Three-scale asymptotics for a diffusion problem coupled with the wave equation

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Abstract

We present an asymptotic analysis of waves of elastic stress in an infinite solid whose boundary is subject to a rapid thermal load. The problem under consideration couples the wave equation and the heat equation, and the asymptotic approximation of the solution requires three scaled variables. The asymptotic approximation is supplied with a rigorous remainder estimate and illustrated numerically.

1 Introduction

Despite the fact that the heat equation is a classical topic discussed in every textbook on partial differential equations, there are issues related to rapid thermal processes, which are still considered in the modern research literature. Examples include waves associated with dynamic thermal stress and strain, and effects of rapid heating of boundaries, as discussed in [2] and [3]. Another example is the paper [4], which addresses the problem of thermal stress and transmission of heat in ceramic materials under thermal shock loading. Thermoelastic waves in thick elastic plates, including thermo-mechanical and thermal relaxation effects, are dealt with in the articles [5] and [6].

A problem of uncoupled thermoelasticity for an elastic half-space under instantaneous thermal load is referred to as the thermal shock elastic problem; a formula for the solution of this problem was derived in [7]. Needless to say, every thermal load related to a realistic physical problem is sustained during a non-zero interval of time. It is conventional (see [8]) to introduce a characteristic "thermal time" defined by $t_T = L^2/\kappa$, where L is the characteristic length and κ is the thermal diffusivity constant. When t_T is fairly small the effects of stress concentration become significant near the boundaries of solids subjected to rapidly varying thermal loads[1].

In the present paper we handle the one-dimensional problem of uncoupled thermoelasticity on a semi-axis. It is assumed that the boundary is subjected to a thermal load during a small time interval ε. Although the solution of the problem can be found analytically in the form of a multiple integral, this representation is quite cumbersome and the asymptotic behaviour of the solution (as $\varepsilon \to 0$) cannot be seen easily. In the present paper we give such an asymptotic analysis.

The rigorous asymptotic formulae presented here give a three-scale asymptotic approximation for a solution of the thermo-elasticity problem associated with a rapid thermal loading on the boundary of an elastic half-space $x > 0$. We show that the thermal stress admits the following representation

$$\sigma(x, t, \varepsilon) = sH(t - x)\eta((t - x)/\varepsilon)$$

$$-\frac{sx}{2\sqrt{\pi}} \int_0^t \eta((t - p)/\varepsilon) p^{-3/2} e^{-x^2/4p} dp + \rho_\varepsilon(x, t),$$

[1]This is especially pronounced for materials with relatively high thermal diffusivity, for example germanium ($\kappa = 0.36 cm^2/s$), gallium nitride ($\kappa = 0.43 cm^2/s$) or gallium arsenide ($\kappa = 0.31 cm^2/s$) widely used in modern electronics applications (see, for example, http://www.carondelet.pvt.k12.ca.us/Family/Science/GroupIVA/germanium.htm). The speed a of dilatational waves in germanium is around $5400 m/s$. Hence, one can deduce that even if the duration of thermal loading is of order 10^{-12} of a second, such a time interval should not be treated as a negligibly small quantity if represented in terms of the normalised time-variable $\tau = a^2 t/\kappa$.

where s is a constant coefficient, H is the Heaviside function, and ε is a small non-dimensional parameter characterising the length of the time-interval associated with the thermal load. By η we denote a function defining the profile of the thermal load, and ρ_ε stands for the small remainder. The first term in the above formula represents a plane wave of thermal stress propagating away from the boundary, whereas the second term plays the role of the boundary layer.

This work may prove useful for asymptotic studies of thermoelastic fields in an elastic domain subject to a rapid fully three-dimensional thermal load. Specifically, a similar scaling of space and time variables should occur in the three-dimensional analysis of thermo-elasticity problems.

The plan of the paper is as follows. We begin with the formulation for a rapid, impulse type thermal loading. A multi-scale asymptotic approximation of thermal stress is formally constructed and rigorously justified. It is shown that this solution can be used in the analysis of thermal stress associated with a rapid increase of the boundary temperature, and the details of this analysis are given in Section 3. The asymptotic solutions are supplied with illustrative numerical examples and physical interpretation, which are included in Section 4. Finally, Section 5 contains concluding remarks on the range of applicability of the asymptotic formulae obtained in this paper.

