Potentials of Gaussians and Approximate Wavelets

Vladimir Maz'ya[1] and **Gunther Schmidt**[*2]

[1] Department of Mathematics, University of Linköping, 581 83 Linköping, Sweden
Department of Mathematics, Ohio State University, 231 W 18th Avenue, Columbus, OH 43210, USA
Department of Mathematical Sciences, M&O Building, University of Liverpool, Liverpool L69 3BX, UK
[2] Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany

Received 5 December 2005, revised xxxx , accepted yyyyy
Published online aaaaaa

Key words Multidimensional cubature, wavelets, harmonic, diffraction elastic, hydrodynamic potentials
MSC (2000) Primary: 41A55; Secondary: 41A63, 65D30

Dedidated to Frank-Olme Speck on the occasion of his 60-th birthday

We derive new formulas for harmonic, diffraction, elastic, and hydrodynamic potentials acting on anisotropic Gaussians and approximate wavelets. These formulas can be used to construct accurate cubature formulas for these potentials.

1 Introduction

The present paper is devoted to the approximation of integral operators of mathematical physics by using *approximate approximations*. These approximation procedures use linear combinations of smooth, rapidly decaying functions to approximate a given function with high order within a prescribed accuracy, but they do not converge as the mesh size tends to zero. The lack of convergence is compensated for by the flexibility in the choice of approximating functions, which makes it easier to find approximants for which the action of a given pseudodifferential operator can be effectively determined.

The concept of approximate approximations and first related results were published by V. Maz'ya in [11, 12]. Various aspects of a general theory of these approximations were systematically investigated in [14, 15, 16, 17, 3]. Some applications of approximate approximations to numerical algorithms of solving linear and nonlinear pseudodifferential equations of mathematical physics were studied for example in [5, 6, 7, 10].

New classes of cubature formulas for important integral operators of mathematical physics by using approximate approximations were studied in [13]. They are based on replacing the density $u(\mathbf{x})$ of the integral operator

$$\mathcal{K}u(\mathbf{x}) = \int\limits_{\mathbb{R}^n} k(\mathbf{x}-\mathbf{y})\, u(\mathbf{y})\, d\mathbf{y}$$

by a quasi-interpolant of the form

$$\mathcal{M}_{h,\mathcal{D}}u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} u_{\mathbf{m}} \eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big), \tag{1.1}$$

where the generating function η is chosen such that $\mathcal{K}\eta$ can be computed efficiently. This is, for example, the case for harmonic, elastic and diffraction potentials acting on the isotropic Gaussian $\mathrm{e}^{-|\mathbf{x}|^2}$ and related functions. Then the linear combination

$$\mathcal{K}(\mathcal{M}_{h,\mathcal{D}}u)(\mathbf{x}) = h^n \sum_{\mathbf{m}\in\mathbb{Z}^n} u_{\mathbf{m}}\, \widetilde{\mathcal{K}}\eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)\,, \tag{1.2}$$

* Corresponding author: e-mail: schmidt@wias-berlin.de, Phone: +049 30 20372-456, Fax: +049 30 2044975

where the integrals

$$\widetilde{\mathcal{K}}\eta(\mathbf{x}) = \int_{\mathbb{R}^n} k(\sqrt{\mathcal{D}}h(\mathbf{x}-\mathbf{y}))\eta(\mathbf{y})\,d\mathbf{y}$$

can be taken either analytically or transformed to a simple one-dimensional integral, is taken as cubature formula for the integral operator $\mathcal{K}$. For example, the n-dimensional harmonic potential of the Gaussian can be transformed to

$$\frac{\Gamma(\frac{n}{2}-1)}{4\pi^{n/2}} \int_{\mathbb{R}^n} \frac{\mathrm{e}^{-|\mathbf{y}|^2}}{|\mathbf{x}-\mathbf{y}|^{n-2}}\,d\mathbf{y} = \frac{1}{4|\mathbf{x}|^{n-2}} \int_0^{|\mathbf{x}|^2} t^{n/2-2}\,\mathrm{e}^{-t}\,dt\,, \tag{1.3}$$

which can be expressed either by the exponential or the error function. It was shown that formulas of the type (1.2) can provide high order approximation rates within a prescribed accuracy.

One aim of the present paper is to derive similar formulas for potentials of anisotropic Gaussians. Then it is straightforward to extend the cubature formulas to the inverse of anisotropic elliptic operators or one can use anisotropic Gaussians and related functions in cubature procedures, as discussed in [3]. We obtain, for example, the following one-dimensional integral representation for the harmonic potential of the tensor product Gaussian

$$\frac{\Gamma(\frac{n}{2}-1)}{4\pi^{n/2}} \int_{\mathbb{R}^n} \frac{\mathrm{e}^{-(y_1^2/a_1+\cdots+y_n^2/a_n)}}{|\mathbf{x}-\mathbf{y}|^{n-2}}\,d\mathbf{y} = \frac{1}{4}\int_0^\infty \prod_{j=1}^n \frac{\mathrm{e}^{-x_j^2/(a_j+t)}}{\sqrt{1+t/a_j}}\,dt\,. \tag{1.4}$$

Closely connected with the problem to find simple formulas for potentials of anisotropic Gaussians is the efficient summation of the cubature formulas (1.2) by using a wavelet decomposition of the densities. It is well known that wavelet expansions lead to a sparse representation of smooth functions within a prescribed accuracy. Moreover, due to the high number of vanishing moments of wavelets the action of integral operators is localized. This can reduce considerably the numerical expenses in the approximation of those operators. Unfortunately, it is rather difficult to find the values of multi-dimensional integral operators acting on wavelets accurately.

It was observed in [15, 3], that many of the basis functions used in approximate approximations satisfy refinement equations approximatively, which can be used to perform a multi-resolution analysis within a prescribed accuracy. Such a construction with Gaussians as generating functions led in [15] to the introduction of *approximate wavelets*, which satisfy the basic requirements in an approximate, but controlled way. More precisely, for fixed $k \in \mathbb{N}$ and $\mathcal{D} > 1$ any element of the L_2-closure of the linear span

$$\mathbf{V}_k := \left\{ \mathrm{e}^{-|2^k\mathbf{x}-\mathbf{m}|^2/\mathcal{D}} : \mathbf{x} \in \mathbb{R}^n,\, \mathbf{m} \in \mathbb{Z}^n \right\}$$

can be approximated by elements of the direct sum (which in fact is almost orthogonal)

$$\mathbf{X}_k := \mathbf{V}_0 \dot{+} \mathbf{W}_0 \dot{+} \ldots \dot{+} \mathbf{W}_{k-1} \tag{1.5}$$

with a relative error $c\,\mathrm{e}^{-3\pi^2\mathcal{D}/4}$, where the constant c depends only on the space dimension n and the number k of refinement levels. The space of approximate wavelets $\mathbf{W}_0$ is spanned by all integer shifts of the collection of 2^n-1 functions given on $\mathbb{R}^n$ as tensor products

$$\Phi_{\mathbf{v}}(\mathbf{x}) = f_{v_1}(x_1)\cdots f_{v_n}(x_n)\,, \quad \mathbf{v} = (v_1,\ldots,v_n) \in \mathcal{V}' = \mathcal{V}\backslash\{\mathbf{0}\}\,, \tag{1.6}$$

with $\mathcal{V}$ denoting the set of vertices of the cube $[0,1]^n$. Here f_0 denotes the one-dimensional generating function

$$f_0(x) = \phi_{\mathcal{D}}(x) := e^{-x^2/\mathcal{D}}\,, \tag{1.7}$$

and f_1 the one-dimensional mother wavelet

$$f_1(x) = \psi_{\mathcal{D}}(x) := \mathrm{e}^{-(2x-1)^2/6\mathcal{D}} \cos\frac{5\pi}{6}(2x-1)\,. \tag{1.8}$$

The principal shift invariant spaces $\{\Phi_{\mathbf{v}}(\cdot-\mathbf{m}),\ \mathbf{m}\in\mathbb{Z}^n\}$, $\mathbf{v}\in\mathcal{V}$, are mutually orthogonal and

$$\mathbf{V}_0=\{\Phi_{\mathbf{0}}(\cdot-\mathbf{m}),\ \mathbf{m}\in\mathbb{Z}^n\}\,,\quad \mathbf{W}_0=\bigoplus_{\mathbf{v}\in\mathcal{V}'}\{\Phi_{\mathbf{v}}(\cdot-\mathbf{m}),\ \mathbf{m}\in\mathbb{Z}^n\}\,.$$

As usual, the wavelet spaces $\mathbf{W}_j$ in (1.5) are obtained by scaling $\mathbf{W}_j=\{\psi(2^j\cdot):\psi\in\mathbf{W}_0\}$.

