PATHOLOGICAL SOLUTIONS TO ELLIPTIC PROBLEMS IN DIVERGENCE FORM WITH CONTINUOUS COEFFICIENTS

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RÉSUMÉ. We construct a function $u \in W_{\text{loc}}^{1,1}(B(0,1))$ which is a solution to $\operatorname{div}(A\nabla u) = 0$ in the sense of distributions, where A is continuous and $u \notin W_{\text{loc}}^{1,p}(B(0,1))$ for $p > 1$. We also give a function $u \in W_{\text{loc}}^{1,1}(B(0,1))$ such that $u \in W_{\text{loc}}^{1,p}(B(0,1))$ for every $p < \infty$, u satisfies $\operatorname{div}(A\nabla u) = 0$ with A continuous but $u \notin W_{\text{loc}}^{1,\infty}(B(0,1))$.

1. INTRODUCTION

Consider the equation

$$(1) \qquad -\operatorname{div} A\nabla u = 0 \qquad\qquad\qquad \text{in } \Omega,$$

for $\Omega \subset \mathbf{R}^n$. If $A : \Omega \to \mathbf{R}^{n \times n}$ is bounded, measurable and elliptic, i.e., there exists $\lambda, \Lambda \in \mathbf{R}_*$ such that for every $x \in \Omega$, $A(x)$ is a symmetric matrix, and

$$|\xi|^2 \le (A(x)\xi) \cdot \xi \le \Lambda |\xi|^2,$$

then one can define a weak solution $u \in W_{\text{loc}}^{1,1}(\Omega)$ by requiring that for every $\varphi \in C_c^1(\Omega)$,

$$\int_\Omega (A\nabla u) \cdot \nabla\varphi = 0.$$

We are interested in the regularity properties of u. A fundamental result of E. De Giorgi [3] states that if $u \in W_{\text{loc}}^{1,2}(\Omega)$, then u is locally Hölder continuous. In particular, u is then locally bounded. In the same direction, N. G. Meyers [8] also proved that $u \in W_{\text{loc}}^{1,p}(\Omega)$ for some $p > 2$.

J. Serrin [9] showed that the assumption $u \in W_{\text{loc}}^{1,2}(\Omega)$ is essential in E. De Giorgi's result by constructing for every $p \in (1, 2)$ a function $u \in W_{\text{loc}}^{1,p}(\Omega)$ that solves such an elliptic equation but which is not locally bounded. In these counterexamples A is not continuous. J. Serrin [9] conjectured that if A was Hölder continuous, then any weak solution $u \in W_{\text{loc}}^{1,1}(\Omega)$ is in $W_{\text{loc}}^{1,2}(\Omega)$, and one can then apply E. De Giorgi's theory.

This conjectured was confirmed for $u \in W^{1,p}(\Omega)$ by R. A. Hager and J. Ross [4] and for $u \in W^{1,1}(\Omega)$ by H. Brezis [1, 2]. The proof of Brezis extends to the case where the modulus of continuity of A

$$(2) \qquad\qquad \omega_A(t) = \sup_{\substack{x,y\in\Omega \\ |x-y|\le t}} |A(x) - A(y)|,$$

satisfies the Dini condition

$$(3) \qquad \int_0^1 \frac{\omega_A(s)}{s}\, ds < \infty.$$

In the case where A is merely continuous, H. Brezis obtained the following result

Theorem 1.1 (H. Brezis [1, 2])**.** *Assume that $A \in C(\Omega; \mathbf{R}^{n \times n})$ is elliptic. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ solves (1), then for every $q \in [p, +\infty)$, one has $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$.*

H. Brezis asked two questions about the cases $p = 1$ and $q = \infty$ in the previous theorem. We answer both questions, with a negative answer. First we have

Proposition 1.2. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n \times n})$ such that u solves (1), but $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p > 1$.*

As a byproduct, we obtain

Proposition 1.3. *There exists $A \in C(B(0,1); \mathbf{R}^{n \times n})$ such that the problem*

$$(4) \qquad \begin{cases} -\operatorname{div}(A \nabla u) = 0 & \text{in } B(0,1), \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

has a nontrivial solution.

Our construction in Proposition 1.2 allows in fact to show that the counterexamples can be improved

Proposition 1.4. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n \times n})$ such that u solves (1), $Du \in (L \log L)_{\mathrm{loc}}(B(0,1))$ but $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p > 1$.*

In particular, in this case, Du belongs locally to the Hardy space $\mathcal{H}^1$ (see [10]).

Concerning the possibility of Lipschitz estimates, we have

Proposition 1.5. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n \times n})$ such that u solves (1), $Du \in W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p > 1$, $Du \in \mathrm{BMO}_{\mathrm{loc}}(B(0,1))$ but $u \notin W^{1,\infty}_{\mathrm{loc}}((B(0,1))$.*

This shows that $Du \in L^p(B(0,1))$ does not imply $Du \in L^\infty(B(0,1/2))$, one can wonder whether it implies that $Du \in \mathrm{BMO}(B(0,1/2))$. The answer is still negative

Proposition 1.6. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n \times n})$ such that u solves (1), $u \in W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p \in (1, \infty)$ but $Du \notin \mathrm{BMO}_{\mathrm{loc}}(B(0,1))$.*

The construction of the counterexamples are made by explicit formulas, inspired by the construction of J. Serrin [9]. They can also be obtained from asymptotic formulas of V. Kozlov and V. Maz'ya [6, 7].

