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Sharp Real Part Theorems

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Preface

We present a unified approach to various sharp pointwise inequalities for analytic functions in a disk with the real part of the function on the circumference as the right-hand side. We refer to these inequalities as "real part theorems" in concert with the first assertion of such a kind, the celebrated Hadamard's real part theorem (1892). The inequalities in question are frequently used in the theory of entire functions and in the analytic number theory.

We hope that collecting these inequalities in one place, as well as generalizing and refining them, may prove useful for various applications. In particular, one can anticipate rich opportunities of extension of these inequalities to analytic functions of several complex variables and solutions of partial differential equations.

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Introduction

Estimates for analytic functions and their derivatives play an important role in complex analysis and its applications. Among these estimates which enjoy a great variety, there are the following two closely related classes having a wide range of applications.

The estimates of the first class contain only modulus of the analytic function in the majorant part of an inequality. In particular, they embrace Cauchy's inequalities, maximum modulus principle, Schwarz lemma, Hadamard three circles theorem (see, for example, Titchmarsh [86], Ch. 2, 5), Bohr's theorem [17], estimates for derivatives due to Landau, Lindelöf, F. Wiener (see Jensen [50]), Makintyre and Rogosinski [68], Rajagopal [78, 79], Szász [85]. In addition to that, the first class embraces estimates of Schwarz-Pick type for derivatives of arbitrary order obtained by Anderson and Rovnyak [10], Avkhadiev and Wirths [11], Bénéteau, Dahlner and Khavinson [12], MacCluer, Stroethoff and Zhao [65, 66], Ruscheweyh [82]. Among other known estimates of the same nature are generalizations on analytic operator-valued functions of a Schwarz-Pick type inequality for any order derivatives by Anderson and Rovnyak [10] and Carathéodory's inequality for the first derivative by Yang [89, 90] .

During last years the so called Bohr's inequality attracted a lot of attention. A refined form of Bohr's result [17], as stated by M. Riesz, I. Schur, F. Wiener (see Landau [61], K. I, § 4), claims that any function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{1}$$

analytic and bounded in the disk $D_R = \{z \in \mathbb{C} : |z| < R\}$, obeys the inequality

$$\sum_{n=0}^{\infty} |c_n z^n| \le \sup_{|\zeta|<R} |f(\zeta)|, \tag{2}$$

where $|z| \le R/3$. Moreover, the value $R/3$ of the radius cannot be improved.

Multi-dimensional analogues and other generalizations of Bohr's theorem are treated in the papers by Aizenberg [1, 2, 9], Aizenberg, Aytuna and Djakov [3, 4], Aizenberg and Tarkhanov [5], Aizenberg, Liflyand and Vidras [7], Aizenberg and Vidras [8], Boas and Khavinson [14], Boas [15], Defant, Garcia and Maestre [29], Dineen and Timoney [31, 32], Djakov and Ramanujan [34], Kaptanoğlu [51].

Various interesting recent results related to Bohr's inequalities were obtained by Aizenberg, Grossman and Korobeinik [6], Bénéteau and Korenblum [13], Bénéteau, Dahlner and Khavinson [12], Bombieri and Bourgain [19], Guadarrama [42], Defant and Frerick [30].

Certain problems of functional analysis connected with Bohr's theorem are examined by Defant, Garcia and Maestre [28], Dixon [33], Glazman and Ljubič [39], Nikolski [71], Paulsen, Popescu and Singh [72], Paulsen and Singh [73].

In estimates of the second class the majorant involves the real part of the analytic function. Among these inequalities are the Hadamard-Borel-Carathéodory inequality for analytic functions in D_R with $\Re f$ bounded from above

$$|f(z) - f(0)| \leq \frac{2r}{R-r} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\}, \tag{3}$$

frequently called the Borel-Carathéodory inequality, and the Carathéodory-Plemelj inequality for analytic functions in D_R with bounded $\Re f$

$$|\Im f(z) - \Im f(0)| \leq \frac{2}{\pi} \log\left(\frac{R+r}{R-r}\right) ||\Re\{f - f(0)\}||_\infty \tag{4}$$

(see, for example, Burckel [22], Ch. 5, 6 and references there), where $|z| = r < R$. The same class includes Carathéodory's inequality for derivatives at the center of a disk [23], M. Riesz' theorem on conjugate harmonic functions [80] and many other estimates (see, for example, Jensen [50], Koebe [52], Rajagopal [77]). The sharp constant in M. Riesz inequality for analytic functions in the half-plane was obtained by Gohberg and Krupnik [40], Pichorides [74] and Cole (see Gamelin [36]). Note that sharp constants in parametric M. Riesz inequalities for analytic functions in the half-plane and in the disk were found in the paper of Hollenbeck, Kalton and Verbitsky [47], where a wide range of questions relating Fourier and Hilbert transforms was treated.

We note that different sources give different formulations of inequalities containing the real part as a majorant. In fact, Cartwright ([26], Ch. 1), Holland ([46], Ch. 3), Levin ([62], Lect. 2), Titchmarsh ([86], Ch. 5) formulate the Hadamard-Borel-Carathéodory inequality for functions which are analytic in $\overline{D}_R$. Unlike them, in the books by Burckel ([22], Ch. 6), Ingham ([49], Ch. 3), Littlewood ([64], Ch. 1) and Polya and Szegö ([75], III, Ch. 6) the same estimate is derived for functions which are analytic in D_R and have the real part bounded from above.

The Hadamard-Borel-Carathéodory inequality is used in an essential fashion in the theory of entire functions (see, e.g. the books Boas [16], Ch. 1 and Holland [46], Ch. 4). In particular, this inequality and its variants are applied for factorization of entire functions (see Hadamard [43]), in the proof of the Little Picard theorem (see Borel [20], Zalcman [91]) and in approximation of entire functions (see Elkins [35]).

The Hadamard-Borel-Carathéodory inequality is of use also in the analytic number theory (see Ingham [49], Ch. 3) and in mathematical physics (see Maharana [67]).

During the last years, generalizations of the Hadamard real part theorem (the first form of the Hadamard-Borel-Carathéodory inequality) for holomorphic functions in domains on a complex manifold (see Aizenberg, Aytuna and Djakov [3]), the Carathéodory inequality for derivatives (see Aizenberg [9]) in several complex variables, and an extension of the Hadamard-Borel-Carathéodory inequality for analytic multifunctions (see Chen [27]) appeared.

The estimates in one of the classes mentioned above have their analogues in the other class. For example, this relates Bohr's theorem as well as its analogues containing the real part (see Aizenberg, Aytuna and Djakov [3], Paulsen, Popescu and Singh [72], Sidon [84], Tomić [87]).

Sharp pointwise estimates, being a classical object of analysis, occupy a special place in analytic function theory. In a way, they provide the best description of the pointwise behaviour of analytic functions from a given space.

The subject matter of this book is sharp pointwise estimates for analytic functions and their derivatives in a disk in terms of the real part of the function on the boundary circle. We consider various inequalities of this type from one point of view which reveals their intimate relations.

All inequalities to be obtained result from the analysis of Schwarz integral representation

$$f(z) = i\,\Im f(0) + \frac{1}{2\pi R}\int_{|\zeta|=R} \frac{\zeta+z}{\zeta-z}\,\Re f(\zeta)|d\zeta|,$$

where $|z| < R$. The sharp estimates for the increment of an analytic function are written in a parametric form, where the role of the parameter is played by an arbitrary real valued function $\alpha(z)$ in D_R.

The book contains seven chapters.

In Chapter 1 we obtain sharp estimate for analytic functions in D_R with $\Re f$ bounded from above

$$\Re\{e^{i\alpha(z)}(f(z)-f(0))\} \le \frac{2r(R-r\cos\alpha(z))}{R^2-r^2} \sup_{|\zeta|<R} \Re\big\{f(\zeta)-f(0)\big\}, \qquad (5)$$

where $r = |z| < R$, and α is a real valued function on D_R. This estimate implies various forms of the Hadamard-Borel-Carathéodory inequality and some other similar inequalities. The sharpness of inequality (5) is proved with the help of a parameter dependent family of test functions, each of them being analytic in $\overline{D}_R$.

Chapter 2 deals with a sharp estimate of $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ by the L_p-norm of $\Re f - \Re f(0)$ on the circle $|\zeta| = R$, where $|z| < R, 1 \le p \le \infty$, and α is a real valued function on D_R. In particular, we give explicit formulas for sharp constants in inequalities for $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ with $p = 1, 2, \infty$. We find also the sharp constant in the upper estimate of $|\Im f(z) - \Im f(0)|$ by $||\Re f - \Re f(0)||_p$ for $1 \le p \le \infty$ which generalizes the classical Carathéodory-Plemelj estimate (4) with $p = \infty$. The evaluation of sharp constants is reduced to finding the minimum value of integrals depending on a real parameter entering the integrand.

In Chapter 3 we give sharp estimates of $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ by the L_p-norm of $\Re f - c$ on the circle $|\zeta| = R$, where $|z| < R, 1 \le p \le \infty$, and α is a real valued function on D_R. Here c is a real constant. More specifically, we obtain similar sharp estimates formulated in terms of the best approximation of $\Re f$ by a real constant on the circle $|\zeta| = R$. As corollaries, we give explicit formulas for sharp constants in inequalities for $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ with $p = 1, 2, \infty$. In particular, an estimate containing $||\Re f - c||_1$ in the right-hand side implies

$$|\Re\{e^{i\alpha}(z)(f(z)-f(0))\}| \le \frac{2r(R + r|\cos\alpha(z)|)}{R^2 - r^2} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\},$$

which contains Hadamard-Borel-Carathéodory inequality (3) and similar estimates for the real and imaginary parts.

Other corollaries of the main results in Chapters 2 and 3 are estimates for $\big|\log|f(z)|\big|, |z| < R$, given in terms of L_p-norm of $\log|f|$ on the circle $|\zeta| = R$, where f is an analytic zero-free function in D_R. The results of Chapters 1-3 also imply sharp inequalities for $|f'(z)|$ in terms of various characteristics of the real part of f on the disk.

Using previous results, in Chapter 4 we obtain sharp estimates for directional derivatives (in particular, for the modulus of the gradient) of a harmonic function in and outside the disk D_R, and in the half-plane. Here the majorants contain either characteristics of a harmonic function (interior estimates for derivatives), or characteristics of its directional derivative. In the last case we differ between estimates with a fixed and with a varying direction. In particular, using an estimate for $|f'(z)|$ inside of the disk D_R, obtained in Chapter 3, we derive a refined inequality (see, for comparison, Protter and Weinberger, [76], Ch. 2, Sect. 13) for the gradient of a harmonic function inside of the bounded domain.

In Chapter 5 we find estimates with the best constants of $|f^{(n)}(z)|$ for $n \ge 1$ by the L_p-norm of $\Re\{f - \mathcal{P}_m\}$ on the circle $|\zeta| = R$, where $\mathcal{P}_m$ is a polynomial of degree $m \le n-1$, $|z| < R, 1 \le p \le \infty$. For $z = 0$ explicit sharp constants are found for all $p \in [1, \infty]$. In particular, from the above mentioned sharp estimates for $|f^{(n)}(z)|$ with $p = 1$, we derive inequalities analogous to the Hadamard real part theorem, as well as to the Carathéodory and Landau inequalities. Sharp inequality for $|f^{(n)}(z)|$ similar to Hadamard's real part theorem is known (see, for example, Ingham [49], Ch. 3 and Rajagopal [77]). Unlike the approach used in these works, the method developed in Chapter 5 yields sharp estimates for the modulus of derivative formulated in terms of L_p-characteristics of the real part.

In Chapter 6 we show that given a function (1) with $\Re f$ in the Hardy space $h_1(D_R)$ of harmonic functions on D_R, the inequality

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}} \, ||\Re f||_1$$

holds with the sharp constant, where $r = |z| < R$, $m \geq 1$, $q \in (0, \infty]$. This estimate implies sharp inequalities for the l_q-norm (quasi-norm for $0 < q < 1$) of the Taylor series remainder for bounded analytic functions, analytic functions with bounded $\Re f$, analytic functions with $\Re f$ bounded from above, as well as for analytic functions with $\Re f > 0$. Each of these estimates, specified for $q = 1$ and $m = 1$, improves a certain sharp Hadamard-Borel-Carathéodory type inequality. As corollaries, we obtain some sharp Bohr's type modulus and real part inequalities.

Chapter 7 is devoted to sharp estimates of $|f^{(n)}(z) - f^{(n)}(0)|$ for $n \geq 0$ by the L_p-norm of $\Re\{f - \mathcal{P}_m\}$ on the circle $|\zeta| = R$, where $\mathcal{P}_m$ is a polynomial of degree $m \leq n$, $|z| < R, 1 \leq p \leq \infty$. In particular, from the estimate for $|f^{(n)}(z) - f^{(n)}(0)|$ by the value $||\Re\{f - \mathcal{P}_m\}||_1$ in the right-hand side we obtain sharp estimates for the increment of derivatives of the type similar to Hadamard-Borel-Carathéodory, Carathéodory and Landau inequalities.

The sharpness of estimates for derivatives, similar to the Hadamard-Borel-Carathéodory, the Carathéodory and the Landau inequalities is proved in Chapters 5 and 7 using a family of test functions, analytic in $\overline{D}_R$. Besides, in these chapters, sharp pointwise estimates for the modulus of the derivatives and their increments are formulated in terms of the best approximation of the real part of f by the real part of polynomials $\mathcal{P}_m$ in the norm of $L_p(\partial D_R)$. In particular, for $p = 2$ the best constants are given in an explicit form.

The index and list of symbols are given at the end of the book.

The reader we have in mind should be familiar with the basics in complex function theory. The references are limited to works mentioned in the text.

1

Estimates for analytic functions bounded with respect to their real part

1.1 Introduction

Hadamard's real part theorem is the following inequality

$$|f(z)| \le \frac{Cr}{R-r} \max_{|\zeta|=R} \Re f(\zeta), \tag{1.1.1}$$

where $|z| = r < R$ and f is an analytic function on the closure $\overline{D}_R$ of the disk $D_R = \{z : |z| < R\}$ and vanishing at $z = 0$. This inequality was first obtained by Hadamard with $C = 4$ in 1892 [43].

A more general estimate for $|f(z)|$ with $f(0) \neq 0$ was obtained by Borel [20] and applied in his proof of Picard's theorem independent of modular functions. The inequality

$$|f(z)| \le |\Im f(0)| + |\Re f(0)| \frac{R+r}{R-r} + \frac{2r}{R-r} \max_{|\zeta|=R} \Re f(\zeta)$$

was found by Carathéodory (see Landau [59], pp. 275-277, [60], pp. 191-194). A detailed historic survey on these and other fundamental inequalities for analytic functions can be found in the paper by Jensen [50].

The following generalization of the real part theorem with $C = 2$ resulting from (1.1.1) after replacing $f(z)$ by $f(z) - f(0)$,

$$|f(z) - f(0)| \le \frac{2r}{R-r} \max_{|\zeta|=R} \Re \{f(\zeta) - f(0)\}, \tag{1.1.2}$$

and its corollary

$$|f(z)| \le \frac{R+r}{R-r} |f(0)| + \frac{2r}{R-r} \max_{|\zeta|=R} \Re f(\zeta), \tag{1.1.3}$$

are often called the Borel-Carathéodory inequalities.

Sometimes, (1.1.2) and (1.1.3), as well as the related inequality for $\Re f$

$$\Re f(z) \le \frac{R-r}{R+r}\, \Re f(0) + \frac{2r}{R+r} \max_{|\zeta|=R} \Re f(\zeta), \tag{1.1.4}$$

are called Hadamard-Borel-Carathéodory inequalities (see, e.g., Burckel [22], Ch. 6 and references there).

In this chapter we obtain sharp estimates for

$$\Re\{e^{i\alpha(z)}(f(z)-f(0))\}$$

by the upper (or lower) bound of $\Re f$ on the disk D_R, where α is an arbitrary real valued function on D_R.

In Section 1.2 we give three known proofs of the real part theorem: based on a conformal representation and the Schwarz lemma, on the Schwarz integral representation, and on a series expansion.

Section 1.3 is auxiliary. Using a lemma proved in Section 1.3, in Section 1.4 we derive the following sharp pointwise estimate

$$\Re\{e^{i\alpha(z)}(f(z)-f(0))\} \le \frac{2r(R-r\cos\alpha(z))}{R^2-r^2} \max_{|\zeta|=R} \Re\left\{f(\zeta)-f(0)\right\}, \tag{1.1.5}$$

where f is analytic in $\overline{D}_R$ and $|z|=r<R$.

The lower estimate for the constant in (1.1.5) is obtained with the help of a family of test functions which are analytic in $\overline{D}_R$. As a corollary of (1.1.5) we obtain the inequality with the same sharp constant for analytic functions f in D_R with $\Re f$ bounded from above

$$\Re\{e^{i\alpha(z)}(f(z)-f(0))\} \le \frac{2r(R-r\cos\alpha(z))}{R^2-r^2} \sup_{|\zeta|<R} \Re\big\{f(\zeta)-f(0)\big\}. \tag{1.1.6}$$

Sections 1.5-1.7 contain various corollaries of estimate (1.1.6). Among them, there are Hadamard-Borel-Carathéodory inequalities for the modulus as well as for the real and imaginary part of an analytic function, Harnack inequalities, and analogues of (1.1.6) for $\Re\{e^{i\alpha(z)}(f(z)-f(\xi))\}$ in the case of a disk and the half-plane.

1.2 Different proofs of the real part theorem

Proofs of (1.1.1) with $C=2$ or (1.1.3) are given in Borel [21], Burckel ([22], Ch. 6), Cartwright ([26], Ch. 1), Holland [46], Ingham ([49], Ch. 3), Levin ([62], L. 11), Littlewood ([64], Ch. 1), Maz'ya and Shaposhnikova ([69], Ch. 9), Polya and Szegö ([75], III, Ch. 6), Rajagopal [77], Titchmarsh ([86], Ch. 5), Zalcman [91].

In this section we provide three different proofs of the real part theorem with the constant $C = 2$. In all these proofs we assume that $f = u + iv$ is an analytic function in $\overline{D}_R$ with $f(0) = 0$. We introduce the notation

$$\mathcal{A}_f(R) = \sup_{|z|<R} \Re f(z) \tag{1.2.1}$$

to be used henceforth.

We recall that according to the Schwarz lemma, every analytic function f in D_R with $|f(z)| \le M$ and $f(0) = 0$ satisfies

$$|f(z)| \le MR^{-1}|z| \quad \text{for} \quad |z| < R$$

(see, for example, Littlewood [64], p. 112).

A combination of conformal mappings and the Schwarz lemma form the basis of the so called subordination principle, used, in particular, in the proof of the Hadamard-Borel-Carathéodory inequality and similar estimates (see Burckel [22], Ch. 6, § 5, Polya and Szegö [75], III, Ch. 6, § 2). The following proof is of the same nature.

Proof based on a conformal mapping and the Schwarz lemma (see Littlewood [64], pp. 113-114, Titchmarsh [86], p. 174-175). When proving the inequality

$$|f(z)| \le \frac{2r}{R-r} \max_{|\zeta|=R} \Re f(\zeta), \tag{1.2.2}$$

we may assume that $f \not\equiv 0$. Then, by the maximum principle for harmonic functions, $\mathcal{A}_f(R) > u(0) = 0$. The function

$$w = \psi(\zeta) = -2\mathcal{A}_f(R)\frac{\zeta}{1-\zeta}$$

performs the conformal mapping of the disk $|\zeta| < 1$ onto the half-plane $\Re w < \mathcal{A}_f(R)$ so that, $\psi(0) = 0$. Using the inverse mapping

$$\varphi(w) = \frac{w}{w - 2\mathcal{A}_f(R)},$$

consider the function

$$\omega(z) = \varphi(f(z)) = \frac{f(z)}{f(z) - 2\mathcal{A}_f(R)}, \qquad |z| < R. \tag{1.2.3}$$

According to the conformal representation theory, the function ω is analytic in D_R and $|\omega(z)| \le 1$. These properties of ω can be also justified by other arguments. The function ω is analytic in D_R, since the denominator in the right-hand side of (1.2.3) does not vanish. Furthermore, since

$$-2\mathcal{A}_f(R) + u(z) \le u(z) \le 2\mathcal{A}_f(R) - u(z),$$

then $|u(z)| \leq 2\mathcal{A}_f(R) - u(z)$ and hence

$$|\omega(z)|^2 = \frac{u^2(z) + v^2(z)}{\{2\mathcal{A}_f(R) - u(z)\}^2 + v^2(z)} \leq 1.$$

Note that $\omega(0) = 0$ because $f(0) = 0$. Thus, by the Schwarz lemma,

$$|\omega(z)| \leq \frac{r}{R}.$$

Now, taking into account (1.2.3), we find

$$|f(z)| = \left| \frac{2\mathcal{A}_f(R)\omega(z)}{1 - \omega(z)} \right| \leq \frac{2\mathcal{A}_f(R)r}{R - r},$$

which proves (1.2.2). □

Another proof of the real part theorem is based on the integral representation of analytic functions in a disk.

Proof based on the Schwarz formula (see Levin [62], p. 75, Rajagopal [77]). Consider the Schwarz formula

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\psi}) \frac{Re^{i\psi} + z}{Re^{i\psi} - z} d\psi, \qquad |z| < R.$$

Combining it with

$$0 = u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\psi}) d\psi,$$

we obtain

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\psi}) \frac{2z}{Re^{i\psi} - z} d\psi.$$

Using the equality

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi}{Re^{i\psi} - z} = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{d\zeta}{\zeta(\zeta - z)} = 0,$$

we find

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \big\{ u(Re^{i\psi}) - \mathcal{A}_f(R) \big\} \frac{2z}{Re^{i\psi} - z} d\psi.$$

Hence,

$$|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \big\{ \mathcal{A}_f(R) - u(Re^{i\psi}) \big\} \frac{2r}{R - r} d\psi,$$

which implies (1.2.2). □

There are proofs of the Hadamard-Borel-Carathéodory inequality, based on series expansions (see Borel [20, 21], Cartwright [26], Ch. 1, Ingham [49], Ch. 3, Littlewood [64], Ch. 1, Zalcman [91]). We now give a proof of the real part theorem, close to that proposed by Hadamard, but with the sharp constant $C = 2$ in (1.1.1).

Proof based on the series analysis (see Maz'ya and Shaposhnikova [69], p. 277-278). Put

$$f(z) = \sum_{n\geq 1} a_n z^n,$$

which means that we assume $f(0) = 0$. Define the maximum term

$$\mu(\rho) = \max_n |a_n|\rho^n$$

of $f(z)$. Clearly,

$$|f(z)| \leq \mu(R) \sum_{n\geq 1} \left(\frac{r}{R}\right)^n = \frac{r}{R-r}\mu(R). \tag{1.2.4}$$

One verifies that

$$a_n R^n = \frac{1}{\pi}\int_0^{2\pi} u(R,\vartheta)e^{-in\vartheta}d\vartheta.$$

Since

$$\int_0^{2\pi} u(R,\vartheta)d\vartheta = 0,$$

and, for any $\zeta \in \mathbb{C}$,

$$|\zeta| = \max_{\varphi\in[0,2\pi]} \Re\big(e^{i\varphi}\zeta\big),$$

we obtain

$$|a_n|R^n = \frac{1}{\pi}\max_{\varphi\in[0,2\pi]}\int_0^{2\pi}\big(1+\cos(n\vartheta-\varphi)\big)u(R,\vartheta)d\vartheta.$$

Using the identity

$$\frac{1}{\pi}\int_0^{2\pi}\big(1+\cos(n\vartheta-\varphi)\big)d\vartheta = 2,$$

we arrive at the inequality

$$\mu(R) \leq 2\max_{\vartheta\in[0,2\pi]} u(R,\vartheta),$$

which, together with (1.2.4), yields estimate (1.2.2). □

1.3 Extremal values of the real part of the rotated Schwarz kernel

In what follows, by $h_p(D_R)$, $1 \le p \le \infty$, we mean the Hardy space of harmonic functions in D_R which are represented by the Poisson integral with a density in $L_p(\partial D_R)$. We shall adopt the notation $\Delta f(z) = f(z) - f(0)$, $|z| = r < R$, $\gamma = r/R$.

It is well known (see, for example, Levin [62], L. 2), by Schwarz formula

$$f(z) = i\Im f(0) + \frac{1}{2\pi R}\int_{|\zeta|=R} \frac{\zeta+z}{\zeta-z}\,\Re f(\zeta)|d\zeta| \tag{1.3.1}$$

one can restore any analytic function f in D_R with $\Re f$ continuous in $\overline{D}_R$.

We show that (1.3.1) can be extended to analytic functions f in D_R with $\Re f \in h_p(D_R)$, $1 \le p \le \infty$. It is known (see Hoffman [45], Ch. 3, 6 and Koosis [53], Ch. 1, 2) that nontangential limit values of the Poisson integral

$$u(z) = \frac{1}{2\pi R}\int_{|\zeta|=R} \Re\left\{\frac{\zeta+z}{\zeta-z}\right\} g(\zeta)|d\zeta|$$

with density $g \in L_p(\partial D_R)$ coincide with $g(\zeta)$ almost everywhere on ∂D_R. The last equality can be written as the representation

$$u(z) = \frac{1}{2\pi R}\int_{|\zeta|=R} \Re\left\{\frac{\zeta+z}{\zeta-z}\right\} u(\zeta)|d\zeta| \tag{1.3.2}$$

for $u \in h_p(D_R)$. Then formula (1.3.1) for an analytic function $f = u + iv$ with $\Re f \in h_p(D_R)$ results from (1.3.2). In fact, by (1.3.2), the real parts of the right and left- hand sides of (1.3.1) coincide in D_R. Hence, these functions may differ only by pure imaginary constant, and $f(0) = u(0)+iv(0)$ by (1.3.1).

The following two lemmas will be used in the next three chapters.

Lemma 1.1. *Let f be analytic on D_R with $\Re f \in h_p(D_R), 1 \le p \le \infty$. For any real function α on D_R the relation*

$$\Re\{e^{i\alpha(z)}\Delta f(z)\} = \frac{1}{\pi R}\int_{|\zeta|=R} \Re\left(\frac{e^{i\alpha(z)}z}{\zeta-z}\right)\Re f(\zeta)|d\zeta| \tag{1.3.3}$$

holds.

Proof. By (1.3.1), we obtain

$$\Delta f(z) = \frac{1}{\pi R}\int_{|\zeta|=R} \frac{z}{\zeta-z}\,\Re f(\zeta)|d\zeta|,$$

which implies (1.3.3). □

In the next Lemma, we find the extremal values of the function

$$G_{z,\alpha(z)}(\zeta) = \Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right) \tag{1.3.4}$$

on the circle $|\zeta| = R$, where α is a real function on D_R.

Since z plays the role of a parameter in what follows, we frequently do not mark the dependence of α on z.

Lemma 1.2. *For any point z with $|z| = r < R$ and an arbitrary real function α on D_R, the relations*

$$\min_{|\zeta|=R} \Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right) = \frac{r(r\cos\alpha(z) - R)}{R^2 - r^2},$$

$$\max_{|\zeta|=R} \Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right) = \frac{r(r\cos\alpha(z) + R)}{R^2 - r^2}$$

hold.

Proof. With the notation $\zeta = Re^{it}, z = re^{i\tau}, \gamma = r/R$ one has

$$\frac{e^{i\alpha}z}{\zeta - z} = \frac{e^{i\alpha}re^{i\tau}}{Re^{it} - re^{i\tau}} = \frac{\gamma e^{i\alpha}}{e^{i(t-\tau)} - \gamma}. \tag{1.3.5}$$

Setting $\varphi = t - \tau$ in (1.3.5), we obtain

$$G_{z,\alpha}(\zeta) = \Re\left(\frac{e^{i\alpha}z}{\zeta - z}\right) = \Re\left(\frac{\gamma e^{i\alpha}}{e^{i\varphi} - \gamma}\right) = \frac{\gamma(\cos(\varphi - \alpha) - \gamma\cos\alpha)}{1 - 2\gamma\cos\varphi + \gamma^2}. \tag{1.3.6}$$

Consider the function

$$g(\varphi) = \frac{\cos(\varphi - \alpha) - \gamma\cos\alpha}{1 - 2\gamma\cos\varphi + \gamma^2}, \quad |\varphi| \le \pi. \tag{1.3.7}$$

We have

$$g'(\varphi) = \frac{(\gamma^2 - 1)\cos\alpha\sin\varphi + (\gamma^2 + 1)\sin\alpha\cos\varphi - 2\gamma\sin\alpha}{(1 - 2\gamma\cos\varphi + \gamma^2)^2}.$$

Solving the equation $g'(\varphi) = 0$, we find

$$\sin\varphi_+ = \frac{(1 - \gamma^2)\sin\alpha}{1 + 2\gamma\cos\alpha + \gamma^2}, \qquad \cos\varphi_+ = \frac{2\gamma + (1 + \gamma^2)\cos\alpha}{1 + 2\gamma\cos\alpha + \gamma^2}, \tag{1.3.8}$$

and

$$\sin\varphi_- = -\frac{(1 - \gamma^2)\sin\alpha}{1 - 2\gamma\cos\alpha + \gamma^2}, \qquad \cos\varphi_- = \frac{2\gamma - (1 + \gamma^2)\cos\alpha}{1 - 2\gamma\cos\alpha + \gamma^2}, \tag{1.3.9}$$

where φ_+ and φ_- are critical points of $g(\varphi)$. Setting (1.3.8) and (1.3.9) into (1.3.7), we arrive at

$$g(\varphi_+) = \frac{\gamma \cos\alpha + 1}{1-\gamma^2}, \quad g(\varphi_-) = \frac{\gamma \cos\alpha - 1}{1-\gamma^2}.$$

It follows from (1.3.7) that

$$g(-\pi) = g(\pi) = -\frac{\cos\alpha}{1+\gamma} = \frac{\gamma\cos\alpha - \cos\alpha}{1-\gamma^2}.$$

Since $g(\varphi_+) > g(\varphi_-)$ and

$$g(-\pi) = g(\pi) = \frac{\gamma\cos\alpha - \cos\alpha}{1-\gamma^2} \le \frac{\gamma\cos\alpha + 1}{1-\gamma^2} = g(\varphi_+),$$

$$g(-\pi) = g(\pi) = \frac{\gamma\cos\alpha - \cos\alpha}{1-\gamma^2} \ge \frac{\gamma\cos\alpha - 1}{1-\gamma^2} = g(\varphi_-),$$

it follows from (1.3.6), (1.3.7) that

$$\max_{|\zeta|=R} \Re\left(\frac{e^{i\alpha}z}{\zeta - z}\right) = \gamma g(\varphi_+) = \gamma\frac{\gamma\cos\alpha + 1}{1-\gamma^2} = \frac{r(r\cos\alpha + R)}{R^2 - r^2},$$

$$\min_{|\zeta|=R} \Re\left(\frac{e^{i\alpha}z}{\zeta - z}\right) = \gamma g(\varphi_-) = \gamma\frac{\gamma\cos\alpha - 1}{1-\gamma^2} = \frac{r(r\cos\alpha - R)}{R^2 - r^2}.$$

The proof of Lemma is complete. □

1.4 Upper estimate of $\Re\{e^{i\alpha}\Delta f\}$ by the supremum of $\Re\Delta f$

We start with the following assertion.