Besides the potentials of anisotropic Gaussians in this paper we obtain also formulas for potentials of the basis functions $\Phi_{\mathbf{v}}$ in $\mathbb{R}^n$, $n\geq 3$. The approach, which is based on the solution of auxiliary non-stationary problems with generalized Gaussians as initial values, enables us to transform various integral operators with singular kernel function k to one-dimensional integrals

$$\int\limits_{\mathbb{R}^n}k(\mathbf{x}-\mathbf{y})\,\Phi_{\mathbf{v}}(\mathbf{y})\,d\mathbf{y}=\int\limits_0^\infty f_{\mathbf{v}}(\mathbf{x},t)\,dt \tag{1.9}$$

with smooth $f_{\mathbf{v}}$. The last integral can be evaluated efficiently using standard quadrature methods or, in some special cases, can be taken analytically.

So it is possible to combine the advantages of well–established wavelet methods in numerical analysis with the efficient computation of important integral operators within the round–off required. Indeed, let the function u is be approximated by a quasi-interpolant (1.1) with $\eta(\mathbf{x})=\mathrm{e}^{-|\mathbf{x}|^2}$ and $h=2^{-k}$, i.e. an element $\mathcal{M}_{2^{-k},\mathcal{D}}u\in\mathbf{V}_k$. We take the ortho-projection $\varphi_k=P(\mathcal{M}_{2^{-k},\mathcal{D}}u)\in\mathbf{X}_k$ and obtain in this way a multivariate wavelet expansion of u

$$\varphi_k(\mathbf{x})=\sum_{\mathbf{m}\in\mathbb{Z}^n}a_{\mathbf{m}}\Phi_{\mathbf{0}}(\mathbf{x}-\mathbf{m})+\sum_{j=0}^{k-1}\sum_{\mathbf{m}\in\mathbb{Z}^n}\sum_{\mathbf{v}\in\mathcal{V}'}a_{j,\mathbf{m}}^{\mathbf{v}}\Phi_{\mathbf{v}}(2^j\mathbf{x}-\mathbf{m})\in\mathbf{X}_k\,. \tag{1.10}$$

The construction of the coefficients is discussed in [15]. Note that most of the wavelet coefficients are small and can be omitted without violating the required accuracy. For compactly supported smooth functions u we obtain therefore a cubature of the integral $\mathcal{K}u$ in the form

$$\begin{aligned}&\sum_{\mathbf{m}\in\mathbb{Z}^n}a_{\mathbf{m}}\int\limits_{\mathbb{R}^n}k(\mathbf{x}-\mathbf{m}-\mathbf{y})\,\Phi_{\mathbf{0}}(\mathbf{y})\,d\mathbf{y}\\&\quad+\sum_{j=0}^{k-1}2^{-jn}\sum_{\mathbf{m}\in\mathbb{Z}^n}\sum_{\mathbf{v}\in\mathcal{V}'}a_{j,\mathbf{m}}^{\mathbf{v}}\int\limits_{\mathbb{R}^n}k(2^{-j}(2^j\mathbf{x}-\mathbf{m}-\mathbf{y}))\,\Phi_{\mathbf{v}}(\mathbf{y})\,d\mathbf{y}\end{aligned} \tag{1.11}$$

with a relatively small number of nonzero coefficients $a_{\mathbf{m}}$, $a_{j,\mathbf{m}}^{\mathbf{v}}$, and where in view of (1.9) the integrals can be computed efficiently.

The outline of the paper is as follows. In Section 2 the approximation of the Hilbert transform is considered as a simple example. The Hilbert transform of approximate wavelets has a very accurate functional representation, which implies an efficient computational formula of this integral transform of a given function with known wavelet expansion. In Section 3 we determine the solution of second order elliptic equations with generalized Gaussians in the right-hand side. This enables us to obtain the values of harmonic and other potentials of anisotropic Gaussians and approximate wavelets as simple special cases. This approach is extended in Section 4 to obtain one-dimensional integral representations for three-dimensional elastic and hydrodynamic potentials of these functions.

2 One-dimensional example

Here we discuss some properties of approximate approximations on the simple quadrature of the Hilbert transform. Let u be a compactly supported function belonging to the Sobolev space $H^4(\mathbb{R})$, i.e. the derivatives of u up to the fourth order are square integrable. Then the quasi-interpolant

$$\mathcal{M}_{h,\mathcal{D}}u(x):=\frac{1}{\sqrt{\pi\mathcal{D}}}\sum_{m\in\mathbb{Z}}u_m\,\mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2} \tag{2.1}$$

with the coefficients

$$u_m = u(hm)\Big(1+\frac{h\mathcal{D}}{2}\Big) - \big(u(h(m+1)) + u(h(m-1))\big)\frac{h\mathcal{D}}{4}$$

approaches u with

$$\|u - \mathcal{M}_{h,\mathcal{D}}u\|_{L_2} \le c_1 \mathcal{D}^2 h^4 \|u\|_{H^4} + c_2 \,\mathrm{e}^{-\pi^2\mathcal{D}}\, \|u\|_{H^3} \tag{2.2}$$

and the constants c_1, c_2 do not depend on u, h, and $\mathcal{D}$, see [14]. The approximation error consists of a fourth order term in h and a saturation error, not depending on h, but smaller than any prescribed positive number if the parameter $\mathcal{D}$ is sufficiently large. Thus, for any $\varepsilon > 0$ one can fix $\mathcal{D}$ sufficiently large in order to obtain a fourth order approximation within this prescribed accuracy ε. Moreover, in view of the rapid decay of the generating function $e^{-x^2/\mathcal{D}h^2}$ the computation of $\mathcal{M}_{h,\mathcal{D}}u(x)$ for given x requires only the summation of a few terms

$$\frac{1}{\sqrt{\pi\mathcal{D}}} \sum_{|x-mh|<\kappa h} u_m \,\mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2}$$

with some fixed κ, depending ε.

As mentioned above, one of the reasons to study such non converging approximations is the possibility to obtain explicit formulae for values of various integral and pseudodifferential operators. For example, the Hilbert transform

$$\mathcal{H}u(x) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(y)}{y-x}\,dy \tag{2.3}$$

is a bounded operator in Sobolev spaces, and therefore the difference $\|\mathcal{H}u - \mathcal{H}(\mathcal{M}_{h,\mathcal{D}}u)\|_{L_2}$ satisfies the estimate (2.2). The Hilbert transform of the Gaussian is a special function,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-y^2}}{y-x}\,dy = -\,\mathrm{e}^{-x^2}\,\mathrm{erfi}(x) \qquad \text{with } \mathrm{erfi}(\tau) = -i\,\mathrm{erf}(i\tau) = \frac{2}{\sqrt{\pi}}\int_0^{\tau} \mathrm{e}^{t^2}\,dt\,,$$

hence we obtain the semi–analytic quadrature of the integral operator

$$\mathcal{H}_{h,\mathcal{D}}u(x) = -\frac{1}{\sqrt{\pi\mathcal{D}}}\sum_{m\in\mathbb{Z}} u_m \,\mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2}\,\mathrm{erfi}\Big(\frac{x-mh}{\sqrt{\mathcal{D}}h}\Big)\,, \tag{2.4}$$

which is of fourth order modulo saturation terms. Note that quasi-interpolation procedures of the form (2.1) can be extended easily to higher approximation rates and multi-dimensional cases ([12, 14]).

Because of $|\,\mathrm{e}^{-x^2}\,\mathrm{erfi}(x)| = O(|x|^{-1})$ the quadrature (2.4) does not have a similar local character as the quasi-interpolation formula (2.1). The computation of $\mathcal{H}_{h,\mathcal{D}}u(x)$ for given x requires the summation of all terms in this sum. This drawback can be overcome by using the wavelet decomposition of u. Suppose that $h = 2^{-k}$ for some $k \in \mathbb{N}$ and take the best L_2 approximation

$$\varphi_k(x) = \sum_{m\in\mathbb{Z}} a_m \phi_{\mathcal{D}}(x-m) + \sum_{j=0}^{k-1}\sum_{m\in\mathbb{Z}} a_{j,m}\psi_{\mathcal{D}}(2^j x - m) \in \mathbf{X}_k \tag{2.5}$$

of the quasi-interpolant $\mathcal{M}_{h,\mathcal{D}}u$ with $\|\mathcal{M}_{h,\mathcal{D}}u - \varphi_k\|_{L_2} \le c\,\mathrm{e}^{-3\pi^2\mathcal{D}/4}\,\|u\|_{L_2}$. Furthermore, in (2.5) we omit all small coefficients $a_m, a_{j,m}$ such that the truncated sum $\widetilde{\varphi}_k$ satisfies the estimate $\|\widetilde{\varphi}_k - \varphi_k\|_{L_2} \le c\,\mathrm{e}^{-3\pi^2\mathcal{D}/4}\,\|u\|_{L_2}$.

