x)\big| \leq \frac{C_n}{d_x}\ ||\boldsymbol{u}||_{[h^\infty(\mathbb{R}^n_\xi)]^m}\ ,$$

where $C_n$ is given by (2.2). Then, using the inequality

$$||\boldsymbol{u}||_{[h^\infty(\mathbb{R}^n_\xi)]^m} \leq \sup_\Omega |\boldsymbol{u}|\ , \tag{3.8}$$

we arrive at (3.7). □

Any element of $[h^\infty(\mathbb{B})]^m$ can be represented as the Poisson integral

$$\boldsymbol{u}(x) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \frac{1-r^2}{|y-x|^n} \boldsymbol{u}(y) d\sigma_y \tag{3.9}$$

with boundary values in $[L^\infty(\partial\mathbb{B})]^m$.

In the next assertion we find a representation for the sharp coefficient $\mathcal{K}_{m,n}(x)$ in the inequality

$$\max_{|\boldsymbol{\ell}|=1} \big|(\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x)\big| \leq \mathcal{K}_{m,n}(x) ||\boldsymbol{u}||_{[L^\infty(\partial\mathbb{B})]^m}\ . \tag{3.10}$$

**Lemma 4.** *Let $\boldsymbol{u} \in [h^{\infty}(\mathbb{B})]^m$, and let $x$ be an arbitrary point in $\mathbb{B}$. The sharp coefficient $\mathcal{K}_{m,n}(x)$ in* (3.10) *is given by*

$$\mathcal{K}_{m,n}(x) = \frac{1}{\omega_n} \sup_{|\boldsymbol{\ell}|=1} \int_{\mathbb{S}^{n-1}} \frac{\left|\left(n\left(1-r^2\right)(y-x) - 2|y-x|^2 x, \boldsymbol{\ell}\right)\right|}{|y-x|^{n+2}}\, d\sigma_y \ . \tag{3.11}$$

*Proof.* Fix a point $x \in \mathbb{B}$. By (3.9)

$$\frac{\partial \boldsymbol{u}}{\partial x_j} = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \left[\frac{-2x_j}{|y-x|^n} + \frac{n\left(1-r^2\right)(y_j - x_j)}{|y-x|^{n+2}}\right] \boldsymbol{u}(y) d\sigma_y,$$

that is

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\ell}} = (\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x) = \frac{1}{\omega_n} \sum_{j=1}^{n} \ell_j \int_{\mathbb{S}^{n-1}} \frac{n\left(1-r^2\right)(y_j - x_j) - 2|y-x|^2 x_j}{|y-x|^{n+2}}\, \boldsymbol{u}(y) d\sigma_y.$$

For any $\boldsymbol{z} \in \mathbb{S}^{m-1}$ we have

$$\big((\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x), \boldsymbol{z}\big) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \frac{\left(n\left(1-r^2\right)(y-x) - 2|y-x|^2 x, \boldsymbol{\ell}\right)}{|y-x|^{n+2}} \big(\boldsymbol{u}(y), \boldsymbol{z}\big) d\sigma_y \ ,$$

which implies (3.11). □

The next assertion is a generalization of Theorem 2.

**Proposition 3.** *Let $\Omega$ be a domain in $\mathbb{R}^n$. Let $\mathfrak{U}(\Omega)$ be the set of $m$-component vector-valued functions $\boldsymbol{u}$ whose components are harmonic in $\Omega$, with $\sup_{\Omega} |\boldsymbol{u}| \le 1$. Assume that a point $\xi \in \partial\Omega$ can be touched by an interior ball $B$. Then*

$$\limsup_{x \to \xi} \sup_{\boldsymbol{u} \in \mathfrak{U}(\Omega)} \max_{|\boldsymbol{\ell}|=1} |x-\xi| \big|(\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x)\big| \le C_n \ , \tag{3.12}$$

*where $x$ is a point of the radius of $B$ directed from the center to $\xi$. Here the constant $C_n$ is the same as in Theorem* 1.