Proposition 1.1. *Let f be analytic on $\overline{D}_R$. Then for any z with $|z| = r < R$ and an arbitrary real valued function α on D_R the inequality*

$$\Re\{e^{i\alpha(z)}\Delta f(z)\} \le \frac{2r(R - r\cos\alpha(z))}{R^2 - r^2} \max_{|\zeta|=R} \Re\Delta f(\zeta) \tag{1.4.1}$$

holds with the sharp constant.

Proof. 1. Proof of inequality (1.4.1). We fix a point z with $|z| = r < R$. Let

$$\mu = \min_{|\zeta|=R} \Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right).$$

By the inequality

$$\Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right) - \mu \geq 0,$$

for all ζ on the circle $|\zeta| = R$ there holds

$$\left\{\Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right) - \mu\right\}\Re f(\zeta) \leq \left\{\Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right) - \mu\right\}\max_{|\zeta|=R}\Re f(\zeta). \qquad (1.4.2)$$

It follows from $d\zeta/i\zeta = |d\zeta|/R$ that for all $z \in D_R$

$$\int_{|\zeta|=R}\frac{e^{i\alpha(z)}z}{\zeta - z}|d\zeta| = -iRe^{i\alpha(z)}z\int_{|\zeta|=R}\frac{d\zeta}{(\zeta - z)\zeta} = 0.$$

Consequently,

$$\int_{|\zeta|=R}\Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right)|d\zeta| = 0 \qquad (1.4.3)$$

and therefore, (1.4.2) implies

$$\frac{1}{\pi R}\int_{|\zeta|=R}\left\{\Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right) - \mu\right\}\Re f(\zeta)|d\zeta| \leq -2\mu\max_{|\zeta|=R}\Re f(\zeta).$$

Taking into account the mean value theorem

$$\frac{1}{2\pi R}\int_{|\zeta|=R}\Re f(\zeta)|d\zeta| = \Re f(0),$$

we rewrite the last inequality as

$$\frac{1}{\pi R}\int_{|\zeta|=R}\Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right)\Re f(\zeta)|d\zeta| \leq -2\mu\max_{|\zeta|=R}\Re\{f(\zeta) - f(0)\}.$$

Hence, and by relation (1.3.3), one has

$$\Re\{e^{i\alpha(z)}\Delta f(z)\} \leq -2\mu\max_{|\zeta|=R}\Re\Delta f(\zeta).$$

Using Lemma 1.2 and setting in the last inequality

$$-2\mu = -2\min_{|\zeta|=R}\Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right) = \frac{2r(R - r\cos\alpha(z))}{R^2 - r^2},$$

we arrive at (1.4.1).

2. Sharpness of the constant in inequality (1.4.1). Let us show that the constant

$$\mathcal{C}(\alpha(z)) = \frac{2r(R - r\cos\alpha(z))}{R^2 - r^2}$$

in (1.4.1) is the best possible. Owing to (1.4.1), the estimate

$$\Re\{e^{i\alpha(z)}\Delta f(z)\} \le C_1(\alpha(z)) \max_{|\zeta|=R} \Re\Delta f(\zeta) \tag{1.4.4}$$

holds, where

$$C_1(\alpha(z)) \le \mathcal{C}(\alpha(z)). \tag{1.4.5}$$

We verify that the constant in estimate (1.4.1) is sharp, i.e. $C_1(\alpha(z)) \ge \mathcal{C}(\alpha(z))$.

Let $\xi = \rho e^{i\vartheta}$, where $\rho > R$. Consider the family of analytic functions in $\overline{D}_R$

$$f_\xi(z) = \frac{z}{z-\xi}, \tag{1.4.6}$$

depending on a complex parameter ξ. Putting $z = \zeta = Re^{it}$ in (1.4.6), we find

$$\Re\Delta f_\xi(\zeta) = \Re\left(\frac{\zeta}{\zeta-\xi}\right) = \Re\left(\frac{Re^{it}}{Re^{it} - \rho e^{i\vartheta}}\right)$$
$$= \frac{R(R-\rho\cos(t-\vartheta))}{\rho^2 - 2\rho R\cos(t-\vartheta) + R^2} = \frac{1}{2}\left(1 - \frac{\rho^2 - R^2}{\rho^2 - 2\rho R\cos(t-\vartheta) + R^2}\right).$$

This implies

$$\max_{|\zeta|=R} \Re\Delta f_\xi(\zeta) = \frac{1}{2}\max_t \left(1 - \frac{\rho^2 - R^2}{\rho^2 - 2\rho R\cos(t-\vartheta) + R^2}\right)$$
$$= \frac{1}{2}\left(1 - \frac{\rho^2 - R^2}{\rho^2 + 2\rho R + R^2}\right) = \frac{R}{\rho+R}. \tag{1.4.7}$$

Further, by (1.4.6),

$$\Re\left\{e^{i\alpha(z)}\Delta f_\xi(z)\right\} = \Re\left(\frac{e^{i\alpha(z)}z}{z-\xi}\right) = \Re\left(\frac{e^{i(\alpha(z)+\pi)}z}{\xi - z}\right). \tag{1.4.8}$$

By Lemma 1.2,

$$\max_{|\xi|=\rho}\left\{\Re\left(\frac{e^{i(\alpha(z)+\pi)}z}{\xi-z}\right)\right\} = \frac{r(\rho - r\cos\alpha(z))}{\rho^2 - r^2}.$$

We fix $z_0, |z_0| = r$, and choose a point $\xi_0 = \rho e^{i\vartheta_0}$ so that

$$\Re\left(\frac{e^{i(\alpha(z_0)+\pi)}z_0}{\xi_0 - z_0}\right) = \frac{r(\rho - r\cos\alpha(z_0))}{\rho^2 - r^2}.$$

Then, using (1.4.8), we find

$$\Re\left\{e^{i\alpha(z_0)}\Delta f_{\xi_o}(z_0)\right\} = \frac{r(\rho - r\cos\alpha(z_0))}{\rho^2 - r^2}. \tag{1.4.9}$$

It follows from (1.4.4), (1.4.7), and (1.4.9) that

$$C_1(\alpha(z_0)) \geq \frac{r(\rho - r\cos\alpha(z_0))}{\rho^2 - r^2} \cdot \frac{\rho + R}{R}.$$

Passing to the limit as $\rho \downarrow R$ in the last inequality, we obtain

$$C_1(\alpha(z_0)) \geq \frac{2r(R - r\cos\alpha(z_0))}{R^2 - r^2} = \mathcal{C}(\alpha(z_0)).$$

Hence, by the arbitrariness of the point z_0 on the circle $|z| = r$, we arrive at $C_1(\alpha(z)) \geq C(\alpha(z))$, which together with (1.4.5), proves the sharpness of the constant in estimate (1.4.1). □

The main objective of this chapter is

Theorem 1.1. *Let f be analytic on D_R with $\Re f$ bounded from above. Then for any z with $|z| = r < R$, and for an arbitrary real valued function α on D_R the sharp inequality*

$$\Re\{e^{i\alpha(z)}(f(z) - f(0))\} \leq \frac{2r(R - r\cos\alpha(z))}{R^2 - r^2} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\} \tag{1.4.10}$$

holds.

Proof. Let z be a fixed point in D_R, and let $\varrho \in (r, R)$. Then, by Proposition 1.1,

$$\Re\{e^{i\alpha(z)}\big(f(z) - f(0)\big)\} \leq \frac{2r(\varrho - r\cos\alpha(z))}{\varrho^2 - r^2}\left\{\mathcal{A}_f(\varrho) - \Re f(0)\right\},$$

where

$$\mathcal{A}_f(\varrho) = \max_{|\zeta|=\varrho} \Re f(\zeta).$$

Replacing $\mathcal{A}_f(\varrho)$ by the upper bound of $\Re f$ on D_R in the last inequality, we obtain

$$\Re\{e^{i\alpha(z)}\big(f(z) - f(0)\big)\} \leq \frac{2r(\varrho - r\cos\alpha(z))}{\varrho^2 - r^2} \sup_{|\zeta|<R} \Re\big\{f(\zeta) - f(0)\big\},$$

Passing here to the limit as $\varrho \uparrow R$, we arrive at (1.4.10). The sharpness of the constant in (1.4.10) follows from Proposition 1.1. □

1.5 Two-sided estimates of $\Re\{e^{i\alpha}\Delta f\}$ by upper and lower bounds of $\Re\Delta f$

The upper estimate for $\Re\{e^{i\alpha}\Delta f\}$ in Theorem 1.1 can be equivalently written as a lower estimate. Namely, replacing $\alpha(z)$ by $\alpha(z)+\pi$ in (1.4.10), we obtain

Corollary 1.1. *Let f be analytic on D_R with $\Re f$ bounded from above. Then for any z with $|z| = r < R$, and an arbitrary real valued function α on D_R the inequality*

$$\Re\{e^{i\alpha(z)}(f(z)-f(0))\} \geq -\frac{2r(R+r\cos\alpha(z))}{R^2-r^2}\sup_{|\zeta|<R}\Re\big\{f(\zeta)-f(0)\big\} \quad (1.5.1)$$

holds with the sharp constant.

Inequalities (1.4.10) and (1.5.1) provide the two-sided estimate of

$$\Re\{e^{i\alpha(z)}\big(f(z)-f(0)\big)\}$$

in terms of

$$\sup_{|\zeta|<R}\Re\Delta f(\zeta) = \sup_{|\zeta|<R}\Re\{f(\zeta)-f(0)\} = \sup_{|\zeta|<R}\Re f(\zeta)-\Re f(0).$$

Replacing f by $-f$ in (1.4.10) and (1.5.1), we can obtain similar inequalities in terms of

$$\sup_{|\zeta|<R}\{-\Re\Delta f(\zeta)\} = \sup_{|\zeta|<R}\Re\{f(0)-f(\zeta)\} = \Re f(0) - \inf_{|\zeta|<R}\Re f(\zeta).$$

Corollary 1.2. *Let f be analytic on D_R with $\Re f$ bounded from below. Then for any z with $|z| = r < R$, and an arbitrary real valued function α on D_R the inequalities*

$$\Re\{e^{i\alpha(z)}\Delta f(z)\} \geq -\frac{2r(R-r\cos\alpha(z))}{R^2-r^2}\{\Re f(0) - \inf_{|\zeta|<R}\Re f(\zeta)\}, \quad (1.5.2)$$

$$\Re\{e^{i\alpha(z)}\Delta f(z)\} \leq \frac{2r(R+r\cos\alpha(z))}{R^2-r^2}\{\Re f(0) - \inf_{|\zeta|<R}\Re f(\zeta)\} \quad (1.5.3)$$

hold with the sharp constants.

1.6 Inequalities for the modulus, real and imaginary parts

We note that various Hadamard-Borel-Carathéodory type inequalities (see Burckel [22], Ch. 6, § 5, Jensen [50], Polya and Szegö [75], III, Ch. 6, § 2, Rajagopal [77, 78] and the bibliography in Burckel [22]) are corollaries of Proposition 1.1.

1.6.1. Estimates for the real and imaginary parts

Introducing the notation

$$\mathcal{A}_f(R) = \sup_{|\zeta|<R} \Re f(\zeta), \qquad \mathcal{B}_f(R) = \inf_{|\zeta|<R} \Re f(\zeta), \tag{1.6.1}$$

and letting $\alpha(z) = 0$ in (1.4.10), (1.5.1) and (1.5.2), (1.5.3), we arrive at two-sided estimates

$$\frac{R+r}{R-r}\Re f(0) - \frac{2r}{R-r}\mathcal{A}_f(R) \leq \Re f(z) \leq \frac{R-r}{R+r}\Re f(0) + \frac{2r}{R+r}\mathcal{A}_f(R), \tag{1.6.2}$$

$$\frac{R-r}{R+r}\Re f(0) + \frac{2r}{R+r}\mathcal{B}_f(R) \leq \Re f(z) \leq \frac{R+r}{R-r}\Re f(0) - \frac{2r}{R-r}\mathcal{B}_f(R). \tag{1.6.3}$$

Inequality (1.6.2) was obtained by Jensen [50] (see also Rajagopal [77]).

If $\Re f(\zeta) > 0$ on the disk D_R, then $\mathcal{B}_f(R) \geq 0$ and (1.6.3) imply the classical Harnack inequality

$$\frac{R-r}{R+r}\Re f(0) \leq \Re f(z) \leq \frac{R+r}{R-r}\Re f(0).$$

Another Harnack's inequality

$$-\frac{2r}{R-r}\mathcal{A}_f(R) \leq \Re f(z) \leq \frac{2r}{R+r}\mathcal{A}_f(R)$$

(see, e.g. Koebe [52]) follows from (1.6.2) where $\Re f(0) = 0$.

Putting $\alpha(z) = \pi/2$ in (1.4.10) and (1.5.1), we arrive at the inequality for imaginary part of analytic functions

$$|\Im \Delta f(z)| \leq \frac{2Rr}{R^2 - r^2} \sup_{|\zeta|<R} \Re \Delta f(\zeta). \tag{1.6.4}$$

This inequality was formulated by Rajagopal [78].

1.6.2 Estimates for the modulus

Putting $\alpha(z) = -\arg \Delta f(z)$ in Theorem 1.1, we arrive at the following assertion.

Corollary 1.3. *Let f be analytic on D_R with $\Re f$ bounded from above. Then for any z with $|z| = r < R$ the sharp inequality*

$$|\Delta f(z)| \leq \frac{2r(R - r\cos\arg \Delta f(z))}{R^2 - r^2} \sup_{|\zeta|<R} \Re \Delta f(\zeta) \tag{1.6.5}$$

holds.

From (1.6.5) one gets the estimate for $\Delta f(z) = f(z) - f(0)$ on the circle $|z| = r < R$ called the Hadamard-Borel-Carathéodory inequality

$$|\Delta f(z)| \leq \frac{2r}{R-r} \sup_{|\zeta|<R} \Re \Delta f(\zeta). \tag{1.6.6}$$

As its consequence we obtain the inequality

$$|f(z)| \leq \frac{R+r}{R-r}|f(0)| + \frac{2r}{R-r} \sup_{|\zeta|<R} \Re f(\zeta), \tag{1.6.7}$$

where $|z| = r$, also called Hadamard-Borel-Carathéodory inequality.

The next two-sided estimate is well known (see, for example, Polya and Szegö [75], III, Ch. 6, § 2, Rajagopal [77], Levin [62], L.2).

Corollary 1.4. *Let f be a bounded analytic and zero-free function on D_R with $f(0) = 1$ and let*

$$\mathcal{M}_f(R) = \sup_{|\zeta|<R} |f(\zeta)|.$$

Then for any z with $|z| = r < R$ the two-sided inequality

$$\mathcal{M}_f(R)^{-\frac{2r}{R-r}} \leq |f(z)| \leq \mathcal{M}_f(R)^{\frac{2r}{R+r}} \tag{1.6.8}$$

holds.

Proof. Let z be a fixed point in D_R, and let $r \in (\varrho, R)$. Applying estimate (1.6.2) to $\log f(z)$, where $f(z) \neq 0$ for $|z| \leq \varrho$, $f(0) = 1$, we obtain

$$\mathcal{M}_f(\varrho)^{-\frac{2r}{\varrho-r}} \leq |f(z)| \leq \mathcal{M}_f(\varrho)^{\frac{2r}{\varrho+r}}.$$

Passing here to the limit as $\varrho \uparrow R$, we arrive at (1.6.8). □

Remark 1.1. A similar estimate for $|f(z)|$ with $f(0) \neq 1$ can be obtained from (1.6.8) with $f(z)$ replaced by $f(z)/f(0)$.

1.7 Variants and extensions

We present some corollaries of Theorem 1.1 which can be obtained via conformal mapping.

Consider a bounded domain G in $\mathbb{C}$, bounded by a Jordan line. Given an arbitrary point ξ in G, let $z = \Phi(w)$ be the conformal mapping of the disk $D_1 = \{w \in \mathbb{C} : |w| < 1\}$ onto G such that $\Phi(0) = \xi$ and let $w = \Psi(z)$ denote the inverse mapping.

In what follows, we adopt the notation $\Delta_\xi f(z) = f(z) - f(\xi)$ and write $\Delta f(z)$ in place of $\Delta_0 f(z)$.

1.7.1 Upper estimate of $\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}$ by the supremum of $\Re\Delta_\xi f(\zeta)$ in a domain. Estimate for the first derivative

Let $f(z)$ be an analytic function in G with real part bounded from above. Then $F(w) = f(\Phi(w))$ is analytic in D_1 and has $\Re f$ bounded from above.

Given an arbitrary real valued function α in G, we introduce $\vartheta(w) = \alpha(\Phi(w))$. By Theorem 1.1, the function $F(w)$ satisfies

$$\Re\{e^{i\vartheta(w)}\Delta F(w)\} \le \frac{2|w|(1-|w|\cos\vartheta(w))}{1-|w|^2} \sup_{|w|<1} \Re\Delta F(w).$$

Hence, going back to the variable z, we arrive at a generalization of (1.4.10) for a domain G and an arbitrary fixed ξ

$$\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\} \le \frac{2|\Psi(z)|(1-|\Psi(z)|\cos\alpha(z))}{1-|\Psi(z)|^2} \sup_{\zeta\in G} \Re\Delta_\xi f(\zeta). \tag{1.7.1}$$

Putting here $\alpha(z) = -\arg\Delta_\xi f(z)$, we obtain

$$|\Delta_\xi f(z)| \le \frac{2|\Psi(z)|(1-|\Psi(z)|\cos\arg\Delta_\xi f(z))}{1-|\Psi(z)|^2} \sup_{\zeta\in G} \Re\Delta_\xi f(\zeta). \tag{1.7.2}$$

Dividing both sides of (1.7.2) by $|z-\xi|$ and using $\Psi(\xi) = 0$, we pass to the limit as $z \to \xi$ in the resulting inequality and thus arrive at the estimate

$$|f'(\xi)| \le 2|\Psi'(\xi)| \sup_{\zeta\in G} \Re\{f(\zeta) - f(\xi)\} \tag{1.7.3}$$

with the sharp constant.

1.7.2 Upper estimate of $\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}$ by the supremum of $\Re\Delta_\xi f(\zeta)$ and an estimate for the first derivative in the disk

Suppose, $G = D_R$ and $\Phi(w) = R(\xi - Rw)/(R - \overline{\xi}w)$. Then $\Psi(z) = R(\xi - z)/(R^2 - z\overline{\xi})$ and (1.7.1) becomes

$$\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\} \le \frac{2q(z,\xi)(1-q(z,\xi)\cos\alpha(z))}{1-q^2(z,\xi)} \sup_{|\zeta|<R} \Re\Delta_\xi f(\zeta), \tag{1.7.4}$$

where $q(z,\xi) = R|\xi - z|/|R^2 - z\overline{\xi}|$. The last estimate coincides with (1.4.10) for $\xi = 0$.

Note that in the case of the disk D_R, inequality (1.7.3) becomes

$$|f'(z)| \le \frac{2R}{R^2-|z|^2} \sup_{|\zeta|<R} \{\Re f(\zeta) - \Re f(z)\}, \tag{1.7.5}$$

where z is an arbitrary point of D_R.

The last estimate was previously obtained by Lindelöf [63] (see also Jensen [50], p. 24).

1.7.3 Upper estimate of $\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}$ by the supremum of $\Re\Delta_\xi f(\zeta)$ and an estimate for the first derivative in the half-plane

Consider the class of functions f analytic in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ such that $\Re f$ is bounded from above.

Given a fixed point $\xi \in \mathbb{C}_+$, we map D_1 onto $\mathbb{C}_+$ using the mapping $z = (\xi - \overline{\xi}w)/(1-w)$ whose inverse is $w = (z-\xi)/(z-\overline{\xi})$. The analogue of (1.4.10) for $\mathbb{C}_+$ is

$$\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\} \leq \frac{2s(z,\xi)(1-s(z,\xi)\cos\alpha(z))}{1-s^2(z,\xi)} \sup_{\zeta\in\mathbb{C}_+} \Re\Delta_\xi f(\zeta), \tag{1.7.6}$$

where $s(z,\xi) = |z-\xi|/|z-\overline{\xi}|$.

An immediate corollary of this inequality is the estimate of the first derivative

$$|f'(z)| \leq \frac{1}{\Im z} \sup_{\zeta\in\mathbb{C}_+} \{\Re f(\zeta) - \Re f(z)\} \tag{1.7.7}$$

of a function f, analytic in $\mathbb{C}_+$ with $\Re f$ bounded from above.

2

Estimates for analytic functions with respect to the L_p-norm of $\Re \Delta f$ on the circle

2.1 Introduction

There exist inequalities for analytic functions with the bounded real part in the disk D_R with $||\Re f - \Re f(0)||_\infty$ as a majorant. One of them, generally known as the Schwarz Arcustangens Formula, is

$$|\Re f(z) - \Re f(0)| \le \frac{4}{\pi} \arctan\left(\frac{r}{R}\right) ||\Re f - \Re f(0)||_\infty, \tag{2.1.1}$$

(see Schwarz [83], p. 190 and pp. 361-362). The following inequality

$$|\Im f(z) - \Im f(0)| \le \frac{2}{\pi} \log\left(\frac{R+r}{R-r}\right) ||\Re f - \Re f(0)||_\infty, \tag{2.1.2}$$

is due to Carathéodory and Plemelj (see Carathéodory [24], p. 21). For the proofs of these estimates see Burckel ([22], Ch. 6, § 5), Carathéodory ([25], Ch. IV, § 76), Koebe [52], Polya and Szegö ([75], III, Ch. 6, § 2).

One more known inequality is

$$|f(z) - f(0)| \le \frac{2}{\pi} \log\left(\frac{R+r}{R-r}\right) ||\Re f - \Re f(0)||_\infty \tag{2.1.3}$$

(see Burckel [22], Ch. 6, § 5).

Estimates (2.1.1)-(2.1.3) are particular cases of more general sharp estimates presented in this chapter, which is devoted to sharp pointwise estimates for

$$|\Re\{e^{i\alpha(z)}(f(z) - f(0))\}|$$

by the L_p-norm of $\Re f - \Re f(0)$ on the circle ∂D_R, where f is an analytic function on D_R with $\Re f \in h_p(D_R)$, α is a real valued function on D_R, $|z| = r < R$, and $1 \le p \le \infty$.

In Section 2.2 we prove a general but somewhat implicit representation of the best constant $\mathcal{C}_p(z, \alpha(z))$ in the inequality

$$|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}| \le \mathcal{C}_p\left(z,\alpha(z)\right)||\Re f-\Re f(0)||_p. \qquad (2.1.4)$$

Namely, we find the representation for the constant in (2.1.4)

$$\mathcal{C}_p(z,\alpha(z)) = R^{-1/p}C_p(r/R,\alpha(z)),$$

where

$$C_p(\gamma,\alpha)=\frac{\gamma}{\pi}\min_{\lambda\in\mathbb{R}}\left\{\int_{-\pi}^{\pi}\left|\frac{\cos(\varphi-\alpha)-\gamma\cos\alpha}{1-2\gamma\cos\varphi+\gamma^2}-\lambda\right|^q d\varphi\right\}^{1/q}, \qquad (2.1.5)$$

$1/q+1/p=1$.

As a corollary of (2.1.4) with $\alpha=0$, for analytic and zero-free functions f on D_R such that $f(0)=1$ and $\log|f|\in h_p(D_R)$, we deduce the two-sided estimate

$$\exp\left\{-\mathcal{C}_p(z,0)\big|\big|\log|f|\,\big|\big|_p\right\}\le\big|f(z)\big|\le\exp\left\{\mathcal{C}_p(z,0)\big|\big|\log|f|\,\big|\big|_p\right\}.$$

Section 2.3 contains the explicit formulas

$$C_1(\gamma,\alpha)=\frac{\gamma}{\pi(1-\gamma^2)}, \qquad C_2(\gamma,\alpha)=\frac{\gamma}{\sqrt{\pi(1-\gamma^2)}}.$$

In Section 2.4 we prove that the constant $C_\infty(\gamma,\alpha)$ is equal to

$$\frac{2}{\pi}\left\{\sin\alpha\ \log\frac{2\gamma\sin\alpha+\sqrt{(1-\gamma^2)^2+4\gamma^2\sin^2\alpha}}{1-\gamma^2}+\cos\alpha\ \arcsin\left(\frac{2\gamma\cos\alpha}{1+\gamma^2}\right)\right\}.$$

Inequality (2.1.4) for $p=\infty$, and the formula for $C_\infty(\gamma,\alpha)$ imply classical estimates (2.1.1)-(2.1.3).

In Section 2.5 we show that the inequality

$$|\Im f(z)-\Im f(0)|\le R^{-1/p}C_p\left(r/R,\pi/2\right)||\Re f-\Re f(0)||_p \qquad (2.1.6)$$

holds with the sharp constant defined by

$$C_p(\gamma,\pi/2)=\frac{\varkappa(\gamma)}{2\pi}\left\{2\int_{-1}^{1}\frac{(1-t^2)^{(q-1)/2}}{[1-\varkappa(\gamma)t]^q}dt\right\}^{1/q}, \qquad (2.1.7)$$

where $q=p/(p-1),\ \varkappa(\gamma)=(2\gamma)/(1+\gamma^2)$. We note also that the constant $C_p(\gamma,\pi/2)$ can be written in the form

$$C_p(\gamma,\pi/2)=\frac{\varkappa(\gamma)}{2\pi}\left\{2\left[1-\varkappa^2(\gamma)\right]^{\frac{1}{2(1-p)}}\sum_{n=0}^{\infty}B\left(\frac{2p-1}{2p-2},\frac{2n+1}{2}\right)\varkappa^{2n}(\gamma)\right\}^{\frac{p-1}{p}},$$

where $B(u,v)$ is the Beta-function. Inequality (2.1.6) with the sharp constant $C_p(\gamma,\pi/2)$ is a generalization of the classical Carathéodory-Plemelj estimate

(2.1.2) for any $p \geq 1$. For natural values of q, the value $C_p(\gamma, \pi/2)$ is expressed in elementary functions. For instance,

$$C_{4/3}(\gamma, \pi/2) = \gamma \left\{ \frac{3-\gamma^2}{4\pi^3(1-\gamma^2)^3} \right\}^{1/4},$$

and

$$C_{3/2}(\gamma, \pi/2) = \frac{1}{\pi} \left\{ \frac{\gamma(1+\gamma^2)}{(1-\gamma^2)^2} - \frac{1}{2} \log \frac{1+\gamma}{1-\gamma} \right\}^{1/3}.$$

The concluding Section 2.6 contains analogues of inequality (2.1.4) with $p = \infty$ for $|\Re\{e^{i\alpha(z)}(f(z) - f(\xi))\}|$ with sharp constants in the case of a disk and a half-plane. As corollaries we obtain sharp estimates for $|f'(z)|$ in these two domains. In particular, we prove the sharp inequality

$$|f'(z)| \leq \frac{4R}{\pi(R^2 - |z|^2)} \sup_{|\zeta|<R} |\Re f - \Re f(z)|,$$

where z is an arbitrary point in D_R. The last estimate is similar to Lindelöf's inequality (1.7.5).

2.2 Estimate of $|\Re\{e^{i\alpha}\Delta f\}|$ by the L_p-norm of $\Re\Delta f$ on the circle. General case

For real valued functions g_1 and g_2 defined on the circle $|\zeta| = R$, we set

$$(g_1, g_2) = \int_{|\zeta|=R} g_1(\zeta) g_2(\zeta) |d\zeta|$$

and by $||g||_p$ we denote the L_p-norm, $1 \leq p \leq \infty$, of g on the circle $|\zeta| = R$.

The following assertion is basic in the present chapter.

Proposition 2.1. *Let f be analytic on D_R with $\Re f \in h_p(D_R)$, $1 \leq p \leq \infty$. Further, let $\alpha(z)$ be a real valued function, $|z| < R$. Then for any fixed point $z, |z| = r < R$, there holds*

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \leq \mathcal{C}_p(z, \alpha(z)) \, ||\Re\Delta f||_p \tag{2.2.1}$$

with the sharp constant $\mathcal{C}_p(z, \alpha(z))$, where

$$\mathcal{C}_p(z, \alpha) = \frac{1}{R^{1/p}} C_p\left(\frac{r}{R}, \alpha\right), \tag{2.2.2}$$

and the factor

$$C_p(\gamma, \alpha) = \frac{\gamma}{\pi} \min_{\lambda \in \mathbb{R}} \left\{ \int_{-\pi}^{\pi} \left| \frac{\cos(\varphi - \alpha) - \gamma \cos\alpha}{1 - 2\gamma\cos\varphi + \gamma^2} - \lambda \right|^q d\varphi \right\}^{1/q}, \tag{2.2.3}$$

is even π-periodic function of α, $1/q + 1/p = 1$.