The Hilbert transform of the approximate mother wavelet can be found explicitly

$$\mathcal{H}\psi_{\mathcal{D}}(x) = -\,e^{-(2x-1)^2/6\mathcal{D}}\sin\frac{5\pi}{6}(2x-1) + \mathrm{e}^{-\,25\pi^2\mathcal{D}/24}\;\mathrm{Im}\,\mathrm{w}\Big(\frac{2x-1+5\pi i\mathcal{D}/2}{\sqrt{6\mathcal{D}}}\Big)\,,$$

where

$$\mathrm{w}(z) := \mathrm{e}^{-z^2}\,\operatorname{erfc}(-iz) = e^{-z^2}\frac{2}{\sqrt{\pi}}\int\limits_{-iz}^{\infty} \mathrm{e}^{-t^2}\,dt \tag{2.6}$$

is the *scaled complementary error function* satisfying for fixed $y > 0$

$$|\operatorname{Im}\mathrm{w}(x+iy)| < 1 \quad \text{and} \quad \operatorname{Im}\mathrm{w}(x+iy) = \frac{1}{\sqrt{\pi}x} + O(x^{-2}) \quad \text{as} \quad |x| \to \infty\,.$$

Hence $\mathcal{H}\psi_{\mathcal{D}}$ coincides modulo $e^{-25\pi^2\mathcal{D}/24}$ with the shifted wavelet $\psi_{\mathcal{D}}$. That the image of the wavelet under the integral transform is a very fast decaying function is due to the small moments of $\psi_{\mathcal{D}}$. Therefore, within that accuracy the Hilbert transform of the truncated wavelet expansion $\widetilde{\varphi}_k$ coincides with the function

$$\begin{aligned} H(x) = &-\sum_{m\in\mathbb{Z}} a_m\,\mathrm{e}^{-(x-m)^2/\mathcal{D}}\,\operatorname{erfi}\Big(\frac{x-m}{\sqrt{\mathcal{D}}}\Big) \\ &-\sum_{j=0}^{k-1}\sum_{m\in\mathbb{Z}} a_{j,m}\,e^{-(2^{j+1}x-2m-1)^2/6\mathcal{D}}\sin\frac{5\pi}{6}(2^{j+1}x-2m-1)\,. \end{aligned}$$

Note that in order to compute $H(x)$ for a given x one has to take into account only a small number of terms in the second sum. Since $\widetilde{\varphi}_k$ approximates u with the order $O(2^{-4k})$ within the accuracy $\mathrm{e}^{-3\pi^2\mathcal{D}/4}$ we conclude that $H(x)$ is an efficient quadrature of $\mathcal{H}u$ of the same order within that accuracy.

3 Harmonic potentials of Gaussians and related functions

Here we solve elliptic equations in $\mathbb{R}^n$ with generalized Gaussians in the right-hand side. For the special cases of Poisson and Helmholtz equation this leads to explicit representations of harmonic and diffraction potentials.

3.1 Gaussians

First we solve the differential equation

$$-\sum_{j,k=1}^{n} b_{jk}\partial_{x_j}\partial_{x_k}u(\mathbf{x}) = \mathrm{e}^{-\langle A^{-1}(\mathbf{x}+\mathbf{z}),\mathbf{x}+\mathbf{z}\rangle}\,, \quad \mathbf{x}\in\mathbb{R}^n\,,\ n\geq 3\,, \tag{3.1}$$

where the constant matrices A and $B := \|b_{jk}\|_{j,k=1}^n$ are supposed to be nonsingular and complex symmetric satisfying $\operatorname{Re}A > 0$, $\operatorname{Re}B \geq 0$, and $\mathbf{z}\in\mathbb{C}^n$ is an arbitrary constant vector. Here and in the following we use for vectors $\mathbf{y},\mathbf{z}\in\mathbb{C}^n$ the notation

$$\langle\mathbf{y},\mathbf{z}\rangle = \sum_{j=1}^{n} y_j z_j\,, \quad |\mathbf{z}| = \sqrt{\langle\mathbf{z},\mathbf{z}\rangle}\,, \tag{3.2}$$

where the branch of the square root is chosen such that $|\mathbf{z}|$ is the Euclidean norm if $\mathbf{z}\in\mathbb{R}^n$ and its branch-cut is $(-\infty, 0)$. Since for $\mathbf{z} = \mathbf{u}+i\mathbf{v}$, $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$, and $A = A_R + iA_I$ with real symmetric matrices $A_R > 0$ and A_I the relation

$$\operatorname{Re}\langle A(\mathbf{x}+\mathbf{z}),\mathbf{x}+\mathbf{z}\rangle = \langle A_R(\mathbf{x}+\mathbf{u}-A_R^{-1}A_I\mathbf{v}),\mathbf{x}+\mathbf{u}-A_R^{-1}A_I\mathbf{v}\rangle - \langle (A_R + A_I A_R^{-1}A_I)\mathbf{v},\mathbf{v}\rangle$$

is valid, the right hand side of (3.1) belongs to $L_2(\mathbb{R}^n)$ for any $\mathbf{z}\in\mathbb{C}^n$. The Fourier transform of the Gaussian $e^{-\langle A^{-1}\mathbf{x},\mathbf{x}\rangle}$ has the form

$$\mathcal{F}\big(e^{-\langle A^{-1}\cdot,\cdot\rangle}\big)(\boldsymbol{\lambda}) := \int\limits_{\mathbb{R}^n} \mathrm{e}^{-\langle A\mathbf{x},\mathbf{x}\rangle}\,\mathrm{e}^{-2\pi i\langle\mathbf{x},\boldsymbol{\lambda}\rangle}\,d\mathbf{x} = \pi^{n/2}\sqrt{\det A}\,\mathrm{e}^{-\pi^2\langle A\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle}\,, \tag{3.3}$$

(see [2, Theorem 7.6.1]). Here the square root of $\det A$ is defined as the value of the analytic branch of $(\det C)^{1/2}$ on the convex set of symmetric matrices $\{C\}$ with positive real part which satisfies $(\det C)^{-1/2} > 0$ for real C.

The solution of (3.1) can be represented as a one-dimensional integral.

Theorem 3.1 *Suppose that the complex $n \times n$ matrices A and B are nonsingular, symmetric and satisfy $\operatorname{Re} A > 0$, $\operatorname{Re} B \geq 0$ and let $\mathbf{z} \in \mathbb{C}^n$. Then for $n \geq 3$ the function*

$$u(\mathbf{x}) = \frac{1}{4} \int_0^\infty \frac{\mathrm{e}^{-\langle (A+tB)^{-1}(\mathbf{x}+\mathbf{z}), \mathbf{x}+\mathbf{z} \rangle}}{\sqrt{\det(A+tB)}}\, dt \tag{3.4}$$

is a bounded solution of the equation

$$-\langle B\nabla, \nabla \rangle u(\mathbf{x}) = \frac{\mathrm{e}^{-\langle A^{-1}(\mathbf{x}+\mathbf{z}), \mathbf{x}+\mathbf{z} \rangle}}{\sqrt{\det A}}\,, \quad \mathbf{x} \in \mathbb{R}^n\,. \tag{3.5}$$

P r o o f. Consider the Cauchy problem for the parabolic equation in $\mathbb{R}^n$

$$\partial_t v(\mathbf{x},t) - \langle B\nabla_\mathbf{x}, \nabla_\mathbf{x} \rangle\, v(\mathbf{x},t) = 0\,, \quad t > 0\,, \quad v(\mathbf{x},0) = \frac{\mathrm{e}^{-\langle A^{-1}\mathbf{x}, \mathbf{x} \rangle}}{\pi^{n/2}\sqrt{\det A}}\,. \tag{3.6}$$

Applying the Fourier transformation to (3.6) we conclude from (3.3) that $\hat{v}(\boldsymbol{\lambda},t) = \mathcal{F}_{\mathbf{x}\to\boldsymbol{\lambda}} v(\cdot,t)$ satisfies the differential equation

$$\partial_t \hat{v} + 4\pi^2 \langle B\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \hat{v} = 0\,, \quad \hat{v}(\boldsymbol{\lambda},0) = \mathrm{e}^{-\pi^2 \langle A\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle}\,, \quad \boldsymbol{\lambda} \in \mathbb{R}^n\,,$$

which gives $\hat{v}(\boldsymbol{\lambda},t) = e^{-\pi^2 \langle (A+4tB)\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle}$. Since $\operatorname{Re}(A+4tB) > 0$ it follows from (3.3) that

$$v(\mathbf{x},t) = \frac{\mathrm{e}^{-\langle (A+4tB)^{-1}\mathbf{x}, \mathbf{x} \rangle}}{\pi^{n/2}\sqrt{\det(A+4tB)}}\,. \tag{3.7}$$