2 Impulse type thermal loading

2.1 Governing equations

Consider a vector-function $(T(x,t), \sigma(x,t))$, defined for non-negative x and t. The functions T and σ are assumed to satisfy the diffusion equation

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = 0, \quad t > 0, \; x > 0, \tag{1}$$

and the coupled equation of motion

$$\frac{\partial^2 \sigma}{\partial x^2} - \frac{\partial^2 \sigma}{\partial t^2} = s \frac{\partial^2 T}{\partial t^2}, \quad t > 0, \; x > 0, \tag{2}$$

where s is a positive constant coefficient; compared to equations that occur in physics the above equations have been normalised, so that the coefficients have a particularly simple form. We note that t is considered as a time-like variable, whereas x is a spatial variable.

In addition, we impose the following initial and boundary conditions

$$T(x,0)=0, \quad x>0, \tag{3}$$

$$T(0,t)=T_0\eta(t/\varepsilon), \quad t>0, \tag{4}$$

for the function T, and

$$\sigma(x,0)=0, \quad \frac{\partial\sigma}{\partial t}(x,0)=0, \quad x>0, \tag{5}$$

$$\sigma(0,t)=0, \quad t>0, \tag{6}$$

for the function σ. Here ε denotes a small positive non-dimensional parameter, and η is a measurable function vanishing for $\tau>1$, and subject to the inequality $|\eta(\tau)|\le 1|$ for almost all $\tau>0$. Thus, we deal with the short time change of temperature on the boundary.

Remark. For readers convenience, we note that the above equation (2) follows from a classical setting of uncoupled thermo-elasticity. Indeed, let us assume that the stress $\sigma(x,t)$, displacement $u(x,t)$ and temperature $T(x,t)$ satisfy the equation of motion

$$\frac{\partial\sigma}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2}, \tag{7}$$

and the constitutive equation

$$\sigma=E\frac{\partial u}{\partial x}-\gamma T, \tag{8}$$

with positive coefficients ρ, E, γ being the material mass density, Young's modulus and a thermo-elastic constant, respectively. By differentiating (7) with respect to x, differentiating (8) twice with respect to t, and subtracting the second equation from the first one, we deduce

$$\frac{\partial^2\sigma}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\sigma}{\partial t^2}=\frac{\rho\gamma}{E}\frac{\partial^2 T}{\partial t^2}. \tag{9}$$

Normalising the time variable t in such a way that $c=1$ and using the notation $s=\rho\gamma/E$ we obtain (2). □

We seek a solution $\{\sigma, T\}$ of problem (1)–(6), uniformly bounded in $\{(x,t): x\ge 0, t\ge 0\}$, with the main aim of revealing its asymptotic structure for the case of a rapid thermal loading.

2.2 Asymptotic solution

The subsequent asymptotic analysis will require the following three scaled variables defined by

$$\xi = x/\sqrt{\varepsilon}, \tau = t/\varepsilon \text{ and } X = x/\varepsilon. \tag{10}$$

We formulate and prove our main result.

Theorem 1. *The solution* (T, σ) *of problem* (1)–(6) *can be represented in the form*

$$T = T(\xi, \tau) = \frac{T_0}{2\pi} \int_0^{\tau/\xi^2} \eta(\tau - \xi^2 p) p^{-3/2} e^{-1/4p} dp, \tag{11}$$

and

$$\sigma = \sigma(x, t, \varepsilon) = sT_0 H(\tau - X)\eta(\tau - X)$$
$$-\frac{T_0 s\xi}{2\sqrt{\pi}} \int_0^{\tau} \eta(\tau - p) p^{-3/2} e^{-\xi^2/4p} dp + \rho_\varepsilon(x, t), \tag{12}$$

where the remainder term ρ_ε *is* $O(\varepsilon)$ *uniformly in* t *and* x*.*

Proof. (i) *Evaluation of the temperature* T*.* In the new scaled variables, equation (1) becomes

$$\frac{\partial T}{\partial \tau} - \frac{\partial^2 T}{\partial \xi^2} = 0, \tag{13}$$

and is accompanied by the initial and boundary conditions

$$T(\xi, 0) = 0 \quad \text{when} \quad \xi \geq 0, \tag{14}$$

and

$$T(0, \tau) = T_0 \eta(\tau) \quad \text{when} \quad \tau \geq 0. \tag{15}$$

Here T_0 is a given constant, which defines the maximum value of the boundary temperature. Solving this problem, we obtain (11) (see [1], Chapter 1).