In particular,

$$|\Delta f(z)| \le \mathcal{C}_p\left(z, \arg \Delta f(z)\right) ||\Re \Delta f||_p. \tag{2.2.4}$$

Remark 2.1. The case $p = 1$ ($q = \infty$) in (2.2.3) is understood in the sense of the relation

$$||g||_\infty = \lim_{q \to \infty} ||g||_q.$$

Note also that all inequalities in the present chapter with $\Delta f(z)$ in the left-hand side and $||\Re \Delta f||_p$ in the right-hand side can be written in an equivalent form provided f is subject to the condition $f(0) = 0$. For example, inequality (2.2.1) takes the form

$$|\Re\{e^{i\alpha(z)} f(z)\}| \le \mathcal{C}_p\left(z, \alpha(z)\right) ||\Re f||_p,$$

where $f(0) = 0$.

Applying Proposition 2.1 with $\alpha(z) = 0$ to $\log f(z)$, where $f(z) \neq 0$ for $|z| < R, f(0) = 1$, and $\log |f| \in h_p(D_R)$, we obtain

Corollary 2.1. *Let f be an analytic and zero-free function on D_R with $f(0) = 1$, and let $\log |f| \in h_p(D_R)$, $1 \le p \le \infty$. Then for any z with $|z| = r < R$ there holds*

$$\big| \log |f(z)| \big| \le \mathcal{C}_p(z, 0) \big|\big| \log |f| \big|\big|_p \tag{2.2.5}$$

with the constant $\mathcal{C}_p(z, 0)$ given by (2.2.2), (2.2.3) *with $\alpha = 0$.*

We can write inequality (2.2.5) in the equivalent form

$$\exp\left\{-\mathcal{C}_p(z, 0) \big|\big| \log |f| \big|\big|_p\right\} \le \big|f(z)\big| \le \exp\left\{\mathcal{C}_p(z, 0) \big|\big| \log |f| \big|\big|_p\right\}, \tag{2.2.6}$$

which is the L_p-analogue of (1.6.8).

The following standard assertion (see, for instance, Korneichuk [54], Sect. 1.4) follows from the Hahn-Banach theorem and will be used in the proof of Proposition 2.1.

Lemma 2.1. *Let g and h be fixed elements of $L_q(\partial D_R)$, $1 \le q \le \infty$, and let the functional $F_g(\psi) = (g, \psi)$ be defined on the subspace*

$$L = \{\psi \in L_p(\partial D_R) : (h, \psi) = 0\}$$

of $L_p(\partial D_R)$, where $1/p + 1/q = 1$. Then

$$||F_g||_L = \min_{\lambda \in \mathbb{R}} ||g - \lambda h||_q. \tag{2.2.7}$$

Proof. It suffices to consider the functions h novanishing on a set of positive measure. Let λ be an arbitrary real constant. Consider the extension of F_g from L onto the whole space $L_p(\partial D_R)$ given by

$$\boldsymbol{F}_g(\psi) = (g, \psi) - \lambda(h, \psi) = (g - \lambda h, \psi). \tag{2.2.8}$$

Hence, by $||F_g||_L \leq ||\boldsymbol{F}_g||$ and due to the arbitrariness of λ

$$||F_g||_L \leq \min_{\lambda \in \mathbb{R}} \sup \big\{ \big|(g - \lambda h, \psi)\big| : ||\psi||_p \leq 1 \big\}. \tag{2.2.9}$$

We show that any linear continuous functional extending F_g from L onto $L_p(\partial D_R)$ is of the form (2.2.8). Given $u \in L_p(\partial D_R)$ with $(h, u) = 1$, we have $\varphi = \psi - (h, \psi)u \in L$ for any $\psi \in L_p(\partial D_R)$. This means that any $\psi \in L_p(\partial D_R)$ admits the representation

$$\psi = \varphi + (h, \psi)u, \tag{2.2.10}$$

where φ is a certain element of L. Let $\boldsymbol{\Phi}_g$ denote an extension of F_g from L onto $L_p(\partial D_R)$. By (2.2.10) and $\boldsymbol{\Phi}_g(\varphi) = F_g(\varphi)$ with $\varphi \in L$ we obtain

$$\begin{aligned}\boldsymbol{\Phi}_g(\psi) &= F_g(\varphi) + (h, \psi)\boldsymbol{\Phi}_g(u) = F_g\big(\psi - (h, \psi)u\big) + (h, \psi)\boldsymbol{\Phi}_g(u) \\ &= \big(g, \psi - (h, \psi)u\big) + (h, \psi)\boldsymbol{\Phi}_g(u) = \big(g, \psi\big) - \{(g, u) - \boldsymbol{\Phi}_g(u)\}\,(h, \psi),\end{aligned}$$

which proves the representation (2.2.8) for $\boldsymbol{\Phi}_g$ with the constant $\lambda = (g, u) - \boldsymbol{\Phi}_g(u)$.

Then by the Hahn-Banach theorem, there exists a constant μ such that

$$||F_g||_L = \sup \big\{ \big|(g, h) - \mu(h, \psi)\big| : ||\psi||_p \leq 1 \big\}.$$

This and (2.2.9) imply

$$||F_g||_L = \min_{\lambda \in \mathbb{R}} \sup \big\{ \big|(g - \lambda h, \psi)\big| : ||\psi||_p \leq 1 \big\},$$

which is equivalent to (2.2.7). □

Proof of Proposition 2.1. *1. Representation of the sharp constant in inequality* (2.2.1). Using (1.3.3) and notation (1.3.4), we have

$$\Re\{e^{i\alpha}\Delta f(z)\} = \frac{1}{\pi R} \int_{|\zeta|=R} G_{z,\alpha}(\zeta)\, \Re f(\zeta) |d\zeta|. \tag{2.2.11}$$

Suppose first that

$$\Re f(0) = \frac{1}{2\pi R} \int_{|\zeta|=R} \Re f(\zeta) |d\zeta| = 0. \tag{2.2.12}$$

Therefore, applying Lemma 2.1 to the functional $F_g(\psi)$ with

$$g(\zeta) = (\pi R)^{-1} G_{z,\alpha}(\zeta)$$

and $h = 1, \psi = \Re f$, we arrive at the representation

$$\mathcal{C}_p(z,\alpha) = \frac{1}{\pi R} \min_{\lambda \in \mathbb{R}} ||G_{z,\alpha} - \lambda||_q \tag{2.2.13}$$

for the sharp constant $\mathcal{C}_p(z,\alpha)$ in

$$|\Re\{e^{i\alpha}\Delta f(z)\}| \le \mathcal{C}_p\,(z,\alpha)\,||\Re f||_p. \tag{2.2.14}$$

Setting $f(z) - \Re f(0)$ instead of $f(z)$ in (2.2.14), we conclude that (2.2.1) holds with the sharp constant (2.2.13).

Now, suppose $1 < p \le \infty$. Combining (2.2.13) with (1.3.4) and (1.3.5) we have

$$\mathcal{C}_p(z,\alpha) = \frac{1}{\pi R} \min_{\lambda \in \mathbb{R}} \left\{ \int_{-\pi+\tau}^{\pi+\tau} \left| \Re\left(\frac{\gamma e^{i\alpha}}{e^{i(t-\tau)} - \gamma} \right) - \lambda \right|^q R dt \right\}^{1/q},$$

which after the change of variable $\varphi = t - \tau$ becomes

$$\mathcal{C}_p(z,\alpha) = \frac{1}{\pi R^{1/p}} \min_{\lambda \in \mathbb{R}} \left\{ \int_{-\pi}^{\pi} \left| \Re\left(\frac{\gamma e^{i\alpha}}{e^{i\varphi} - \gamma} \right) - \lambda \right|^q d\varphi \right\}^{1/q}. \tag{2.2.15}$$

Using the notation

$$C_p(\gamma,\alpha) = \frac{1}{\pi} \min_{\lambda \in \mathbb{R}} \left\{ \int_{-\pi}^{\pi} \left| \Re\left(\frac{\gamma e^{i\alpha}}{e^{i\varphi} - \gamma} \right) - \lambda \right|^q d\varphi \right\}^{1/q}, \tag{2.2.16}$$

we rewrite (2.2.15) as

$$\mathcal{C}_p(z,\alpha) = \frac{1}{R^{1/p}} C_p\left(\frac{r}{R},\alpha\right), \tag{2.2.17}$$

which together with (1.3.6) proves (2.2.2) and (2.2.3) for $1 < p \le \infty$.

The case $p = 1$ ($q = \infty$) in (2.2.3) is handled by passage to the limit.

2. Properties of $C_p(\gamma,\alpha)$ *and inequality* (2.2.4). We show that $C_p(\gamma,-\alpha) = C_p(\gamma,\alpha)$. Let $1 < p \le \infty$. By (2.2.16),

$$C_p(\gamma,\alpha) = \frac{\gamma}{\pi} \min_{\lambda \in \mathbb{R}} \left\{ \int_0^{2\pi} \left| \Re\left(\frac{e^{-i\alpha}}{e^{-i\varphi} - \gamma} \right) - \lambda \right|^q d\varphi \right\}^{1/q},$$

which after the change of variable $\varphi = 2\pi - \psi$ becomes

$$C_p(\gamma,\alpha) = \frac{\gamma}{\pi} \min_{\lambda \in \mathbb{R}} \left\{ \int_0^{2\pi} \left| \Re\left(\frac{e^{-i\alpha}}{e^{i\psi} - \gamma} \right) - \lambda \right|^q d\psi \right\}^{1/q} = C_p(\gamma,-\alpha).$$

The equality $C_p(\gamma, \alpha + \pi) = C_p(\gamma, \alpha)$ follows also from (2.2.16). Obviously, $C_p(\gamma, \alpha)$ remains even and π-periodic in the limiting case $p = 1$ as well.

Putting $\alpha(z) = -\arg \Delta f(z)$ in (2.2.1) and taking into account that $C_p(\gamma, \alpha)$ is even in α, we arrive at (2.2.4). □

The next two subsections contain explicit formulas for the sharp constant $\mathcal{C}_p(z, \alpha)$ in inequality (2.2.1) with $p = 1, 2, \infty$. As corollaries, we obtain sharp estimates for the real and imaginary parts of an analytic function in the disk $|z| < R$, as well as for its modulus, in terms of its real part on the circle $|z| = R$.

2.3 The cases $p = 1$ and $p = 2$

The next assertion specifies Proposition 2.1 for $p = 1$.

Corollary 2.1. *Let f be analytic on D_R with $\Re f \in h_1(D_R)$. Further, let $\alpha(z)$ be a real valued function, $|z| < R$. Then for any fixed point $z, |z| = r < R$, the sharp inequality*

$$|\Re\{e^{i\alpha(z)} \Delta f(z)\}| \le \frac{r}{\pi(R^2 - r^2)} ||\Re \Delta f||_1 \tag{2.3.1}$$

holds.

Proof. By (2.2.13),

$$\mathcal{C}_1(z, \alpha) = \frac{1}{\pi R} \min_{\lambda \in \mathbb{R}} \max_{|\zeta|=R} |G_{z,\alpha}(\zeta) - \lambda| \,. \tag{2.3.2}$$

Since λ is subject to one of the three alternatives

$$\lambda \le \min_{|\zeta|=R} G_{z,\alpha}(\zeta), \quad \min_{|\zeta|=R} G_{z,\alpha}(\zeta) < \lambda < \max_{|\zeta|=R} G_{z,\alpha}(\zeta), \quad \lambda \ge \max_{|\zeta|=R} G_{z,\alpha}(\zeta),$$

it follows that the minimum with respect to λ in (2.3.2) is attained at

$$\lambda = \frac{1}{2} \left\{ \min_{|\zeta|=R} G_{z,\alpha}(\zeta) + \max_{|\zeta|=R} G_{z,\alpha}(\zeta) \right\},$$

which by 1.3.4 and Lemma 1.2 implies

$$\lambda = \frac{r^2 \cos \alpha}{R^2 - r^2}.$$

Putting the value of λ into (2.3.2) and using Lemma 1.2 we obtain

$$\mathcal{C}_1(z, \alpha) = \frac{r}{\pi(R^2 - r^2)}, \tag{2.3.3}$$

which proves (2.3.1). □

Corollary 2.1 with $p = 1$ and formula (2.3.3) imply

Corollary 2.2. *If f is an analytic and zero-free function on D_R with $f(0) = 1$, and $\log|f| \in h_1(D_R)$, then for any z with $|z| = r < R$ the inequality*

$$\big|\log|f(z)|\big| \le \frac{r}{\pi(R^2 - r^2)}\big|\big|\log|f|\big|\big|_1 \tag{2.3.4}$$

holds.

Note, that the inclusion $\log|f| \in h_1(D_R)$ holds for $f \in H_p(D_R), 1 \le p \le \infty$ (see, for example, Koosis [53]).

The next assertion specifies Proposition 2.1 for $p = 2$.

Corollary 2.3. *Let f be analytic on D_R with $\Re f \in h_2(D_R)$. Further, let $\alpha(z)$ be a real valued function, $|z| < R$. Then for any fixed point $z, |z| = r < R$, the sharp inequality*

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \le \frac{r}{\sqrt{\pi R(R^2 - r^2)}}||\Re\Delta f||_2 \tag{2.3.5}$$

holds.

Proof. Combining

$$\int_{|\zeta|=R} G_{z,\alpha}(z)|d\zeta| = \Re\left\{\int_{|\zeta|=R}\frac{e^{i\alpha}z}{\zeta - z}|d\zeta|\right\} = \Re\left\{\int_{|\zeta|=R}\frac{Re^{i\alpha}z}{i(\zeta - z)\zeta}d\zeta\right\} = 0$$

with (2.2.13) for $p = 2$, we have

$$\mathcal{C}_2(z, \alpha) = \frac{1}{\pi R}||G_{z,\alpha}||_2. \tag{2.3.6}$$

Let us calculate $||G_{z,\alpha}||_2$. Using (1.3.5) and setting $\varphi = t - \tau$ we obtain

$$\begin{aligned} ||G_{z,\alpha}||_2^2 &= \int_{|\zeta|=R}\left[\Re\left(\frac{e^{i\alpha}z}{\zeta - z}\right)\right]^2|d\zeta| = \gamma^2\int_{-\pi+\tau}^{\pi+\tau}\left[\Re\left(\frac{e^{i\alpha}}{e^{i(t-\tau)} - \gamma}\right)\right]^2 Rdt \\ &= \frac{r^2}{R}\int_{-\pi}^{\pi}\frac{(\cos(\varphi - \alpha) - \gamma\cos\alpha)^2}{(1 - 2\gamma\cos\varphi + \gamma^2)^2}d\varphi, \end{aligned} \tag{2.3.7}$$

and making elementary calculations, we arrive at

$$\int_{-\pi}^{\pi}\frac{(\cos(\varphi - \alpha) - \gamma\cos\alpha)^2}{(1 - 2\gamma\cos\varphi + \gamma^2)^2}d\varphi = \frac{\pi}{1 - \gamma^2},$$

which together with (2.3.7) gives

$$||G_{z,\alpha}||_2^2 = \frac{\pi r^2 R}{R^2 - r^2}. \tag{2.3.8}$$

Hence and by (2.3.6) we conclude

$$\mathcal{C}_2(z, \alpha) = \frac{r}{\sqrt{\pi R(R^2 - r^2)}}. \tag{2.3.9}$$

□

The following assertion results directly from Corollary 2.1 with $p = 2$ and formula (2.3.9).

Corollary 2.4. *If f is an analytic and zero-free function on D_R with $f(0) = 1$, and $\log|f| \in h_2(D_R)$, then for any z with $|z| = r < R$ the inequality*

$$\big|\log|f(z)|\big| \le \frac{r}{\sqrt{\pi R(R^2 - r^2)}} \big|\big|\log|f|\big|\big|_2 \tag{2.3.10}$$

holds.

2.4 The case $p = \infty$

The next assertion gives a sharp constant in (2.2.3) for $p = \infty$.

Theorem 2.1. *Let f be analytic on D_R with bounded $\Re f$. Further, let $\alpha(z)$ be a real valued function, $|z| < R$. Then for any fixed point $z, |z| = r < R$, the estimate*

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \le C_\infty\left(\frac{r}{R}, \alpha(z)\right) ||\Re\Delta f||_\infty \tag{2.4.1}$$

holds with the sharp constant

$$C_\infty(\gamma, \alpha) = \frac{2}{\pi}\left\{\sin\alpha \log\frac{2\gamma\sin\alpha\sqrt{(1-\gamma^2)^2 + 4\gamma^2\sin^2\alpha}}{1-\gamma^2}\right. \\ \left. + \cos\alpha \arcsin\left(\frac{2\gamma\cos\alpha}{1+\gamma^2}\right)\right\}. \tag{2.4.2}$$

Proof. The representation for the sharp constant in (2.4.1)

$$C_\infty(\gamma, \alpha) = \frac{\gamma}{\pi}\min_{\lambda\in\mathbb{R}}\int_{-\pi}^{\pi}\left|\Re\left(\frac{e^{i\alpha}}{e^{i\varphi} - \gamma}\right) - \lambda\right| d\varphi \tag{2.4.3}$$

results by (2.2.16) and (2.2.17) with $q = 1$ ($p = \infty$).

1. Solution of the extremal problem in (2.4.3). Let the function α and the point z with $|z| = r < R$ be fixed. Suppose that $\lambda = \lambda_0$ is a solution of the equation

$$\int_{-\pi}^{\pi} \operatorname{sign}\left\{\Re\left(\frac{e^{i\alpha}}{e^{i\varphi}-\gamma}\right)-\lambda\right\} d\varphi = 0. \tag{2.4.4}$$

Let

$$g(\varphi) = \Re\left(\frac{e^{i\alpha}}{e^{i\varphi}-\gamma}\right) - \lambda_0. \tag{2.4.5}$$

Then, for any $\mu \in \mathbb{R}$,

$$\int_{-\pi}^{\pi} |g(\varphi)| d\varphi = \int_{-\pi}^{\pi} \Big(g(\varphi)-\mu\Big) \operatorname{sign} g(\varphi) d\varphi \le \int_{-\pi}^{\pi} |g(\varphi)-\mu| d\varphi,$$

which together with (2.4.3) and (2.4.5) leads to

$$C_\infty(\gamma,\alpha) = \frac{\gamma}{\pi}\int_{-\pi}^{\pi} \left|\Re\left(\frac{e^{i\alpha}}{e^{i\varphi}-\gamma}\right)-\lambda_0\right| d\varphi. \tag{2.4.6}$$

We show now that (2.4.4) holds with $\lambda = -\gamma(1+\gamma^2)^{-1}\cos\alpha$, where $\gamma \in [0,1)$. We rewrite the left-hand side of the equation

$$\Re\left(\frac{e^{i\alpha}}{e^{i\varphi}-\gamma}\right) + \frac{\gamma\cos\alpha}{1+\gamma^2} = 0 \tag{2.4.7}$$

as

$$\Re\left(\frac{e^{i\alpha}}{e^{i\varphi}-\gamma}\right) + \frac{\gamma\cos\alpha}{1+\gamma^2} = \frac{1}{1+\gamma^2}\cdot\frac{(1-\gamma^2)\cos\varphi\cos\alpha + (1+\gamma^2)\sin\varphi\sin\alpha}{1-2\gamma\cos\varphi+\gamma^2}. \tag{2.4.8}$$

Let ϑ be the solution of the system

$$\cos\vartheta = \frac{(1-\gamma^2)}{k(\alpha,\gamma)}\cos\alpha, \qquad \sin\vartheta = \frac{(1+\gamma^2)}{k(\alpha,\gamma)}\sin\alpha, \tag{2.4.9}$$

where

$$k(\alpha,\gamma) = [(1-\gamma^2)^2\cos^2\alpha + (1+\gamma^2)^2\sin^2\alpha]^{1/2}. \tag{2.4.10}$$

From (2.4.8), (2.4.9) we obtain

$$\Re\left(\frac{e^{i\alpha}}{e^{i\varphi}-\gamma}\right) + \frac{\gamma\cos\alpha}{1+\gamma^2} = \frac{k(\alpha,\gamma)}{1+\gamma^2}\cdot\frac{\cos(\varphi-\vartheta)}{1-2\gamma\cos\varphi+\gamma^2}. \tag{2.4.11}$$

Thus, the equation (2.4.7) with unknown φ is reduced to $\cos(\varphi-\vartheta)=0$.

The distance between two successive roots $\varphi_n = \vartheta - \pi/2 + \pi n,\ n = 0, \pm 1, \pm 2, \dots,$ of the equation $\cos(\varphi-\vartheta)=0$ is equal to π. We put $\zeta_0 = e^{i\varphi_0},\ \zeta_1 = e^{i\varphi_1}$ with $\varphi_0 = \vartheta - \pi/2,\ \varphi_1 = \vartheta + \pi/2$. Then

$$\Re\left(\frac{e^{i\alpha}}{\zeta_0 - \gamma}\right) + \frac{\gamma\cos\alpha}{1+\gamma^2} = \Re\left(\frac{e^{i\alpha}}{\zeta_1 - \gamma}\right) + \frac{\gamma\cos\alpha}{1+\gamma^2} = 0.$$

Thus, for fixed $\gamma \in [0, 1)$ and α, the points ζ_0 and ζ_1 divide the circle $|\zeta| = 1$ into two half-circles such that the left-hand side of (2.4.7) is positive on one of them and negative on another one. Hence (2.4.4) holds with $\lambda = -\gamma(1+\gamma^2)^{-1}\cos\alpha$ and, therefore, by (2.4.6),

$$C_\infty(\gamma, \alpha) = \frac{\gamma}{\pi}\int_{-\pi}^{\pi}\left|\Re\left(\frac{e^{i\alpha}}{e^{\varphi} - \gamma}\right) + \frac{\gamma}{1+\gamma^2}\cos\alpha\right| d\varphi. \tag{2.4.12}$$

2. Calculation of $C_\infty(\gamma, \alpha)$ *by* (2.4.12). The last equality and (2.4.11) imply

$$C_\infty(\gamma, \alpha) = \frac{\gamma k(\alpha, \gamma)}{\pi(1+\gamma^2)}\int_{-\pi}^{\pi}\frac{|\cos(\varphi - \vartheta)|}{1 - 2\gamma\cos\varphi + \gamma^2}d\varphi, \tag{2.4.13}$$

where $k(\alpha, \gamma)$ is defined by (2.4.10) and ϑ is the solution of (2.4.9) in $(-\pi, \pi]$.

We denote, for brevity

$$\mathcal{J} = \int_{-\pi}^{\pi}\frac{|\cos(\varphi - \vartheta)|}{1 - 2\gamma\cos\varphi + \gamma^2}d\varphi. \tag{2.4.14}$$

Equality (2.4.14) can be written as

$$\mathcal{J} = \int_{\vartheta-\pi/2}^{\vartheta+\pi/2}\frac{\cos(\varphi - \vartheta)}{1 - 2\gamma\cos\varphi + \gamma^2}d\varphi - \int_{\vartheta+\pi/2}^{\vartheta+3\pi/2}\frac{\cos(\varphi - \vartheta)}{1 - 2\gamma\cos\varphi + \gamma^2}d\varphi.$$

In the first integral we make the change of variable $\psi = -\varphi$ and in the second integral we put $\eta = \pi - \varphi$. Then

$$\mathcal{J} = \int_{-\pi/2-\vartheta}^{\pi/2-\vartheta}\frac{\cos(\psi + \vartheta)}{1 - 2\gamma\cos\psi + \gamma^2}d\psi + \int_{-\pi/2-\vartheta}^{\pi/2-\vartheta}\frac{\cos(\eta + \vartheta)}{1 + 2\gamma\cos\eta + \gamma^2}d\eta,$$

which implies

$$\mathcal{J} = 2(1+\gamma^2)\int_{-\pi/2-\vartheta}^{\pi/2-\vartheta}\frac{\cos\psi\cos\vartheta - \sin\psi\sin\vartheta}{(1+\gamma^2)^2 - 4\gamma^2\cos^2\psi}d\psi. \tag{2.4.15}$$

Substituting the integrals

$$\int_{-\pi/2-\vartheta}^{\pi/2-\vartheta}\frac{\cos\psi}{(1+\gamma^2)^2 - 4\gamma^2\cos^2\psi}d\psi = \frac{1}{\gamma(1-\gamma^2)}\arctan\left(\frac{2\gamma\cos\vartheta}{1-\gamma^2}\right),$$

$$\int_{-\pi/2-\vartheta}^{\pi/2-\vartheta}\frac{\sin\psi}{(1+\gamma^2)^2 - 4\gamma^2\cos^2\psi}d\psi = -\frac{1}{2\gamma(1+\gamma^2)}\log\frac{1+\gamma^2+2\gamma\sin\vartheta}{1+\gamma^2-2\gamma\sin\vartheta}$$

into (2.4.15) we obtain

$$\mathcal{J} = \frac{2(1+\gamma^2)}{\gamma}\left\{\frac{\cos\vartheta}{1-\gamma^2}\arctan\frac{2\gamma\cos\vartheta}{1-\gamma^2} + \frac{\sin\vartheta}{2(1+\gamma^2)}\log\frac{1+\gamma^2+2\sin\vartheta}{1+\gamma^2-2\gamma\sin\vartheta}\right\}.$$

Hence, taking into account (2.4.9), (2.4.10) and (2.4.13), (2.4.14), as well as the identity $\arctan[x(1-x^2)^{-1/2}] = \arcsin x$, we arrive at formula (2.4.2). □

Remark 2.1. Using the equality

$$\sup_{|\zeta|<R} |g(\zeta)| = ||g||_\infty,$$

which is valid for bounded harmonic functions g in D_R, we can replace $||\Re\Delta f||_\infty$ in (2.4.1) and its corollaries by

$$\sup_{|\zeta|<R} |\Re f(\zeta) - \Re f(0)|.$$

The next assertion contains particular cases of (2.4.1), well known estimates (2.1.1)-(2.1.3) for $|\Re\Delta f(z)|, |\Im\Delta f(z)|$ and $|\Delta f(z)|$ by $||\Re\Delta f||_\infty$.

Corollary 2.4. *Let f be analytic on D_R with bounded $\Re f$. Then for any fixed point $z, |z| = r < R$, the sharp inequalities*

$$|\Re\Delta f(z)| \le \frac{4}{\pi}\arctan\left(\frac{r}{R}\right)||\Re\Delta f||_\infty, \tag{2.4.16}$$

$$|\Im\Delta f(z)| \le \frac{2}{\pi}\log\frac{R+r}{R-r}||\Re\Delta f||_\infty, \tag{2.4.17}$$

$$|\Delta f(z)| \le \frac{2}{\pi}\log\frac{R+r}{R-r}||\Re\Delta f||_\infty \tag{2.4.18}$$

hold.

Proof. 1. Inequalities for $|\Re\Delta f(z)|$. Estimate (2.4.16) follows from (2.4.1) and (2.4.2) with $\alpha = 0$. In fact, by (2.4.2),

$$C_\infty(\gamma, 0) = \frac{2}{\pi}\arcsin\left(\frac{2\gamma}{1+\gamma^2}\right) = \frac{4}{\pi}\arctan\gamma. \tag{2.4.19}$$

2. Inequality for $|\Im\Delta f(z)|$. Inequality (2.4.17) stems from (2.4.1) and (2.4.2) with $\alpha = \pi/2$. In fact, by (2.4.2),

$$C_\infty(\gamma, \pi/2) = \frac{2}{\pi}\log\frac{1+\gamma}{1-\gamma}.$$

3. Inequality for $|\Delta f(z)|$. Since $C_\infty(\gamma, \alpha)$ is an even π-periodic function in α, it follows that

$$\max\{C_\infty(\gamma,\alpha) : -\pi \le \alpha \le \pi\} = \max\{C_\infty(\gamma,\alpha) : 0 \le \alpha \le \pi/2\}. \quad (2.4.20)$$

We show that $C_\infty(\gamma,\alpha)$ is an increasing function of α on $[0,\pi/2]$ and hence

$$\max\{C_\infty(\gamma,\alpha) : 0 \le \alpha \le \pi/2\} = C_\infty(\gamma,\pi/2) = \frac{2}{\pi}\log\frac{1+\gamma}{1-\gamma}. \quad (2.4.21)$$

Let us consider $C_\infty(\gamma,\alpha)$ for $0 \le \alpha \le \pi/2$. By (2.4.2) we have

$$\frac{\partial C_\infty(\gamma,\alpha)}{\partial\alpha} = \frac{2}{\pi}\left\{\cos\alpha\log\frac{2\gamma\sin\alpha + \sqrt{(1-\gamma^2)^2 + 4\gamma^2\sin^2\alpha}}{1-\gamma^2}\right. \quad (2.4.22)$$

$$\left. - \sin\alpha\arcsin\left(\frac{2\gamma\cos\alpha}{1+\gamma^2}\right)\right\}.$$

Note that the relations

$$\log\frac{2\gamma\sin\alpha + \sqrt{(1-\gamma^2)^2 + 4\gamma^2\sin^2\alpha}}{1-\gamma^2} = \int_0^{2\gamma\sin\alpha}\frac{dt}{\sqrt{(1-\gamma^2)^2 + t^2}},$$

$$\arcsin\left(\frac{2\gamma\cos\alpha}{1+\gamma^2}\right) = \int_0^{2\gamma(1+\gamma^2)^{-1}\cos\alpha}\frac{dt}{\sqrt{1-t^2}},$$

and the mean value theorem imply

$$\cos\alpha\int_0^{2\gamma\sin\alpha}\frac{dt}{\sqrt{(1-\gamma^2)^2+t^2}} > \frac{2\gamma\cos\alpha\sin\alpha}{[(1-\gamma^2)^2 + 4\gamma^2\sin^2\alpha]^{1/2}},$$

$$\sin\alpha\int_0^{2\gamma(1+\gamma^2)^{-1}\cos\alpha}\frac{dt}{\sqrt{1-t^2}} < \frac{2\gamma\cos\alpha\sin\alpha}{[(1-\gamma^2)^2 + 4\gamma^2\sin^2\alpha]^{1/2}},$$

where $\alpha \in (0,\pi/2)$. Therefore, it follows from (2.4.22) that

$$\frac{\partial C_\infty(\gamma,\alpha)}{\partial\alpha} > 0.$$

Thus, $C_\infty(\gamma,\alpha)$ increases on the interval $[0,\pi/2]$ and by (2.4.1), (2.4.20) and (2.4.21) we arrive at (2.4.18). □

As a corollary, we give an estimate for $\big|\log|f(z)|\big|$ in terms of $\big|\big|\log|f|\big|\big|_\infty$.