Integrating (3.6) in t we arrive at

$$\int_0^T \partial_t v(\mathbf{x},t)\, dt = \int_0^T \langle B\nabla_\mathbf{x}, \nabla_\mathbf{x} \rangle\, v(\mathbf{x},t)\, dt = \langle B\nabla_\mathbf{x}, \nabla_\mathbf{x} \rangle \int_0^T v(\mathbf{x},t)\, dt = v(\mathbf{x},T) - v(\mathbf{x},0)\,.$$

The asymptotics $|v(\mathbf{x},t)| = O(t^{-n/2})$ as $t \to \infty$, which is uniform in $\mathbf{x}$, implies that for $n \geq 3$ and $T \to \infty$

$$-\langle B\nabla, \nabla \rangle \int_0^\infty \frac{\mathrm{e}^{-\langle (A+4tB)^{-1}\mathbf{x}, \mathbf{x} \rangle}}{\pi^{n/2}\sqrt{\det(A+4tB)}}\, dt = \frac{\mathrm{e}^{-\langle A^{-1}\mathbf{x}, \mathbf{x} \rangle}}{\pi^{n/2}\sqrt{\det A}}\,,$$

which establishes the assertion if $\mathbf{z} = \mathbf{0}$ and in view of (3.2) for any fixed $\mathbf{z} \in \mathbb{C}^n$. □

The approach of Theorem 3.1 can be extended to other second order elliptic equations.

Corollary 3.2 *Under the above assumptions on A and B a solution of the elliptic equation*

$$-\langle B\nabla, \nabla \rangle u(\mathbf{x}) + a u(\mathbf{x}) = \frac{\mathrm{e}^{-\langle A^{-1}(\mathbf{x}+\mathbf{z}), \mathbf{x}+\mathbf{z} \rangle}}{\sqrt{\det A}}\,, \quad \mathbf{x} \in \mathbb{R}^n\,, \tag{3.8}$$

with constant $a \in \mathbb{C}$, $\operatorname{Re} a > 0$, and $\mathbf{z} \in \mathbb{C}^n$ is given by the integral

$$u(\mathbf{x}) = \frac{1}{4} \int_0^\infty \frac{\mathrm{e}^{-\langle (A+tB)^{-1}(\mathbf{x}+\mathbf{z}), \mathbf{x}+\mathbf{z} \rangle}}{\sqrt{\det(A+tB)}}\, \mathrm{e}^{-at/4}\, dt\,. \tag{3.9}$$

If $\operatorname{Re} a = 0$, then the solution formula (3.9) holds for $n \geq 3$.

P r o o f. Let v be the solution of (3.6). Then

$$w(\mathbf{x},t)=v(\mathbf{x},t)\ \mathrm{e}^{-at}=\frac{\mathrm{e}^{-\langle(A+4tB)^{-1}\mathbf{x},\mathbf{x}\rangle}\ \mathrm{e}^{-at}}{\pi^{n/2}\sqrt{\det(A+4tB)}}$$

satisfies obviously

$$\partial_t w-\langle B\nabla_{\mathbf{x}},\nabla_{\mathbf{x}}\rangle w+a\,w=0\,,\quad t>0\,,\quad w(\mathbf{x},0)=\frac{\mathrm{e}^{-\langle A^{-1}\mathbf{x},\mathbf{x}\rangle}}{\pi^{n/2}\sqrt{\det A}},\quad \mathbf{x}\in\mathbb{R}^n\,. \tag{3.10}$$

and hence

$$(\langle B\nabla_{\mathbf{x}},\nabla_{\mathbf{x}}\rangle-a)\int\limits_0^T w(\mathbf{x},t)\,dt=w(\mathbf{x},T)-w(\mathbf{x},0)\,.$$

Note that in the case $\mathrm{Re}\,a>0$ the limits as $T\to\infty$ exists for space dimension $n<3$. □

3.2 Some special cases

Let us determine the solution of (3.1) for some special cases.

1. If $B=I$ and $a=0$, then (3.1) with some right-hand side $\varphi\in L_2(\mathbb{R}^n)$ has a unique solution satisfying $u(\mathbf{x})\to 0$ as $|\mathbf{x}|\to\infty$. This solution is given as the harmonic potential

$$\mathcal{L}_n\varphi(\mathbf{x}):=\frac{\Gamma(\frac{n}{2}-1)}{4\pi^{n/2}}\int\limits_{\mathbb{R}^n}\frac{\varphi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-2}}\,d\mathbf{y}\ ,\quad n\geq 3\ .$$

Hence, the harmonic potential of $e^{-\langle A^{-1}\mathbf{x},\mathbf{x}\rangle}$ can be obtained from

$$\mathcal{L}_n\big(e^{-\langle A^{-1}\cdot,\cdot\rangle}\big)(\mathbf{x})=\frac{1}{4}\int\limits_0^\infty\frac{\mathrm{e}^{-\langle(A+tI)^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det(I+tA^{-1})}}\,dt\,. \tag{3.11}$$

In particular, for diagonal $A=\mathrm{diag}(a_1,\dots,a_n)$, $a_j>0$, we derive (1.4).

2. If $B=iI$ and $a=\varepsilon-ik^2$, with $\varepsilon>0$, then (3.8) is the Helmholtz equation

$$\Delta u+(k^2+i\varepsilon)u=\frac{i\,\mathrm{e}^{-\langle A^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det A}}$$

and from (3.9) the L_2 solution has the form

$$u_\varepsilon(\mathbf{x})=\frac{1}{4}\int\limits_0^\infty\frac{\mathrm{e}^{-(\varepsilon-ik^2)t/4}\ \mathrm{e}^{-\langle(A+itI)^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det(A+itI)}}\,dt\,. \tag{3.12}$$

If $n\geq 3$, then

$$\lim_{\varepsilon\to 0}u_\varepsilon(\mathbf{x})=\frac{1}{4}\int\limits_0^\infty\frac{\mathrm{e}^{ik^2t/4}\ \mathrm{e}^{-\langle(A+itI)^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det(A+itI)}}\,dt:=u(\mathbf{x})\,, \tag{3.13}$$

which satisfies, by the limiting absorption principle [1], the equation

$$\Delta u+k^2u=\frac{i\,\mathrm{e}^{-\langle A^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det A}} \tag{3.14}$$

and Sommerfeld's radiation condition

$$\frac{\partial u(\mathbf{x})}{\partial r} - ik\, u(\mathbf{x}) = o\big(|\mathbf{x}|^{(1-n)/2}\big) \quad \text{as} \quad |\mathbf{x}| \to \infty\,.$$

Note that the solution of

$$\Delta u + k^2 u = -\varphi(\mathbf{x})$$

is given by the diffraction potential

$$\mathcal{S}_n\varphi(\mathbf{x}) := \int\limits_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}-\mathbf{y})\,\varphi(\mathbf{y})\,d\mathbf{y}$$

with the fundamental solution

$$\mathcal{E}_k(\mathbf{x}) = \frac{i}{4}\Big(\frac{k}{2\pi|\mathbf{x}|}\Big)^{n/2-1} H^{(1)}_{n/2-1}(k|\mathbf{x}|)\,, \tag{3.15}$$

here $H^{(1)}_n = J_n + iY_n$ is the n^{th} order Hankel function of the first kind. In particular, the diffraction potential of the anisotropic Gaussian can be given as

$$\int\limits_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}-\mathbf{y})\;\mathrm{e}^{-(y_1^2/a_1+\ldots+y_n^2/a_n)}\;d\mathbf{y} = \frac{i}{4}\int\limits_0^\infty \mathrm{e}^{ik^2t/4}\prod_{j=1}^n \frac{\mathrm{e}^{-x_j^2/(a_j+it)}}{\sqrt{1+it/a_j}}\,dt\,.$$

In the special case $n=3$ and $a_j = 1$ this integral can be expressed by

$$\frac{1}{4\pi}\int\limits_{\mathbb{R}^3} \frac{\mathrm{e}^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} e^{-|\mathbf{y}|^2}\,d\mathbf{y} = \frac{\sqrt{\pi}}{2}\frac{\mathrm{e}^{-|\mathbf{x}|^2}}{4|\mathbf{x}|}\Big(\mathrm{w}\Big(\frac{k}{2}-i|\mathbf{x}|\Big) - \mathrm{w}\Big(\frac{k}{2}+i|\mathbf{x}|\Big)\Big)\,,$$

using the scaled complementary error function w defined by (2.6).