(ii) *Evaluation of the stress.* The stress σ is sought in the form

$$\sigma(x, t, \varepsilon) = -sT(x/\sqrt{\varepsilon}, t/\varepsilon) + sS(x/\varepsilon, t/\varepsilon) + \rho_\varepsilon(x, t), \tag{16}$$

where T is the function defined by (11), and $S(X, \tau)$ is a bounded solution of the problem

$$\frac{\partial^2 S}{\partial X^2} - \frac{\partial^2 S}{\partial \tau^2} = 0, \quad (X, \tau) \in \Sigma, \tag{17}$$

with the boundary condition

$$S = \eta(\tau) \;\; \text{when} \;\; X = 0, \tau > 0, \tag{18}$$

and the initial conditions

$$S = \frac{\partial S}{\partial \tau} = 0 \;\; \text{when} \;\; \tau = 0, X > 0. \tag{19}$$

We will show that ρ_ε in (16) plays the role of the remainder term in the asymptotic approximation of the stress.

The solution of (17)–(19) is a plane wave defined by

$$S(X, \tau) = H(\tau - X)\eta(\tau - X), \tag{20}$$

where H denotes the Heaviside function.

Thus, the stress σ has the representation

$$\sigma(x, t, \varepsilon) = sT_0 H(t - x)\eta((t - x)/\varepsilon)$$

$$-\frac{T_0 s x}{2\sqrt{\pi}} \int_0^t \eta((t - p)/\varepsilon) p^{-3/2} e^{-x^2/4p} dp + \rho_\varepsilon(x, t), \tag{21}$$

which is equivalent to (12). The remainder estimate is still to be given.

(iii) *Remainder estimate, for the case* $0 < t \leq 2\varepsilon$. We would like to prove that $\rho_\varepsilon = O(\varepsilon)$, uniformly in x and t. For the function ρ_ε, we deduce the inhomogeneous wave equation accompanied by homogeneous initial and boundary conditions:

$$\frac{\partial^2 \rho_\varepsilon}{\partial x^2} - \frac{\partial^2 \rho_\varepsilon}{\partial t^2} = s\frac{\partial T_\varepsilon}{\partial t},$$

and

$$\rho_\varepsilon = 0 \;\; \text{at} \; x = 0,$$

$$\rho_\varepsilon = \partial \rho_\varepsilon / \partial t = 0 \;\; \text{at} \; t = 0.$$

Here $T_\varepsilon(x, t) = T(x/\sqrt{\varepsilon}, t/\varepsilon)$.

It is verified directly that

$$s^{-1}\rho_\varepsilon(x, t) = -\frac{1}{2}\int_0^t dt_1 \int_{x-t+t_1}^{x+t-t_1} \frac{\partial T_\varepsilon(x_1, t_1)}{\partial t_1} dx_1, \tag{22}$$

where

$$T_\varepsilon(x_1,t_1)=\frac{1}{2\pi}\int_0^{t_1/x_1^2}\eta((t_1-x_1^2p)/\varepsilon)p^{-3/2}\exp(-1/4p)dp. \tag{23}$$

The derivative $\partial T_\varepsilon/\partial t_1$ is extended as an odd function for negative values of x_1. The formula (22) can be written in the equivalent form

$$s^{-1}\rho_\varepsilon(x,t)=-\frac{1}{2}\left(\int_{x-t}^{x}dx_1\int_0^{x_1-x+t}\frac{\partial T_\varepsilon}{\partial t_1}dt_1+\int_x^{x+t}dx_1\int_0^{x-x_1+t}\frac{\partial T_\varepsilon}{\partial t_1}dt_1\right)$$

$$=-\frac{1}{2}\left(\int_{x-t}^{x-t+2\varepsilon}T_\varepsilon(x_1,x_1-x+t)dx_1+\int_{x-t+2\varepsilon}^{x}T_\varepsilon(x_1,x_1-x+t)dx_1\right.$$

$$\left.+\int_x^{x+t-2\varepsilon}T_\varepsilon(x_1,x-x_1+t)dx_1+\int_{x+t-2\varepsilon}^{x+t}T_\varepsilon(x_1,x-x_1+t)dx_1\right). \tag{24}$$

Each of the above integrals will be estimated separately.

We recall that $\sup|\eta|=1$, and hence

$$|T(z,y)|\leq\frac{1}{2\sqrt{\pi}}\int_0^\infty p^{-3/2}e^{-1/4p}dp=1.$$

The first and the fourth integrals in (24) have their absolute values within the interval $(0,2\varepsilon)$. The second and third integrals have their absolute values within the same range, provided $0<t\leq 2\varepsilon$. We need only to verify that the second and third integrals in (24) are small when $t>2\varepsilon$.

(iv) *The case* $t>2\varepsilon$. Let us prove additional, more refined estimates for the function $T(x,t)$.