*Proof.* By Lemma 4, $\mathcal{K}_{m,n}(x)$ does not depend on $m$ and therefore $\mathcal{K}_{m,n}(x) = \mathcal{K}_{1,n}(x) = \mathcal{K}_n(x)$, where $\mathcal{K}_n(x)$ is the sharp coefficient in (2.6). Hence (3.10) can be written in the form

$$\max_{|\boldsymbol{\ell}|=1} \big|(\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x)\big| \le \mathcal{K}_n(x) ||\boldsymbol{u}||_{[L^{\infty}(\partial\mathbb{B})]^m} \ .$$

By dilation in the last inequality we obtain the analogue of Lemma 2

$$\max_{|\boldsymbol{\ell}|=1} \big|(\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x)\big| \le \mathcal{K}_{n,R}(x) ||\boldsymbol{u}||_{[L^{\infty}(\partial\mathbb{B}_R)]^m} \ , \tag{3.13}$$

where $x \in \mathbb{B}_R$ and $\boldsymbol{u} \in [h^{\infty}(\mathbb{B}_R)]^m$. Now, (3.13) along with the representation of $\mathcal{K}_{n,R}(x)$ from Lemma 2 leads to the inequality

$$\limsup_{|x| \to R} \sup \left\{ (R-|x|) \big|(\boldsymbol{\ell}, \nabla)\boldsymbol{u}(x)\big| : |\boldsymbol{\ell}| = 1,\ ||\boldsymbol{u}||_{[h^{\infty}(\mathbb{B}_R)]^m} \le 1 \right\} \le \lim_{|x| \to R} (R-|x|)\mathcal{K}_{n,R}(x) = C_n \ ,$$

where $C_n$ is given by (2.30). The proof is completed in the same way as that of Theorem 2, with the only difference that (2.33) is replaced by the inequality

$$||\boldsymbol{u}||_{[h^{\infty}(B)]^m} \le \sup_{\Omega} |\boldsymbol{u}|. \tag{3.14}$$

□

## 4 Estimates for the divergence of a vector field with harmonic components

Let $\boldsymbol{u}(x)=(u_1(x),\dots,u_n(x))$ be a vector field with $n$ bounded harmonic components in $\Omega\subset\mathbb{R}^n$.

**Proposition 4.** *Let $\boldsymbol{u}\in[h^\infty(\mathbb{R}^n_+)]^n$, and let $x$ be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{D}_n(x)$ in the inequality*

$$|\mathrm{div}\ \boldsymbol{u}(x)|\le\mathcal{D}_n(x)||\boldsymbol{u}||_{[L^\infty(\partial\mathbb{R}^n_+)]^n} \tag{4.1}$$

*is given by*

$$\mathcal{D}_n(x)=D_n x_n^{-1}\ , \tag{4.2}$$

*where*

$$D_n=\frac{2\omega_{n-1}}{\omega_n}\int_0^{\pi/2}\left[1+n(n-2)\cos^2\vartheta\right]^{1/2}\sin^{n-2}\vartheta d\vartheta\ . \tag{4.3}$$

*In particular,*

$$D_2=1\ ,\qquad D_3=1+\frac{\sqrt{3}}{6}\ln\left(2+\sqrt{3}\right)\ .$$

*Proof.* By (3.2),

$$\begin{aligned}\mathrm{div}\ \boldsymbol{u}\ &=\ \frac{2}{\omega_n}\sum_{j=1}^n\int_{\partial\mathbb{R}^n_+}u_j(y)\frac{\partial}{\partial x_j}\left(\frac{x_n}{|y-x|^n}\right)dy'=\\ &\frac{2}{\omega_n}\sum_{j=1}^n\int_{\partial\mathbb{R}^n_+}\left(\frac{\delta_{jn}}{|y-x|^n}+\frac{nx_n(y_j-x_j)}{|y-x|^{n+2}}\right)u_j(y)dy'=\\ &\frac{2}{\omega_n}\sum_{j=1}^n\int_{\partial\mathbb{R}^n_+}\left(\frac{\delta_{jn}-n(\boldsymbol{e}_{xy},\boldsymbol{e}_n)(\boldsymbol{e}_{xy},\boldsymbol{e}_j)}{|y-x|^n}\right)u_j(y)dy'\ ,\end{aligned} \tag{4.4}$$

which implies

$$\mathrm{div}\ \boldsymbol{u}=\frac{2}{\omega_n}\int_{\partial\mathbb{R}^n_+}\frac{\left(\boldsymbol{e}_n-n(\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{u}(y)\right)}{|y-x|^n}dy'\ . \tag{4.5}$$

This equality shows that the sharp coefficient $\mathcal{D}_n(x)$ in (4.1) is represented in the form

$$\mathcal{D}_n(x)=\frac{2}{\omega_n}\int_{\partial\mathbb{R}^n_+}\frac{\left|\boldsymbol{e}_n-n(\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy}\right|}{|y-x|^n}dy'\ .$$