3.3 Harmonic and diffraction potentials of approximate wavelets

Here we apply Theorem 3.1 to obtain formulas for the harmonic and diffraction potential of the functions $\Phi_{\mathbf{v}}$, $\mathbf{v}\in\mathcal{V}$, which are defined in (1.6). To this end we write $\Phi_{\mathbf{v}}$ as a Gaussian function. Denote by e_j, $j=1,\ldots,n$, the vectors in $\mathbb{R}^n$ with the components δ_{jk}, $k=1,\ldots,n$, and set $\mathbf{e} = e_1+\ldots+e_n$. Recall that $\mathcal{V}$ is the set of vertices of the cube $[0,1]^n$. For given $\mathbf{v}\in\mathcal{V}$ we denote by Q the projection matrix satisfying $Q\mathbf{e} = \mathbf{v}$ and let $P = I - Q$. Then from (1.6) follows that

$$\Phi_{\mathbf{v}}(\mathbf{x}) = \mathrm{e}^{-|P\mathbf{x}|^2/\mathcal{D}}\;\mathrm{e}^{-|Q(2\mathbf{x}-\mathbf{e})|^2/6\mathcal{D}}\prod_{j=1}^n \cos\frac{5\pi}{6}\langle Q(2\mathbf{x}-\mathbf{e}), e_j\rangle\,. \tag{3.16}$$

Using the relation

$$\prod_{j=1}^n \cos\langle Q\mathbf{y}, e_j\rangle = 2^{-m}\sum_{\mathbf{u}\in Q(\mathbb{R}^n)} \cos\langle Q\mathbf{y},\mathbf{u}\rangle,$$

where $\mathbf{u} = (\pm1,\ldots,\pm1)\in\mathbb{R}^n$ and $m = \operatorname{rank} Q$, we write $\Phi_{\mathbf{v}}$ as the sum

$$\begin{aligned}\Phi_{\mathbf{v}}(\mathbf{x}) &= 2^{-m} e^{-|P\mathbf{x}|^2/\mathcal{D}}\, e^{-|Q(2\mathbf{x}-\mathbf{e})|^2/6\mathcal{D}}\sum_{\mathbf{u}\in Q(\mathbb{R}^n)} \cos\frac{5\pi}{6}\langle Q(2\mathbf{x}-\mathbf{e}),\mathbf{u}\rangle \\ &= \frac{\mathrm{e}^{-25\pi^2\mathcal{D}m/24}}{2^m}\sum_{\mathbf{u}\in Q(\mathbb{R}^n)} f_{\mathbf{u}}(\mathbf{x})\,,\end{aligned} \tag{3.17}$$

where

$$f_{\mathbf{u}}(\mathbf{x}) := \mathrm{e}^{-|P\mathbf{x}|^2/\mathcal{D}}\;\mathrm{e}^{-|Q(2\mathbf{x}-\mathbf{e}+5\pi i\mathcal{D}\mathbf{u}/2)|^2/6\mathcal{D}} = \mathrm{e}^{-\langle A^{-1}(\mathbf{x}-\mathbf{z}),\mathbf{x}-\mathbf{z}\rangle} \tag{3.18}$$

with the matrix $A = \mathcal{D}P + \dfrac{3\mathcal{D}}{2}Q$ and the vector $\mathbf{z} = Q\Big(\dfrac{\mathbf{e}}{2} - \dfrac{5\pi i\mathcal{D}\mathbf{u}}{4}\Big)$.

3.3.1 Harmonic potentials

By Theorem 3.1 the harmonic potential of the function $f_{\mathbf{u}}$ equals to

$$\mathcal{L}_n f_{\mathbf{u}}(\mathbf{x}) = \frac{\mathcal{D}^{(n-m)/2}(3\mathcal{D})^{m/2}}{4} \int_0^\infty \frac{\mathrm{e}^{-|P\mathbf{x}|^2/(\mathcal{D}+t)}\, e^{-|Q(2\mathbf{x}-\mathbf{e}+5\pi i\mathcal{D}\mathbf{u}/2)|^2/(6\mathcal{D}+4t)}}{(\mathcal{D}+t)^{(n-m)/2}(3\mathcal{D}+2t)^{m/2}}\, dt$$
$$= \frac{\mathcal{D}}{4} \int_0^\infty \frac{\mathrm{e}^{-|P\mathbf{x}|^2/\mathcal{D}(1+t)}\, e^{-|Q(2\mathbf{x}-\mathbf{e}+5\pi i\mathcal{D}\mathbf{u}/2)|^2/\mathcal{D}(6+4t)}}{(1+t)^{(n-m)/2}(1+2t/3)^{m/2}}\, dt\,,$$

which gives the harmonic potential of the basis wavelet functions

$$\begin{aligned} \mathcal{L}_n \Phi_{\mathbf{v}}(\mathbf{x}) =& \frac{\mathcal{D}\,\mathrm{e}^{-25\pi^2\mathcal{D}m/24}}{2^{m+2}} \sum_{\mathbf{u}\in Q(\mathbb{R}^n)} \int_0^\infty \frac{\mathrm{e}^{-|P\mathbf{x}|^2/\mathcal{D}(1+t)}\, e^{-|Q(2\mathbf{x}-\mathbf{e}+5\pi i\mathcal{D}\mathbf{u}/2)|^2/\mathcal{D}(6+4t)}}{(1+t)^{(n-m)/2}(1+2t/3)^{m/2}}\, dt \\ =& \frac{\mathcal{D}}{4} \int_0^\infty \frac{\mathrm{e}^{-|P\mathbf{x}|^2/\mathcal{D}(1+t)}\ \mathrm{e}^{-|Q(2\mathbf{x}-\mathbf{e})|^2/\mathcal{D}(6+4t)}}{(1+t)^{(n-m)/2}(1+2t/3)^{m/2}} \prod_{j=1}^n \cos\frac{5\pi}{6+4t}\langle Q(2\mathbf{x}-\mathbf{e}), e_j\rangle\, dt\,, \end{aligned} \tag{3.19}$$

where we use that

$$\mathrm{e}^{-25\pi^2\mathcal{D}m/24}\left(\mathrm{e}^{-|Q(2\mathbf{x}-\mathbf{e}+5\pi i\mathcal{D}\mathbf{u}/2)|^2/\mathcal{D}(6+4t)} + \mathrm{e}^{-|Q(2\mathbf{x}-\mathbf{e}-5\pi i\mathcal{D}\mathbf{u}/2)|^2/\mathcal{D}(6+4t)}\right)$$
$$= 2\,\mathrm{e}^{-25\pi^2\mathcal{D}mt/6(6+4t)}\ \mathrm{e}^{-|Q(2\mathbf{x}-\mathbf{e}|^2/\mathcal{D}(6+4t)}\cos\frac{5\pi\langle Q(2\mathbf{x}-\mathbf{e}),\mathbf{u}\rangle}{6+4t}\,.$$

Similar to (1.6) the integral (3.19) can be written in a compact form

$$\mathcal{L}_n \Phi_{\mathbf{v}}(\mathbf{x}) = \frac{\mathcal{D}}{4}\int_0^\infty g_{v_1}(x_1,t)\dots g_{v_n}(x_n,t)\, dt\,, \tag{3.20}$$

where the components v_j of the vector $\mathbf{v}\in\mathcal{V}$ are either 0 or 1 and the functions g_0, g_1 are given by

$$\begin{aligned} g_0(x,t) &= \frac{\mathrm{e}^{-x^2/\mathcal{D}(1+t)}}{\sqrt{1+t}}\,, \\ g_1(x,t) &= \frac{\mathrm{e}^{-25\pi^2\mathcal{D}t/6(6+4t)}}{\sqrt{1+2t/3}}\ \mathrm{e}^{-(2x-1)^2/\mathcal{D}(6+4t)}\cos\frac{5\pi}{6+4t}(2x-1)\,. \end{aligned} \tag{3.21}$$

Using formula (1.3) for $\Phi_{\mathbf{0}}(\mathbf{x}) = \mathrm{e}^{-|\mathbf{x}|^2/\mathcal{D}}$ and (1.11) we conclude that the harmonic potential of a function u with the wavelet expansion (1.10) is approximated by

$$\begin{aligned} \mathcal{L}_{n,h}u(\mathbf{x}) =& \sum_{\mathbf{m}\in\mathbb{Z}^n} a_{\mathbf{m}}\,\frac{\mathcal{D}^{n/2}}{4|\mathbf{x}-\mathbf{m}|^{n-2}} \int_0^{|\mathbf{x}-\mathbf{m}|^2/\mathcal{D}} t^{n/2-2}\,\mathrm{e}^{-t}\ dt \\ &+ \sum_{j=0}^{k-1}\frac{\mathcal{D}}{2^{2j+2}}\sum_{\mathbf{m}\in\mathbb{Z}^n}\sum_{\mathbf{v}\in\mathcal{V}'} a_{j,\mathbf{m}}^{\mathbf{v}} \int_0^\infty g_{v_1}(2^j x_1 - m_1, t)\dots g_{v_n}(2^j x_n - m_n, t)\, dt\,. \end{aligned} \tag{3.22}$$