We introduce the new variable $p_1=x^2p/t$ and use the fact that $\sup|\eta|=1$ and $\eta(\tau)=0$ when $\tau\geq 1$. Based on the representation (23), we derive

$$|T_\varepsilon(x,t)|=\frac{|x|}{2\sqrt{\pi t}}\int_0^1|\eta(t(1-p_1)/\varepsilon)|p_1^{-3/2}e^{-x^2/4tp_1}dp_1$$

$$=\frac{|x|}{2\sqrt{\pi t}}\int_{1-\varepsilon/t}^1|\eta(t(1-p_1)/\varepsilon)|p_1^{-3/2}e^{-x^2/4tp_1}dp_1$$

$$\leq\frac{\varepsilon|x|}{2\sqrt{\pi}t^{3/2}}\max_{1-\varepsilon/t\leq p_1\leq 1}p_1^{-3/2}e^{-x^2/4tp_1}. \tag{25}$$

Let

$$\phi(x,t,p_1) := p_1^{-3/2} e^{-x^2/4tp_1}.$$

In what follows, estimating this function for $1-\varepsilon/t_1 \le p_1 \le 1$ and $t > 2\varepsilon$, we obtain

$$|T_\varepsilon(x,t)| \le \sqrt{\frac{2}{\pi}} \frac{\varepsilon x}{t^{3/2}} e^{-x^2/4t}, \tag{26}$$

We note that for fixed x and t, the function $\phi(x,t,p_1)$ has the point of maximum at $p_1 = p_{max} = x^2/6t$, and its maximum value is given by

$$\phi(x,t,p_{max}) = (6tx^{-2})^{3/2} e^{-3/2}.$$

Consider the following three cases of possible locations of the point p_{max}: (a) $p_{max} \le 1-\varepsilon/t$; (b) $1-\varepsilon/t \le p_{max} \le 1$; (c) $p_{max} \ge 1$.

(a) When $p_{max} \le 1-\varepsilon/t$ the function $\phi(x,t,p_1)$ (where x and t are fixed) is decreasing on the interval $(1-\varepsilon/t, 1)$. In this case

$$\max_{1-\varepsilon/t \le p_1 \le 1} \phi(x,t,p_1) = \phi(x,t,1-\varepsilon/t)$$

$$= (1-\varepsilon/t)^{-3/2} e^{-x^2/t(1-\varepsilon/t)} \le 2^{3/2} e^{-x^2/4t}.$$

(b) For the case $1-\varepsilon/t \le p_{max} \le 1$ we have

$$\max_{1-\varepsilon/t \le p_1 \le 1} \phi(x,t,p_1) \le (1-\varepsilon/t)^{-3/2} e^{-x^2/4t} \le 2^{3/2} e^{-x^2/4t}.$$

(c) Finally, when $p_{max} \ge 1$ the quantity $\phi(x,t,p)$ increases (as a function of p) on the whole range $1-\varepsilon/t \le p \le 1$, and it attains its maximum value at $p=1$.

Taking into account the estimates just obtained and the inequality (25), we deduce (26).

Let

$$I_1 = \int_{x-t+2\varepsilon}^{x} |x_1|(x_1 - x + t)^{-3/2} e^{-x_1^2/4(x_1-x+t)} dx_1,$$

and

$$I_2 = \int_{x}^{x+t-2\varepsilon} |x_1|(x + t - x_1)^{-3/2} e^{-x_1^2/4(x+t-x_1)} dx_1.$$

After the substitution $y = x_1 - x + t$, the integral I_1 takes the form

$$I_1 = \int_{2\varepsilon}^{t} |y + x - t| y^{-3/2} \exp(-(y+x-t)^2/4y) dy.$$

First, consider the case when $x - t > 0$. Then $y + x - t$ is positive on the whole interval of integration, and

$$I_1 = \int_{2\varepsilon}^{t} y^{-1/2} \exp(-(y + (x-t)^2/y + 2(x-t))/4) dy$$

$$+ \int_{2\varepsilon}^{t} y^{-3/2}(x-t) \exp(-(y + (x-t)^2/y + 2(x-t))/4) dy$$

$$\leq \int_{2\varepsilon}^{t} y^{-1/2} e^{-y/4} dy + \int_{2\varepsilon}^{t} (x-t) y^{-3/2} e^{-(x-t)^2/4y} dy.$$