Then

$$\mathcal{D}_n(x)=\frac{2}{\omega_n x_n}\int_{\partial\mathbb{R}^n_+}\left|\boldsymbol{e}_n-n(\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy}\right|\frac{x_n}{|y-x|^n}dy'=\frac{D_n}{x_n}\ , \tag{4.6}$$

where

$$D_n=\frac{2}{\omega_n}\int_{\mathbb{S}^{n-1}_-}\left|\boldsymbol{e}_n-n(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma\right|d\sigma=\frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}}\left|\boldsymbol{e}_n-n(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma\right|d\sigma\ . \tag{4.7}$$

The identity

$$\left|\boldsymbol{e}_n-n(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma\right|^2=1+n(n-2)(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)^2\ , \tag{4.8}$$

along with (4.7) leads to the formula

$$D_n = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \Big(1 + n(n-2)(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)^2\Big)^{1/2} d\sigma \ . \tag{4.9}$$

Using

$$\int_{\mathbb{S}^{n-1}} f\big((\boldsymbol{\xi}, \boldsymbol{y})\big) d\sigma_y = \omega_{n-1} \int_{-1}^{1} f\big(|\boldsymbol{\xi}|t\big)(1-t^2)^{(n-3)/2} dt \tag{4.10}$$

(see, e.g., [14], **4.3.2(2)**) and the change of variable $t = \cos\vartheta$, we obtain

$$\int_{\mathbb{S}^{n-1}} \Big(1 + n(n-2)(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)^2\Big)^{1/2} d\sigma = 2\omega_{n-1} \int_0^{\pi/2} \big[1 + n(n-2)\cos^2\vartheta\big]^{1/2} \sin^{n-2}\vartheta d\vartheta \ . \tag{4.11}$$

By (4.6), (4.9) and (4.11), we arrive at (4.2) and (4.3). □

The next assertion is analogous to Proposition 2. Here the divergence replaces the directional derivative.

**Proposition 5.** *Let $\Omega = \mathbb{R}^n \backslash \overline{G}$, where $G$ be a convex subdomain of $\mathbb{R}^n$, and let $\boldsymbol{u}$ be a $n$-component vector-valued function with bounded harmonic components in $\Omega$. Then for any point $x \in \Omega$ the inequality*

$$|\text{div } \boldsymbol{u}(x)| \le \frac{D_n}{d_x} \sup_{\Omega} |\boldsymbol{u}| \tag{4.12}$$

*holds, where the constant $D_n$ is the same as in Proposition* 4.

*Proof.* Let $\xi \in \partial\Omega$ be a point at $\partial\Omega$ nearest to $x \in \Omega$. Let the notation $\mathbb{R}^n_\xi$ be the same as in the proof of Theorem 1. By Proposition 4,

$$|\text{div } \boldsymbol{u}(x)| \le \frac{D_n}{d_x} \ ||\boldsymbol{u}||_{[h^\infty(\mathbb{R}^n_\xi)]^n} \ ,$$

where $D_n$ is defined by (4.3). Then by (3.8) with $m = n$, we arrive at (4.12). □

**Lemma 5.** *Let $\boldsymbol{u} \in [h^\infty(\mathbb{B})]^n$, and let $x$ be an arbitrary point in $\mathbb{B}$. The sharp coefficient $\mathcal{T}_n(x)$ in the inequality*

$$|\text{div } \boldsymbol{u}| \le \mathcal{T}_n(x) ||\boldsymbol{u}||_{[L^\infty(\partial\mathbb{B})]^n} \tag{4.13}$$

*is given by*

$$\mathcal{T}_n(x) = \frac{2^{n-1}\omega_{n-1}}{\omega_n(1+r)^{n-1}(1-r)} \int_0^{\pi/2} \frac{\Big[\big(n-(n-2)r\big)^2 + 4n(n-2)r\cos^2\vartheta\Big]^{1/2}}{\Big[1+\left(\frac{1-r}{1+r}\right)^2 \tan^2\vartheta\Big]^{(n-2)/2}} \sin^{n-2}\vartheta \ d\vartheta \ . \tag{4.14}$$

*In particular,*

$$\mathcal{T}_2(x) = \frac{2}{1-r^2} \ , \qquad \mathcal{T}_3(x) = \frac{1}{1-r^2}\left(2 + \frac{3-r^2}{2\sqrt{3}r} \ln \frac{\sqrt{3}+r}{\sqrt{3}-r}\right) \ .$$

*Proof.* Let us fix a point $x \in \mathbb{B}$. By (3.9) we have

$$\frac{\partial u_j}{\partial x_j} = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \left[ \frac{-2x_j}{|y-x|^n} + \frac{n\left(1-r^2\right)\left(y_j - x_j\right)}{|y-x|^{n+2}} \right] u_j(y) d\sigma_y.$$