3.3.2 Diffraction potentials

To determine the diffraction potential

$$\mathcal{S}_n\Phi_{\mathbf{v}}(\mathbf{x}) := \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}-\mathbf{y})\,\Phi_{\mathbf{v}}(\mathbf{y})\, d\mathbf{y}$$

with the fundamental solution given by (3.15) we use (3.18) and formula (3.13) to derive

$$\mathcal{S}_n f_{\mathbf{u}}(\mathbf{x}) = \frac{i\mathcal{D}^{(n-m)/2}(3\mathcal{D})^{m/2}}{4} \int_0^\infty \frac{\mathrm{e}^{ik^2t/4}\,\mathrm{e}^{-|P\mathbf{x}|^2/(\mathcal{D}+it)}\,\mathrm{e}^{-|Q(2\mathbf{x}-\mathbf{e}+5\pi i\mathcal{D}\mathbf{u}/2)|^2/(6\mathcal{D}+4ti)}}{(\mathcal{D}+it)^{(n-m)/2}(3\mathcal{D}+2it)^{m/2}}\,dt$$
$$= \frac{i3^{m/2}\mathcal{D}}{4} \int_0^\infty \mathrm{e}^{ik^2t/4}\,\frac{\mathrm{e}^{-|P\mathbf{x}|^2/\mathcal{D}(1+it)}\,\mathrm{e}^{-|Q(2\mathbf{x}-\mathbf{e}+5\pi i\mathcal{D}\mathbf{u}/2)|^2/\mathcal{D}(6+4it)}}{(1+it)^{(n-m)/2}(3+2it)^{m/2}}\,dt\,.$$

Because of

$$\mathrm{e}^{-25\pi^2\mathcal{D}m/24}\left(\mathrm{e}^{-|Q(2\mathbf{x}-\mathbf{e}+5\pi i\mathcal{D}\mathbf{u}/2)|^2/\mathcal{D}(6+4it)} + \mathrm{e}^{-|Q(2\mathbf{x}-\mathbf{e}-5\pi i\mathcal{D}\mathbf{u}/2)|^2/\mathcal{D}(6+4it)}\right)$$
$$= \mathrm{e}^{-25\pi^2 i\mathcal{D}mt/6(6+4it)}\,\mathrm{e}^{-|Q(2\mathbf{x}-\mathbf{e}|^2/\mathcal{D}(6+4it)}\cos\frac{5\pi\langle Q(2\mathbf{x}-\mathbf{e}),\mathbf{u}\rangle}{6+4it}$$

one obtains the compact form of the diffraction potentials

$$\mathcal{S}_n\Phi_{\mathbf{v}}(\mathbf{x}) = \frac{i\mathcal{D}}{4}\int_0^\infty \mathrm{e}^{ik^2t/4}\,g_{v_1}(x_1,it)\ldots g_{v_n}(x_n,it)\,dt \tag{3.23}$$

with g_0, g_1 defined by (3.21).

4 Elastic and hydrodynamic potentials

Here we consider volume potentials which arise in the solution of three-dimensional problems in elasticity and hydrodynamics, see the monographs [8] and [9]. The solution of the Lamé system

$$\mu\,\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = \mathbf{f} \tag{4.1}$$

with $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{f} = (f_1, f_2, f_3)$, is given by the volume potentials

$$u_j(\mathbf{x}) = \sum_{k=1}^{3}\int_{\mathbb{R}^3}\Gamma_{jk}(\mathbf{x}-\mathbf{y})\,f_k(\mathbf{y})\,d\mathbf{y}\,,$$

where $\|\Gamma_{jk}\|_{j,k=1}^3$ is the Kelvin-Somigliana fundamental matrix with

$$\Gamma_{jk}(\mathbf{x}) = -\frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\left(\frac{\lambda+3\mu}{\lambda+\mu}\frac{\delta_{jk}}{|\mathbf{x}|} + \frac{x_j\,x_k}{|\mathbf{x}|^3}\right). \tag{4.2}$$

For the Stokes problem in $\mathbb{R}^3$

$$\nu\Delta\mathbf{u} - \mathrm{grad}\,p = \mathbf{f}\,,\quad \mathrm{div}\,\mathbf{u} = 0\,, \tag{4.3}$$

the solution can be expressed by the hydrodynamic potentials

$$u_j(\mathbf{x}) = \sum_{k=1}^{3}\int_{\mathbb{R}^3}\Psi_{jk}(\mathbf{x}-\mathbf{y})\,f_k(\mathbf{y})\,d\mathbf{y}\,,\quad p(\mathbf{x}) = \int_{\mathbb{R}^3}\langle\Theta(\mathbf{x}-\mathbf{y}),\mathbf{f}(\mathbf{y})\rangle d\mathbf{y} \tag{4.4}$$

with the fundamental solution of the Stokes system

$$\Psi_{jk}(\mathbf{x}) = -\frac{1}{8\pi\nu}\Big(\frac{\delta_{jk}}{|\mathbf{x}|} + \frac{x_jx_k}{|\mathbf{x}|^3}\Big)\,,\quad \Theta(\mathbf{x}) = -\frac{\mathbf{x}}{4\pi|\mathbf{x}|^3}\,. \tag{4.5}$$

In the following we determine the value of the integral operators with the kernels (4.2) and (4.5) acting on the general Gaussian function. Noting that

$$\frac{x_j x_k}{|\mathbf{x}|^3} = -\partial_{x_j}\partial_{x_k}|\mathbf{x}| + \frac{\delta_{jk}}{|\mathbf{x}|}\,, \quad \frac{\mathbf{x}}{|\mathbf{x}|^3} = -\nabla\frac{1}{|\mathbf{x}|}\,, \tag{4.6}$$

and that the harmonic potentials are already known from (3.20), it suffices to find the integrals

$$\frac{\partial_{x_j}\partial_{x_k}}{8\pi}\int\limits_{\mathbb{R}^3}|\mathbf{x}-\mathbf{y}|\ \mathrm{e}^{-\langle A^{-1}(\mathbf{y}+\mathbf{z}),\mathbf{y}+\mathbf{z}\rangle}\ d\mathbf{y} = \frac{1}{8\pi}\int\limits_{\mathbb{R}^3}|\mathbf{x}-\mathbf{y}|\,\partial_{y_j}\partial_{y_k}\ \mathrm{e}^{-\langle A^{-1}(\mathbf{y}+\mathbf{z}),\mathbf{y}+\mathbf{z}\rangle}\ d\mathbf{y}\,.$$

Note the function $|\mathbf{x}|/8\pi$ is the fundamental solution of the bi-Laplace equation in $\mathbb{R}^3$, i.e. for $\varphi \in L_2(\mathbb{R}^3)$

$$-\Delta^2\int\limits_{\mathbb{R}^3}|\mathbf{x}-\mathbf{y}|\,\varphi(\mathbf{y})\,d\mathbf{y} = 8\pi\varphi(\mathbf{x})\,.$$

4.1 Biharmonic potential of derivatives of Gaussians

Theorem 4.1 *Let $n \geq 3$ and the $n\times n$ matrix A satisfy the assumptions of Theorem 3.1. The unique solution w_{jk} of the bi-Laplace equation*

$$-\Delta^2 w\,(\mathbf{x}) = \frac{\partial^2}{\partial x_k \partial x_l}\,\frac{\mathrm{e}^{-\langle A^{-1}(\mathbf{x}+\mathbf{z}),\mathbf{x}+\mathbf{z}\rangle}}{\sqrt{\det A}}\,, \quad \mathbf{x}\in\mathbb{R}^n\,, \tag{4.7}$$

satisfying $w\,(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ is given by the one-dimensional integral

$$w_{kl}(\mathbf{x}) = -\frac{1}{16}\int\limits_0^\infty t\,\frac{\partial^2}{\partial x_k\partial x_l}\frac{\mathrm{e}^{-\langle (A+tI)^{-1}(\mathbf{x}+\mathbf{z}),\mathbf{x}+\mathbf{z}\rangle}}{\sqrt{\det(A+tI)}}\ dt\,. \tag{4.8}$$

P r o o f. Similar to the proof of Theorem 3.1 we find the solution of (4.7) by solving the Cauchy problem

$$\begin{aligned} &\partial_t^2 v(\mathbf{x},t) + \Delta_{\mathbf{x}}^2 v(\mathbf{x},t) = 0\,, \quad t>0\,, \quad \mathbf{x}\in\mathbb{R}^n\,,\\ &v(\mathbf{x},0) = \frac{\partial_{x_j}\partial_{x_k}\,e^{-\langle A^{-1}\mathbf{x},\mathbf{x}\rangle}}{\pi^{n/2}\sqrt{\det A}}\,, \quad \partial_t v(\mathbf{x},0) = 0\,. \end{aligned} \tag{4.9}$$