Corollary 2.5. *If f is an analytic and zero-free function on D_R with $f(0) = 1$, and $\log|f| \in h_\infty(D_R)$, then for any z with $|z| = r < R$ the inequality*

$$\big|\log|f(z)|\big| \le \frac{4}{\pi}\arctan\left(\frac{r}{R}\right)\big|\big|\log|f|\big|\big|_\infty \quad (2.4.23)$$

holds.

Proof. Estimate (2.4.23) results from (2.2.5) combined with (2.4.19) and (2.2.2) with $p = \infty$. □

2.5 Generalization of the Carathéodory and Plemelj inequality

2.5.1. The general case $p \in [1, \infty]$

The next assertion contains a sharp inequality which is a generalization for $1 \leq p < \infty$ of the classical estimate

$$|\Im f(z) - \Im f(0)| \leq \frac{2}{\pi} \log\left(\frac{R+r}{R-r}\right) ||\Re f - \Re f(0)||_\infty$$

due to Carathéodory and Plemelj (see Carathéodory [24], p.21, Burckel [22], Ch. 5, § 3 and Notes to Ch. 5).

Corollary 2.6. *Let f be analytic on D_R with $\Re f \in h_p(D_R)$, $1 \leq p \leq \infty$. Then for any fixed $z, |z| = r < R$, there holds*

$$|\Im \Delta f(z)| \leq R^{-1/p} C_{\Im,p}(r/R) ||\Re \Delta f||_p \tag{2.5.1}$$

with the sharp constant

$$C_{\Im,1}(\gamma) = \frac{\gamma}{\pi(1-\gamma^2)}, \tag{2.5.2}$$

and

$$C_{\Im,p}(\gamma) = \frac{\varkappa(\gamma)}{2\pi}\left\{2\int_{-1}^{1} \frac{(1-t^2)^{(q-1)/2}}{[1-\varkappa(\gamma)t]^q} dt\right\}^{1/q} \tag{2.5.3}$$

$$= \frac{\varkappa(\gamma)}{2\pi}\left\{2\left[1-\varkappa^2(\gamma)\right]^{1/(2-2p)} \sum_{n=0}^{\infty} B\left(\frac{2p-1}{2p-2}, \frac{2n+1}{2}\right) \varkappa^{2n}(\gamma)\right\}^{(p-1)/p},$$

for $1 < p \leq \infty$, where $\varkappa(\gamma) = (2\gamma)/(1+\gamma^2)$ and $B(u, v)$ is the Beta-function.

In particular,

$$C_{\Im,2}(\gamma) = \frac{\gamma}{\sqrt{\pi(1-\gamma^2)}}. \tag{2.5.4}$$

Proof. By Proposition 2.1, inequality (2.5.1) holds with the sharp constant

$$C_{\Im,p}(\gamma) = C_p(\gamma, \pi/2), \tag{2.5.5}$$

where

$$C_p(\gamma, \pi/2) = \frac{\gamma}{\pi} \min_{\lambda \in \mathbb{R}} \left\{\int_{-\pi}^{\pi} \left|\frac{\sin\varphi}{1-2\gamma\cos\varphi+\gamma^2} - \lambda\right|^q d\varphi\right\}^{1/q}. \tag{2.5.6}$$

The formulae (2.5.2), (2.5.4) were obtained in Corollaries 2.1 and 2.3, respectively.

Let $1 < p \le \infty$ $(1 \le q < \infty)$. We fix $\gamma \in [0, 1)$ and put

$$g(\varphi) = \frac{\sin\varphi}{1 - 2\gamma\cos\varphi + \gamma^2}. \tag{2.5.7}$$

Taking into account the equality

$$\int_{-\pi}^{\pi} |g(\varphi)|^{q-1} \text{sign } g(\varphi) d\varphi = 0,$$

which holds since g is odd, for any $\mu \in \mathbb{R}$ we have

$$||g||_q^q = \int_{-\pi}^{\pi} |g(\varphi)|^q d\varphi = \int_{-\pi}^{\pi} \Big(g(\varphi) - \mu\Big)|g(\varphi)|^{q-1} \text{sign } g(\varphi) d\varphi.$$

This implies

$$||g||_q^q \le \int_{-\pi}^{\pi} \big|g(\varphi) - \mu\big||g(\varphi)|^{q-1} d\varphi. \tag{2.5.8}$$

Further, by Hölder's inequality

$$\int_{-\pi}^{\pi} \big|g(\varphi) - \mu\big||g(\varphi)|^{q-1} d\varphi \le ||g - \mu||_q ||g||_q^{q-1}.$$

Hence, by (2.5.8),

$$||g||_q \le ||g - \mu||_q.$$

Combining this with (2.5.6) and (2.5.7), we arrive at

$$C_p(\gamma, \pi/2) = \frac{\gamma}{\pi}\left\{\int_{-\pi}^{\pi}\left(\frac{\sin\varphi}{1 - 2\gamma\cos\varphi + \gamma^2}\right)^q d\varphi\right\}^{1/q},$$

which in view of (2.5.5), results at

$$C_{\Im,p}(\gamma) = \frac{\gamma}{\pi}\left\{2\int_{0}^{\pi}\left(\frac{\sin\varphi}{1 - 2\gamma\cos\varphi + \gamma^2}\right)^q d\varphi\right\}^{1/q}. \tag{2.5.9}$$

Making the change of variable $t = \cos\varphi$ and setting $\varkappa(\gamma) = (2\gamma)/(1+\gamma^2)$, we arrive at the first equality for the constant $\mathcal{C}_{\Im,p}(\gamma)$ in (2.5.3).

We shall write the sharp constant $\mathcal{C}_{\Im,p}(\gamma)$ in (2.5.1) in a different form. Using the equality (see, for example, Gradshtein and Ryzhik [41], **3.665**)

$$\int_{0}^{\pi}\left(\frac{\sin\varphi}{1 - 2\gamma\cos\varphi + \gamma^2}\right)^q d\varphi = B\left(\frac{q+1}{2}, \frac{1}{2}\right) F\left(q, \frac{q}{2}; \frac{q+2}{2}; \gamma^2\right),$$

where $F(a, b; c; x)$ is the hypergeometric Gauss function, and the relation

$$F(a,b;a-b+1;x) = (1-x)^{1-2b}(1+x)^{2b-a-1}F\left(\frac{a+1}{2}-b,\frac{a}{2}+1-b;a-b+1;\frac{4x}{(1+x)^2}\right),$$

we conclude by (2.5.9) that $C_{\Im,p}(\gamma)$ is equal to

$$\frac{\varkappa(\gamma)}{2\pi}\left\{2\left[1-\varkappa^2(\gamma)\right]^{(1-q)/2} B\left(\frac{q+1}{2},\frac{1}{2}\right) F\left(\frac{1}{2},1;\frac{q+2}{2};\varkappa^2(\gamma)\right)\right\}^{1/q}$$

$$=\frac{\varkappa(\gamma)}{2\pi}\left\{2\left[1-\varkappa^2(\gamma)\right]^{1/(2-2p)}\sum_{n=0}^{\infty} B\left(\frac{2p-1}{2p-2},\frac{2n+1}{2}\right)\varkappa^{2n}(\gamma)\right\}^{(p-1)/p},$$

that is, we arrive at the second equality for $\mathcal{C}_{\Im,p}(\gamma)$ in (2.5.3). □

The integral

$$\mathcal{I}_q(\varkappa) = \int_{-1}^{1}\frac{(1-t^2)^{(q-1)/2}}{(1-\varkappa t)^q}dt$$

in (2.5.3) is the sum of each of two series

$$\sum_{m=0}^{\infty}(-1)^m\binom{(q-1)/2}{m}\int_{-1}^{1}\frac{t^{2m}}{(1-\varkappa t)^q}dt$$

and

$$\sum_{m=0}^{\infty}(-1)^m\binom{q/2}{m}\int_{-1}^{1}\frac{t^{2m}}{(1-\varkappa t)^q(1-t^2)^{1/2}}dt.$$

The first of these series becomes a finite sum for odd q and the second one for even q.

2.5.2. The case of odd q

For odd q, the recurrence relation

$$\mathcal{I}_{2n+1}(\varkappa) = \frac{2(2n-2)!!}{(2n-1)!!}\frac{1}{\varkappa^2(1-\varkappa^2)^n} - \frac{1}{\varkappa^2}\mathcal{I}_{2n-1}(\varkappa)$$

with

$$\mathcal{I}_1(\varkappa) = \frac{1}{\varkappa}\log\frac{1+\varkappa}{1-\varkappa}$$

implies

$$\mathcal{I}_{2n+1}(\varkappa) = \frac{2}{\varkappa^{2n+2}}\sum_{k=1}^{n}\frac{(-1)^{n+k}(2k-2)!!}{(2k-1)!!}\left(\frac{\varkappa^2}{1-\varkappa^2}\right)^k + \frac{(-1)^n}{\varkappa^{2n+1}}\log\frac{1+\varkappa}{1-\varkappa}.$$

Hence, putting $\varkappa = (2\gamma)/(1+\gamma^2)$ in the last equality and taking into account (2.5.3), we find

$$C_{\Im,\,\frac{2n+1}{2n}}(\gamma) = \frac{1}{2\pi}\left\{4(-1)^n \log\frac{1+\gamma}{1-\gamma}\right.$$
$$\left. + \frac{2(1+\gamma^2)}{\gamma}\sum_{k=1}^{n}\frac{(-1)^{n+k}(2k-2)!!}{(2k-1)!!}\left(\frac{2\gamma}{1-\gamma^2}\right)^{2k}\right\}^{\frac{1}{2n+1}}.$$

For example,

$$C_{\Im,\,3/2}(\gamma) = \frac{1}{\pi}\left\{\frac{\gamma(1+\gamma^2)}{(1-\gamma^2)^2} - \frac{1}{2}\log\frac{1+\gamma}{1-\gamma}\right\}^{1/3}.$$

2.5.3. The case of even q

For even q, the recurrence relation

$$\mathcal{I}_{2n+2}(\varkappa) = \frac{\pi(2n-1)!!}{(2n)!!}\frac{1}{\varkappa^2(1-\varkappa^2)^{(2n+1)/2}} - \frac{1}{\varkappa^2}\mathcal{I}_{2n}(\varkappa)$$

with

$$\mathcal{I}_2(\varkappa) = \frac{\pi\left(1-\sqrt{1-\varkappa^2}\right)}{\varkappa^2\sqrt{1-\varkappa^2}}$$

leads to

$$\mathcal{I}_{2n+2}(\varkappa) = \frac{\pi}{\varkappa^{2n+3}}\sum_{k=1}^{n}\frac{(-1)^{n+k}(2k-1)!!}{(2k)!!}\left(\frac{\varkappa^2}{1-\varkappa^2}\right)^{(2k+1)/2}$$
$$+\frac{\pi(-1)^n}{\varkappa^{2n+2}}\frac{\left(1-\sqrt{1-\varkappa^2}\right)}{\sqrt{1-\varkappa^2}}.$$

Hence, putting $\varkappa = (2\gamma)/(1+\gamma^2)$ in the last equality and using (2.5.3), we obtain

$$C_{\Im,\,\frac{2n+2}{2n+1}}(\gamma) = \frac{1}{2\pi}\left\{\frac{4(-1)^n\pi\gamma^2}{1-\gamma^2}\right.$$
$$\left. + \frac{\pi(1+\gamma^2)}{\gamma}\sum_{k=1}^{n}\frac{(-1)^{n+k}(2k-1)!!}{(2k)!!}\left(\frac{2\gamma}{1-\gamma^2}\right)^{2k+1}\right\}^{\frac{1}{2n+2}}.$$

In particular,

$$C_{\Im,\,4/3}(\gamma) = \gamma\left\{\frac{3-\gamma^2}{4\pi^3(1-\gamma^2)^3}\right\}^{1/4}.$$

2.6 Variants and extensions

Here we collect some estimates which result from Theorem 2.1 by conformal mappings.

Similar to Section 1.7, we assume that G is a bounded domain in $\mathbb{C}$, bounded by a Jordan curve. By ξ we denote an arbitrary fixed point of G. Let $z = \Phi(w)$ be a conformal mapping of $D_1 = \{w \in \mathbb{C} : |w| < 1\}$ onto G such that $\Phi(0) = \xi$ and let $w = \Psi(z)$ stand for the inverse mapping.

We preserve the notation $\Delta_\xi f(z) = f(z) - f(\xi)$ introduced in Chapter 1 and write $\Delta f(z)$ in place of $\Delta_0 f(z)$.

2.6.1 Estimate of $|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}|$ by the supremum of $|\Re\Delta_\xi f(\zeta)|$ in a domain. Estimate for the first derivative

Let $f(z)$ be an analytic function in G with bounded $\Re f$. Then $F(w) = f(\Phi(w))$ is an analytic function in D_1 with bounded $\Re F$.

Given an arbitrary real valued function α in G, we introduce $\vartheta(w) = \alpha(\Phi(w))$. By Theorem 2.1 we have

$$|\Re\{e^{i\vartheta(w)}\Delta F(w)\}| \le C_\infty\left(|w|,\ \vartheta(w)\right) \sup_{|w|<1} |\Re\Delta F(w)|,$$

where $C_\infty(\gamma, \alpha)$ is defined by (2.4.2).

Coming back to the variable z, we obtain the following generalization of (2.4.1) for a domain G and arbitrary ξ

$$|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}| \le C_\infty\left(|\Psi(z)|,\ \alpha(z)\right) \sup_{\zeta\in G} |\Re\Delta_\xi f(\zeta)|. \tag{2.6.10}$$

Putting here $\alpha(z) = -\arg \Delta_\xi f(z)$, we obtain

$$|\Delta_\xi f(z)| \le C_\infty\left(|\Psi(z)|,\ -\arg \Delta_\xi f(z)\right) \sup_{\zeta\in G} |\Re\Delta_\xi f(\zeta)|.$$

After dividing this inequality by $|z - \xi|$ and taking into account (2.4.2) and $\Psi(\xi) = 0$ we make the limit passage as $z \to \xi$. As a result, we arrive at the sharp estimate

$$|f'(\xi)| \le \frac{4|\Psi'(\xi)|}{\pi} \sup_{\zeta\in G} |\Re f(\zeta) - \Re f(\xi)|. \tag{2.6.11}$$

2.6.2 Estimate of $|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}|$ by the supremum of $|\Re\Delta_\xi f(\zeta)|$ and an estimate for the first derivative in the disk

Let $G = D_R$ and $\Phi(w) = R(\xi - Rw)/(R - \overline{\xi}w)$. Then $\Psi(z) = R(\xi - z)/(R^2 - z\overline{\xi})$. In concert with (2.6.10),

$$|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}| \le C_\infty\left(\frac{R|z-\xi|}{|R^2 - \overline{\xi}z|},\ \alpha(z)\right) \sup_{|\zeta|<R} |\Re\Delta_\xi f(\zeta)|, \tag{2.6.12}$$

which coincides with (2.4.1) for $\xi = 0$.

In this case the sharp inequality (2.6.11) takes the form

$$|f'(z)| \le \frac{4R}{\pi(R^2 - |z|^2)} \sup_{|\zeta|<R} |\Re f(\zeta) - \Re f(\xi)|, \tag{2.6.13}$$

where z is an arbitrary point of D_R.

2.6.3 Estimate of $|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}|$ by the supremum of $|\Re\Delta_\xi f(\zeta)|$ and an estimate for the first derivative in the half-plane

Consider a function f, analytic in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ with bounded $\Re f$.

Given a fixed point $\xi \in \mathbb{C}_+$, we map D_1 onto $\mathbb{C}_+$ using the mapping $z = (\xi - \overline{\xi}w)/(1-w)$ whose inverse is $w = (z-\xi)/(z-\overline{\xi})$. An analogue of (2.6.10) for $\mathbb{C}_+$ is

$$|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}| \le C_\infty \left(\frac{|z-\xi|}{|z-\overline{\xi}|},\ \alpha(z) \right) \sup_{\zeta\in\mathbb{C}_+} |\Re\Delta_\xi f(\zeta)|. \tag{2.6.14}$$

Hence, in the same way as in Section 2.6.1, we obtain the sharp inequality

$$|f'(z)| \le \frac{2}{\pi \Im z} \sup_{\zeta\in\mathbb{C}_+} |\Re f(\zeta) - \Re f(z)|, \tag{2.6.15}$$

where z is an arbitrary point of $\mathbb{C}_+$.

3

Estimates for analytic functions by the best L_p-approximation of $\Re f$ on the circle

3.1 Introduction

Along with (2.1.1) and (2.1.2), there exist other inequalities for the increment of the real and imaginary parts at zero of a function f which is analytic in the disk D_R and has a bounded real part. In particular, we mean the inequalities

$$|\Re f(z) - \Re f(0)| \leq \frac{4}{\pi} \arcsin\left(\frac{r}{R}\right) ||\Re f||_\infty, \tag{3.1.1}$$

and

$$|\Im f(z) - \Im f(0)| \leq \frac{2}{\pi} \log\left(\frac{R+r}{R-r}\right) ||\Re f||_\infty \tag{3.1.2}$$

(see, for example, Hurwitz and Courant [48], III, § 9). The first inequality is known as the Schwarz Arcussinus Formula.

The estimates (3.1.1), (3.1.2) are special cases of more general sharp inequalities presented in this chapter. Here we deal with sharp pointwise estimate for

$$|\Re\{e^{i\alpha(z)}(f(z) - f(0))\}|$$

by the L_p-norm of $\Re f$ on the circle ∂D_R and its corollaries, where f is an analytic function on D_R with $\Re f \in h_p(D_R)$, α is a real valued function on D_R, $|z| = r < R$, and $1 \leq p \leq \infty$.

In Section 3.2 we obtain a general representation for the best constant $\mathcal{K}_p(z, \alpha(z))$ in the inequality

$$|\Re\{e^{i\alpha(z)}(f(z) - f(0))\}| \leq \mathcal{K}_p\left(z, \alpha(z)\right) ||\Re f - c||_p, \tag{3.1.3}$$

where c is an arbitrary real constant. Namely, we find the representation for the sharp constant in (3.1.3)

$$\mathcal{K}_p(z, \alpha(z)) = R^{-1/p} K_p(r/R, \alpha(z)), \tag{3.1.4}$$

where

$$K_p(\gamma,\alpha)=\frac{\gamma}{\pi}\left\{\int_{-\pi}^{\pi}\left|\frac{\cos(\varphi-\alpha)-\gamma\cos\alpha}{1-2\gamma\cos\varphi+\gamma^2}\right|^q d\varphi\right\}^{1/q}, \tag{3.1.5}$$

$1/q+1/p=1$. The value $||\Re f-c||_p$ in the right-hand side of (3.1.3) can be replaced by $E_p(\Re f)$, where

$$E_p(g)=\min_{c\in\mathbb{R}}||g-c||_p. \tag{3.1.6}$$

stands for the best approximation of g by a real constant in the norm of $L_p(\partial D_R)$.

As a corollary of (3.1.3) with $\alpha=0$ one gets the two-sided estimate

$$\exp\left\{-\mathcal{K}_p(z,0)\,E_p\big(\log|f|\big)\right\}\le\big|f(z)\big|\le\exp\left\{\mathcal{K}_p(z,0)\,E_p\big(\log|f|\big)\right\},$$

where f is an analytic and zero-free function on D_R with $f(0)=1$, $\log|f|\in h_p(D_R)$, $1\le p\le\infty$.

The explicit formulas

$$K_1(\gamma,\alpha)=\frac{\gamma(1+\gamma|\cos\alpha|)}{\pi(1-\gamma^2)},\qquad K_2(\gamma,\alpha)=\frac{\gamma}{\sqrt{\pi(1-\gamma^2}}$$

are derived in Section 3.3. In particular, for $p=1$ and $c=\sup\{\Re f(\zeta):|\zeta|<R\}$ inequality (3.1.3) and formula for $K_1(\gamma,\alpha)$ imply the Hadamard-Borel-Carathéodory inequality (1.1.2).

In Section 3.4 we show that the constant $K_\infty(\gamma,\alpha)$ is equal to

$$\frac{4}{\pi}\left\{\sin\alpha\log\frac{\gamma\sin\alpha+\big(1-\gamma^2\cos^2\alpha\big)^{1/2}}{(1-\gamma^2)^{1/2}}+\cos\alpha\arcsin\left(\gamma\cos\alpha\right)\right\}.$$

We note, that by (3.1.3) with $c=0,p=\infty$ and by formula for $K_\infty(\gamma,\alpha)$ with $\alpha=0$, and $\alpha=\pi/2$ one gets the estimates (3.1.1), (3.1.2). Besides, (3.1.3) with $c=0,p=\infty$ and the above formula for $K_\infty(\gamma,\alpha)$ imply the sharp inequality

$$|f(z)-f(0)|\le\frac{2}{\pi}\log\left(\frac{R+r}{R-r}\right)||\Re f||_\infty. \tag{3.1.7}$$

A direct corollary of (3.1.3) with $p=\infty$ and the formula for $K_\infty(\gamma,\alpha)$ is the explicit sharp estimate for $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ in terms of $\mathcal{O}_{\Re f}(D_R)$ which stands for the oscillation of the real part on the disk D_R.

Section 3.5 contains corollaries of (3.1.3), (3.1.4) and (3.1.5) giving estimates for the modulus of the increment at zero of the real or imaginary parts of an analytic function. In particular, we find an explicit formula for $K_p(\gamma,0)$ with $p=2n/(2n-1)$. For instance,

$$K_{4/3}(\gamma, 0) = \gamma \left\{ \frac{3 + 7\gamma^2}{4\pi^3(1-\gamma^2)^3} \right\}^{1/4}.$$

For $\alpha = \pi/2$ and any z with $|z| = r < R$, inequality (3.1.3) and formulas (3.1.4), (3.1.5) imply the estimate

$$|\Im f(z) - \Im f(0)| \le R^{-1/p} K_p\,(r/R, \pi/2)\, E_p(\Re f). \qquad (3.1.8)$$

In general, (2.1.5) and (3.1.5) lead to the inequality $C_p(\gamma, \alpha) \le K_p(\gamma, \alpha)$ which becomes equality for some values of p and α. In particular, this is the case for $p = 2$. We also show that $K_p(\gamma, \pi/2) = C_p(\gamma, \pi/2)$, that is the inequality (3.1.8) holds with the sharp constant defined by (2.1.7).

In Section 3.6 we deduce a sharp estimate for the oscillation of $\Re\{e^{i\alpha(z)} f(z)\}$ on a subset of D_R stated in terms of the oscillation of the real part on D_R. The constant in that estimate is specified for symmetric with respect to the origin subset of D_R. This, in turn, leads to sharp inequalities for the supremum modulus of the increment of an analytic function, as well as for the oscillation of the real or imaginary parts by $\mathcal{O}_{\Re f}(D_R)$. Such estimates for the oscillation of the real and imaginary parts in the disk $D_r, r < R$, are well known (see Koebe [52], Polya and Szegö [75], III, Ch. 6, § 2).

The last Section 3.7 contains analogues of (3.1.3) with $p = \infty$ for $|\Re\{e^{i\alpha(z)}(f(z) - f(\xi))\}|$ in the disk and the half-plane. These estimates imply sharp inequalities for $|f'(z)|$ with explicit constants. In particular, if f is analytic function in D_R with bounded $\Re f$, then for any point $z \in D_R$

$$|f'(z)| \le \frac{2R}{\pi(R^2 - |z|^2)} \mathcal{O}_{\Re f}(D_R). \qquad (3.1.9)$$

This inequality is used in the next chapter. Note that (3.1.9) is an analogue of (1.7.5) and (2.6.13).

3.2 Estimate of $|\Re\{e^{i\alpha}\Delta f\}|$ by the L_p-norm of $\Re f - c$ on the circle. General case

The next assertion is basic in this chapter. It contains a representation of the best constant in the estimate of $|\Re\{e^{i\alpha(z)}\Delta f(z)\}|,\ |z| < R$, by the L_p-norm of $\Re f - c$ on the circle $|z| = R$, where c is an arbitrary real constant. As a direct corollary, we obtain sharp estimates for $|\Re\{e^{i\alpha(z)}\Delta f(z)\}|$ with the best approximation $E_p(\Re f)$ of $\Re f$ by a real constant in the norm of $L_p(\partial D_R)$ in the right-hand side.

Proposition 3.1. *Let f be analytic on D_R with $\Re f \in h_p(D_R)$, $1 \le p \le \infty$. Further, let $\alpha(z)$ be a real valued function, $|z| < R$, and let c be a real constant. Then for any fixed point $z, |z| = r < R$, we have*

$$|\Re\{e^{i\alpha(z)}\varDelta f(z)\}| \le \mathcal{K}_p\left(z,\alpha(z)\right)||\Re f - c||_p \tag{3.2.1}$$

with the sharp constant $\mathcal{K}_p(z,\alpha(z))$, *where*

$$\mathcal{K}_p(z,\alpha) = \frac{1}{R^{1/p}} K_p\left(\frac{r}{R},\alpha\right), \tag{3.2.2}$$

and the factor

$$K_p(\gamma,\alpha) = \frac{\gamma}{\pi}\left\{\int_{-\pi}^{\pi}\left|\frac{\cos(\varphi-\alpha)-\gamma\cos\alpha}{1-2\gamma\cos\varphi+\gamma^2}\right|^q d\varphi\right\}^{1/q} \tag{3.2.3}$$

is an even π*-periodic function of* α, $1/p + 1/q = 1$.

In particular,

$$|\Re\{e^{i\alpha(z)}\varDelta f(z)\}| \le \mathcal{K}_p\left(z,\alpha(z)\right)E_p(\Re f), \tag{3.2.4}$$

where the notation $E_p(g)$ *is defined by* (3.1.6).

Proof. 1. Representation of the sharp constant in inequality (3.2.1). By (1.3.3), (1.3.4) and (1.4.3), we have

$$\Re\{e^{i\alpha}\varDelta f(z)\} = \frac{1}{\pi R}\int_{|\zeta|=R} G_{z,\alpha}(\zeta)\{\Re f(\zeta)-c\}|d\zeta|, \tag{3.2.5}$$

where c is an arbitrary real constant. By (3.2.5) we obtain the formula

$$\mathcal{K}_p(z,\alpha) = \frac{1}{\pi R}\,||G_{z,\alpha}||_q \tag{3.2.6}$$

for the sharp constant $\mathcal{K}_p(z,\alpha)$ in

$$|\Re\{e^{i\alpha}\varDelta f(z)\}| \le \mathcal{K}_p\left(z,\alpha\right)||\Re f - c||_p.$$

Now, suppose $1 < p \le \infty$. Combining (3.2.6) with (1.3.4) and (1.3.5) we have

$$\mathcal{K}_p(z,\alpha) = \frac{1}{\pi R}\left\{\int_{-\pi+\tau}^{\pi+\tau}\left|\Re\left(\frac{\gamma e^{i\alpha}}{e^{i(t-\tau)}-\gamma}\right)\right|^q R dt\right\}^{1/q},$$

which, after the change of variable $\varphi = t - \tau$, becomes

$$\mathcal{K}_p(z,\alpha) = \frac{1}{\pi R^{1/p}}\left\{\int_{-\pi}^{\pi}\left|\Re\left(\frac{\gamma e^{i\alpha}}{e^{i\varphi}-\gamma}\right)\right|^q d\varphi\right\}^{1/q}. \tag{3.2.7}$$

Using the notation

$$K_p(\gamma,\alpha) = \frac{1}{\pi}\left\{\int_{-\pi}^{\pi}\left|\Re\left(\frac{\gamma e^{i\alpha}}{e^{i\varphi}-\gamma}\right)\right|^q d\varphi\right\}^{1/q}, \tag{3.2.8}$$

we write (3.2.7) as

$$\mathcal{K}_p(z,\alpha) = \frac{1}{R^{1/p}} K_p\left(\frac{r}{R}, \alpha\right),$$

which together with (1.3.6) proves (3.2.2) and (3.2.3) for $1 < p \le \infty$.

The equality (3.2.3) with $p = 1$ $(q = \infty)$ results from the limit relation

$$||g||_\infty = \lim_{q\to\infty} ||g||_q.$$

Inequality (3.2.4) follows directly from (3.2.1).

2. Properties of $K_p(\gamma,\alpha)$. The π-periodicity of $K_p(\gamma,\alpha)$ in α follows directly from (3.2.8). One shows that $K_p(\gamma,\alpha)$ is an even function of α in the same way as in Proposition 2.1. □

Writing (3.2.8) as

$$K_p(\gamma,\alpha) = \frac{\gamma}{\pi}\left\{\int_{|\zeta|=1}\left|\Re\left(\frac{e^{i\alpha}}{\zeta-\gamma}\right)\right|^q |d\zeta|\right\}^{1/q}$$

and making the change of variable $\zeta = (w+\gamma)/(1+\gamma w)$, we obtain

$$K_p(\gamma,\alpha) = \frac{\gamma}{\pi(1-\gamma^2)^{1/p}}\left\{\int_{|w|=1}\frac{\left|\Re\left\{e^{i\alpha}(\gamma+\overline{w})\right\}\right|^q}{|1+\gamma w|^2}|dw|\right\}^{1/q}.$$

Hence, $K_p(\gamma,\alpha)$ can be given in the form different from (3.2.3):

$$K_p(\gamma,\alpha) = \frac{\gamma}{\pi(1-\gamma^2)^{1/p}}\left\{\int_{-\pi}^{\pi}\frac{|\cos(\psi-\alpha)-\gamma\cos\alpha|^q}{1-2\gamma\cos\psi+\gamma^2}d\psi\right\}^{1/q}. \qquad (3.2.9)$$

We give two more corollaries of Proposition 3.1. The first of them follows by putting $\alpha(z) = -\arg\Delta f(z)$ in Proposition 3.1 and taking into account that $K_p(\gamma,\alpha)$ is even in α.