Therefore,

$$\text{div } \boldsymbol{u} = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \left( \frac{-2x}{|y-x|^n} + \frac{n\left(1-r^2\right)(y-x)}{|y-x|^{n+2}},\ \boldsymbol{u}(y) \right) d\sigma_y.$$

This implies that the sharp coefficient $\mathcal{T}_n(x)$ in (4.13) has the form

$$\mathcal{T}_n(x) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \frac{\big| -2x|y-x|^2 + n(1-r^2)(y-x) \big|}{|y-x|^{n+2}} d\sigma_y\ ,$$

which leads to the formula

$$\mathcal{T}_n(x) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \frac{\left(4r^2 + a^2(r) - 4a(r)(x,y)\right)^{1/2}}{\left(1 - 2(x,y) + r^2\right)^{(n+1)/2}} d\sigma_y\ , \tag{4.15}$$

where $a(r) = 2r^2 + n(1-r^2)$. Transforming the integral in (4.15) with help of (4.10), we obtain

$$\mathcal{T}_n(x) = \frac{\omega_{n-1}}{\omega_n} \int_{-1}^{1} \frac{\left(4r^2 + a^2(r) - 4ra(r)t\right)^{1/2}}{\left(1 - 2rt + r^2\right)^{(n+1)/2}} (1-t^2)^{(n-3)/2} dt\ .$$

Changing the variable $t = \cos\theta$, we derive

$$\mathcal{T}_n(x) = \frac{\omega_{n-1}}{\omega_n} \int_{0}^{\pi} \frac{\left(4r^2 + a^2(r) - 4ra(r)\cos\theta\right)^{1/2}}{\left(1 - 2r\cos\theta + r^2\right)^{(n+1)/2}} \sin^{n-2}\theta d\theta\ . \tag{4.16}$$

Finally, setting

$$\theta = 2\arctan\left( \frac{1-r}{1+r} \tan\vartheta \right)$$

in (4.16) and using (2.23)-(2.25), we arrive at (4.14). □

By dilation in Lemma 5, we obtain

**Lemma 6.** *Let* $\boldsymbol{u} \in [h^\infty(\mathbb{B}_R)]^n$, *and let* $x$ *be an arbitrary point in* $\mathbb{B}_R$. *The sharp coefficient* $\mathcal{T}_{n,R}(x)$ *in the inequality*

$$|\text{div } \boldsymbol{u}(x)| \le \mathcal{T}_{n,R}(x) ||\boldsymbol{u}||_{[L^\infty(\partial\mathbb{B}_R)]^n}$$

*is given by*

$$\mathcal{T}_{n,R}(x) = \frac{2^{n-1}\omega_{n-1}R^{n-1}}{\omega_n(R+|x|)^{n-1}(R-|x|)} \int_0^{\pi/2} Q_n\left(\vartheta;\ \frac{|x|}{R}\right) \sin^{n-2}\vartheta\ d\vartheta\ ,$$

*where*

$$Q_n(\vartheta; r) = \frac{\left[ \left(n - (n-2)r\right)^2 + 4n(n-2)r\cos^2\vartheta \right]^{1/2}}{\left[ 1 + \left(\frac{1-r}{1+r}\right)^2 \tan^2\vartheta \right]^{(n-2)/2}}\ .$$

**Proposition 6.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $\mathfrak{U}(\Omega)$ be the set of $n$-component vector-valued functions $\boldsymbol{u}$ whose components are harmonic in $\Omega$, and $\sup_\Omega |\boldsymbol{u}| \le 1$. Suppose that a point $\xi \in \partial\Omega$ can be touched by an interior ball $B$. Then*

$$\limsup_{x\to\xi} \sup_{\boldsymbol{u}\in\mathfrak{U}(\Omega)} |x-\xi|\big|\mathrm{div}\ \boldsymbol{u}(x)\big| \le D_n\ ,$$

*where $x$ is a point of the radius of $B$ directed from the center to $\xi$. Here the constant $D_n$ is the same as in Proposition 4.*

*Proof.* By Lemma 6, the relations

$$\limsup_{|x|\to R} \sup\left\{(R-|x|)\big|\mathrm{div}\ \boldsymbol{u}(x)\big| : ||\boldsymbol{u}||_{[h^\infty(\mathbb{B}_R)]^n} \le 1\right\} \le \lim_{|x|\to R}(R-|x|)\mathcal{T}_{n,R}(x) = D_n \tag{4.17}$$

hold, where $D_n$ is the same as in Proposition 4.