Setting $y = z^2$ and $(x-t)^2/y = z^2$ in the first and second integrals in the right-hand side of the above inequality, we find

$$I_1 \leq 2 \int_{\sqrt{2\varepsilon}}^{\sqrt{t}} e^{-z^2/4} dz + 2 \int_{(x-t)/\sqrt{t}}^{(x-t)/\sqrt{2\varepsilon}} e^{-z^2/4} dz$$

$$\leq 4 \int_{0}^{\infty} e^{-z^2/4} dz = 4\sqrt{\pi}.$$

Next, let $x - t < 0$. In this case,

$$I_1 = \int_{2\varepsilon}^{t-x} (t - x - y) y^{-3/2} e^{-(t-x-y)^2/4y} dy$$

$$+ \int_{t-x}^{t} (y - t + x) y^{-3/2} e^{-(y-t+x)^2/4y} dy$$

$$\leq \int_{2\varepsilon}^{t-x} (t-x) y^{-3/2} e^{-(t-x-y)^2/4y} dy + \int_{t-x}^{t} y^{-1/2} e^{-(y-t+x)^2/4y} dy$$

$$= 2 \int_{\sqrt{t-x}}^{(t-x)/\sqrt{2}} e^{-(z-(t-x)/z)^2/4} dz + 2 \int_{\sqrt{t-x}}^{\sqrt{t}} e^{-(z-(t-x)/z)^2/4}$$

$$\leq 4 \int_{0}^{\infty} e^{-(z-(t-x)/z)^2/4} dz = 4 \int_{-\infty}^{\infty} e^{-y^2/4} = 8\sqrt{\pi}$$

Consequently, the integral I_1 is uniformly bounded in x and t.

The integral I_2 can be handled in a similar way:

$$I_2 = \int_{2\varepsilon}^{t} y^{-3/2} (x + t - y) e^{-(x+t-y)^2/4y} dy$$

$$\leq 2 \int_{-\infty}^{\infty} (1 + (x+t)/z^2)^{-1} e^{-y^2/4} dy$$

$$\leq 4\int_0^\infty e^{-y^2/4}dy = 8\sqrt{\pi},$$

which implies the uniform boundness of I_2 in x, t.

It follows from (22) and (26) that the remainder ρ_ε can be estimated in the form

$$|\rho_\varepsilon| \leq \frac{\varepsilon s}{2}\Big(4 + 2^{3/2}(I_1 + I_2)\Big). \tag{27}$$

Since both integrals I_1 and I_2 in the right-hand side of (27) are uniformly bounded with respect to t and x, the remainder ρ_ε is $O(\varepsilon)$ uniformly in t and x. The proof is complete. □

3 Gradient field near the boundary of the half-space under a rapid increase of the temperature

Previous analysis of impulse type loading enables one to construct an asymptotic approximation of a solution to a problem involving a rapid increase of the temperature.

The boundary condition for temperature. We modify the boundary condition in problem (1), (3), (4) by assuming that the temperature on the boundary of the half-space increases from 0 to T_0, within a short time interval, and then it remains constant:

$$T(0,t) = \phi(t/\varepsilon)T_0, \tag{28}$$

where the boundary temperature $\phi(\tau)$ is assumed to be smooth for all $\tau \geq 0$, and

$$\phi(\tau) = 0, \text{ when } \tau \leq 0; \; \phi(\tau) = 1 \text{ when } \tau \geq 1,$$
$$\max|\phi(\tau)| = 1. \tag{29}$$

The solution T of this problem can be represented as the sum $T = T_{sh} - T_{imp}$ where T_{imp} is the temperature associated with the short-time thermal impulse, as described in Section 2, and T_{sh} is the term generated by an instant change of the boundary temperature (we shall call it the thermal shock temperature). The boundary values of T_{sh} and T_{imp} are given by the formulae

$$T_{sh}(0,t) = H(t)T_0, \tag{30}$$

where H is the Heaviside function, and

$$T_{imp}(0,t) = \eta(t/\varepsilon) = (H(t) - \phi(t/\varepsilon))T_0. \tag{31}$$

We note that the function $\eta(\tau)$ is discontinuous at $\tau = 0$. The problem for the field T_{sh}, modelling a thermal shock, is straightforward, and its solution is

$$T_{sh} = T_0 \text{erfc}(\frac{X}{2\sqrt{\tau}}). \tag{32}$$

The field T_{imp} is given by

$$T_{imp}(X,\tau) = \frac{T_0}{\sqrt{2\pi}} \int_0^{\tau/\xi^2} \eta(\tau - X^2\xi)\xi^{-3/2}e^{-1/4\xi}d\xi. \tag{33}$$