Corollary 3.1. *Let f be analytic on D_R with $\Re f \in h_p(D_R)$, $1 \le p \le \infty$. Further, let c be a real constant. Then for any fixed point $z, |z| = r < R$, the estimate*

$$|\Delta f(z)| \le \mathcal{K}_p(z, \arg\Delta f(z))\, ||\Re f - c||_p \qquad (3.2.10)$$

holds with the sharp constant $\mathcal{K}_p(z,\alpha)$, given by (3.2.2) *and* (3.2.3).

The next corollary is an analogue of (2.2.5). It results from (3.2.4) with $\alpha(z) = 0$ and $\log f$ in place of f.

Corollary 3.2. *Let f be an analytic and zero-free function on D_R with $f(0) = 1$, and let $\log|f| \in h_p(D_R), 1 \le p \le \infty$. Then for any z with $|z| = r < R$ there holds*

$$\big|\log|f(z)|\,\big| \le \mathcal{K}_p(z,0)\; E_p\big(\log|f|\big) \tag{3.2.11}$$

or, in equivalent form,

$$\exp\big\{-\mathcal{K}_p(z,0)\; E_p\big(\log|f|\big)\big\} \le \big|f(z)\big| \le \exp\big\{\mathcal{K}_p(z,0)\; E_p\big(\log|f|\big)\big\}\,,$$

where the constant $\mathcal{K}_p(z,0)$ is given by (3.2.2) *and* (3.2.3) *with $\alpha = 0$, and $E_p(g)$ is defined by* (3.1.6).

The values of $\mathcal{K}_p(z,\alpha)$ for $p = 1, 2, \infty$ and corresponding inequalities from Proposition 3.1, Corollaries 3.1 and 3.2 will be given in the next sections of this chapter.

3.3 The cases $p = 1$ and $p = 2$

Next, we present the inequalities from Proposition 3.1 and Corollary 3.2 with explicit constants for $p = 1$. They follow from (3.2.1) and (3.2.11), combined with (3.2.6) for $q \to \infty$ as well as (1.3.4) and Lemma 1.2.

Corollary 3.3. *Let f be analytic on D_R with $\Re f \in h_1(D_R)$. Further, let $\alpha(z)$ be a real valued function, $|z| < R$, and c be a real constant. Then for any fixed point $z, |z| = r < R$, the sharp inequality*

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \le \frac{r(R + r|\cos\alpha(z)|)}{\pi R(R^2 - r^2)}||\Re f - c||_1 \tag{3.3.1}$$

holds.

Corollary 3.4. *If f is an analytic and zero-free function on D_R with $|f(0)| = 1$, $\log|f| \in h_1(D_R)$, then for any z with $|z| = r < R$ the inequality*

$$\big|\log|f(z)|\big| \le \frac{r}{\pi R(R-r)}E_1\big(\log|f|\big) \tag{3.3.2}$$

holds.

Note that inequalities (1.6.4) and (1.6.6) are consequences of (3.3.1). In fact, putting

$$c = \sup_{|\zeta|<R} \Re f(\zeta)$$

into (3.3.1) with $\rho \in (r, R)$ and passing to the limit as $\rho \uparrow R$, we arrive at

$$|\Re\{e^{i\alpha}\Delta f(z)\}| \le \frac{2r(R + r|\cos\alpha|)}{R^2 - r^2} \sup_{|\zeta|<R} \Re\Delta f(\zeta).$$

The last inequqlity contains both (1.6.4) and (1.6.6).

Consider the case $p = 2$ in Proposition 3.1 and Corollary 3.2. By (3.2.6) with $p = 2$ and (2.3.8),

$$\mathcal{K}_2(z, \alpha) = \frac{r}{\sqrt{\pi R(R^2 - r^2)}}. \tag{3.3.3}$$

Note also that

$$E_2(\Re f) = ||\Re f - \Re f(0)||_2. \tag{3.3.4}$$

Indeed, we have

$$||\Re f - c||_2 = \left\{ R \int_{-\pi}^{\pi} [\Re f(Re^{i\varphi}) - c]^2 d\varphi \right\}^{1/2},$$

which implies the representation

$$E_2(\Re f) = \min_{c \in \mathbb{R}} ||\Re f - c||_2 = \left\{ R \int_{-\pi}^{\pi} \left[\Re f(Re^{i\varphi}) - A_0\right]^2 d\varphi \right\}^{1/2},$$

where

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re f(Re^{i\varphi}) d\varphi = \Re f(0)$$

which gives (3.3.4).

By (3.3.3) and (3.3.4), inequalities (3.2.4) and (3.2.11) with $p = 2$ coincide with (2.3.5) and (2.3.10), respectively. Hence estimates (2.3.5) and (2.3.10) are corollaries of the next assertion which follows from Proposition 3.1 and (3.3.3).

Corollary 3.5. *Let f be analytic on D_R with $\Re f \in h_2(D_R)$. Further, let $\alpha(z)$ be a real valued function, $|z| < R$, and c be a real constant. Then for any fixed point $z, |z| = r < R$, the sharp inequality*

$$|\Re\{e^{i\alpha(z)} \Delta f(z)\}| \leq \frac{r}{\sqrt{\pi R(R^2 - r^2)}} ||\Re f - c||_2 \tag{3.3.5}$$

holds.

3.4 The case $p = \infty$

The next theorem contains sharp constants in (3.2.1) and (3.2.4) for $p = \infty$. When using (3.2.4) with $p = \infty$ we take into account that

$$E_\infty(\Re f) = \frac{1}{2} \mathcal{O}_{\Re f}(D_R), \tag{3.4.1}$$

where $\mathcal{O}_{\Re f}(D_R)$ is the oscillation of $\Re f$ on the disk D_R. Note that (3.4.1) is true since the minimal value in

$$E_\infty(\Re f) = \min_{c\in\mathbb{R}} ||\Re f - c||_\infty = \min_{c\in\mathbb{R}} \sup_{|\zeta|<R} |\Re f(\zeta) - c|$$

is attained at $c = \big(\mathcal{A}_f(R) + \mathcal{B}_f(R)\big)/2$, where $\mathcal{A}_f(R)$ and $\mathcal{B}_f(R)$ are the supremum and the infimum of $\Re f$ on the disk D_R, respectively.

Theorem 3.1. *Let f be analytic on D_R with bounded $\Re f$. Further, let $\alpha(z)$ be a real valued function, $|z| < R$, and c be a real constant. Then for any fixed point $z, |z| = r < R$, the inequality*

$$|\Re\{e^{i\alpha(z)}\varDelta f(z)\}| \le K_\infty\left(\frac{r}{R}, \alpha(z)\right) ||\Re f - c||_\infty \tag{3.4.2}$$

holds with the sharp constant

$$K_\infty(\gamma,\alpha) = \frac{4}{\pi}\left\{\sin\alpha \log \frac{\gamma \sin\alpha \left(1-\gamma^2\cos^2\alpha\right)^{1/2}}{(1-\gamma^2)^{1/2}} \right. \\ \left. + \cos\alpha \arcsin\left(\gamma\cos\alpha\right)\right\}. \tag{3.4.3}$$

In particular,

$$|\Re\{e^{i\alpha(z)}\varDelta f(z)\}| \le \frac{1}{2}K_\infty\left(\frac{r}{R}, \alpha(z)\right) \mathcal{O}_{\Re f}(D_R). \tag{3.4.4}$$

Proof. By Proposition 3.1,

$$\mathcal{K}_\infty(z,\alpha) = K_\infty(r/R, \alpha), \tag{3.4.5}$$

where

$$K_\infty(\gamma,\alpha) = \frac{\gamma}{\pi}\int_{-\pi}^{\pi} \frac{|\cos(\varphi-\alpha) - \gamma\cos\alpha|}{1-2\gamma\cos\varphi+\gamma^2} d\varphi, \tag{3.4.6}$$

which implies (3.4.2) with the sharp constant (3.4.6). Estimate (3.4.4) results from (3.2.4) with $p = \infty$ together with (3.4.1) and (3.4.5).

We evaluate the integral in (3.4.6). By Proposition 3.1, $K_\infty(\gamma,\alpha)$ is a π-periodic and even function of α, therefore we take $0 \le \alpha \le \pi/2$.

Since for any $z, |z| < R$, there holds

$$\int_{|\zeta|=1} \frac{d\zeta}{(\zeta - z)\zeta} = 0,$$

it follows that

$$0 = \Re\left\{\int_{|\zeta|=1} \frac{e^{i\alpha} z}{i(\zeta - z)\zeta} d\zeta\right\} = \gamma\int_{-\pi}^{\pi} \frac{\cos(\varphi-\alpha) - \gamma\cos\alpha}{1-2\gamma\cos\varphi+\gamma^2} d\varphi.$$

Using the last equality and (3.4.6), we conclude

$$K_\infty(\gamma,\alpha) = \frac{2\gamma}{\pi}\int_{\psi_1(\alpha)}^{\psi_2(\alpha)} \frac{\cos(\varphi-\alpha)-\gamma\cos\alpha}{1-2\gamma\cos\varphi+\gamma^2}d\varphi, \tag{3.4.7}$$

where

$$\psi_1(\alpha) = \alpha - \arccos(\gamma\cos\alpha), \qquad \psi_2(\alpha) = \alpha + \arccos(\gamma\cos\alpha). \tag{3.4.8}$$

By (3.4.7) we have

$$K_\infty(\gamma,\alpha) = \frac{1}{\pi}\Big\{2\gamma I_1(\gamma,\alpha)\sin\alpha + \Big((1-\gamma^2)I_2(\gamma,\alpha) - I_3(\gamma,\alpha)\Big)\cos\alpha\Big\}, \tag{3.4.9}$$

where

$$I_1(\gamma,\alpha) = \int_{\psi_1(\alpha)}^{\psi_2(\alpha)} \frac{\sin\varphi}{1-2\gamma\cos\varphi+\gamma^2}d\varphi, \tag{3.4.10}$$

$$I_2(\gamma,\alpha) = \int_{\psi_1(\alpha)}^{\psi_2(\alpha)} \frac{d\varphi}{1-2\gamma\cos\varphi+\gamma^2},$$

$$I_3(\gamma,\alpha) = \int_{\psi_1(\alpha)}^{\psi_2(\alpha)} d\varphi = 2\arccos(\gamma\cos\alpha). \tag{3.4.11}$$

We evaluate the integral (3.4.10)

$$I_1(\gamma,\alpha) = \frac{2}{\gamma}\log\frac{\gamma\sin\alpha + \big(1-\gamma^2\cos^2\alpha\big)^{1/2}}{(1-\gamma^2)^{1/2}}. \tag{3.4.12}$$

Next, note that

$$I_2(\gamma,\alpha) = \frac{2}{1-\gamma^2}\arctan\left(\frac{1+\gamma}{1-\gamma}\tan\frac{\varphi}{2}\right)\Big|_{\psi_1(\alpha)}^{\psi_2(\alpha)},$$

because

$$\arctan x - \arctan y = \pi + \arctan\frac{x-y}{1+xy}. \tag{3.4.13}$$

Hence

$$I_2(\gamma,\alpha) = \frac{2}{1-\gamma^2}\left\{\pi + \arctan\frac{\frac{1+\gamma}{1-\gamma}\left(\tan\frac{\psi_2(\alpha)}{2} - \tan\frac{\psi_2(\alpha)}{2}\right)}{1+\left(\frac{1+\gamma}{1-\gamma}\right)^2\tan\frac{\psi_2(\alpha)}{2}\tan\frac{\psi_2(\alpha)}{2}}\right\}. \tag{3.4.14}$$

The conditions $x > 0$ and $xy < -1$, necessary for (3.4.13) to hold, are satisfied, since by (3.4.8) and $0 \le \alpha \le \pi/2$,

$$\tan\frac{\psi_2(\alpha)}{2} > 0, \qquad \tan\frac{\psi_2(\alpha)}{2}\tan\frac{\psi_1(\alpha)}{2} = -\frac{1-\gamma}{1+\gamma}.$$

Combining this with

$$\tan\frac{\psi_2(\alpha)}{2} - \tan\frac{\psi_1(\alpha)}{2} = \frac{2\sin\big(\arccos(\gamma\cos\alpha)\big)}{(1+\gamma)\cos\alpha},$$

and (3.4.14) we find

$$I_2(\gamma,\alpha) = \frac{2}{1-\gamma^2}\left\{\pi + \arctan\frac{\sin\big(\arccos(\gamma\cos\alpha)\big)}{-\gamma\cos\alpha}\right\},$$

that is

$$I_2(\gamma,\alpha) = \frac{2}{1-\gamma^2}\Big(\pi - \arccos(\gamma\cos\alpha)\Big).$$

This and (3.4.11) imply

$$(1-\gamma^2)I_2(\gamma,\alpha) - I_3(\gamma,\alpha) = 2\Big(\pi - 2\arccos(\gamma\cos\alpha)\Big),$$

which together with $\arcsin x = (\pi/2) - \arccos x$ gives

$$(1-\gamma^2)I_2(\gamma,\alpha) - I_3(\gamma,\alpha) = 4\arcsin(\gamma\cos\alpha).$$

The last inequality together with (3.4.9) and (3.4.12) leads to (3.4.3). □

The next corollary specifies inequality (3.4.2). Namely, it contains the estimates for $|\Re\Delta f(z)|, |\Im\Delta f(z)|$ and $|\Delta f(z)|$ by $||\Re f - c||_\infty$, where c is an arbitrary real constant. In particular, two first inequalities below imply (for $c = 0$) estimates (3.1.1) and (3.1.2). We show that the right-hand side of the inequality for $|\Im\Delta f(z)|$ is, in fact, the sharp majorant for $|\Delta f(z)|$.

Corollary 3.6. *Let f be analytic on D_R with bounded $\Re f$. Further, let c be a real constant. Then for any fixed point $z, |z| = r < R$, the inequalities with sharp constants*

$$|\Re\Delta f(z)| \le \frac{4}{\pi}\arcsin\left(\frac{r}{R}\right)||\Re f - c||_\infty, \tag{3.4.15}$$

$$|\Im\Delta f(z)| \le \frac{2}{\pi}\log\frac{R+r}{R-r}||\Re f - c||_\infty, \tag{3.4.16}$$

$$|\Delta f(z)| \le \frac{2}{\pi}\log\frac{R+r}{R-r}||\Re f - c||_\infty \tag{3.4.17}$$

hold.

Proof. 1. Inequalities for $|\Re \Delta f(z)|$ *and* $|\Im f(z)|$. Inequalities (3.4.15) and (3.4.16) follow from (3.4.2) and (3.4.3) with $\alpha(z) = 0$ and $\alpha(z) = \pi/2$, respectively.

2. Inequality for $|\Delta f(z)|$. Since $K_\infty(\gamma, \alpha)$ is an even and π-periodic in α, we have

$$\max\{K_\infty(\gamma, \alpha) : -\pi \le \alpha \le \pi\} = \max\{K_\infty(\gamma, \alpha) : 0 \le \alpha \le \pi/2\}. \quad (3.4.18)$$

We show that $K_\infty(\gamma, \alpha)$ is an increasing function of α on $[0, \pi/2]$ and hence

$$\max\{K_\infty(\gamma, \alpha) : 0 \le \alpha \le \pi/2\} = K_\infty(\gamma, \pi/2) = \frac{2}{\pi} \log \frac{1+\gamma}{1-\gamma}. \quad (3.4.19)$$

Let us consider $K_\infty(\gamma, \alpha)$ for $0 \le \alpha \le \pi/2$. In view of (3.4.3),

$$\frac{\partial K_\infty(\gamma, \alpha)}{\partial \alpha} = \frac{4}{\pi} \Big\{ \cos\alpha \log \frac{\gamma \sin\alpha + \left(1 - \gamma^2 \cos^2\alpha\right)^{1/2}}{(1-\gamma^2)^{1/2}} - \sin\alpha \arcsin(\gamma \cos\alpha) \Big\}. \quad (3.4.20)$$

Using the equalities

$$\cos\alpha \log \frac{\gamma \sin\alpha + (1 - \gamma^2 \cos^2\alpha)^{1/2}}{(1-\gamma^2)^{1/2}} = \cos\alpha \int_0^{\gamma \sin\alpha} \frac{dt}{\sqrt{1 - \gamma^2 + t^2}},$$

$$\sin\alpha \arcsin(\gamma \cos\alpha) = \sin\alpha \int_0^{\gamma \cos\alpha} \frac{dt}{\sqrt{1 - t^2}},$$

and the estimates

$$\cos\alpha \int_0^{\gamma \sin\alpha} \frac{dt}{\sqrt{1 - \gamma^2 + t^2}} > \frac{\gamma \sin\alpha \cos\alpha}{\sqrt{1 - \gamma^2 + \gamma^2 \sin^2\alpha}},$$

$$\sin\alpha \int_0^{\gamma \cos\alpha} \frac{dt}{\sqrt{1 - t^2}} < \frac{\gamma \sin\alpha \cos\alpha}{\sqrt{1 - \gamma^2 \cos^2\alpha}} = \frac{\gamma \sin\alpha \cos\alpha}{\sqrt{1 - \gamma^2 + \gamma^2 \sin^2\alpha}},$$

which follow from the mean value theorem for $\alpha \in (0, \pi/2)$, we obtain from (3.4.20)

$$\frac{\partial \mathcal{K}_\infty(z, \alpha)}{\partial \alpha} > 0.$$

Thus, $K_\infty(\gamma, \alpha)$ increases on the interval $[0, \pi/2]$, and by (3.4.2), (3.4.18) and (3.4.19) we arrive at (3.4.17). □

The next assertion contains particular cases of (3.4.4) giving estimates for $|\Re \Delta f(z)|, |\Im \Delta f(z)|$ and $|\Delta f(z)|$.

Corollary 3.7. *Let f be analytic on D_R with bounded $\Re f$. Then for any fixed point $z, |z| = r < R$, the sharp inequalities*

$$|\Re \Delta f(z)| \leq \frac{2}{\pi} \arcsin\left(\frac{r}{R}\right) \mathcal{O}_{\Re f}(D_R), \tag{3.4.21}$$

$$|\Im \Delta f(z)| \leq \frac{1}{\pi} \log \frac{R+r}{R-r} \mathcal{O}_{\Re f}(D_R), \tag{3.4.22}$$

$$|\Delta f(z)| \leq \frac{1}{\pi} \log \frac{R+r}{R-r} \mathcal{O}_{\Re f}(D_R) \tag{3.4.23}$$

hold.

Proof. Inequalities (3.4.21)-(3.4.23) follow from relations between sharp constants in (3.4.2) and (3.4.4) together with Corollary 3.6. □

Next, we specify (3.2.11) for $p = \infty$ in terms of

$$\mathcal{M}_f(R) = \sup_{|\zeta|<R} |f(\zeta)| \quad \text{and} \quad m_f(R) = \inf_{|\zeta|<R} |f(\zeta)|,$$

where f is an analytic and zero-free function on D_R.

Corollary 3.8. *If f is an analytic and zero-free function on D_R with $f(0) = 1$, and $\log|f|$ is bounded, then for any z with $|z| = r < R$ the inequality*

$$\big|\log|f(z)|\big| \leq \frac{2}{\pi} \arcsin\left(\frac{r}{R}\right) \log\left(\frac{\mathcal{M}_f(R)}{m_f(R)}\right) \tag{3.4.24}$$

holds.

Proof. Inequality (3.4.24) results from (3.4.21) after replacing f by $\log f$ with $f(z) \neq 0$ for $|z| < R$, $|f(0)| = 1$, together with

$$\mathcal{O}_{\log|f|}(D_R) = \log \mathcal{M}_f(R) - \log m_f(R).$$

□

Estimate (3.4.24) is a particular case of (3.2.11) for $p = \infty$, where

$$E_\infty(\log|f|) = \frac{1}{2} \mathcal{O}_{\log|f|}(D_R).$$

Inequality (3.4.21) is a corollary of the Schwarz Arcussinus Formula. In fact, putting $f - c$ in place of f in inequality (3.1.1) and minimizing in c, we arrive at (3.4.21).

Note that estimate (3.4.21) with $f(0) = 0$ does not coincide with the inequality

$$|\Re f(z)| \le \frac{2}{\pi} \arctan\left(\frac{r}{R}\right) \mathcal{O}_{\Re f}(D_R), \tag{3.4.25}$$

obtained by Koebe (see [52], p. 70) for functions f vanishing at $z = 0$.

The next section contains corollaries of Proposition 3.1, generalizing (3.4.21) and (3.4.22) for $p \in [1, \infty]$ in view of $E_\infty(\Re f) = \mathcal{O}_{\Re f}(D_R)/2$. A different proof of (3.4.22) was given by Koebe in the above mentioned paper.

3.5 Inequalities for the real and imaginary parts

3.5.1. Sharp constant in an inequality for the real part

The next assertion follows from Proposition 3.1 and (3.2.9) with $\alpha = 0$ together with Corollaries 3.3, 3.5 and 3.6. We shall use the notation $K_{\Re,p}(\gamma)$ in place of $K_p(\gamma, 0)$.

Corollary 3.9. *Let f be analytic on D_R with $\Re f \in h_p(D_R)$, $1 \le p \le \infty$. Then for any fixed point $z, |z| = r < R$, there holds*

$$|\Re \Delta f(z)| \le R^{-1/p} K_{\Re,p}\,(r/R)\, E_p(\Re f)$$

with the sharp constant

$$K_{\Re,1}(\gamma) = \frac{\gamma}{\pi(1-\gamma)},$$

and

$$K_{\Re,p}(\gamma) = \frac{\gamma}{\pi(1-\gamma^2)^{1/p}} \left\{ 2\int_0^\pi \frac{|\cos\varphi - \gamma|^q}{1 - 2\gamma\cos\varphi + \gamma^2} d\varphi \right\}^{1/q} \tag{3.5.1}$$

for $1 < p \le \infty$.

In particular,

$$K_{\Re,2}(\gamma) = \frac{\gamma}{\sqrt{\pi(1-\gamma^2)}}, \qquad K_{\Re,\infty}(\gamma) = \frac{4}{\pi} \arcsin\gamma.$$

Note that (3.5.1) can be written as

$$K_{\Re,p}(\gamma) = \frac{(1-\gamma^2)^{1/q}}{2\pi} \left\{ 2\int_0^\pi \left| 1 - \frac{1 - 2\gamma\cos\varphi + \gamma^2}{1-\gamma^2} \right|^q \frac{d\varphi}{1 - 2\gamma\cos\varphi + \gamma^2} \right\}^{1/q}.$$

Hence, straightforward calculations for $q = 2n$ imply

$$K_{\Re,\frac{2n}{2n-1}}(\gamma) = (2\pi)^{\frac{1-2n}{2n}} \left\{ 1 + \sum_{k=1}^{2n} \sum_{m=0}^{k-1} (-1)^k \binom{2n}{k} \binom{k-1}{m}^2 \frac{\gamma^{2m}}{(1-\gamma^2)^{k-1}} \right\}^{\frac{1}{2n}}.$$

For example,

$$K_{\Re,\,4/3}(\gamma) = \gamma \left\{ \frac{3+7\gamma^2}{4\pi^3(1-\gamma^2)^3} \right\}^{1/4}.$$

3.5.2. Sharp constant in an inequality for the imaginary part

Comparing the formulas (2.2.2), (2.2.3) and (3.2.2), (3.2.3) we conclude that the sharp constants in the inequalities

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \le \mathcal{C}_p\,(z, \alpha(z))\, ||\Re \Delta f||_p,$$

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \le \mathcal{K}_p\,(z, \alpha(z))\, ||\Re f - c||_p$$

are related in general as $\mathcal{C}_p(z,\alpha) \le \mathcal{K}_p(z,\alpha)$. For example, by (2.3.1) and (3.3.1),

$$\mathcal{C}_1(z,\alpha) = \frac{r}{\pi(R^2-r^2)}, \qquad \mathcal{K}_1(z,\alpha) = \frac{r(R+r|\cos\alpha|)}{\pi R(R^2-r^2)}.$$

However, for certain values of p and α the equality $\mathcal{C}_p(z,\alpha) = \mathcal{K}_p(z,\alpha)$ may hold. This is, clearly, the case for $p=2$ in view of (2.3.5) and (3.3.5).

Another case of equality is

$$\mathcal{C}_p(z,\pi/2) = \mathcal{K}_p(z,\pi/2) \tag{3.5.2}$$

for any $p \in [1,\infty]$. Indeed, in the proof of Corollary 2.6 it was shown that for $\alpha = \pi/2$ the minimum in λ in (2.2.3) is attained at $\lambda = 0$. Hence, comparing (2.2.3) and (3.2.3) for $\alpha = \pi/2$ and taking into account (2.2.2), (3.2.2) we arrive at (3.5.2).

Thus, by Corollary 2.6 and (3.5.2), together with Proposition 3.1, we obtain the following inequality for $|\Im \Delta f(z)|$ in terms of the best approximation $E_p(\Re f)$ of f by a constant on the circle $|\zeta| = R$ in the norm of $L_p(\partial D_R)$. We shall denote $K_p(\gamma,\pi/2)$ by $K_{\Im,p}(\gamma)$.

Corollary 3.10. *Let f be analytic on D_R with $\Re f \in h_p(D_R)$, $1 \le p \le \infty$. Then for any fixed point $z, |z| = r < R$, the inequality*

$$|\Im \Delta f(z)| \le R^{-1/p} K_{\Im,p}\,(r/R)\, E_p(\Re f)$$

holds with the sharp constant

$$K_{\Im,1}(\gamma) = \frac{\gamma}{\pi(1-\gamma^2)},$$

and

$$K_{\Im,p}(\gamma) = \frac{\varkappa(\gamma)}{2\pi} \left\{ 2\int_{-1}^{1} \frac{(1-t^2)^{(q-1)/2}}{[1-\varkappa(\gamma)t]^q}\,dt \right\}^{1/q}$$

$$= \frac{\varkappa(\gamma)}{2\pi} \left\{ 2\left[1-\varkappa^2(\gamma)\right]^{1/(2-2p)} \sum_{n=0}^{\infty} B\left(\frac{2p-1}{2p-2}, \frac{2n+1}{2}\right) \varkappa^{2n}(\gamma) \right\}^{(p-1)/p},$$

for $1 < p \leq \infty$, *where* $\varkappa(\gamma) = (2\gamma)/(1+\gamma^2)$ *and* $B(u,v)$ *is the Beta-function.*
In particular,

$$K_{\Im,2}(\gamma) = \frac{\gamma}{\sqrt{\pi(1-\gamma^2)}}, \qquad K_{\Im,\infty}(\gamma) = \frac{2}{\pi}\log\frac{1+\gamma}{1-\gamma}.$$

Formulas for $K_{\Im,p}(\gamma) = C_{\Im,p}(\gamma)$ for even and odd values of $q = p/(p-1)$ were given in Sect. 2.5.

3.6 Estimate for the oscillation of $\Re\{e^{i\alpha}f\}$ and its corollaries

In this subsection we obtain a sharp estimate for the oscillation of $\Re\{e^{i\alpha}f\}$ on a set $G \subset D_R$ by the oscillation $\mathcal{O}_{\Re f}(D_R)$ of the function $\Re f$ on the disk D_R. The following assertion holds.

Theorem 3.2. *Let* f *be analytic on* D_R *with bounded* $\Re f$*. Further, let* G *be a subset of* D_R*. Then for any real* α *the inequality*

$$\sup_{z,\xi\in G}\left|\Re\left\{e^{i\alpha}\big(f(z)-f(\xi)\big)\right\}\right| \leq \frac{1}{2}\,K_\infty\left(\sup_{z,\xi\in G}|\Psi(z,\xi)|,\ \alpha\right)\mathcal{O}_{\Re f}(D_R) \quad (3.6.1)$$

holds with the sharp constant, where $K_\infty(\gamma,\alpha)$ *is given by* (3.4.3)*, and*

$$\Psi(z,\xi) = \frac{R(\xi-z)}{R^2 - z\bar{\xi}}. \quad (3.6.2)$$

Proof. Let ξ be a fixed point of the disk D_R and let $z = R(\xi - Rw)/(R - \bar{\xi}w)$, $|w| \leq 1$. If f obeys the conditions of the theorem, the function

$$F(w) = f\left(\frac{R(\xi-Rw)}{(R-\bar{\xi}w)}\right)$$

is analytic in D_1 and its real part is bounded in D_1.

By Corollary 3.1, $F(w)$ satisfies

$$|\Re\{e^{i\alpha}\Delta F(w)\}| \leq K_\infty\left(|w|,\ \alpha\right)||\Re F - c||_\infty,$$

where $K_\infty(\gamma,\alpha)$ is defined by (3.4.3). Hence, returning back to the variable z, we find

$$|\Re\{e^{i\alpha}\big(f(z)-f(\xi)\big)\}| \leq K_\infty\big(|\Psi(z,\xi)|,\ \alpha\big)||\Re f - c||_\infty, \quad (3.6.3)$$

with $\Psi(z,\xi)$ defined by (3.6.2).

It follows from (3.4.3) that

$$\frac{\partial K_\infty(\gamma,\alpha)}{\partial \gamma} = \frac{4}{\pi}\frac{1}{\sqrt{1-\gamma^2\cos^2\alpha}}\left(\frac{\sin^2\alpha}{1-\gamma^2}+\cos^2\alpha\right) > 0.$$

Since $K_\infty(\gamma,\alpha)$ is an increasing function of γ, the sharp inequality (3.6.1) results from $E_\infty(\Re f) = \mathcal{O}_{\Re f}(D_R)/2$, and (3.6.3). □

In particular, for a set G, symmetric with respect to the origin, we arrive at the following assertion.