Using the notation introduced in Theorem 2, by (4.17) and (3.14) with $m = n$ the result follows. □

# 5 Estimates for the divergence of an elastic displacement field and the pressure in a fluid

Let $[\mathrm{C_b}(\partial\mathbb{R}^n_+)]^n$ be the space of vector-valued functions with $n$ components which are bounded and continuous on $\partial\mathbb{R}^n_+$. This space is endowed with the norm $||\boldsymbol{u}||_{[\mathrm{C_b}(\partial\mathbb{R}^n_+)]^n} = \sup\{|\boldsymbol{u}(x)| : x \in \partial\mathbb{R}^n_+\}$.

In the half-space $\mathbb{R}^n_+$, $n \ge 2$, consider the Lamé system

$$\Delta\boldsymbol{u} + (1-2\sigma)^{-1}\ \mathrm{grad\ div}\ \boldsymbol{u} = \boldsymbol{0}\ , \tag{5.1}$$

and the Stokes system

$$\Delta\boldsymbol{u} - \mathrm{grad}\ p = \boldsymbol{0}\ , \qquad \mathrm{div}\ \boldsymbol{u} = 0\ , \tag{5.2}$$

with the boundary condition

$$\boldsymbol{u}\big|_{x_n=0} = \boldsymbol{f}, \tag{5.3}$$

where $\sigma$ is the Poisson coefficient, $\boldsymbol{f} \in [\mathrm{C_b}(\partial\mathbb{R}^n_+)]^n$, $\boldsymbol{u} = (u_1, \dots, u_n)$ is the displacement vector of an elastic medium or the velocity vector of a fluid, and $p(x)$ is the pressure in the fluid vanishing as $x_n \to \infty$.

We assume that $\sigma \in (-\infty, 1/2) \cup (1, +\infty)$ which means the strong ellipticity of system (5.1). By $\lambda$ and $\mu$ we denote the Lamé constants. Since $\sigma = \lambda/2(\lambda+\mu)$ the strong ellipticity is equivalent to the inequalities $\mu > 0, \lambda+\mu > 0$ and $-\mu < \lambda+\mu < 0$.

A unique solution $\boldsymbol{u} \in [\mathrm{C}^2(\mathbb{R}^n_+)]^n \cap [\mathrm{C_b}(\overline{\mathbb{R}^n_+})]^n$ of problem (5.1), (5.3) and the vector component $\boldsymbol{u} \in [\mathrm{C}^2(\mathbb{R}^n_+)]^n \cap [\mathrm{C_b}(\overline{\mathbb{R}^n_+})]^n$ of a solution $\{\boldsymbol{u}, p\}$ to problem (5.2), (5.3) admit the representation (see, e.g., [10], pp. 64-65)

$$\boldsymbol{u}(x) = \int_{\partial\mathbb{R}^n_+} \mathcal{H}\left(\frac{y-x}{|y-x|}\right)\frac{x_n}{|y-x|^n}\boldsymbol{f}(y')dy', \tag{5.4}$$

where $x \in \mathbb{R}^n_+$, $y = (y', 0)$, $y' \in \mathbb{R}^{n-1}$. Here $\mathcal{H}$ is the $(n\times n)$-matrix-valued function on $\mathbb{S}^{n-1}$ with elements

$$\frac{2}{\omega_n}\left((1-\kappa)\delta_{jk} + n\kappa\frac{(y_j-x_j)(y_k-x_k)}{|y-x|^2}\right), \tag{5.5}$$

where $\kappa = 1$ for the Stokes system and $\kappa = (3-4\sigma)^{-1}$ for the Lamé system.

**Proposition 7.** (i) *Let* $\boldsymbol{u} \in [\mathrm{C}^2(\mathbb{R}^n_+)]^n \cap [\mathrm{C_b}(\overline{\mathbb{R}^n_+})]^n$ *be a solution of the Lamé system in* $\mathbb{R}^n_+$. *The sharp coefficient* $\mathcal{E}_n(x)$ *in the inequality*

$$|\mathrm{div}\ \boldsymbol{u}(x)| \leq \mathcal{E}_n(x)||\boldsymbol{u}||_{[\mathrm{C_b}(\partial\mathbb{R}^n_+)]^n} \tag{5.6}$$