Asymptotic approximation of stress. The associated stress field can be found as

$$\sigma = \sigma_{sh} - \sigma_{imp}, \tag{34}$$

where σ_{sh} and σ_{imp} are the stresses produced by the temperatures T_{sh} and T_{imp}, respectively. According to [7], the stress σ_{sh} is

$$\sigma_2 = -\frac{sT_0}{2}e^t\Big(\text{erfc}(\frac{x}{2\sqrt{t}} - \sqrt{t})e^{-x} + \text{erfc}(\frac{x}{2\sqrt{t}} + \sqrt{t})e^{x}\Big) + sT_0H(t-x)e^{t-x}. \tag{35}$$

The stress field σ_{imp} can be written in the form (11). Hence we arrive at the asymptotic representation for the whole stress field

$$\sigma(x,t,\varepsilon) = sT_0H(t-x)\Big(e^{t-x} - 1 + \varphi((t-x)/\varepsilon)\Big) \tag{36}$$

$$-\tfrac{1}{2}sT_0e^t\Big(\text{erfc}(\frac{x}{2\sqrt{t}} - \sqrt{t})e^{-x} + \text{erfc}(\frac{x}{2\sqrt{t}} - \sqrt{t})e^{x}\Big)$$

$$+\frac{sT_0x}{2\sqrt{\pi}}\int_0^t \Big(1 - \varphi(\frac{t-p}{\varepsilon})\Big)p^{-3/2}e^{-x^2/4p}dp + \rho_\varepsilon(x,t)$$

The remainder estimate in (36) follows the same pattern as in Section 2.

4 Physical interpretation and numerical experiments

The impulse type loading. The formula (12) has a clear physical interpretation. In fact, the first term in (12) is the plane wave of stress, which propagates away from the boundary of the half-space. The profile of the wave, modulo the constant factor s, is the same as the profile η of the boundary temperature (see (4)).

The second term in (12) is proportional to the temperature field T in the half-space. According to (26), it can be neglected when $t > \text{const}$, since

$$|T_\varepsilon(x,t)| \le (2/\pi)^{1/2}\varepsilon t^{-1} \max(ye^{-y^2/4}) = 2\varepsilon(\pi^{1/2}e^{1/2}t)^{-1}. \tag{37}$$

Furthermore, the second term in (12) is $O(\varepsilon)$ for $x \ge \text{Const}\varepsilon^{1/2}$, as follows from the estimate

$$|T_\varepsilon| \le (2\xi^2\sqrt{\pi})^{-1} \max(y^3 e^{-y^2/4}) = 3(6/\pi)^{1/2}e^{-3/2}x^{-2}\varepsilon. \tag{38}$$

Hence, this term may be essential near the boundary ($x = o(1)$) only during a short period of time ($t = o(1)$).

An advantage of the asymptotic formula (12), in contrast with the integral representation of the exact solution, is that it shows explicitly the wave and diffusion components of the thermal stress. In particular, one can see that these components have opposite signs if the function η does not change its sign.

We see from (37) and (38) that the stress produced by the thermal impulse coincides asymptotically with the stress wave produced by the mechanical pressure $sT_0\eta$ on the boundary of the half-space, provided the distance from the boundary is sufficiently large ($x > \text{const}$).

Consider an example where the function η is given by

$$\eta(\tau) = \tfrac{1}{2}(1 - \cos(2\pi\tau)) \quad \text{as } \tau \in (0,1), \tag{39}$$

and $\eta(\tau) = 0$ otherwise. Let the value of the small parameter ε to be 0.1.

In Fig. 1, we display $(sT_0)^{-1}\sigma_{asymp}$, where σ_{asymp} is the asymptotic solution

$$\sigma_{asymp}(x,t,\varepsilon) = sS(x/\varepsilon, t/\varepsilon) - sT(x/\sqrt{\varepsilon}, t/\varepsilon), \tag{40}$$

(see (12) and (16)).

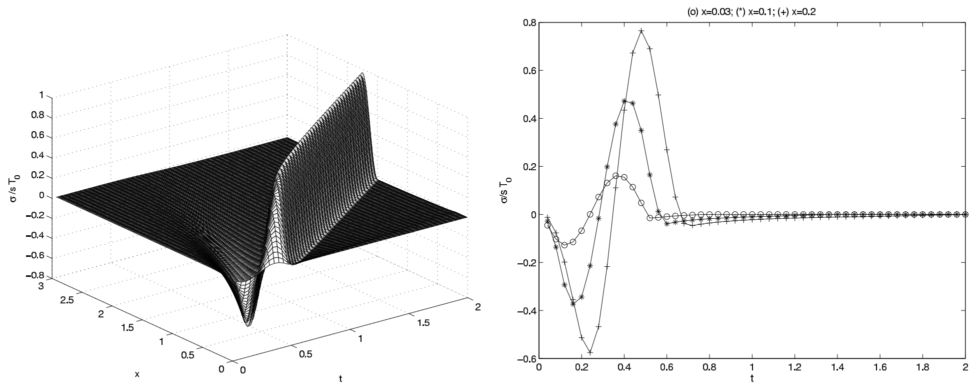

Figure 1: Left: The surface plot of $(sT_0)^{-1}\sigma_{asymp}$. Right: Cross-sections of the surface plot for fixed x.