Corollary 3.11. *Let f be analytic on D_R with bounded $\Re f$. Let G be a subset of D_R such that $z \in G$ implies $-z \in G$, and let* diam $G = 2d$*. Then for any real α there holds*

$$\sup_{z,\xi\in G}\left|\Re\left\{e^{i\alpha}\big(f(z)-f(\xi)\big)\right\}\right| \le \frac{1}{2}\,K_\infty\left(\frac{2dR}{R^2+d^2},\ \alpha\right)\mathcal{O}_{\Re f}(D_R) \tag{3.6.4}$$

with the sharp constant, where $K_\infty(\gamma,\alpha)$ is given by (3.4.3).

In particular,

$$\sup_{z,\xi\in G}\left|\Re f(z)-\Re f(\xi)\right| \le \frac{4}{\pi}\arctan\left(\frac{d}{R}\right)\mathcal{O}_{\Re f}(D_R), \tag{3.6.5}$$

$$\sup_{z,\xi\in G}\left|\Im f(z)-\Im f(\xi)\right| \le \frac{2}{\pi}\log\left(\frac{R+d}{R-d}\right)\mathcal{O}_{\Re f}(D_R) \tag{3.6.6}$$

and

$$\sup_{z,\xi\in G}\left|f(z)-f(\xi)\right| \le \frac{2}{\pi}\log\left(\frac{R+d}{R-d}\right)\mathcal{O}_{\Re f}(D_R). \tag{3.6.7}$$

Proof. Since for some ϑ the point $de^{i\vartheta}$ belongs to G, and

$$|\Psi(z,\xi)|^2 = \frac{R^2\big(|z|^2-2\Re(\overline{z}\xi)+|\xi|^2\big)}{R^4-2R^2\Re(\overline{z}\xi)+|\xi|^2|z|^2} = 1-\frac{\big(R^2-|z|^2\big)\big(R^2-|\xi|^2\big)}{R^4-2R^2\Re(\overline{z}\xi)+|\xi|^2|z|^2},$$

the maximum of $|\Psi|$ on $G\times G$ is attained at $z = de^{i\vartheta}, \xi = -de^{i\vartheta}$, i.e.

$$\max_{z,\xi\in\mathbb{G}}|\Psi(z,\xi)| = \left\{1-\frac{(R^2-d^2)^2}{(R^2+d^2)^2}\right\}^{1/2} = \frac{2dR}{R^2+d^2},$$

which by Theorem 3.2 proves (3.6.4).

It follows from (3.4.3) that

$$K_\infty\left(\frac{2dR}{R^2+d^2},\ 0\right) = \frac{4}{\pi}\arcsin\left(\frac{2dR}{R^2+d^2}\right) = \frac{8}{\pi}\arctan\left(\frac{d}{R}\right),$$

which together with (3.6.4) gives (3.6.5). Inequality (3.6.6) follows from (3.6.4) and (3.4.3) with $\alpha(z) = \pi/2$. The estimate (3.6.7) results from(3.6.4) and (3.4.3) combined with (3.4.18) and (3.4.19). □

Inequality (3.6.5) for the oscillation of the real part in the disk $D_r, r < R$, was obtained by Neumann (see [70], p. 415). The estimate (3.6.6) with $G = D_r$ for the oscillation of the imaginary part of the analytic function was found by Koebe [52].

3.7 Variants and extensions

Next we derive some estimates which follow from Theorem 3.1 by conformal mapping.

As in Sections 1.7 and 2.6, we assume that G is a bounded domain in $\mathbb{C}$, bounded by a Jordan curve. Given an arbitrary point ξ of G, by $z = \Phi(w)$ we denote a function which maps $D_1 = \{w \in \mathbb{C} : |w| < 1\}$ conformly onto G so that $\Phi(0) = \xi$, and let $w = \Psi(z)$ denote the inverse mapping.

We keep the notation $\Delta_\xi f(z) = f(z) - f(\xi)$, introduced in Chapter 1 and, as before, write $\Delta f(z)$ instead of $\Delta_0 f(z)$.

3.7.1 Estimate of $|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}|$ by the supremum of $|\Re f(\zeta) - c|$ in a domain. Estimate for the first derivative

By $f(z)$ we denote a function analytic in G with bounded $\Re f$. Then $F(w) = f(\Phi(w))$ is an analytic function in D_1 whose real part is bounded in D_1.

Let α be an arbitrary real-valued function in G and let $\vartheta(w) = \alpha(\Phi(w))$. By Theorem 3.1,

$$|\Re\{e^{i\vartheta(w)}\Delta F(w)\}| \le K_\infty\left(|w|,\ \vartheta(w)\right) \sup_{|w|<1} |\Re F(w) - c|,$$

where $K_\infty(\gamma, \alpha)$ is defined by (3.4.3). Hence, returning to the variable z, we find a generalization of (3.4.2)

$$|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}| \le K_\infty\left(|\Psi(z)|,\ \alpha(z)\right) \sup_{\zeta\in G} |\Re f(\zeta) - c|. \tag{3.7.8}$$

Putting here $\alpha(z) = -\arg \Delta_\xi f(z)$, we obtain

$$|\Delta_\xi f(z)| \le K_\infty\left(|\Psi(z)|,\ -\arg \Delta_\xi f(z)\right) \sup_{\zeta\in G} |\Re f(\zeta) - c|.$$

Then we divide both sides by $|z - \xi|$, use (3.4.3) and $\Psi(\xi) = 0$, and make passage to the limit as $z \to \xi$. As a result we obtain the inequality

$$|f'(\xi)| \le \frac{4|\Psi'(\xi)|}{\pi} \sup_{\zeta\in G} |\Re f(\zeta) - c| \tag{3.7.9}$$

with the sharp factor in front of the maximum.

Taking into account

$$\min_{c\in\mathbb{R}} \sup_{\zeta\in G} |\Re f(\zeta) - c| = \frac{1}{2}\mathcal{O}_{\Re f}(G),$$

where $\mathcal{O}_{\Re f}(G)$ is the oscillation of $\Re f$ on G, by (3.7.9) we obtain

$$|f'(\xi)| \le \frac{2|\Psi'(\xi)|}{\pi}\mathcal{O}_{\Re f}(G). \tag{3.7.10}$$

3.7.2 Estimate of $|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}|$ by the supremum of $|\Re f(\zeta) - c|$ and an estimate for the first derivative in the disk

Let $G = D_R$ and $\Phi(w) = R(\xi - Rw)/(R - \overline{\xi}w)$. Then

$$\Psi(z) = R(\xi - z)/(R^2 - z\overline{\xi})$$

and (3.7.8) implies

$$|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}| \le K_\infty\left(\frac{R|z-\xi|}{|R^2 - \overline{\xi}z|},\ \alpha(z)\right) \sup_{|\zeta|<R} |\Re f(\zeta) - c|. \tag{3.7.11}$$

The last estimate coincides with (3.4.2) for $\xi = 0$.

Now, the sharp estimate (3.7.9) takes the form

$$|f'(z)| \le \frac{4R}{\pi(R^2 - |z|^2)} \sup_{|\zeta|<R} |\Re f(\zeta) - c|, \tag{3.7.12}$$

where z is an arbitrary point of D_R. A corollary of the last inequality

$$|f'(z)| \le \frac{2R}{\pi(R^2 - |z|^2)}\mathcal{O}_{\Re f}(D_R) \tag{3.7.13}$$

is a particular case of (3.7.10) for the disk.

3.7.3 Estimate of $|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}|$ by the supremum of $|\Re f(\zeta) - c|$ and an estimate for the first derivative in the half-plane

Consider the class of functions f analytic in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ such that $\Re f$ is bounded in $\mathbb{C}_+$.

Given a fixed point $\xi \in \mathbb{C}_+$, we map D_1 onto $\mathbb{C}_+$ using the mapping $z = (\xi - \overline{\xi}w)/(1 - w)$ whose inverse is $w = (z - \xi)/(z - \overline{\xi})$.

The analogue of (3.7.8) for $\mathbb{C}_+$ is

$$|\Re\{e^{i\alpha(z)}\Delta_\xi f(z)\}| \le K_\infty\left(\frac{|z-\xi|}{|z-\overline{\xi}|},\ \alpha(z)\right) \sup_{\zeta\in\mathbb{C}_+} |\Re f(\zeta) - c|. \tag{3.7.14}$$

Hence, in the same way as in Section 3.7.1, we obtain the inequality

$$|f'(z)| \leq \frac{2}{\pi \Im z} \sup_{\zeta \in \mathbb{C}_+} |\Re f(\zeta) - c| \tag{3.7.15}$$

with the sharp factor in front of the supremum, where z is an arbitrary point in $\mathbb{C}_+$.

A direct corollary of the last inequality is the sharp estimate

$$|f'(z)| \leq \frac{1}{\pi \Im z} \mathcal{O}_{\Re f}(\mathbb{C}_+), \tag{3.7.16}$$

where

$$\mathcal{O}_{\Re f}(\mathbb{C}_+) = \sup_{\zeta \in \mathbb{C}_+} \Re f(\zeta) - \inf_{\zeta \in \mathbb{C}_+} \Re f(\zeta).$$

4

Estimates for directional derivatives of harmonic functions

4.1 Introduction

In the present chapter we deduce various estimates for directional derivatives (in particular, for the modulus of the gradient) of harmonic functions inside of a planar domain in terms of various characteristics of harmonic functions and their directional derivatives in the domain or on the boundary. These inequalities follow from the estimates for analytic functions obtained in Chapters 1-3. Henceforth in this chapter we assume that a real valued function is defined on a set of points $z = (x, y)$ of the real plane, while a complex valued function is defined on a set of points $z = x + iy$ of the complex plane. However, the sets in $\mathbb{R}^2$ and in $\mathbb{C}$, which differ in notations of points, will be denoted in the same way.

In Section 4.2 we obtain sharp pointwise estimates for the gradient of a harmonic function inside of a bounded domain $G \subset \mathbb{R}^2$ with Jordan boundary in terms of certain characteristics of the function itself on G. Such characteristics are the supremum of the increment, the supremum of the modulus of the increment, and the oscillation of the function on G. Particular cases of these estimates are given for the disk D_R. Similar sharp inequalities are given for the upper half-plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.

In particular, we show that for any harmonic function u in D_R and any point $z \in D_R$ the sharp estimates hold

$$|\nabla u(z)| \leq \frac{2R}{R^2 - |z|^2} \sup_{|\zeta|<R} \{u(\zeta) - u(z)\},$$

$$|\nabla u(z)| \leq \frac{4R}{\pi(R^2 - |z|^2)} \sup_{|\zeta|<R} |u(\zeta) - u(z)|,$$

$$|\nabla u(z)| \leq \frac{2R}{\pi(R^2 - |z|^2)} \mathcal{O}_u(D_R),$$

where $\mathcal{O}_u(D_R)$ is the oscillation of a function u defined on D_R. As corollaries of sharp pointwise estimates for the gradient inside of the disk, we obtain precise interior estimates for the gradient of a harmonic function in any bounded domain.

Section 4.3 contains sharp estimates for the directional derivative of a harmonic function inside of a domain stated in terms of certain characteristics of the derivative with respect to any fixed direction on the boundary. These inequalities imply sharp estimates for the modulus of the increment of the gradient with right-hand sides containing different characteristics of the derivative in the domain with respect to a fixed direction. In particular, we obtain the sharp estimates

$$|\nabla u(z) - \nabla u(\xi)| \leq \frac{2|z-\xi|}{|z-\overline{\xi}| - |z-\xi|} \sup_{\zeta \in \mathbb{R}^2_+} \left\{ \frac{\partial u(\zeta)}{\partial \ell} - \frac{\partial u(\xi)}{\partial \ell} \right\},$$

$$|\nabla u(z) - \nabla u(\xi)| \leq \frac{2}{\pi} \log \left(\frac{|z-\overline{\xi}| + |z-\xi|}{|z-\overline{\xi}| - |z-\xi|} \right) \sup_{\zeta \in \mathbb{R}^2_+} \left| \frac{\partial u(\zeta)}{\partial \ell} - \frac{\partial u(\xi)}{\partial \ell} \right|,$$

where $z, \xi \in \mathbb{R}^2_+$, and ℓ is an arbitrary fixed unit vector.

Section 4.4 is devoted to estimates for directional derivatives and, in particular, for the gradient of a harmonic function in the disk. We show, for instance, that a harmonic function u with bounded directional derivative $\partial u/\partial l$ in D_R obeys the following sharp inequality at any point z with $|z| = r < R$

$$|\nabla u(z)| \leq \frac{2R}{\pi r} \log \left(\frac{R+r}{R-r} \right) \left\| \frac{\partial u}{\partial l} \right\|_\infty,$$

where l is a unit vector such that the angle between l and the radial direction is constant. In particular, l can be directed either normally or tangentially to ∂D_R. A related theorem from Hile and Stanoyevitch [44] states that $|\nabla u(z)|$ has logarithmic growth as z approaches the smooth boundary ∂G of a bounded domain G under the assumption that the boundary values of a harmonic function are Lipschitz.

4.2 Interior estimates for derivatives in a domain

In this section we are concerned with sharp or improved pointwise interior estimates for the gradient of a harmonic function formulated in terms of some characteristics of the function.

In the next assertion we assume that G is a bounded domain in $\mathbb{R}^2$, bounded by a Jordan curve. By ξ we denote an arbitrary fixed point of G. Let $z = \Phi(w)$ be a conformal mapping of $D_1 = \{w \in \mathbb{C} : |w| < 1\}$ onto G such that $\Phi(0) = \xi$ and let $w = \Psi(z)$ stand for the inverse mapping.

Given harmonic function u in G , we put $f'(z) = u'_x - iu'_y$ in (1.7.3), (2.6.11), (3.7.9) and (1.7.5), (2.6.13), (3.7.12). The result is contained in the following assertion.

Corollary 4.1. *For any harmonic function u in G and any real constant c the inequalities*

$$|\nabla u(\xi)| \le 2|\Psi'(\xi)| \sup_{\zeta \in G} \{u(\zeta) - u(\xi)\},$$

$$|\nabla u(\xi)| \le \frac{4}{\pi} |\Psi'(\xi)| \sup_{\zeta \in G} |u(\zeta) - u(\xi)|, \tag{4.2.1}$$

$$|\nabla u(\xi)| \le \frac{4}{\pi} |\Psi'(\xi)| \sup_{\zeta \in G} |u(\zeta) - c| \tag{4.2.2}$$

with the sharp coefficients hold. In particular, for any $z \in D_R$,

$$|\nabla u(z)| \le \frac{2R}{R^2 - |z|^2} \sup_{|\zeta| < R} \{u(\zeta) - u(z)\}, \tag{4.2.3}$$

$$|\nabla u(z)| \le \frac{4R}{\pi(R^2 - |z|^2)} \sup_{|\zeta| < R} |u(\zeta) - u(z)|, \tag{4.2.4}$$

$$|\nabla u(z)| \le \frac{4R}{\pi(R^2 - |z|^2)} \sup_{|\zeta| < R} |u(\zeta) - c|. \tag{4.2.5}$$

As a particular case of (4.2.5) *one has*

$$|\nabla u(z)| \le \frac{2R}{\pi(R^2 - |z|^2)} \mathcal{O}_u(D_R). \tag{4.2.6}$$

Now, let G be a bounded domain in $\mathbb{R}^2$, $z \in G$, and let $d_z = \text{dist}(z, \partial G)$. According to Protter and Weinberger $\big($[76], Chapt. 2, Sect. 13$\big)$,

$$|\nabla u(z)| \le \frac{2}{\pi d_z} \mathcal{O}_u(G) \tag{4.2.7}$$

for any harmonic function u in G, where $\mathcal{O}_u(G)$ is the oscillation of the function u on G.

The estimate (4.2.7) can be improved. This simple application of inequality (4.2.6) is given in the next assertion.

Corollary 4.2. *Let z be a fixed point in a bounded domain $G \subset \mathbb{R}^2$, and let η be a point on ∂G for which $|\eta - z| = d_z$. Further, let R be the radius of the largest disk lying entirely in G with center on the straight line L passing through z and η. Then, for any harmonic function u in G*

$$|\nabla u(z)| \le \frac{2R}{\pi d_z (2R - d_z)} \mathcal{O}_u(G). \tag{4.2.8}$$

The last estimate follows from (4.2.6) with $|z| = R - d_z,\ R \geq d_z$, and the inequality $\mathcal{O}_u(D_R) \leq \mathcal{O}_u(G)$. We note that the factor

$$\frac{2R}{\pi(2R - d_z)}$$

before $1/d_z$ in (4.2.8) tends to $1/\pi$, as $z \to \eta,\ z \in L$, whereas the similar coefficient in (4.2.7) is equal to the constant $2/\pi$.

Analogously, from (4.2.5) we obtain the interior estimate for the gradient

$$|\nabla u(z)| \leq \frac{4R}{\pi d_z(2R - d_z)} \sup_G |u|,$$

where $z \in G$ and u is harmonic in D. The last inequality is a refinement of the estimate

$$|\nabla u(z)| \leq \frac{2}{d_z} \sup_G |u| \tag{4.2.9}$$

which can be found in Gilbarg and Trudinger ([38], Ch. 2, Sect. 2.7). We note also that (4.2.3) implies

$$|\nabla u(z)| \leq \frac{2R}{d_z(2R - d_z)} \Big\{ \sup_G u - u(z) \Big\},$$

which improves the inequality

$$|\nabla u(z)| \leq \frac{2}{d_z} \Big\{ \sup_G u - u(z) \Big\} \tag{4.2.10}$$

(see [38], Ch. 2, p. 29).

Putting $f'(z) = u'_x - iu'_y$ in (1.7.7), (2.6.15) and (3.7.15), we arrive at

Corollary 4.3. *For any harmonic function u in $\mathbb{R}^2_+$ and any real constant c the sharp inequalities*

$$|\nabla u(z)| \leq \frac{1}{y} \sup_{\zeta \in \mathbb{R}^2_+} \{u(\zeta) - u(z)\},$$

$$|\nabla u(z)| \leq \frac{2}{\pi y} \sup_{\zeta \in \mathbb{R}^2_+} |u(\zeta) - u(z)|,$$

$$|\nabla u(z)| \leq \frac{2}{\pi y} \sup_{\zeta \in \mathbb{R}^2_+} |u(\zeta) - c| \tag{4.2.11}$$

hold at any point $z = (x, y) \in \mathbb{R}^2_+$.

As a particular case of (4.2.11) *one has*

$$|\nabla u(z)| \leq \frac{1}{\pi y}\, \mathcal{O}_u(\mathbb{R}^2_+)$$

for every $z \in \mathbb{R}^2_+$.

4.3 Estimates for directional derivatives with constant direction

In this section we reformulate inequalities for analytic functions obtained in the three preceding chapters for directional derivatives of harmonic functions.

We are going to present sharp estimates for the increment of the directional derivative and the gradient of a harmonic function inside of a domain in terms of various characteristics of directional derivatives.Within this section by ℓ_ϑ we mean a unit vector at an angle ϑ with respect to the x-axis.

In the next statement we assume that G is a bounded domain in $\mathbb{R}^2$, bounded by a Jordan curve. Putting $\alpha(z) = \alpha = \text{const}$ and

$$f(z) = e^{i\beta}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) \tag{4.3.1}$$

in (1.7.1), (2.6.10) and (3.7.8), and using the equalities

$$\max_\alpha C_\infty(\gamma, \alpha) = \max_\alpha K_\infty(\gamma, \alpha) = \frac{2}{\pi}\log\frac{1+\gamma}{1-\gamma}$$

(see Corollaries 2.4, 3.6), we arrive at

Corollary 4.4. *For any harmonic function u in G any real constant c and any points $z, \xi \in G$ the sharp inequalities*

$$\frac{\partial u(z)}{\partial \ell_{\alpha+\beta}} - \frac{\partial u(\xi)}{\partial \ell_{\alpha+\beta}} \leq \frac{2|\Psi(z)|\left(1 - |\Psi(z)|\cos\alpha\right)}{1 - |\Psi(z)|^2} \sup_{\zeta\in G}\left\{\frac{\partial u(\zeta)}{\partial \ell_\beta} - \frac{\partial u(\xi)}{\partial \ell_\beta}\right\},$$

$$\left|\frac{\partial u(z)}{\partial \ell_{\alpha+\beta}} - \frac{\partial u(\xi)}{\partial \ell_{\alpha+\beta}}\right| \leq C_\infty\Big(|\Psi(z)|,\ \alpha\Big) \sup_{\zeta\in G}\left|\frac{\partial u(\zeta)}{\partial \ell_\beta} - \frac{\partial u(\xi)}{\partial \ell_\beta}\right|,$$

$$\left|\frac{\partial u(z)}{\partial \ell_{\alpha+\beta}} - \frac{\partial u(\xi)}{\partial \ell_{\alpha+\beta}}\right| \leq K_\infty\Big(|\Psi(z)|,\ \alpha\Big) \sup_{\zeta\in G}\left|\frac{\partial u(\zeta)}{\partial \ell_\beta} - c\right|$$

hold with arbitrary constants α, β and the coefficients $C_\infty(\gamma, \alpha)$, $K_\infty(\gamma, \alpha)$ defined by (2.4.2) *and* (3.4.3), *respectively.*

In particular,

$$|\nabla u(z) - \nabla u(\xi)| \leq \frac{2|\Psi(z)|}{1 - |\Psi(z)|} \sup_{\zeta\in G}\left\{\frac{\partial u(\zeta)}{\partial \ell} - \frac{\partial u(\xi)}{\partial \ell}\right\},$$

$$|\nabla u(z) - \nabla u(\xi)| \leq \frac{2}{\pi}\log\left(\frac{1 + |\Psi(z)|}{1 - |\Psi(z)|}\right) \sup_{\zeta\in G}\left|\frac{\partial u(\zeta)}{\partial \ell} - \frac{\partial u(\xi)}{\partial \ell}\right|, \tag{4.3.2}$$

$$|\nabla u(z) - \nabla u(\xi)| \leq \frac{2}{\pi} \log\left(\frac{1+|\Psi(z)|}{1-|\Psi(z)|}\right) \sup_{\zeta \in G} \left|\frac{\partial u(\zeta)}{\partial \ell} - c\right|, \tag{4.3.3}$$

where ℓ *is an arbitrary fixed unit vector.*

As a particular case of (4.3.3) *one has*

$$|\nabla u(z) - \nabla u(\xi)| \leq \frac{1}{\pi} \log\left(\frac{1+|\Psi(z)|}{1-|\Psi(z)|}\right) \mathcal{O}_{\partial u/\partial \ell}(G).$$

Remark 4.1. Each inequality in Corollary 4.4 can be written, in particular, for the disk D_R and an arbitrary point $\xi \in D_R$. We arrive at corresponding estimates putting $\Phi(w) = R(\xi - Rw)/(R - \overline{\xi}w)$ and setting $\Psi(z) = R(\xi - z)/(R^2 - z\overline{\xi})$ in Corollary 4.4. For example, inequality (4.3.2) takes the form

$$|\nabla u(z) - \nabla u(\xi)| \leq \frac{2}{\pi} \log\left(\frac{|R^2 - z\overline{\xi}| + R|\xi - z|}{|R^2 - z\overline{\xi}| - R|\xi - z|}\right) \sup_{|\zeta|<R} \left|\frac{\partial u(\zeta)}{\partial \ell} - \frac{\partial u(\xi)}{\partial \ell}\right|.$$

Next we write explicit sharp estimates for the increment of the directional derivative and the gradient of a harmonic function in the half-plane. Putting $\alpha(z) = \alpha = \text{const}$ and combining (4.3.1) with (1.7.6), (2.6.14), and (3.7.14) we obtain the following assertion.

Corollary 4.5. *For any harmonic function* u *in* $\mathbb{R}^2_+$ *, any real constant* c *and any* $z, \xi \in \mathbb{R}^2_+$ *the sharp inequalities*

$$\frac{\partial u(z)}{\partial \ell_{\alpha+\beta}} - \frac{\partial u(\xi)}{\partial \ell_{\alpha+\beta}} \leq \frac{2|z-\xi|\big(|z-\overline{\xi}| - |z-\xi|\cos\alpha\big)}{|z-\overline{\xi}|^2 - |z-\xi|^2} \sup_{\zeta \in \mathbb{R}^2_+} \left\{\frac{\partial u(\zeta)}{\partial \ell_\beta} - \frac{\partial u(\xi)}{\partial \ell_\beta}\right\},$$

$$\left|\frac{\partial u(z)}{\partial \ell_{\alpha+\beta}} - \frac{\partial u(\xi)}{\partial \ell_{\alpha+\beta}}\right| \leq C_\infty\left(\frac{|z-\xi|}{|z-\overline{\xi}|},\ \alpha\right) \sup_{\zeta \in \mathbb{R}^2_+} \left|\frac{\partial u(\zeta)}{\partial \ell_\beta} - \frac{\partial u(\xi)}{\partial \ell_\beta}\right|,$$

$$\left|\frac{\partial u(z)}{\partial \ell_{\alpha+\beta}} - \frac{\partial u(\xi)}{\partial \ell_{\alpha+\beta}}\right| \leq K_\infty\left(\frac{|z-\xi|}{|z-\overline{\xi}|},\ \alpha\right) \sup_{\zeta \in \mathbb{R}^2_+} \left|\frac{\partial u(\zeta)}{\partial \ell_\beta} - c\right|$$

hold with arbitrary constants α, β *and the coefficients* $C_\infty(\gamma, \alpha)$, $K_\infty(\gamma, \alpha)$ *defined by* (2.4.2) *and* (3.4.3)*, respectively.*

In particular,

$$|\nabla u(z) - \nabla u(\xi)| \leq \frac{2|z-\xi|}{|z-\overline{\xi}| - |z-\xi|} \sup_{\zeta \in \mathbb{R}^2_+} \left\{\frac{\partial u(\zeta)}{\partial \ell} - \frac{\partial u(\xi)}{\partial \ell}\right\},$$

$$|\nabla u(z) - \nabla u(\xi)| \leq \frac{2}{\pi} \log \left(\frac{|z - \overline{\xi}| + |z - \xi|}{|z - \overline{\xi}| - |z - \xi|} \right) \sup_{\zeta \in \mathbb{R}^2_+} \left| \frac{\partial u(\zeta)}{\partial \ell} - \frac{\partial u(\xi)}{\partial \ell} \right|,$$

$$|\nabla u(z) - \nabla u(\xi)| \leq \frac{2}{\pi} \log \left(\frac{|z - \overline{\xi}| + |z - \xi|}{|z - \overline{\xi}| - |z - \xi|} \right) \sup_{\zeta \in \mathbb{R}^2_+} \left| \frac{\partial u(\zeta)}{\partial \ell} - c \right|, \tag{4.3.4}$$

where ℓ is an arbitrary fixed unit vector.

As a special case of (4.3.4) *one has*

$$|\nabla u(z) - \nabla u(\xi)| \leq \frac{1}{\pi} \log \left(\frac{|z - \overline{\xi}| + |z - \xi|}{|z - \overline{\xi}| - |z - \xi|} \right) \mathcal{O}_{\partial u/\partial \ell}(\mathbb{R}^2_+).$$

4.4 Estimates for directional derivatives with varying direction

Here we collect another group of sharp inequalities for directional derivatives of functions harmonic in a disk. Unlike estimates in the previous section, we take directional derivatives on the boundary in the directions having a constant angle with respect to the radial vector.

In this section we put $\alpha(z) = \alpha = \text{const}$. By l_ϑ we denote a unit vector having a constant angle ϑ with the radial direction. As before, we use the notation $|z| = r$.

The corollary below is based on an inequality for analytic functions obtained in Section 1.4. This assertion contains an estimate of the directional derivative of a harmonic function in the disk D_R by the maximum of the directional derivative on ∂D_R. The direction vector in question has a constant angle with the radius.

Corollary 4.6. *Let u be either a harmonic function on D_R in $C^1(\overline{D}_R)$, or a harmonic function on $\mathbb{R}^2 \backslash \overline{D}_R$ in $C^1(\mathbb{R}^2 \backslash D_R)$. There hold sharp inequalities*

$$\frac{\partial u(z)}{\partial l_{\alpha+\beta}} \leq \frac{2R(R - r\cos\alpha)}{R^2 - r^2} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta}, \qquad r < R, \tag{4.4.1}$$

$$\frac{\partial u(z)}{\partial l_{\alpha+\beta}} \leq \frac{2R^2(r - R\cos\alpha)}{r(r^2 - R^2)} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta}, \qquad r > R, \tag{4.4.2}$$

where l_β can be directed, for instance, either normally or tangentially to ∂D_R.

In particular ,

$$|\nabla u(z)| \leq \frac{2R}{R - r} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta}, \qquad r < R, \tag{4.4.3}$$

$$|\nabla u(z)| \le \frac{2R^2}{r(r-R)} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta}, \qquad r > R. \tag{4.4.4}$$

Proof. Inequality (4.4.1) can be deduced from (1.4.10) by putting

$$f(z) = e^{i\beta} z \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right), \tag{4.4.5}$$

where $\beta \in [0, 2\pi]$ and u is a harmonic function on D_R in $C^1(\overline{D}_R)$. It follows from the sharpness of the constant in (1.4.10) that the factor $2R(R - r\cos\alpha)/(R^2 - r^2)$ in (4.4.1) cannot be diminished.