*is given by*

$$\mathcal{E}_n(x) = \frac{1-2\sigma}{3-4\sigma} E_n x_n^{-1}\ , \tag{5.7}$$

*where*

$$E_n = \frac{4\omega_{n-1}}{\omega_n}\int_0^{\pi/2}\left[1+n(n-2)\cos^2\vartheta\right]^{1/2}\sin^{n-2}\vartheta d\vartheta\ . \tag{5.8}$$

*In particular,*

$$E_2 = 2\ , \qquad E_3 = 2\left(1+\frac{\sqrt{3}}{6}\ln\left(2+\sqrt{3}\right)\right)\ .$$

(ii) *Let* $\boldsymbol{u} \in [\mathrm{C}^2(\mathbb{R}^n_+)]^n \cap [\mathrm{C_b}(\overline{\mathbb{R}^n_+})]^n$ *be the vector component of a solution* $\{\boldsymbol{u}, p\}$ *of the Stokes system* (5.2) *in* $\mathbb{R}^n_+$ *and* $p(x)$ *be the pressure vanishing as* $x_n \to \infty$. *The sharp coefficient* $\mathcal{S}_n(x)$ *in the inequality*

$$|p(x)| \leq \mathcal{S}_n(x)||\boldsymbol{u}||_{[\mathrm{C_b}(\partial\mathbb{R}^n_+)]^n} \tag{5.9}$$

*is given by*

$$\mathcal{S}_n(x) = E_n x_n^{-1}\ , \tag{5.10}$$

*where the constant* $E_n$ *is defined by* (5.8).

*Proof.* (i) *Proof of inequality* (5.6). By (5.4) and (5.5),

$$u_j(x) = \frac{2}{\omega_n}\int_{\partial\mathbb{R}^n_+}\left((1-\kappa)\boldsymbol{e}_j + n\kappa\frac{(y_j-x_j)(y-x)}{|y-x|^2},\ \boldsymbol{f}(y')\right)\frac{x_n}{|y-x|^n}dy'\ . \tag{5.11}$$

Noting that $y_n = 0$ in (5.11), we find

$$\sum_{j=1}^n \frac{\partial}{\partial x_j}\left\{\frac{(y_j-x_j)\big(y-x,\ \boldsymbol{f}(y')\big)x_n}{|y-x|^{n+2}}\right\} = \sum_{j=1}^n \frac{(n+2)(y_j-x_j)^2\big(y-x,\ \boldsymbol{f}(y')\big)x_n}{|y-x|^{n+4}} +$$

$$\sum_{j=1}^n \frac{-\big(y-x,\ \boldsymbol{f}(y')\big)x_n + (y_j-x_j)f(y')x_n + (y_j-x_j)\big(y-x,\ \boldsymbol{f}(y')\big)\delta_{nj}}{|y-x|^{n+2}} =$$

$$\frac{-n\big(y-x,\ \boldsymbol{f}(y')\big)x_n - \big(y-x,\ \boldsymbol{f}(y')\big)x_n + (y_n-x_n)\big(y-x,\ \boldsymbol{f}(y')\big) + (n+2)\big(y-x,\ \boldsymbol{f}(y')\big)}{|y-x|^{n+2}} = 0.$$

This together with (5.11) gives

$$\mathrm{div}\ \boldsymbol{u}(x) = \frac{2}{\omega_n}(1-\kappa)\sum_{j=1}^n\int_{\partial\mathbb{R}^n_+} f_j(y')\frac{\partial}{\partial x_j}\left(\frac{x_n}{|y-x|^n}\right)dy'\ .$$

Hence using (4.4), (4.5) and $\kappa = (3-4\sigma)^{-1}$, we have

$$\mathrm{div}\ \boldsymbol{u}(x) = \frac{4(1-2\sigma)}{\omega_n(3-4\sigma)}\int_{\partial\mathbb{R}^n_+}\frac{\big(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{f}(y')\big)}{|y-x|^n}dy'\ . \tag{5.12}$$

Therefore the sharp coefficient $\mathcal{E}_n(x)$ in (5.6) is represented in the form

$$\mathcal{E}_n(x) = \frac{4(1-2\sigma)}{\omega_n(3-4\sigma)} \int_{\partial\mathbb{R}^n_+} \frac{\big|\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy}\big|}{|y-x|^n} dy' \ .$$

Thus,

$$\mathcal{E}_n(x) = \frac{4(1-2\sigma)}{\omega_n(3-4\sigma)x_n} \int_{\partial\mathbb{R}^n_+} \big|\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy}\big| \frac{x_n}{|y-x|^n} dy' = \frac{(1-2\sigma)E_n}{(3-4\sigma)x_n} \ , \tag{5.13}$$

where

$$E_n = \frac{4}{\omega_n} \int_{\mathbb{S}^{n-1}_-} \big|\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma\big| d\sigma = \frac{2}{\omega_n} \int_{\mathbb{S}^{n-1}} \big|\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma\big| d\sigma \ .$$

Using (4.8), we write the last equality as

$$E_n = \frac{2}{\omega_n} \int_{\mathbb{S}^{n-1}} \Big(1 + n(n-2)(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)^2\Big)^{1/2} d\sigma \ . \tag{5.14}$$

By (5.13), (5.14) and (4.11), we arrive at (5.7) and (5.8).