2. The pathological solutions

Our counterexamples rely on the following computation

Lemma 2.1. *Let $v \in C^2((0, R))$ and $\alpha \in C^1((0, R))$. Define $A(x) = (a_{ij}(x))_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$ by*

$$a_{ij}(x) = \delta_{ij} + \alpha(|x|)\Big(\delta_{ij} - \frac{x_i x_j}{|x|^2}\Big).$$

Then for every $x \in B(0, R) \setminus \{0\}$,
(5)
$$\operatorname{div}\big(A(x)\nabla(x_1 v(|x|))\big) = x_1\Big(v''(|x|) + \frac{n+1}{|x|}v'(|x|) - \frac{n-1}{|x|^2}\alpha(|x|)v(|x|)\Big).$$

Remark 1. *If P is a homogeneous harmonic polynomial of degree k, the formula generalizes to*

(6) $\quad \operatorname{div}\big(A(x)\nabla(P(x)v(|x|))\big)$
$$= P(x)\Big(v''(|x|) + \frac{n+2k-1}{|x|}v'(|x|) - \frac{k(n+k-2)}{|x|^2}\alpha(|x|)v(|x|)\Big).$$

Proof of Proposition 1.2. Choose $\beta > 1$, and define for some $r_0 > 1$, for $r \in (0, 1)$,

(7)
$$v(r) = \frac{1}{r^n(\log \frac{r_0}{r})^\beta}.$$

One takes then

(8) $\quad \alpha(r) = \dfrac{r^2 v''(r) + (n+1)rv'(r)}{(n-1)v(r)} = \dfrac{-\beta n}{(n-1)\log \frac{r_0}{r}} + \dfrac{\beta(\beta+1)}{(n-1)\big(\log \frac{r_0}{r}\big)^2}.$

One can take r_0 large enough so that $\alpha \geq -\frac{1}{2}$ on $(0, 1)$; the coefficient matrix A is then uniformly elliptic. Define now $u(x) = x_1 v(|x|)$. One checks that $u \in W^{1,1}(B(0, 1))$ and that u is a weak solution of (1). Indeed, it is a classical solution on $B(0, 1) \setminus \{0\}$ by the previous lemma. Taking, $\varphi \in C_c^1(B(0, 1))$ and $\rho \in (0, 1)$, and integrating by parts we obtain

$$\int_{B(0,1) \setminus B(0,\rho)} \nabla\varphi \cdot (A\nabla u) = -\int_{\partial B(0,\rho)} \varphi \nabla u \cdot \big(A\frac{x}{\rho}\big)$$
$$= -\int_{\partial B(0,\rho)} \varphi \nabla u \cdot \frac{x}{\rho}$$
$$= -\int_{\partial B(0,\rho)} \varphi x_1\Big(\frac{v(\rho)}{\rho} + v'(\rho)\Big)$$
$$= -\int_{\partial B(0,\rho)} (\varphi(x) - \varphi(0))x_1\Big(\frac{v(\rho)}{\rho} + v'(\rho)\Big).$$

Since $\varphi \in C_c^1(B(0, 1))$, one has

$$\Big|\int_{B(0,1) \setminus B(0,\rho)} \nabla\varphi \cdot (A\nabla v)\Big| \leq C\rho^n(|v(\rho)| + \rho|v'(\rho)|),$$

since the right-hand side goes to 0 as $\rho \to 0$, u is a weak solution. $\qquad\square$

Remark 2. The examples constructed in the case of merely measurable co-efficients by J. Serrin [9] to show that a solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ need not be in $W^{1,2}_{\mathrm{loc}}(\Omega)$ and by N. G. Meyers [8] to show that for every $p > 2$, that a solution in $W^{1,2}_{\mathrm{loc}}(\Omega)$ need not be in $W^{1,p}_{\mathrm{loc}}(\Omega)$ can be recovered with the same construction, by taking $v(r) = r^\alpha$. The ellipticity condition requires $\alpha < n-1$ or $\alpha > 1$. This covers all the cases when $n = 2$; a descent argument finishes the construction in higher dimension.