In Fig. 2, we show the wave term $(sT_0)^{-1}S$ with profile (39) and the diffusion term $-T_0^{-1}T$ in representation (40).

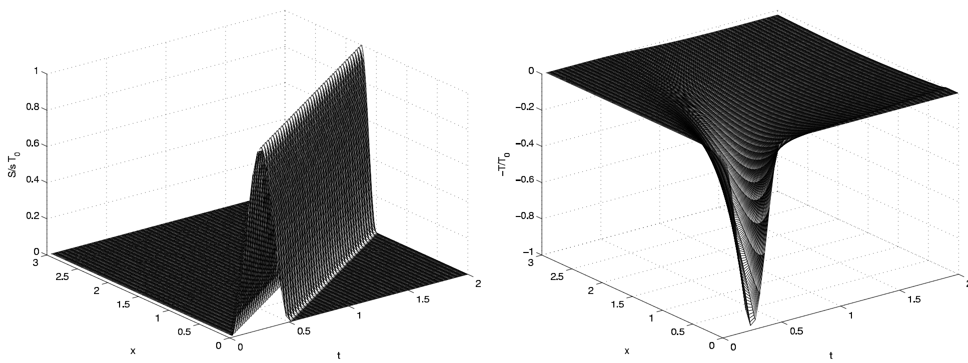

Figure 2: Left: The wave term $(sT_0)^{-1}S$. Right: The diffusion term $-T_0^{-1}T$.

The rapid growth of the boundary temperature. The asymptotic approximation of σ in (36) can be written as

$$\sigma_{asymp}(x,t,\varepsilon) = \sigma_{sh}(x,t) + sT_0 H(t-x)(1-\varphi(\varepsilon^{-1}(t-x)))$$
$$-\frac{2sT_0}{\sqrt{\pi}} \int_{x/2\sqrt{t}}^{\infty} \Big(1-\varphi(\varepsilon^{-1}(t-(x/2z)^2)\Big) e^{-z^2} dz. \qquad (41)$$

The thermal shock stress is discontinuous at $x = t$, i.e. it has the jump sT_0 across the wave front: the stress is compressive as $t < x$, and it becomes tensile when $t > x$. The second term in (41) is a plane elastic wave proportional to the thermal boundary impulse. The third term, of diffusion type, is the same as the term $-sT$ in (12), with $\eta(\tau) = H(\tau)(1-\varphi(\tau))$.

In the numerical example we use

$$\eta(\tau) = \begin{cases} 0, & \text{as } \tau < 0, \\ T_0\tau, & \text{as } 0 \le \tau \le 1, \\ T_0, & \text{as } \tau > 1. \end{cases}$$

In Fig. 3, we display the surface plots for $(sT_0)^{-1}\sigma_{sh}$ and $(sT_0)^{-1}\sigma_{asymp}$, with the latter showing regularisation in the vicinity of $x = t$.

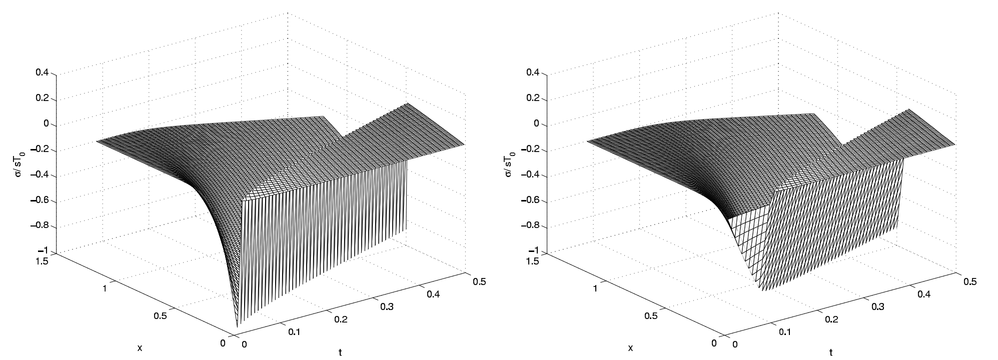

Figure 3: Left: The shock stress $(sT_0)^{-1}\sigma_{sh}$. Right: The asymptotic approximation of the total stress$(sT_0)^{-1}\sigma_{asymp}$.