(ii) *Proof of inequality* (5.9). We write (5.1) as

$$\Delta \boldsymbol{u} - \text{grad}\, p = \boldsymbol{0} \ , \qquad p = -\frac{1}{1-2\sigma} \,\text{div}\, \boldsymbol{u} \ . \tag{5.15}$$

It follows from (5.12) that div $\boldsymbol{u}(x) \to 0$ for every $x \in \mathbb{R}^n_+$ as $\sigma \to 1/2$. We also see that

$$p(x) = -\frac{1}{1-2\sigma} \,\text{div}\, \boldsymbol{u}(x) = -\frac{4}{\omega_n(3-4\sigma)} \int_{\partial\mathbb{R}^n_+} \frac{\big(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{f}(y')\big)}{|y-x|^n} dy'$$

tends to

$$-\frac{4}{\omega_n} \int_{\partial\mathbb{R}^n_+} \frac{\big(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{f}(y')\big)}{|y-x|^n} dy' \ .$$

as $\sigma \to 1/2$. Hence

$$p(x) = -\frac{4}{\omega_n} \int_{\partial\mathbb{R}^n_+} \frac{\big(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{f}(y')\big)}{|y-x|^n} dy' \ .$$

Replacing div $\boldsymbol{u}(x)$ by $(2\sigma-1)p(x)$ in (5.6), and taking the limit as $\sigma \to 1/2$, we arrive at (5.9) with the sharp coefficient (5.10). □

By Proposition 7 with the same argument as in Proposition 5, we derive

**Corollary 1.** *Let* $\Omega = \mathbb{R}^n \backslash \overline{G}$, *where* $G$ *is a convex domain in* $\mathbb{R}^n$. *Let* $\boldsymbol{u} \in [\mathrm{C}^2(\Omega)]^n \cap [\mathrm{C_b}(\overline{\Omega})]^n$ *be a solution of the Lamé system in* $\Omega$. *Then for any point* $x \in \Omega$ *the inequality*

$$|\text{div}\, \boldsymbol{u}(x)| \leq \frac{(1-2\sigma)E_n}{(3-4\sigma)d_x} \sup_{\Omega} |\boldsymbol{u}|$$

*holds, where the constant* $E_n$ *is the same as in Proposition* 7.

**Corollary 2.** *Let $\Omega = \mathbb{R}^n \backslash \overline{G}$, where $G$ is a convex domain in $\mathbb{R}^n$. Let $\boldsymbol{u} \in [\mathrm{C}^2(\Omega)]^n \cap [\mathrm{C_b}(\overline{\Omega})]^n$ be the vector component of a solution $\{\boldsymbol{u}, p\}$ of the Stokes system (5.2) in $\Omega$ and let $p(x)$ be the pressure vanishing as $d_x \to \infty$. Then for any point $x \in \Omega$ the inequality*

$$|p(x)| \leq \frac{E_n}{d_x} \sup_{\Omega} |\boldsymbol{u}|$$

*holds, where the constant $E_n$ is the same as above.*

## 6 Real-part estimates for derivatives of analytic functions

**Theorem 3.** *Let $\Omega = \mathbb{C} \backslash \overline{G}$, where $G$ is a convex domain in $\mathbb{C}$, and let $f$ be a holomorphic function in $\Omega$ with bounded real part. Then for any point $z \in \Omega$ the inequality*

$$\left| f^{(s)}(z) \right| \leq \frac{K_s}{d_z^s} \sup_{\Omega} |\Re f| \, , \qquad s = 1, 2, \dots \, , \tag{6.1}$$

*holds with $d_z = \mathrm{dist}\,(z, \partial\Omega)$, where*

$$K_s = \frac{s!}{\pi} \max_{\alpha} \int_{-\pi/2}^{\pi/2} \left| \cos\big(\alpha + (s+1)\varphi\big) \right| \cos^{s-1} \varphi \, d\varphi \tag{6.2}$$