Proof of Proposition 1.4. One checks that the counterexample constructed in the proof of Proposition 1.4 satisfies $Du \in L \log L(B(0,1))$ when $\beta > 2$. $\qquad\square$

Similar examples can be obtained following the results of V. Kozlov and V. Maz'ya [7]. By (4) therein, if $A \in C(B(0,1); \mathbf{R}^{n \times n})$, $A(Rx) = RA(x)R$ where R is the reflection with respect to the x_1 variable and A satisfies some regularity assumptions, then the equation $-\operatorname{div}(A\nabla u) = 0$ has a solution that is odd with respect to the x_1 variable and that behaves like

$$\frac{x_1}{|x|^n} \exp\Big(\int_{B(0,1)\setminus B(0,|x|)} \mathcal{R}(y)\,dy\Big)$$

around 0, where $\mathcal{R}$ is defined following [7, (3)][1]

(9)
$$\mathcal{R}(x) = \frac{(e_1 \cdot (A(x) - A(0))e_1)(x \cdot A(0)^{-1}x) - n(e_1 \cdot (A(x) - A(0))A(0)^{-1}x)(e_1 \cdot x)}{|\partial B(0,1)| |\det A(0)|^{\frac{1}{2}} (x \cdot A(0)^{-1}x)^{\frac{n}{2}+1}}.$$

Taking A as in Lemma 2.1 with $\lim_{r \to 0} \alpha(r) = 0$, one has $\mathcal{R}(x) = \alpha(|x|)(|x|^2 - x_1^2)/(|\partial B(0,1)||x|^{n+2})$. Therefore, there is a solution that behaves like

$$\frac{x_1}{|x|^n} \exp\Big(\frac{n-1}{n}\int_{|x|}^1 \alpha(r)\frac{dr}{r}\Big).$$

In particular, if one takes $\alpha(r) = -\beta n/((n-1)\log\frac{r_0}{r})$, one obtains a solution that behaves like $\frac{x_1}{|x|^n}(\log\frac{r_0}{r})^{-\beta}$. One could also take $a_{ij}(x) = \delta_{ij} + \kappa(|x|)(\delta_{ij} - n\delta_{i1}\delta_{j1}\frac{x_1^2}{|x|^2})$ and continue the computations with now $\mathcal{R}(x) = \kappa(|x|)(|x|^2 - nx_1^2)^2/(|\partial B(0,1)||x|^{n+2})$.

Proof of Proposition 1.3. Let u be given by the proof of Proposition 1.2. Notice that u is smooth on $\partial B(0,1)$. Since A is bounded and elliptic, the problem

$$\begin{cases} -\operatorname{div}(A\nabla w) = 0 & \text{in } B(0,1), \\ \qquad\qquad w = u & \text{on } \partial B(0,1). \end{cases}$$

has a unique solution in $w \in W^{1,2}(B(0,1))$. Since $u \notin W^{1,2}(B(0,1))$, $u \neq w$. Hence, $u - w \in W^{1,1}(B(0,1))$ is a nontrivial solution of (4). $\qquad\square$

Proof of Proposition 1.5. Take for $r \in (0,1)$,

(10) $$v(r) = \log\frac{r_0}{r}$$

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[1]The reader should correct the misprint in [7, (3)] and read $|S^{n-1}_+|$ instead of $|S^{n-1}|$.

and

$$(11) \qquad \alpha(r) = \frac{1 - (n+1)}{(n-1)\log \frac{r_0}{r}} = \frac{-n}{(n-1)\log \frac{r_0}{r}},$$

where r_0 is chosen so that $\alpha(r) > -\frac{1}{2}$ on $(0,1)$. Defining $u(x) = x_1 v(|x|)$, one checks that $Du \in W^{1,p}_{\mathrm{loc}}(B(0,1))$, $Du \in \mathrm{BMO}(B(0,1))$, $u \notin W^{1,\infty}(B(0,1))$ and that u solves (1) in the sense of distributions. $\qquad\square$

As for the previous singular pathological solutions, similar examples can be obtained from results of V. Kozlov and V. Maz'ya for solutions [6]. By (4) therein if $A \in C(B(0,1); \mathbf{R}^{n\times n})$, $A(Rx) = RA(x)R$ where R is the reflection with respect to the x_1 variable and A satisfies some regularity assumptions, then the equation $-\operatorname{div}(A\nabla u) = 0$ has a solution in $W^{1,2}(B(0,1))$ that is odd with respect to to the x_1 variable and that behaves like

$$x_1 \exp\Big(-\int_{B(0,1)\setminus B(0,|x|)} \mathcal{R}(y)\,dy\Big)$$

around 0, where $\mathcal{R}$ is given by (9). Taking A as in Lemma 2.1 with $\alpha(r) = \frac{-n}{n-1}(\log \frac{r_0}{r})^{-1}$ one recovers the counterexample presented above.

Proof of Proposition 1.6. Define for $r \in (0,1)$,

$$v(r) = \big(\log \frac{r_0}{r}\big)^2.$$

and

$$\alpha(r) = \frac{-2n}{(n-1)\log \frac{r_0}{r}} + \frac{2}{(n-1)(\log \frac{r_0}{r})^2}.$$

Defining $u(x) = x_1 v(|x|)$, one checks that $u \in W^{1,p}(B(0,1))$ for every $p > 1$ and that u solves (1) in the sense of distributions. One checks that for every $c > 0$, $\exp(c|Du|) \notin L^1(B(0,\frac{1}{2}))$; hence by the John–Nirenberg embedding theorem [5] (see also e.g. [11, Chapter 4, §1.3]), $Du \notin \mathrm{BMO}(B(0,\frac{1}{2}))$. $\qquad\square$

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