Let $u(z)$ be a harmonic function on $\mathbb{R}^2 \backslash \overline{D}_R$ in the class $C^1(\mathbb{R}^2 \backslash D_R)$. By $z^* \in \overline{D}_R$ we denote a point symmetric to z with respect to the circle ∂D_R: $z^* = R^2/\,\overline{z}$. The Kelvin transform of $u(z)$ is $u^*(z^*) = u(R^2 z^*/|z^*|^2)$. Let (r, φ) and (r^*, φ) be polar coordinates of z and z^*, and let $\tilde{u}(r, \varphi) = u(z)$, $\tilde{u}^*(r^*, \varphi) = u^*(z^*)$. Since $\tilde{u}(r, \varphi) = \tilde{u}^*(r^*, \varphi)$ and $rr^* = R^2$ we have

$$\tilde{u}^*(r^*, \varphi) = \tilde{u}(R^2/r^*, \varphi).$$

This implies

$$\nabla u^*(z^*) = \frac{\partial \tilde{u}^*}{\partial r^*} e_{r^*} + \frac{1}{r^*} \frac{\partial \tilde{u}^*}{\partial \varphi} e_\varphi = -\frac{r^2}{R^2} \frac{\partial \tilde{u}}{\partial r} e_r + \frac{r}{R^2} \frac{\partial \tilde{u}}{\partial \varphi} e_\varphi$$

and therefore

$$\nabla u^*(z^*) = \frac{r^2}{R^2} \left(-\frac{\partial \tilde{u}}{\partial r} e_r + \frac{1}{r} \frac{\partial \tilde{u}}{\partial \varphi} e_\varphi \right). \tag{4.4.6}$$

Let $\eta = (\eta_1, \eta_2)$, $|\eta| = 1$ and let the unit vector $\eta_1 e_r + \eta_2 e_\varphi$ form a constant angle β with the radial direction, i.e. $\eta_1 e_r + \eta_2 e_\varphi = l_\beta$. Thus, $-\eta_1 e_r + \eta_2 e_\varphi = l_{\pi-\beta}$ and by (4.4.6)

$$\frac{\partial u^*(z^*)}{\partial l_{\pi-\beta}} = \frac{r^2}{R^2} \left(\frac{\partial \tilde{u}}{\partial r}\, \eta_1 + \frac{1}{r} \frac{\partial \tilde{u}}{\partial \varphi}\, \eta_2 \right) = \frac{r^2}{R^2}\, \frac{\partial u(z)}{\partial l_\beta}. \tag{4.4.7}$$

Therefore

$$\frac{\partial u^*(z^*)}{\partial l_{\pi-\beta}}\Big|_{r^*=R} = \frac{\partial u(z)}{\partial l_\beta}\Big|_{r=R}. \tag{4.4.8}$$

By (4.4.1)

$$\frac{\partial u^*(z^*)}{\partial l_{\alpha+\pi-\beta}} \le \frac{2R(R - r^* \cos\alpha)}{R^2 - {r^*}^2} \max_{\partial D_R} \frac{\partial u^*(z^*)}{\partial l_{\pi-\beta}}.$$

This and (4.4.7), (4.4.8) imply

$$\frac{r^2}{R^2}\, \frac{\partial u(z)}{\partial l_{\beta-\alpha}} \le \frac{2R(R - r^* \cos\alpha)}{R^2 - {r^*}^2} \max_{\partial D_R} \frac{\partial u(z)}{\partial l_\beta}.$$

Replacing here α by $-\alpha$ and using $rr^* = R^2$ we arrive at (4.4.2) with the sharp constant.

Inequalities (4.4.3), (4.4.4) follow immediately from (4.4.1), (4.4.2). □

Using the equality

$$\frac{\partial u(z)}{\partial l_{\pi+\vartheta}} = -\frac{\partial u(z)}{\partial l_{\vartheta}},$$

and putting $\pi + \alpha$ in place of α in (4.4.1) and (4.4.2), we obtain the following lower estimates for the directional derivative

$$\frac{\partial u(z)}{\partial l_{\alpha+\beta}} \geq -\frac{2R(R + r\cos\alpha)}{R^2 - r^2} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta}, \qquad r < R, \tag{4.4.9}$$

$$\frac{\partial u(z)}{\partial l_{\alpha+\beta}} \geq -\frac{2R^2(R + r\cos\alpha)}{r(r^2 - R^2)} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta}, \qquad r > R. \tag{4.4.10}$$

Setting $\alpha = 0$ in (4.4.1), (4.4.2) and (4.4.9), (4.4.10), we get the two-sided estimates

$$-\frac{2R}{R-r} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta} \leq \frac{\partial u(z)}{\partial l_\beta} \leq \frac{2R}{R+r} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta}, \qquad r < R, \tag{4.4.11}$$

$$-\frac{2R^2}{r(r-R)} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta} \leq \frac{\partial u(z)}{\partial l_\beta} \leq \frac{2R^2}{r(R+r)} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta}. \qquad r > R, \tag{4.4.12}$$

Similarly, putting $\alpha = \pi/2$ in (4.4.1), (4.4.2) and (4.4.9), (4.4.10), we obtain

$$\left|\frac{\partial u(z)}{\partial l_{\frac{\pi}{2}+\beta}}\right| \leq \frac{2R^2}{R^2 - r^2} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta}, \qquad r < R, \tag{4.4.13}$$

$$\left|\frac{\partial u(z)}{\partial l_{\frac{\pi}{2}+\beta}}\right| \leq \frac{2R^2}{r^2 - R^2} \max_{\partial D_R} \frac{\partial u}{\partial l_\beta}, \qquad r > R. \tag{4.4.14}$$

All inequalities (4.4.9)-(4.4.14) are valid under assumptions in Corollary 4.6.

The next assertion is based on inequalities for analytic functions derived in Sections 2.3-2.5. We give an estimate of the directional derivative of a harmonic function inside or outside the circle ∂D_R by the L_p-norm of the directional derivative on ∂D_R.

Corollary 4.7. *Let u be either a harmonic function on D_R with*

$$\Re\{ze^{i\beta}(u'_x - iu'_y)\} \in h_p(D_R)$$

or a harmonic function on $\mathbb{R}^2\backslash\overline{D}_R$ *with*

$$\Re\{ze^{i\beta}(u'_x - iu'_y)\} \in h_p(\mathbb{R}^2\backslash\overline{D}_R),$$

$1 \le p \le \infty$. *There the sharp inequalities*

$$\left|\frac{\partial u(z)}{\partial l_{\alpha+\beta}}\right| \le \frac{R^{(p-1)/p}}{r} C_p\left(\frac{r}{R},\, \alpha\right)\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_p, \qquad r < R, \tag{4.4.15}$$

$$\left|\frac{\partial u(z)}{\partial l_{\alpha+\beta}}\right| \le \frac{R^{(p-1)/p}}{r} C_p\left(\frac{R}{r},\, \alpha\right)\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_p, \qquad r > R, \tag{4.4.16}$$

hold, where the coefficient $C_p(\gamma, \alpha)$ *is given by* (2.2.3). *Here* l_β *can be directed, for instance, either normally or tangentially to* ∂D_R.

As particular cases of (4.4.15) *and* (4.4.16) *one has*

$$|\nabla u(z)| \le \frac{R}{\pi(R^2 - r^2)}\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_1, \tag{4.4.17}$$

$$|\nabla u(z)| \le \sqrt{\frac{R}{\pi(R^2 - r^2)}}\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_2, \tag{4.4.18}$$

$$|\nabla u(z)| \le \frac{2R}{\pi r}\log\left(\frac{R + r}{R - r}\right)\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_\infty, \tag{4.4.19}$$

for $r < R$, *and*

$$|\nabla u(z)| \le \frac{R}{\pi(r^2 - R^2)}\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_1, \tag{4.4.20}$$

$$|\nabla u(z)| \le \frac{R}{r}\sqrt{\frac{R}{\pi(r^2 - R^2)}}\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_2, \tag{4.4.21}$$

$$|\nabla u(z)| \le \frac{2R}{\pi r}\log\left(\frac{r + R}{r - R}\right)\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_\infty, \tag{4.4.22}$$

for $r > R$.

Proof. Applying inequality (2.2.1) to the function

$$f(z) = e^{i\beta}z\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right),$$

we obtain

$$\left|\frac{\partial u(z)}{\partial l_{\alpha+\beta}}\right| \leq \frac{\mathcal{C}_p\,(z,\alpha)\,R}{|z|}\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_p. \tag{4.4.23}$$

Taking into account (2.2.2) and (2.2.3) we arrive at inequality (4.4.15).

Now we pass to the case of a harmonic function outside a disk. Keeping the notation used in Corollary 4.6 and replacing β by $\pi-\beta$ in (4.4.23) we get

$$\left|\frac{\partial u^*(z^*)}{\partial l_{\alpha+\pi-\beta}}\right| \leq \frac{\mathcal{C}_p\,(z^*,\alpha)\,R}{|z^*|}\left|\left|\frac{\partial u^*}{\partial l_{\pi-\beta}}\right|\right|_p,$$

which together with (4.4.7) and (4.4.8) implies

$$\frac{r^2}{R^2}\left|\frac{\partial u(z)}{\partial l_{\beta-\alpha}}\right| \leq \frac{\mathcal{C}_p\,(z^*,\alpha)\,R}{|z^*|}\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_p.$$

Using the relation $\mathcal{C}_p(z,-\alpha)=\mathcal{C}_p(z,\alpha)$, proved in Proposition 2.1, and the equality $rr^*=R^2$ we rewrite the last inequality as

$$\left|\frac{\partial u(z)}{\partial l_{\beta+\alpha}}\right| \leq \frac{\mathcal{C}_p\,(z^*,\alpha)\,R}{r}\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_p. \tag{4.4.24}$$

Using again (2.2.2), (2.2.3) and the equality $rr^*=R^2$ we deduce

$$\mathcal{C}_p\,(z^*,\alpha)=\frac{1}{R^{1/p}}C_p\left(\frac{R}{r},\alpha\right).$$

Combining the last equality and (4.4.24) we arrive at (4.4.16).

Sharp inequalities for the gradient (4.4.17), (4.4.18) and (4.4.20), (4.4.21) follow directly from (4.4.15), (4.4.16) and (2.3.3), (2.3.9) together with (2.2.2). Finally, sharp estimates (4.4.19), (4.4.22) result from (4.4.15), (4.4.16) together with relation (2.4.21). □

Putting $\alpha=0, p=\infty$ in (4.4.15), (4.4.16) and using (2.4.2) we obtain the sharp estimates

$$\left|\frac{\partial u(z)}{\partial l_\beta}\right| \leq \frac{4R}{\pi r}\arctan\left(\frac{r}{R}\right)\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_\infty, \qquad r<R,$$

$$\left|\frac{\partial u(z)}{\partial l_\beta}\right| \leq \frac{4R}{\pi r}\arctan\left(\frac{R}{r}\right)\left|\left|\frac{\partial u}{\partial l_\beta}\right|\right|_\infty, \qquad r>R. \tag{4.4.25}$$

Note that the constant in (4.4.25) does not exceed the unity, hence $|\partial u/\partial l_\beta|$ obeys the maximum principle outside the disk.

For $\alpha = \pi/2$ the constant $C_p(\gamma, \pi/2) = C_{\Im,p}(\gamma)$ in (4.4.15) and (4.4.16) was found in Section 2.5. In particular, from (4.4.15), (4.4.16) and (2.4.2) we obtain the sharp inequalities

$$\left| \frac{\partial u(z)}{\partial l_{\frac{\pi}{2}+\beta}} \right| \leq \frac{2R}{\pi r} \log\left(\frac{R+r}{R-r} \right) \left\| \frac{\partial u}{\partial l_\beta} \right\|_\infty , \qquad r < R,$$

$$\left| \frac{\partial u(z)}{\partial l_{\frac{\pi}{2}+\beta}} \right| \leq \frac{2R}{\pi r} \log\left(\frac{r+R}{r-R} \right) \left\| \frac{\partial u}{\partial l_\beta} \right\|_\infty , \qquad r > R.$$

5

Estimates for derivatives of analytic functions

5.1 Introduction

In Chapters 1-3 we derived estimates (1.7.5), (2.6.13), (3.7.12) for the modulus of the first derivative of an analytic function in the disk D_R with sharp factors in the right hand-sides. In particular, the estimate

$$|f'(z)| \leq \frac{4R}{\pi(R^2 - |z|^2)} \sup_{|\zeta|=R} |\Re f(\zeta) - c|$$

and its corollary

$$|f'(z)| \leq \frac{2R}{\pi(R^2 - |z|^2)} \mathcal{O}_{\Re f}(D_R)$$

are closely related to the questions we address in the present chapter. Here we obtain sharp pointwise estimates for the modulus of higher derivatives of an analytic function f in the disk D_R. The right-hand sides in these estimates involve the $L_p(\partial D_R)$-norm of the real part of the difference of f and a polynomial. Similar sharp estimates with the $L_p(\partial D_R)$-norm of $|f|$ in the right-hand side were obtained by Makintyre and Rogosinski [68] and Szász [85].

We mention some known estimates for $|f^{(n)}(0)|, n \geq 1$. If the real part of f is positive in D_R, then the following Carathéodory inequality holds

$$|f^{(n)}(0)| \leq \frac{2n!}{R^n} \Re f(0) \tag{5.1.1}$$

(see Carathéodory [23]). Another known estimate

$$|f^{(n)}(0)| \leq \frac{2n!}{R^n} \max_{|\zeta|=R} \Re\big\{f(\zeta) - f(0)\big\} \tag{5.1.2}$$

for functions f analytic on $\overline{D}_R$ (see, for example, Holland [46], Ch. 3, Ingham [49], Ch. 3, Rajagopal [77]), is closely connected with the Carathéodory inequality (5.1.1) and the Landau inequality

$$|f^{(n)}(0)| \leq \frac{2n!}{R^n} \max_{|\zeta|=R} \big\{|f(\zeta)| - |f(0)|\big\} \tag{5.1.3}$$

(see Landau [61], I, § 4, pp. 33-34). Namely, the Carathéodory inequality and the Landau inequality are corollaries of (5.1.2). Indeed, replacing f by $fe^{i\alpha}$ in (5.1.2), estimating then $\Re\{f(\zeta)e^{i\alpha}\}$ by $|f(\zeta)|$ and using the equality $\max\{e^{i\alpha}\Re f(0) : \alpha \in \mathbb{R}\} = |f(0)|$ we deduce (5.1.3). Further, setting $-f$ in place of f in (5.1.2) we obtain

$$|f^{(n)}(0)| \leq \frac{2n!}{R^n} \max_{|\zeta|=R} \Re\big\{f(0) - f(\zeta)\big\}, \tag{5.1.4}$$

which implies (5.1.1) whenever $\Re f(\zeta) \geq 0$ for $|\zeta| = R$. Another proof of (5.1.1) is given by Aizenberg, Aytuna and Djakov [4]. For sharp estimates of $|f^{(n)}(z)|/\Re f(z)$, where f is analytic in D_R with $\Re f > 0$ see Ruscheweyh [81], Yamashita [88].

Among estimates of $|f^{(n)}(z)|$ by values of $\Re f$ on the circle $|\zeta| = R$, there is a corollary of the Hadamard-Borel-Carathéodory inequality (1.1.3) (see, e.g., Holland [46], Ch. 3, Titchmarsh [86], Ch. 5)

$$|f^{(n)}(z)| \leq \frac{2^{n+2} n! R}{(R-r)^{n+1}} \max_{|\zeta|=R} \big\{\Re f(\zeta) + |f(0)|\big\}, \tag{5.1.5}$$

where $|z| = r < R, n \geq 1$, f is an analytic in D_R and $\Re f \in C(\overline{D}_R)$.

We note also that the inequality

$$|f^{(n)}(0)| \leq \frac{n!}{R^n} \frac{[\mathcal{M}_f(R)]^2 - |f(0)|^2}{\mathcal{M}_f(R)}, \tag{5.1.6}$$

with f bounded on D and $\mathcal{M}_f(R)$ being the supremum of $|f(z)|$ on D, was obtained by Landau (see [59], pp. 305-306) for $n = 1$ and by F. Wiener (see Bohr [17], Jensen [50]) for all n. A proof of (5.1.6), different from that given by Wiener was found by Paulsen, Popescu and Singh [72]. A generalization of (5.1.6) is due to Jensen [50].

As corollary of (5.1.6), Rajagopal [78] obtained the inequality

$$|f^{(n)}(z)| \leq \frac{n! R}{(R-r)^{n+1}} \frac{[\mathcal{M}_f(R)]^2 - |f(0)|^2}{\mathcal{M}_f(R)} \tag{5.1.7}$$

for derivatives in any point $z \in D_R$. Another direction of generalizing (5.1.6) is related to the following so called invariant form of Schwarz's lemma due to Pick (see Garnett [37], Ch. 1, § 1 and Jensen [50], Lindelöf [63]):

$$|f'(z)| \leq \frac{R}{R^2 - r^2} \frac{[\mathcal{M}_f(R)]^2 - |f(z)|^2}{\mathcal{M}_f(R)}. \tag{5.1.8}$$

The following sharp estimate for derivatives of f at an arbitrary point of D_R, which includes (5.1.6) and (5.1.8),

$$|f^{(n)}(z)| \le \frac{n!R}{(R+r)(R-r)^n}\,\frac{[\mathcal{M}_f(R)]^2-|f(z)|^2}{\mathcal{M}_f(R)}, \tag{5.1.9}$$

is due to Ruscheweyh [82] who applied classical methods. A different approach to Schwarz-Pick type estimates and their generalizations was worked out by Anderson and Rovnyak [10], Bénéteau, Dahlner and Khavinson [12] as well as by MacCluer, Stroethoff and Zhao [65]. Extensions to several variables can be found in the articles by Bénéteau, Dahlner and Khavinson [12] and MacCluer, Stroethoff and Zhao [65].

In the present chapter, we consider analytic functions in D_R with the real part in $h_p(D_R)$, and obtain estimates with sharp constants for $|f^{(n)}(z)|, n \ge 1$, $z \in D_R$, formulated in terms of various characteristics of $\Re f$ on the circle $|\zeta| = R$. As before, $|z| = r < R$ and $||\cdot||_p$ denote the L_p-norm of a real valued function on the circle $|\zeta| = R$, where $1 \le p \le \infty$.

In Section 2 of this chapter we find a representation for the best constant in the inequality

$$|f^{(n)}(z)| \le \mathcal{H}_{n,p}(z)||\Re\{f-\mathcal{P}_m\}||_p, \tag{5.1.10}$$

where $n \ge 1$, and $\mathcal{P}_m$ is a polynomial of degree $m, m \le n-1$. From (5.1.10) we obtain estimates with right-hand sides containing the best polynomial approximation of $\Re f$ on the circle $|\zeta| = R$ in the $L_p(\partial D_R)$-norm

$$|f^{(n)}(z)| \le \mathcal{H}_{n,p}(z)E_{n-1,p}(\Re f) \tag{5.1.11}$$

with $n \ge 1$. Here and henceforth

$$E_{k,p}(\Re f) = \inf_{\mathcal{P}\in\{\mathfrak{P}_k\}} ||\Re\{f-\mathcal{P}\}||_p,$$

where the infimum is taken over the set $\{\mathfrak{P}_k\}$ of all polynomials of degree not higher than k.

In Section 3 we find the values $\mathcal{H}_{n,p}(0)$ for $1 \le p \le \infty$. For instance,

$$\mathcal{H}_{n,1}(0) = \frac{n!}{\pi R^{n+1}}, \qquad \mathcal{H}_{n,2}(0) = \frac{n!}{\sqrt{\pi}R^{(2n+1)/2}}, \qquad \mathcal{H}_{n,\infty}(0) = \frac{4n!}{\pi R^n}.$$

Section 4 concerns corollaries of inequality (5.1.10) for $p = 1$. First, we prove the equality

$$\mathcal{H}_{n,1}(z) = \frac{n!}{\pi(R-r)^{n+1}},$$

where $|z| = r < R$. From (5.1.10) with $p = 1$ and $m = 0$ we deduce the sharp estimate

$$|f^{(n)}(z)| \le \frac{2n!R}{(R-r)^{n+1}} \sup_{|\zeta|<R} \Re\{f(\zeta)-f(0)\}, \tag{5.1.12}$$

where $n \geq 1$, and z is a fixed point of the circle $|z| = r < R$. The last inequality can be viewed as a generalization of the real part theorem (1.1.1) to higher order derivatives. We also show that the sharp constants in (5.1.5) and (5.1.12) coincide. For other proofs of (5.1.12) see Ingham ([49], Ch. 3) and Rajagopal [77].

Similar sharp estimates are obtained when the right-hand side of (5.1.12) involves the expressions

$$\sup_{|\zeta|<R} \big\{ |\Re f(\zeta)| - |\Re f(0)| \big\}, \qquad \sup_{|\zeta|<R} \big\{ |f(\zeta)| - |f(0)| \big\},$$

as well as $\Re f(0)$ provided that $\Re f(\zeta) > 0$ for $|\zeta| < R$. As particular cases, the estimates just mentioned contain the Landau inequality (5.1.3), and the Carathéodory inequality (5.1.1).

The lower estimates for the constants in (5.1.12) and above mentioned similar estimates are obtained with the help of a family of test functions which are analytic in $\overline{D}_R$.

In Section 5 we deduce corollaries of (5.1.11) for $p = 2$. In particular, we show that the inequality (5.1.11) with $p = 2$ holds with the sharp constant

$$\mathcal{H}_{n,2}(z) = \frac{1}{R^{(2n+1)/2}} H_{n,2}\left(\frac{r}{R}\right),$$

where

$$H_{n,2}(\gamma) = \frac{n!}{\sqrt{\pi}(1-\gamma^2)^{(2n+1)/2}} \left\{ \sum_{k=0}^{n} \binom{n}{k}^2 \gamma^{2k} \right\}^{1/2},$$

and

$$E_{n,2}(\Re f) = \left|\left| \Re \left\{ f - \sum_{k=0}^{n} \frac{f^{(k)}(0)\zeta^k}{k!} \right\} \right|\right|_2 .$$

In Section 6 of the chapter, we deduce corollaries of (5.1.11) for $p = \infty$. They contain an estimate for $|f^{(n)}(z)|$ formulated in terms of

$$\mathcal{O}_{n,\Re f}(D_R) = \inf_{\mathcal{P} \in \{\mathfrak{P}_n\}} \mathcal{O}_{\Re\{f-\mathcal{P}\}}(D_R),$$

where $\mathcal{O}_{\Re f}(D_R)$ is the oscillation of $\Re f$ on the disk D_R.

5.2 Estimate for $|f^{(n)}(z)|$ by $||\Re\{f - \mathcal{P}_m\}||_p$. General case

In the sequel, we use the notation

$$E_{m,p}(\Re f) = \inf_{\mathcal{P} \in \{\mathfrak{P}_m\}} ||\Re\{f - \mathcal{P}\}||_p \tag{5.2.1}$$

for the best approximation of $\Re f$ by the real part of algebraic polynomials in $L_p(\partial D_R)$-norm, where $\{\mathfrak{P}_m\}$ is the set of all polynomials of degree at most m. The notation $E_p(\Re f)$, introduced previously, coincides with $E_{0,p}(\Re f)$.

In what follows by $\mathcal{P}_m$ we denote a polynomial of degree m.

The following lemma will be used in the next chapters, in particular, in the proof of the main assertion of this chapter.

Lemma 5.1. *Let f be analytic on D_R with $\Re f \in h_p(D_R), 1 \le p \le \infty$. Then for all real α and for any point $z \in D_R$ there holds*

$$\Re\{e^{i\alpha} f^{(n)}(z)\} = \frac{n!}{\pi R} \int_{|\zeta|=R} \Re\left\{\frac{\zeta e^{i\alpha}}{(\zeta - z)^{n+1}}\right\} \Re f(\zeta)|d\zeta|, \quad (5.2.2)$$

where $n \ge 1$, and

$$\Re\{e^{i\alpha} \Delta f^{(n)}(z)\} = \frac{n!}{\pi R} \int_{|\zeta|=R} \Re\left\{\frac{\zeta^{n+1} - (\zeta - z)^{n+1}}{(\zeta - z)^{n+1}\zeta^n} e^{i\alpha}\right\} \Re f(\zeta)|d\zeta|, \quad (5.2.3)$$

where $n \ge 0$.

Proof. Differentiating with respect to the parameter z in the right-hand side of (1.3.1) we obtain

$$f^{(n)}(z) = \frac{n!}{\pi R} \int_{|\zeta|=R} \frac{\zeta}{(\zeta - z)^{n+1}} \Re f(\zeta)|d\zeta|, \quad (5.2.4)$$

which leads to (5.2.2).

By (5.2.2), for $n \ge 1$ we have

$$\Re\{e^{i\alpha}(f^{(n)}(z) - f^{(n)}(0)\} = \frac{n!}{\pi R} \int_{|\zeta|=R} \Re\left\{\frac{\zeta^{n+1} - (\zeta - z)^{n+1}}{(\zeta - z)^{n+1}\zeta^n} e^{i\alpha}\right\} \Re f(\zeta)|d\zeta|.$$

In view of Lemma 1.1, the last equality, i.e. (5.2.3), holds also for $n = 0$. $\square$

We introduce the notation

$$\mathcal{G}_{n,z,\alpha}(\zeta) = \Re\left\{\frac{\zeta e^{i\alpha}}{(\zeta - z)^{n+1}}\right\}, \quad (5.2.5)$$

where $|\zeta| = R, |z| < R$ and α is a real constant.

The main objective of this section is

Proposition 5.1. *Let f be analytic on D_R with $\Re f \in h_p(D_R), 1 \le p \le \infty$. $1 \le p \le \infty$. Further, let $n \ge 1$, and let $\mathcal{P}_m$ be a polynomial of degree $m \le n-1$. Then for any fixed point $z, |z| = r < R$, the inequality*

$$|f^{(n)}(z)| \le \mathcal{H}_{n,p}(z)||\Re\{f - \mathcal{P}_m\}||_p \quad (5.2.6)$$

holds with the sharp factor

$$\mathcal{H}_{n,p}(z) = \frac{1}{R^{(np+1)/p}} H_{n,p}\left(\frac{r}{R}\right), \tag{5.2.7}$$

where

$$H_{n,p}(\gamma) = \frac{n!}{\pi} \sup_{\alpha} \left\{ \int_{|\zeta|=1} \left| \Re\left\{ \frac{\zeta e^{i\alpha}}{(\zeta-\gamma)^{n+1}} \right\} \right|^q |d\zeta| \right\}^{1/q}, \tag{5.2.8}$$

and $1/p + 1/q = 1$.

In particular,

$$|f^{(n)}(z)| \le \mathcal{H}_{n,p}(z) E_{n-1,p}(\Re f). \tag{5.2.9}$$

Proof. Using Lemma 5.1 and notation (5.2.5), we have

$$|f^{(n)}(z)| = \frac{n!}{\pi R} \sup_{\alpha} \int_{|\zeta|=R} \mathcal{G}_{n,z,\alpha}(\zeta)\, \Re f(\zeta)\, |d\zeta|. \tag{5.2.10}$$

The last equality implies the representation

$$\mathcal{H}_{n,p}(z) = \frac{n!}{\pi R} \sup_{\alpha} ||\mathcal{G}_{n,z,\alpha}||_q \tag{5.2.11}$$

for the sharp constant $\mathcal{H}_{n,p}(z)$ in

$$|f^{(n)}(z)| \le \mathcal{H}_{n,p}(z) ||\Re f||_p. \tag{5.2.12}$$

Suppose $1 < p \le \infty$. The case $p = 1$ $(q = \infty)$ in (5.2.11) is handled by passage to the limit.

Representation (5.2.11) can be written, in view of (5.2.5), as

$$\mathcal{H}_{n,p}(z) = \frac{n!}{\pi R} \sup_{\beta} \left\{ \int_{|\xi|=R} \left| \Re\left\{ \frac{\xi e^{i\beta}}{(\xi - z)^{n+1}} \right\} \right|^q |d\xi| \right\}^{1/q}. \tag{5.2.13}$$

We rewrite this representation to have it in the form stated in Proposition. Setting $z = re^{i\tau}, \xi = Re^{it}, \varphi = t - \tau$ in (5.2.13), we obtain

$$\begin{aligned}
\mathcal{H}_{n,p}(z) &= \frac{n!}{\pi R} \sup_{\beta} \left\{ \int_{\tau}^{2\pi+\tau} \left| \Re\left\{ \frac{Re^{it} e^{i\beta}}{(Re^{it} - re^{i\tau})^{n+1}} \right\} \right|^q R dt \right\}^{1/q} \\
&= \frac{n!}{\pi R} \sup_{\beta} \left\{ \int_{0}^{2\pi} \left| \Re\left\{ \frac{Re^{i\varphi} e^{i(\beta - n\tau)}}{(Re^{i\varphi} - r)^{n+1}} \right\} \right|^q R d\varphi \right\}^{1/q} \\
&= \frac{n!}{\pi R} \sup_{\beta} \left\{ \int_{|\zeta|=R} \left| \Re\left\{ \frac{\zeta e^{i(\beta - n\tau)}}{(\zeta - r)^{n+1}} \right\} \right|^q |d\zeta| \right\}^{1/q},
\end{aligned}$$

where $\zeta = Re^{i\varphi}$. Putting here $\alpha = \beta - n\tau$ and using 2π-periodicity of the resulting function in α, we find

$$\mathcal{H}_{n,p}(z) = \frac{n!}{\pi R} \sup_{\alpha} \left\{ \int_{|\zeta|=R} \left| \Re \left\{ \frac{\zeta e^{i\alpha}}{(\zeta - r)^{n+1}} \right\} \right|^q |d\zeta| \right\}^{1/q}. \qquad (5.2.14)$$

Adopting the notation

$$H_{n,p}(\gamma) = \frac{n!}{\pi} \sup_{\alpha} \left\{ \int_{|\zeta|=1} \left| \Re \left\{ \frac{\zeta e^{i\alpha}}{(\zeta - \gamma)^{n+1}} \right\} \right|^q |d\zeta| \right\}^{1/q}, \qquad (5.2.15)$$

where $\gamma = r/R$, we rewrite (5.2.14) as

$$\mathcal{H}_{n,p}(z) = \frac{1}{R^{(np+1)/p}} H_{n,p}\left(\frac{r}{R}\right),$$

which together with (5.2.15) proves (5.2.7) and (5.2.8).