*is the best constant in the inequality*

$$|f^{(s)}(z)| \leq \frac{K_s}{(\Im z)^s} \, ||\Re f||_{L^\infty(\partial \mathbb{C}_+)} \tag{6.3}$$

*for holomorphic functions $f$ in the half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ with bounded real part.*

*In particular,*

$$K_{2l+1} = \frac{2[(2l+1)!!]^2}{\pi(2l+1)} \, , \tag{6.4}$$

*and*

$$K_2 = \frac{3\sqrt{3}}{2\pi} \, , \tag{6.5}$$

$$K_4 = \frac{3(16 + 5\sqrt{5})}{4\pi} \, . \tag{6.6}$$

*Proof.* Inequality (6.3) with the best constant (6.2) can be found in [9]. Let $\zeta \in \partial\Omega$ be the point nearest to $z \in \Omega$ and let $T(\zeta)$ be the line containing $\zeta$ and orthogonal to the line passing through $z$ and $\zeta$. By $\mathbb{C}_\zeta$ we denote the half-plane with the boundary $T(\zeta)$ which is contained in $\Omega$. Then by (6.3),

$$\left| f^{(s)}(z) \right| \leq \frac{K_s}{d_z^s} ||\Re f||_{h^\infty(\mathbb{C}_\zeta)} \, , \tag{6.7}$$

where $K_s$ is given by (6.2). Using

$$||\Re f||_{h^\infty(\mathbb{C}_\zeta)} \leq \sup_{\Omega} |\Re f| \, ,$$

we obtain (6.1). □

**Theorem 4.** *Let $\Omega$ be a domain in $\mathbb{C}$, and let $\mathfrak{R}(\Omega)$ be the set of holomorphic functions $f$ in $\Omega$ with $\sup_\Omega |\Re f| \le 1$. Assume that a point $\zeta \in \partial\Omega$ can be touched by an interior disk $D$. Then*

$$\limsup_{z\to\zeta} \sup_{f\in\mathfrak{R}(\Omega)} |z-\zeta|^s \big|f^{(s)}(z)\big| \le K_s\ , \qquad s=1,2,\dots\ ,$$

*where $z$ is a point of the radius of $D$ directed from the center to $\zeta$. Here the constant $K_s$ is the same as in Theorem 3 and cannot be diminished.*

*Proof.* In Theorem 7.1 of paper [9] (see also Corollary 1 in [11]) the limit relation was proved:

$$\lim_{r\to R} (R-r)^s \mathcal{H}_s(z) = K_s, \tag{6.8}$$

where $r=|z|$, $K_s$ is the best constant (6.2) in inequality (6.3), and $\mathcal{H}_s(z)$ is the sharp coefficient in the inequality

$$|f^{(s)}(z)| \le \mathcal{H}_s(z) ||\Re f||_{L^\infty(\partial\mathbb{D}_R)}. \tag{6.9}$$

Here $f$ is an analytic function with bounded real part in the disk $\mathbb{D}_R = \{z\in\mathbb{C} : |z|<R\}$.

Therefore, by (6.8) and (6.9), the relations

$$\limsup_{r\to R} \sup\big\{(R-r)^s|f^{(s)}(z)| : ||\Re f||_{h^\infty(\mathbb{D}_R)} \le 1\big\} \le \lim_{|z|\to R} (R-r|)^s \mathcal{H}_s(z) = K_s \tag{6.10}$$

hold.

Let $R$ be the radius of the interior disk $D$ tangent to $\partial\Omega$ at a point $\zeta$. We place the origin $\mathcal{O}$ at the center of $D$. Let $z$ belong to the interval connecting $\mathcal{O}$ and $\zeta$. Then $R-r=|z-\zeta|$. By (6.10) and the inequality

$$||\Re f||_{h^\infty(\mathbb{D}_R)} \le \sup_\Omega |\Re f|\ ,$$

the result follows. □

**Remark 2**. We note that the estimate

$$\big|f^{(s)}(z)\big| \le \frac{4s!}{\pi d_z^s} \sup_\Omega |\Re f|\ , \qquad s=1,2,\dots\ ,$$

with a rougher constant than in (6.1), holds for an arbitrary domain $\Omega\subset\mathbb{C}$. The estimate follows from the sharp inequality

$$\big|f^{(s)}(0)\big| \le \frac{4s!}{\pi R^s} \sup_{|\zeta|<R} |\Re f(\zeta)|$$

obtained in [7], Section 5.3. Certain estimates for $\big|f^{(s)}(z)\big|$ in an arbitrary complex domain are obtained in [4].

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