The Fourier transformed problem

$$\begin{aligned} &\partial_t^2\hat v(\boldsymbol{\lambda},t) + 16\pi^4|\boldsymbol{\lambda}|^4\hat v(\boldsymbol{\lambda},t) = 0\,, \quad t>0\,, \quad \boldsymbol{\lambda}\in\mathbb{R}^n\,,\\ &\hat v(\boldsymbol{\lambda},0) = -4\pi^2\lambda_j\lambda_k\ \mathrm{e}^{-\pi^2\langle A\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle}\,, \quad \partial_t\hat v(\boldsymbol{\lambda},0) = 0\,, \end{aligned}$$

has the solution

$$\begin{aligned} \hat v_{jk}(\boldsymbol{\lambda},t) &= -4\pi^2\lambda_j\lambda_k\ \mathrm{e}^{-\pi^2\langle A\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle}\cos 4\pi^2|\boldsymbol{\lambda}|^2 t\\ &= -2\pi^2\lambda_j\lambda_k\Big(e^{-\pi^2\langle (A+4itI)\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle} + e^{-\pi^2\langle (A-4itI)\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle}\Big)\,. \end{aligned}$$

Hence, from (3.3) the solution of (4.9) is

$$v_{jk}(\mathbf{x},t) = \frac{\partial_{x_j}\partial_{x_k}}{2\pi^{n/2}}\left(\frac{\mathrm{e}^{-\langle (A+4itI)^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det(A+4itI)}} + \frac{\mathrm{e}^{-\langle (A-4itI)^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det(A-4itI)}}\right).$$

Multiplying the differential equation in (4.9) with t and integrating we arrive at

$$-\int\limits_0^T t\,\Delta_{\mathbf{x}}^2 v_{jk}(\mathbf{x},t)\,dt = \int\limits_0^T t\,\partial_t^2 v_{jk}(\mathbf{x},t)\,dt = T\,\partial_t v_{jk}(\mathbf{x},T) - v_{jk}(\mathbf{x},T) + v_{jk}(\mathbf{x},0)\,.$$

Noting that for $t \to \infty$

$$v_{jk}(\mathbf{x},t) = \begin{cases} O(t^{-n/2-1}) & j = k\,, \\ O(t^{-n/2-2}) & j \neq k\,, \end{cases} \qquad \text{and} \quad \partial_t^\ell v_{jk}(\mathbf{x},t) = O(t^{-\ell} v_{jk}(\mathbf{x},t))\,,\ \ell \in \mathbb{N}\,,$$

uniformly in $\mathbf{x}$ and letting $T \to \infty$ we obtain

$$\Delta^2 \int\limits_0^\infty t\, v_{jk}(\mathbf{x},t)\, dt = -\,\partial_{x_j}\partial_{x_k} \frac{\mathrm{e}^{-\langle A^{-1}\mathbf{x},\mathbf{x}\rangle}}{\pi^{3/2}\sqrt{\det A}}\,.$$

Now we note that

$$\begin{aligned} w_{jk}(\mathbf{x}) &= \int\limits_0^\infty t\, v_{jk}(\mathbf{x},t)\, dt = \frac{1}{2}\int\limits_0^\infty t\ \partial_{x_j}\partial_{x_k}\Big(\frac{\mathrm{e}^{-\langle (A+4itI)^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det(A+4itI)}} + \frac{\mathrm{e}^{-\langle (A-4itI)^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det(A-4itI)}}\Big)\, dt \\ &= -\frac{1}{32}\Big(\int\limits_0^{i\,\infty} \tau\, \partial_{x_j}\partial_{x_k} \frac{\mathrm{e}^{-\langle (A+\tau I)^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det(A+\tau I)}}\, d\tau + \int\limits_0^{-i\,\infty} \tau\, \partial_{x_j}\partial_{x_k} \frac{\mathrm{e}^{-\langle (A+\tau I)^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det(A+\tau I)}}\, d\tau\Big). \end{aligned}$$

The function

$$g(\tau) := \tau\ \partial_{x_j}\partial_{x_k} \frac{\mathrm{e}^{-\langle (A+\tau I)^{-1}\mathbf{x},\mathbf{x}\rangle}}{\sqrt{\det(A+\tau I)}}$$

is holomorphic in the half-plane $\{\mathrm{Re}\,\tau > -\lambda_{\min}\}$, where $\lambda_{\min} > 0$ is the minimal eigenvalue of the symmetric matrix $\mathrm{Re}\,A$. Furthermore, $|g(\tau)| \le cR^{-n/2}$ for $|\tau| = R \to \infty$, hence

$$\int\limits_{\substack{|\tau|=R \\ \mathrm{Re}\,\tau \ge 0}} g(\tau)\, d\tau \to 0 \quad \text{as} \quad R \to \infty\,,$$

if $n \ge 3$. Therefore, by Cauchy's integral theorem

$$\int\limits_0^{\pm i\,\infty} g(\tau)\, d\tau = \int\limits_0^\infty g(\tau)\, d\tau\,,$$

which establishes the assertion for $\mathbf{z} = \mathbf{0}$. □

4.2 Potentials of approximate wavelets

From Theorem 4.1 we derive immediately that the integrals

$$w_{jk}(\mathbf{x}) := \frac{\partial_{x_j}\partial_{x_k}}{8\pi} \int\limits_{\mathbb{R}^3} |\mathbf{x}-\mathbf{y}|\, f_{\mathbf{u}}(\mathbf{y})\, d\mathbf{y} = \frac{1}{8\pi}\int\limits_{\mathbb{R}^3} |\mathbf{x}-\mathbf{y}|\, \partial_{y_j}\partial_{y_k}\, f_{\mathbf{u}}(\mathbf{y})\, d\mathbf{y}$$

with $f_{\mathbf{u}}$ defined by (3.18) can be written in the form

$$w_{jk}(\mathbf{x}) = -\frac{3^{m/2}\mathcal{D}^2}{16} \int\limits_0^\infty t\, \partial_{x_j}\partial_{x_k} \frac{\mathrm{e}^{-|P\mathbf{x}|^2/\mathcal{D}(1+t)}\, e^{-|Q(2\mathbf{x}-\mathbf{e}+5\pi i\mathcal{D}\mathbf{u}/2)|^2/\mathcal{D}(6+4t)}}{(1+t)^{(3-m)/2}(3+2t)^{m/2}}\, dt\,.$$

Together with formula (3.20) the one-dimensional integral representations of the elastic and hydrodynamic potentials of the basis wavelet functions follow immediately. Since by (4.6)

$$\Gamma_{jk}(\mathbf{x}) = \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\,\partial_{x_j}\partial_{x_k}|\mathbf{x}| - \frac{\delta_{jk}}{4\pi\mu|\mathbf{x}|}\,,$$

the elastic potential of $\Phi_{\mathbf{v}}$ can be obtained from

$$\int_{\mathbb{R}^3} \Gamma_{jk}(\mathbf{x}-\mathbf{y})\Phi_{\mathbf{v}}(\mathbf{y})\,d\mathbf{y} = -\frac{\delta_{jk}}{\mu}\mathcal{L}_3(\Phi_{\mathbf{v}})(\mathbf{x}) + \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\int_{\mathbb{R}^3} |\mathbf{x}-\mathbf{y}|\,\partial_{y_j}\partial_{y_k}\,\Phi_{\mathbf{v}}(\mathbf{y})\,d\mathbf{y}$$
$$= -\frac{\mathcal{D}}{4\mu}\int_0^\infty \Big(\delta_{jk} + \frac{t\,\mathcal{D}(\lambda+\mu)}{4(\lambda+2\mu)}\,\partial_{x_j}\partial_{x_k}\Big) g_{v_1}(x_1,t)\dots g_{v_n}(x_n,t)\,dt\,, \tag{4.10}$$

with the functions g_0, g_1 defined by (3.21). Further, from

$$\Psi_{jk}(\mathbf{x}) = \frac{1}{8\pi\nu}\,\partial_{x_j}\partial_{x_k}|\mathbf{x}| - \frac{\delta_{jk}}{4\pi\nu|\mathbf{x}|}$$

we derive the hydrodynamic potential

$$\int_{\mathbb{R}^3} \Psi_{jk}(\mathbf{x}-\mathbf{y})\Phi_{\mathbf{v}}(\mathbf{y})\,d\mathbf{y} = -\frac{\delta_{jk}}{\nu}\mathcal{L}_3(\Phi_{\mathbf{v}})(\mathbf{x}) + \frac{1}{8\pi\nu}\int_{\mathbb{R}^3} |\mathbf{x}-\mathbf{y}|\,\partial_{y_j}\partial_{y_k}\,\Phi_{\mathbf{v}}(\mathbf{y})\,d\mathbf{y}$$
$$= -\frac{\mathcal{D}}{4\nu}\int_0^\infty \Big(\delta_{jk} + \frac{t\,\mathcal{D}}{4}\,\partial_{x_j}\partial_{x_k}\Big) g_{v_1}(x_1,t)\dots g_{v_n}(x_n,t)\,dt\,, \tag{4.11}$$

and from (4.6)

$$-\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x_k-y_k}{|\mathbf{x}-\mathbf{y}|^3}\,\Phi_{\mathbf{v}}(\mathbf{y})\,d\mathbf{y} = \frac{\mathcal{D}}{4}\,\partial_{x_k}\int_0^\infty g_{v_1}(x_1,t)g_{v_2}(x_2,t)g_{v_3}(x_3,t)\,dt. \tag{4.12}$$