The profiles of $(sT_0)^{-1}\sigma_{asymp}(x,t,\varepsilon)$ and $(sT_0)^{-1}\sigma_{sh}(x,t)$, for a fixed x, are presented in Fig. 4.

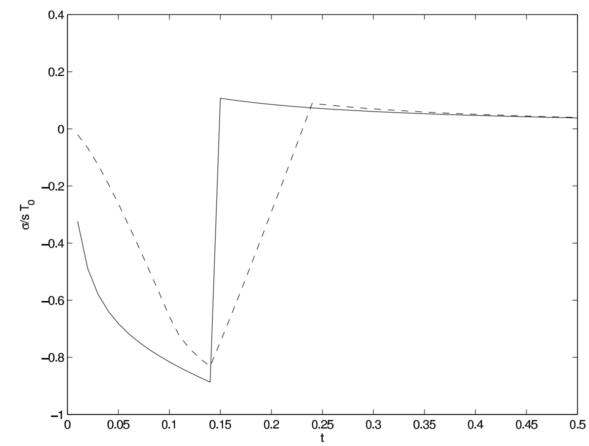

Figure 4: For $x = 0.14$, the solid line corresponds to the thermal shock stress, it is discontinuous at $t = x$. The dashed line gives the continuous function $(sT_0)^{-1}\sigma_{asymp}$.

The magnitude of the stress wave increases while it propagates away from the boundary of the half space until it reaches the value sT_0. At a long distance, away from the boundary the shape of the wave is determined by the profile of the boundary temperature.

5 Concluding remarks on the range of applicability of the asymptotic formula

Equations analysed in this paper are written in normalised variables. Despite the fact that we did not consider particular physical applications, it is worthwhile estimating the range of values for the small parameter ε that would be adequate for real life materials. For instance, the thermal diffusivity of germanium is $\kappa = 0.36cm^2/s$. We look at a normalised time $\kappa t/L^2$, where L is a characteristic length scale. Taking formally $L = 1cm$, we deduce that the case of a fast heating during the time interval of $0.5s$ corresponds to $\varepsilon = 0.18$ in our model.

Although a rigorous remainder estimate has been derived for the asymptotic approximation of stress, it is also interesting to evaluate the error numerically. We shall see that in numerical approximations the asymptotic formula (12) works so well, that it exceeds the expectations based on rigorous estimates of the remainder terms.

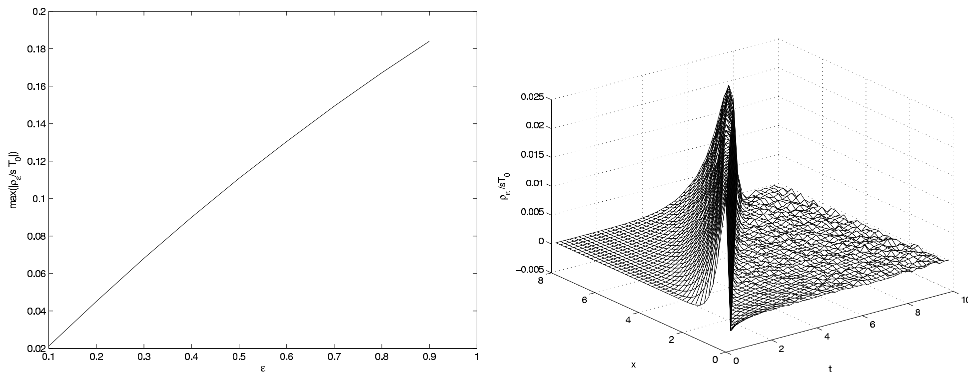

Figure 5: The error of the asymptotic approximation. The diagram on the left shows the maximum error as a function of ε, and the diagram on the right presents ρ_ε as a function of x and t for the case when $\varepsilon = 0.1$.

In Fig. 5 we give results of calculations of the normalised error term ρ_ε / sT_0. The first graph presents $\max(|\rho_\varepsilon / sT_0|)$ as a function of ε; it is worth mentioning that the actual numerical accuracy of the computation is considerably better than the one predicted in Theorem 1. Even for the case $\varepsilon = 0.9$ (which is truly extreme) the absolute value of ρ_ε / sT_0 is less than 0.2. The second graph gives ρ_ε / sT_0 as a function of x and t for $\varepsilon = 0.1$. Here, the error is localised and remains small, well within the limits predicted by the asymptotic analysis of Section 2.

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