4.3 Potential of Gaussians

Explicit expressions for two- and three-dimensional elastic potentials of Gaussians have been obtained in [13] in a rather complicated way. Here we derive the three-dimensional elastic and hydrodynamic potentials of the Gaussian as simple consequences of formulas (4.10) and (4.11). Setting $\mathbf{v}=\mathbf{0}$ and $\mathcal{D}=1$ we obtain from (3.21)

$$\int_{\mathbb{R}^3} \Gamma_{jk}(\mathbf{x}-\mathbf{y})\,\mathrm{e}^{-|\mathbf{y}|^2}\,d\mathbf{y} = -\frac{1}{4\mu}\int_0^\infty \Big(\delta_{jk} + \frac{t\,(\lambda+\mu)}{4(\lambda+2\mu)}\,\partial_{x_j}\partial_{x_k}\Big)\frac{\mathrm{e}^{-|\mathbf{x}|^2/(1+t)}}{(1+t)^{3/2}}\,dt$$
$$= -\frac{1}{4\mu}\int_0^\infty \Big(\frac{\delta_{jk}}{1+t}\Big(1+\frac{t(\lambda+3\mu)}{2(\lambda+2\mu)}\Big) + \frac{x_jx_kt(\lambda+\mu)}{(\lambda+2\mu)(1+t)^2}\Big)\frac{\mathrm{e}^{-|\mathbf{x}|^2/(1+t)}}{(1+t)^{3/2}}\,dt$$
$$= \frac{\lambda+\mu}{8\mu(\lambda+2\mu)|\mathbf{x}|^2}\Big(\frac{\sqrt{\pi}\,\mathrm{erf}(|\mathbf{x}|)}{2|\mathbf{x}|} - \mathrm{e}^{-|\mathbf{x}|^2}\Big)\Big(\frac{3x_jx_k}{|\mathbf{x}|^2} - \delta_{jk}\Big)$$
$$- \frac{1}{4\mu(\lambda+2\mu)}\frac{\sqrt{\pi}\,\mathrm{erf}(|\mathbf{x}|)}{2|\mathbf{x}|}\Big((\lambda+\mu)\frac{x_jx_k}{|\mathbf{x}|^2} + (\lambda+3\mu)\,\delta_{jk}\Big).$$

The hydrodynamic potentials can be found from

$$\int_{\mathbb{R}^3} \Psi_{jk}(\mathbf{x}-\mathbf{y})\Phi_{\mathbf{v}}(\mathbf{y})\,d\mathbf{y} = -\frac{1}{4\nu}\int_0^\infty \Big(\delta_{jk} + \frac{t}{4}\,\partial_{x_j}\partial_{x_k}\Big)\frac{\mathrm{e}^{-|\mathbf{x}|^2/(1+t)}}{(1+t)^{3/2}}\,dt$$
$$= -\frac{1}{4\nu}\int_0^\infty \Big(\frac{\delta_{jk}}{1+t}\Big(1+\frac{t}{2}\Big) + \frac{x_jx_kt}{(1+t)^2}\Big)\frac{\mathrm{e}^{-|\mathbf{x}|^2/(1+t)}}{(1+t)^{3/2}}\,dt$$
$$= \frac{1}{8\nu|\mathbf{x}|^2}\Big(\frac{\sqrt{\pi}\,\mathrm{erf}(|\mathbf{x}|)}{2|\mathbf{x}|} - \mathrm{e}^{-|\mathbf{x}|^2}\Big)\Big(\frac{3x_jx_k}{|\mathbf{x}|^2} - \delta_{jk}\Big) - \frac{1}{4\nu}\frac{\sqrt{\pi}\,\mathrm{erf}(|\mathbf{x}|)}{2|\mathbf{x}|}\Big(\frac{x_jx_k}{|\mathbf{x}|^2} + \delta_{jk}\Big),$$

whereas

$$-\frac{1}{4\pi}\int\limits_{\mathbb{R}^3}\frac{x_k-y_k}{|\mathbf{x}-\mathbf{y}|^3}\,\mathrm{e}^{-|\mathbf{y}|^2}\,d\mathbf{y}=\partial_{x_k}\mathcal{L}_3\big(\mathrm{e}^{-|\cdot|^2}\big)(\mathbf{x})=\frac{x_k}{2|\mathbf{x}|^2}\bigg(\mathrm{e}^{-|\mathbf{x}|^2}-\frac{\sqrt{\pi}\,\mathrm{erf}(|\mathbf{x}|)}{2|\mathbf{x}|}\bigg)$$

can be used to approximate the pressure p in (4.3).

References

[1] D. M. Eidus, On the principle of limiting absorption. (Russian) Mat. Sborn. (N.S.) **57** (99), 13–44 (1962).

[2] L. Hörmander, The Analysis of Linear Partial Differential Operators, vol. I (Springer-Verlag, Berlin 1983).

[3] T. Ivanov, V. Maz'ya, and G. Schmidt, Boundary layer approximate approximations for the cubature of potentials in domains , Adv. Comp. Math. **10**, 311–342 (1999).

[4] S. K. Kanaun, A numerical method for the solution of electromagnetic wave diffraction problems on perfectly conducting screens, J. Comput. Phys. **176**, 170–195 (2002).

[5] V. Karlin and V. Maz'ya, Time-marching algorithms for initial-boundary value problems for semi-linear evolution equations based upon approximate approximations, BIT **35**, 548–560 (1995).

[6] V. Karlin and V. Maz'ya, Time-marching algorithms for non-local evolution equations based upon "approximate approximations", SIAM J. Sci. Comput. **18**,736–752 (1997).

[7] V. Karlin, V. Maz'ya, A. B. Movchan, J. R. Willis, and R. Bullogh, Numerical solution of nonlinear hypersingular integral equations of the Peierls type in dislocation theory, SIAM J. Appl. Math., **60**, No. 2, 664–678 (2000).

[8] V. D. Kupradze, Potential methods in the theory of elasticity (Daniel Davey & Co., New York 1965).

[9] O. A. Ladyzhenskaya, The mathematical theory of viscous incompressible flow (Gordon and Breach Science Publishers, New York 1963).

[10] F. Lanzara, V. Maz'ya, and G. Schmidt, Numerical Solution of the Lippmann–Schwinger Equation by Approximate Approximations, J. Fourier Anal. Appl. **10**, No. 6, 645-660 (2004).

[11] V. Maz'ya, A New Approximation Method and its Applications to the Calculation of Volume Potentials. Boundary Point Method, in: 3. DFG-Kolloqium des DFG-Forschungsschwerpunktes "Randelementmethoden", 1991.

[12] V. Maz'ya, Approximate approximations, in: J. R. Whiteman (ed.), The Mathematics of Finite Elements and Applications. Highlights 1993, 77–104. (Wiley & Sons, 1994).

[13] V. Maz'ya and G. Schmidt "Approximate Approximations" and the cubature of potentials, Rend. Mat. Acc. Lincei **6**, s. 9, 161–184 (1995).

[14] V. Maz'ya and G. Schmidt, On Approximate approximations using Gaussian kernels, IMA J. Num. Anal. **16**, 13–29 (1996).

[15] V. Maz'ya and G. Schmidt, Approximate wavelets and the approximation of pseudodifferential operators, Appl. Comp. Harm. Anal. **6**, 287–313 (1999).

[16] V. Maz'ya and G. Schmidt, Construction of basis functions for high order approximate approximations, in: M. Bonnet, A.-M. Sändig, and W. L. Wendland (eds.), Mathematical Aspects of Boundary Element Methods, vol. 10 of Research Notes in Mathematics, 191–202, (Chapman & Hall/CRC, 1999).

[17] V. Maz'ya and G. Schmidt, On quasi–interpolation with non–uniformly distributed centers on domains and manifolds, J. Approx. Th. **110**, 125–145 (2001).