Replacing f by $f - \mathcal{P}_m$ with $m \le n-1$ in (5.2.12), we arrive at inequality (5.2.6), which leads immediately to (5.2.9). □

5.3 Estimate for $|f^{(n)}(0)|$ by $||\Re\{f-\mathcal{P}_m\}||_p$

The following assertion contains an estimate for $|f^{(n)}(0)|$ with explicit sharp constant as a consequence of the representation for $\mathcal{H}_{n,p}(z)$ given in Proposition 5.1.

Corollary 5.1. *Let f be analytic on D_R with $\Re f \in h_p(D_R), 1 \le p \le \infty$. Further, let $n \ge 1$, and let $\mathcal{P}_m$ be a polynomial of degree $m \le n-1$. Then*

$$|f^{(n)}(0)| \le \mathcal{H}_{n,p}(0) ||\Re\{f - \mathcal{P}_m\}||_p, \qquad (5.3.1)$$

and the exact constant $\mathcal{H}_{n,p}(0)$ is given by

$$\mathcal{H}_{n,p}(0) = \begin{cases} \dfrac{n!}{\pi R^{n+1}} & \text{for} \quad p = 1, \\[2ex] \dfrac{2^{(p-1)/p} n!}{\pi^{(p+1)/(2p)} R^{(np+1)/p}} \left[\dfrac{\Gamma\left(\frac{2p-1}{2p-2}\right)}{\Gamma\left(\frac{3p-2}{2p-2}\right)} \right]^{(p-1)/p} & \text{for} \quad 1 < p < \infty, \\[2ex] \dfrac{4n!}{\pi R^n} & \text{for} \quad p = \infty. \end{cases}$$

In particular,

$$\mathcal{H}_{n,2}(0) = \frac{n!}{\sqrt{\pi} R^{(2n+1)/2}}.$$

As a special case of (5.3.1) *one has*

$$|f^{(n)}(0)| \le \mathcal{H}_{n,p}(0) E_{n-1,p}(\Re f). \tag{5.3.2}$$

Proof. Inequalities (5.3.1) and (5.3.2) are particular cases of (5.2.6) and (5.2.9) for $z = 0$, respectively. We derive explicit formulas for $\mathcal{H}_{n,p}(0)$.

From (5.2.8) with $p = 1$ it follows that

$$H_{n,1}(0) = \sup_{\alpha} \sup_{|\zeta|=1} \left| \Re\left(\frac{e^{i\alpha}}{\zeta^n}\right)\right| = \sup_{|\zeta|=1} \sup_{\alpha} \left| \Re\left(\frac{e^{i\alpha}}{\zeta^n}\right)\right| = \sup_{|\zeta|=1} \frac{1}{|\zeta|^n} = 1,$$

and, by (5.2.7), we obtain

$$\mathcal{H}_{n,1}(0) = \frac{n!}{\pi R^{n+1}}.$$

For $1 < p \le \infty$, by (5.2.7) and (5.2.8) we have

$$\mathcal{H}_{n,p}(0) = \frac{n!}{\pi R^{(np+1)/p}} \sup_{\alpha} \left\{ \int_{|\zeta|=1} \left| \Re\left(\frac{e^{i\alpha}}{\zeta^n}\right)\right|^{p/(p-1)} |d\zeta| \right\}^{(p-1)/p}.$$

Putting here $\zeta = e^{i\varphi}$ we find

$$\mathcal{H}_{n,p}(0) = \frac{n!}{\pi R^{(np+1)/p}} \sup_{\alpha} \left\{ \int_0^{2\pi} |\cos(\alpha - n\varphi)|^{p/(p-1)} d\varphi \right\}^{(p-1)/p}. \tag{5.3.3}$$

Changing the variable $\vartheta = \alpha - n\varphi$ in the last integral, we obtain

$$\int_0^{2\pi} |\cos(\alpha - n\varphi)|^{p/(p-1)} d\varphi = -\frac{1}{n} \int_{\alpha}^{\alpha - 2n\pi} |\cos\vartheta|^{p/(p-1)} d\vartheta$$

$$= \frac{1}{n} \int_{\alpha-2n\pi}^{\alpha} |\cos\vartheta|^{p/(p-1)} d\vartheta = \frac{1}{n} \int_0^{2n\pi} |\cos\vartheta|^{p/(p-1)} d\vartheta$$

$$= \int_0^{2\pi} |\cos\vartheta|^{p/(p-1)} d\vartheta = 4 \int_0^{\pi/2} (\cos\vartheta)^{p/(p-1)} d\vartheta.$$

Taking into account the equality

$$\int_0^{\pi/2} \cos^s x dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+2}{2}\right)}, \quad s > -1,$$

gives

$$\int_0^{2\pi} |\cos(\alpha - n\varphi)|^{p/(p-1)} d\varphi = 2\sqrt{\pi}\, \frac{\Gamma\left(\frac{2p-1}{2(p-1)}\right)}{\Gamma\left(\frac{3p-2}{2(p-1)}\right)},$$

which together with (5.3.3) implies

$$\mathcal{H}_{n,p}(0) = \frac{n!}{\pi R^{(np+1)/p}} \left\{ 2\sqrt{\pi}\, \frac{\Gamma\left(\frac{2p-1}{2(p-1)}\right)}{\Gamma\left(\frac{3p-2}{2(p-1)}\right)} \right\}^{(p-1)/p}.$$

In particular,

$$\mathcal{H}_{n,2}(0) = \frac{\sqrt{2}n!}{\pi^{3/4} R^{(2n+1)/2}} \left[\frac{\Gamma(3/2)}{\Gamma(2)} \right]^{1/2} = \frac{n!}{\sqrt{\pi} R^{(2n+1)/2}},$$

$$\mathcal{H}_{n,\infty}(0) = \frac{2n!}{\sqrt{\pi} R^n} \frac{\Gamma(1)}{\Gamma(3/2)} = \frac{4n!}{\pi R^n}.$$

□

5.4 The case $p = 1$ and its corollaries

5.4.1 Explicit estimate in the case $p = 1$

In this section, we deal with inequality (5.2.6) for $p = 1$ and its consequences. First we derive an explicit representation for $\mathcal{H}_{n,1}(z)$.

Corollary 5.2. *Let f be analytic on D_R with $\Re f \in h_1(D_R)$. Further, let $n \geq 1$, and let $\mathcal{P}_m$ be a polynomial of degree $m \leq n - 1$. Then for any fixed point $z, |z| = r < R$, there holds*

$$|f^{(n)}(z)| \leq \mathcal{H}_{n,1}(z) ||\Re\{f - \mathcal{P}_m\}||_1 \tag{5.4.1}$$

with the sharp constant

$$\mathcal{H}_{n,1}(z) = \frac{n!}{\pi (R - r)^{n+1}}. \tag{5.4.2}$$

In particular,

$$|f^{(n)}(z)| \leq \mathcal{H}_{n,1}(z) E_{n-1,1}(\Re f). \tag{5.4.3}$$

Proof. Inequalities (5.4.1) and (5.4.3) are particular cases of Proposition 5.1. Representation (5.2.11) for $p = 1$ can be written as

$$\mathcal{H}_{n,1}(z) = \frac{n!}{\pi R} \sup_{\alpha} \sup_{|\zeta|=R} \left| \Re\left\{ \frac{\zeta e^{i\alpha}}{(\zeta - z)^{n+1}} \right\} \right| \tag{5.4.4}$$

Permutating the suprema in (5.4.4), we obtain the equality

$$\mathcal{H}_{n,1}(z) = \frac{n!}{\pi R} \sup_{|\zeta|=R} \sup_{\alpha} \left| \Re\left\{ \frac{\zeta e^{i\alpha}}{(\zeta - z)^{n+1}} \right\} \right| = \frac{n!}{\pi R} \sup_{|\zeta|=R} \left| \frac{\zeta}{(\zeta - z)^{n+1}} \right| ,$$

which proves (5.4.2). □

The next four assertions contain corollaries of inequality (5.4.1). They are obtained in the following manner.

We put for brevity $\omega = \Re\mathcal{P}_0(z)$. Letting first $m = 0$, we put in (5.4.1) successively

$$\omega = \sup_{|\zeta|<R} \Re f(\zeta), \quad \omega = \sup_{|\zeta|<R} |\Re f(\zeta)|, \quad \omega = \sup_{|\zeta|<R} |f(\zeta)|,$$

and arrive at inequalities for derivatives of an analytic function with right-hand sides

$$\sup_{|\zeta|<R} \Re\Delta f(\zeta), \quad \sup_{|\zeta|<R} \Delta|\Re f(\zeta)|, \quad \sup_{|\zeta|<R} \Delta|f(\zeta)|,$$

respectively.

The first of the resulting inequalities has the same right-hand side as that in the Hadamard-Borel-Carathéodory inequality (1.1.2). It can be viewed as a generalization to derivatives of Hadamard's real part theorem (1.1.1). We shall also obtain a sharp constant in the related estimate (5.1.5).

The second inequality we shall get is similar to the third which contains the Landau inequality (5.1.3) as a particular case.

Besides, we obtain estimates for derivatives of an analytic function subject to the condition $\Re f(\zeta) > 0$ for $\zeta \in D_R$ with $\Re f(0)$ in the right-hand side. Such inequalities generalize the Carathéodory inequality (5.1.1). We show that all inequalities we get are sharp.

5.4.2 Hadamard's real part theorem for derivatives

The estimate for $|f^{(n)}(z)|$ with $n \geq 1$ below contains the value

$$\sup_{|\zeta|<R} \Re f(\zeta) - \Re f(0)$$

in the right-hand side. In particular, for $f(0) = 0$, this inequality generalizes the Hadamard's real part theorem (1.1.1) for derivatives.

Corollary 5.3. *Let f be analytic on D_R with $\Re f$ bounded from above. Then for any fixed $z, |z| = r < R$, the inequality holds with best constant*

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R-r)^{n+1}} \sup_{|\zeta|<R} \Re\Delta f(\zeta), \tag{5.4.5}$$

where $n \geq 1$.

In particular, for functions f *vanishing at* $z = 0$*, the inequality*

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R-r)^{n+1}} \sup_{|\zeta|<R} \Re f(\zeta) \tag{5.4.6}$$

holds with the sharp constant.

Proof. 1. Proof of inequality (5.4.5). By Corollary 5.2,

$$|f^{(n)}(z)| \leq \frac{n!}{\pi(\rho - r)^{n+1}} \, ||\Re f - \omega||_{L_1(\partial D_\rho)}, \tag{5.4.7}$$

where $\rho \in (r, R)$, ω is a real constant and $n \geq 1$.

We set

$$\omega = \mathcal{A}_f(R) = \sup_{|\zeta|<R} \Re f(\zeta)$$

in (5.4.7). Since, by the mean value theorem,

$$||\Re f - \mathcal{A}_f(R)||_{L_1(\partial D_\rho)} = \int_{|\zeta|=\rho} \{\mathcal{A}_f(R) - \Re f(\zeta)\} \, |d\zeta|$$

$$= 2\pi\rho \, \{\mathcal{A}_f(R) - \Re f(0)\} = 2\pi\rho \sup_{|\zeta|<R} \Re \Delta f(\zeta), \tag{5.4.8}$$

it follows from (5.4.7) that inequality

$$|f^{(n)}(z)| \leq \frac{2n!\rho}{(\rho - r)^{n+1}} \sup_{|\zeta|<R} \Re \Delta f(\zeta)$$

holds, where $n \geq 1$. Passing to the limit as $\rho \uparrow R$ in the last inequality, we obtain (5.4.5).

2. Sharpness of the constant in inequality (5.4.5). Consider the family of analytic functions on $\overline{D}_R$

$$f_\xi(z) = \frac{\xi}{z - \xi}, \tag{5.4.9}$$

where ξ is a complex parameter, $|\xi| > R$.

We are looking for $\max\{\Re \Delta f_\xi(z) : |z| = R\}$. Putting $\xi = \rho e^{i\tau}, z = Re^{it}$ we find

$$\Re f_\xi(z) = \Re \left\{ \frac{\xi}{z - \xi} \right\} = \Re \left\{ \frac{\rho e^{i\tau}(Re^{-it} - \rho e^{-i\tau})}{|Re^{it} - \rho e^{i\tau}|^2} \right\},$$

that is

$$\Re f_\xi(z) = \frac{\gamma(\cos\varphi - \gamma)}{1 - 2\gamma\cos\varphi + \gamma^2}, \tag{5.4.10}$$

where $\gamma = \rho/R, \varphi = t - \tau$. Hence, taking into account that $\gamma > 1$ we obtain

$$\max_{|z|=R} \Re \Delta f_\xi(z) = \max_{0\le\varphi\le 2\pi} \left\{ \frac{\gamma(\cos\varphi - \gamma)}{1 - 2\gamma\cos\varphi + \gamma^2} + 1 \right\}$$
$$= \frac{1}{2} \max_{0\le\varphi\le 2\pi} \left\{ 1 + \frac{1-\gamma^2}{1 - 2\gamma\cos\varphi + \gamma^2} \right\} = \frac{1}{1+\gamma} = \frac{R}{\rho + R}. \quad (5.4.11)$$

Let $z = re^{it}$ be a fixed point in the disk D_R and let $\xi = \rho e^{it}$. For any natural number n by (5.4.9) we have

$$|f_\xi^{(n)}(z)| = \left| \frac{(-1)^n n!\xi}{(z-\xi)^{n+1}} \right| = \frac{n!|\xi|}{|z-\xi|^{n+1}} = \frac{n!\rho}{(\rho - r)^{n+1}}. \quad (5.4.12)$$

Let $\mathcal{H}_n(z)$ denote the best constant in the inequality

$$|f^{(n)}(z)| \le \mathcal{H}_n(z) \sup_{|\zeta|<R} \Re \Delta f(\zeta).$$

By (5.4.5),

$$\mathcal{H}_n(z) \le \frac{2n!R}{(R-r)^{n+1}}. \quad (5.4.13)$$

Using

$$|f_\xi^{(n)}(z)| \le \mathcal{H}_n(z) \max_{|\zeta|=R} \Re \Delta f_\xi(\zeta)$$

together with (5.4.11), (5.4.12), we find

$$\mathcal{H}_n(z) \ge \frac{n!\rho(R+\rho)}{(\rho - r)^{n+1} R}.$$

Passing here to the limit as $\rho \downarrow R$, we conclude that

$$\mathcal{H}_n(z) \ge \frac{2n!R}{(R-r)^{n+1}},$$

which together with (5.4.13) proves sharpness of the constant in (5.4.5).

3. Inequality (5.4.6). This sharp inequality is an immediate consequence of estimate (5.4.5) with the best constant. □

The estimate (5.1.2) follows as a particular case of (5.4.5) with $z = 0$.

Observe also that replacing f by $-f$ in (5.4.5), we deduce

$$|f^{(n)}(z)| \le \frac{2n!R}{(R-r)^{n+1}} \sup_{|\zeta|<R} \Re \Delta\{-f(\zeta)\} \quad (5.4.14)$$

for analytic functions f on D_R with $\Re f$ bounded from below, where $n \geq 1$. Unlike inequality (5.4.5) with

$$\mathbf{A}_f(R) = \sup_{|\zeta|<R} \Re \Delta f(\zeta) = \sup_{|\zeta|<R} \Re f(\zeta) - \Re f(0)$$

in the right-hand side, inequality (5.4.14) contains the expression

$$\mathbf{B}_f(R) = \sup_{|\zeta|<R} \Re \Delta \{-f(\zeta)\} = \Re f(0) - \inf_{|\zeta|<R} \Re f(\zeta).$$

Unifying (5.4.5) with (5.4.14), we arrive at the sharp inequality

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R-r)^{n+1}} \min \{\mathbf{A}_f(R), \mathbf{B}_f(R)\}$$

for analytic functions f on D_R with bounded $\Re f$, where $n \geq 1$.

We conclude this subsection by noting that Corollary 5.3 implies the following inequalities similar to (5.1.5)

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R-r)^{n+1}} \left\{ \sup_{|\zeta|<R} \Re f(\zeta) + |\Re f(0)| \right\} \tag{5.4.15}$$

and

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R-r)^{n+1}} \left\{ \sup_{|\zeta|<R} \Re f(\zeta) + |f(0)| \right\}. \tag{5.4.16}$$

Remark 5.1. The sharpness of the constant in (5.4.15) and (5.4.16) is established in the same way as in inequality (5.4.5).

5.4.3 Landau type inequality

The following assertion contains a sharp estimate for $|f^{(n)}(z)|$ with

$$\sup_{|\zeta|<R} |\Re f(\zeta)| - |\Re f(0)|$$

in the right-hand side. The estimate below is closely related to the Landau inequality (5.1.3).

Corollary 5.4. *Let f be analytic on D_R with bounded $\Re f$. Then for any fixed $z, |z| = r < R$, the inequality*

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R-r)^{n+1}} \sup_{|\zeta|<R} \Delta |\Re f(\zeta)| \tag{5.4.17}$$

holds with the best constant, where $n \geq 1$.

Proof. 1. Proof of inequality (5.4.17). We set

$$\omega = \mathcal{R}_f(R) = \sup_{|\zeta|<R} |\Re f(\zeta)|$$

in (5.4.7). Since

$$||\Re f - \mathcal{R}_f(R)||_{L_1(\partial D_\rho)} = 2\pi\rho \left\{ \mathcal{R}_f(R) - \Re f(0) \right\}, \tag{5.4.18}$$

it follows from (5.4.7) that

$$|f^{(n)}(z)| \leq \frac{2n!\rho}{(\rho - r)^{n+1}} \left\{ \mathcal{R}_f(R) - \Re f(0) \right\},$$

where $n \geq 1$. Passing to the limit as $\rho \uparrow R$ in the last inequality, we obtain

$$|f^{(n)}(z)| \leq \frac{2n!\rho}{(\rho - r)^{n+1}} \left\{ \mathcal{R}_f(R) - \Re f(0) \right\}. \tag{5.4.19}$$

Replacing f by $-f$ in (5.4.19), we have

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R - r)^{n+1}} \left\{ \mathcal{R}_f(R) + \Re f(0) \right\},$$

which together with (5.4.19) implies

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R - r)^{n+1}} \left\{ \mathcal{R}_f(R) - |\Re f(0)| \right\}. \tag{5.4.20}$$

The last inequality proves (5.4.17).

2. Sharpness of the constant in inequality (5.4.17). We show that the constant in (5.4.20), that is in (5.4.17), is sharp.

Introduce the family of analytic functions in $\overline{D}_R$

$$g_\xi(z) = \frac{\xi}{z - \xi} + \frac{|\xi|^2}{|\xi|^2 - R^2}, \tag{5.4.21}$$

depending on a complex parameter $\xi = \rho e^{i\tau},\ \rho > R$.

Let, as before, $\gamma = \rho/R$. Clearly,

$$|\Re g_\xi(0)| = \left| -1 + \frac{\gamma^2}{\gamma^2 - 1} \right| = \frac{1}{\gamma^2 - 1}. \tag{5.4.22}$$

We find $\mathcal{R}_{g_\xi}(R)$. Let $z = Re^{it}, \varphi = t - \tau$. Using (5.4.10) in (5.4.21), we obtain

$$\begin{aligned} \Re g_\xi(z) &= \Re \left\{ \frac{\xi}{z - \xi} \right\} + \frac{\gamma^2}{\gamma^2 - 1} = \frac{\gamma(\cos\varphi - \gamma)}{1 - 2\gamma\cos\varphi + \gamma^2} + \frac{\gamma^2}{\gamma^2 - 1} \\ &= -\frac{1}{2} + \frac{\gamma^2}{\gamma^2 - 1} - \frac{\gamma^2 - 1}{2(1 - 2\gamma\cos\varphi + \gamma^2)}. \end{aligned} \tag{5.4.23}$$

Hence

$$\max_{|\zeta|=R} \Re g_\xi(\zeta) = -\frac{1}{2} + \frac{\gamma^2}{\gamma^2-1} - \frac{\gamma^2-1}{2(\gamma+1)^2} = \frac{\gamma}{\gamma^2-1},$$

$$\min_{|\zeta|=R} \Re g_\xi(\zeta) = -\frac{1}{2} + \frac{\gamma^2}{\gamma^2-1} - \frac{\gamma^2-1}{2(\gamma-1)^2} = -\frac{\gamma}{\gamma^2-1},$$

that is

$$\mathcal{R}_{g_\xi}(R) = \frac{\gamma}{\gamma^2-1}. \tag{5.4.24}$$

Thus, by (5.4.22) and (5.4.24),

$$\mathcal{R}_{g_\xi}(R) - |\Re g_\xi(0)| = \frac{\gamma}{\gamma^2-1} - \frac{1}{\gamma^2-1} = \frac{1}{\gamma+1} = \frac{R}{\rho+R}. \tag{5.4.25}$$

Let $z = re^{it}$ be a fixed point with $r < R$ and let $\xi = \rho e^{it}$. For any natural number n by (5.4.21) we have

$$|g_\xi^{(n)}(z)| = \left|\frac{(-1)^n n!\xi}{(z-\xi)^{n+1}}\right| = \frac{n!|\xi|}{|z-\xi|^{n+1}} = \frac{n!\rho}{(\rho-r)^{n+1}}. \tag{5.4.26}$$

By $\mathcal{H}_n(z)$ we denote the best constant in

$$|f^{(n)}(z)| \le \mathcal{H}_n(z)\{\mathcal{R}_f(R) - |\Re f(0)|\}.$$

In view of (5.4.20),

$$\mathcal{H}_n(z) \le \frac{2n!R}{(R-r)^{n+1}}. \tag{5.4.27}$$

Using

$$|g_\xi^{(n)}(z)| \le \mathcal{H}_n(z)\{\mathcal{R}_{g_\xi}(R) - |\Re g_\xi(0)|\}$$

together with (5.4.25), (5.4.26) we find

$$\mathcal{H}_n(z) \ge \frac{n!\rho(R+\rho)}{(\rho-r)^{n+1}R}.$$

Passing here to the limit as $\rho \downarrow R$ we find

$$\mathcal{H}_n(z) \ge \frac{2n!R}{(R-r)^{n+1}},$$

which, along with (5.4.27), proves sharpness of the constant in inequality (5.4.17). □

Remark 5.2. Note that (5.4.17) can be obtained as a consequence of estimate (5.4.5) without proof of its sharpness.

5.4.4 Generalization of the Landau inequality

The next assertion contains a sharp estimate of $|f^{(n)}(z)|$ with

$$\sup_{|\zeta|<R} |f(\zeta)| - |f(0)|$$

in the right-hand side. This estimate contains the Landau inequality (5.1.3).

Corollary 5.5. *Let f be analytic and bounded on D_R. Then for any fixed $z, |z| = r < R$, the inequality*

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R-r)^{n+1}} \sup_{|\zeta|<R} \Delta|f(\zeta)| \tag{5.4.28}$$

holds with the best constant, where $n \geq 1$.

Proof. 1. Proof of inequality (5.4.28). We put

$$\omega = \mathcal{M}_f(R) = \sup_{|\zeta|<R} |f(\zeta)|.$$

in (5.4.7). Since

$$||\Re f - \mathcal{M}_f(R)||_{L_1(\partial D_\rho)} = 2\pi\rho \left\{\mathcal{M}_f(R) - \Re f(0)\right\}, \tag{5.4.29}$$

it follows from (5.4.7) that

$$|f^{(n)}(z)| \leq \frac{2n!\rho}{(\rho - r)^{n+1}} \left\{\mathcal{M}_f(R) - \Re f(0)\right\}.$$

Passing to the limit as $\rho \uparrow R$ in the last inequality, we obtain

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R-r)^{n+1}} \left\{\mathcal{M}_f(R) - \Re f(0)\right\}. \tag{5.4.30}$$

Replacing f by $fe^{i\alpha}$ in (5.4.30) we arrive at

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R-r)^{n+1}} \left\{\mathcal{M}_f(R) - \Re(f(0)e^{i\alpha})\right\},$$

which due to the arbitrariness of α implies (5.4.28).

2. Sharpness of the constant in inequality (5.4.28). For the analytic function $g_\xi(z)$ defined by (5.4.21) we have

$$|g_\xi(0)| = \left| -1 + \frac{\gamma^2}{\gamma^2 - 1} \right| = \frac{1}{\gamma^2 - 1}. \tag{5.4.31}$$

We are looking for $\mathcal{M}_{g_\xi}(R)$. Let $z = Re^{it}, \xi = \rho e^{i\tau}, \gamma = \rho/R, \varphi = t - \tau$. In view of (5.4.21),

$$\Im g_\xi(z) = \Im\left(\frac{\xi}{z-\xi}\right) = \Im\left(\frac{\rho e^{i\tau}}{Re^{it} - \rho e^{i\tau}}\right) = -\frac{\gamma \sin\varphi}{1 - 2\gamma\cos\varphi + \gamma^2}.$$

Taking into account (5.4.23), we have

$$|g_\xi(z)|^2 = \left[\frac{\gamma(\cos\varphi - \gamma)}{1 - 2\gamma\cos\varphi + \gamma^2} + \frac{\gamma^2}{\gamma^2 - 1}\right]^2 + \left[-\frac{\gamma\sin\varphi}{1 - 2\gamma\cos\varphi + \gamma^2}\right]^2.$$

This simplifies to

$$|g_\xi(z)|^2 = \frac{\gamma^2}{(\gamma^2 - 1)^2}.$$

Thus,

$$\mathcal{M}_{g_\xi}(R) = \frac{\gamma}{\gamma^2 - 1} \tag{5.4.32}$$

and by (5.4.31),

$$\mathcal{M}_{g_\xi}(R) - |g_\xi(0)| = \frac{\gamma}{\gamma^2 - 1} - \frac{1}{\gamma^2 - 1} = \frac{1}{\gamma + 1} = \frac{R}{\rho + R}. \tag{5.4.33}$$

Let n be a natural number and let $\mathcal{H}_n(z)$ denote the best constant in

$$|f^{(n)}(z)| \le \mathcal{H}_n(z)\{\mathcal{M}_f(R) - |f(0)|\}. \tag{5.4.34}$$

As shown above,

$$\mathcal{H}_n(z) \le \frac{2n!R}{(R - r)^{n+1}}. \tag{5.4.35}$$

We take an arbitrary fixed point $z = re^{it}$ with $r < R$, and let $\xi = \rho e^{it}$. By (5.4.26), (5.4.33), and (5.4.34)

$$\mathcal{H}_n(z) \ge \frac{|g_\xi^{(n)}(z)|}{\mathcal{M}_{g_\xi}(R) - |g_\xi(0)|} = \frac{n!\rho(R + \rho)}{(\rho - r)^{n+1}R}.$$

Passing to the limit as $\rho \downarrow R$ in the last inequality, we obtain

$$\mathcal{H}_n(z) \ge \frac{2n!R}{(R - r)^{n+1}},$$

which, along with (5.4.35), proves sharpness of the constant in inequality (5.4.28). □

Remark 5.3. Inequality (5.4.28) follow from (5.4.5) without proof of its sharpness.

5.4.5 Generalization of the Carathéodory inequality

The following assertion contains estimate for $|f^{(n)}(z)|$ in terms of $\Re f(0)$ under the assumption that $\Re f(\zeta) > 0$ for $|\zeta| < R$. This estimate generalizes the Carathéodory inequality (5.1.1).

Corollary 5.6. *Let f be analytic with $\Re f(\zeta) > 0$ on the disk D_R. Then for any fixed $z, |z| = r < R$, the inequality*

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R-r)^{n+1}} \Re f(0) \tag{5.4.36}$$

holds with the best constant, where $n \geq 1$.

Proof. Suppose $\Re f(\zeta) > 0$ for $|\zeta| < R$. We put $\omega = 0$ in (5.4.7). Since

$$||\Re f||_{L_1(\partial D_\rho)} = 2\pi\rho\, \Re f(0),$$

it follows from (5.4.7) that

$$|f^{(n)}(z)| \leq \frac{2n!\rho}{(\rho - r)^{n+1}}\, \Re f(0).$$

Passing to the limit as $\rho \uparrow R$ in the last inequality, we obtain

$$|f^{(n)}(z)| \leq \frac{2n!\rho}{(\rho - r)^{n+1}}\, \Re f(0). \tag{5.4.37}$$

We show that the constant in (5.4.37) is sharp. Introduce the family of analytic functions in $\overline{D}_R$

$$h_\xi(z) = \frac{\xi}{\xi - z} - \frac{|\xi|}{|\xi| + R}, \tag{5.4.38}$$

which depend on the complex parameter $\xi = \rho e^{i\tau},\ \rho > R$. Putting $z = Re^{it}, \gamma = r/R, \varphi = t - \tau$ and taking into account (5.4.10), we find

$$\begin{aligned} \Re h_\xi(z) &= \Re\left(\frac{\xi}{\xi - z}\right) - \frac{\gamma}{\gamma+1} = \frac{\gamma(\gamma - \cos\varphi)}{1 - 2\gamma\cos\varphi + \gamma^2} - \frac{\gamma}{\gamma+1} \\ &= \frac{1}{2} - \frac{\gamma}{\gamma+1} + \frac{\gamma^2 - 1}{2(1 - 2\gamma\cos\varphi + \gamma^2)}. \end{aligned}$$

Hence

$$\min_{|\zeta|=R} \Re h_\xi(z) = \frac{1}{2} - \frac{\gamma}{\gamma+1} + \frac{\gamma^2 - 1}{2(\gamma+1)^2} = 0,$$

that is

$$\Re h_\xi(\zeta) \geq 0, \quad |\zeta| = R. \tag{5.4.39}$$

According to (5.4.38),

$$\Re h_\xi(0) = 1 - \frac{\gamma}{\gamma+1} = \frac{1}{\gamma+1} = \frac{R}{\rho+R}. \tag{5.4.40}$$

Let n be a natural number. By $\mathcal{H}_n(z)$ we denote the sharp constant in

$$|f^{(n)}(z)| \le \mathcal{H}_n(z) \Re f(0) \tag{5.4.41}$$

where $\Re f(\zeta) \ge 0$ for $|\zeta| = R$. By (5.4.37) we have

$$\mathcal{H}_n(z) \le \frac{2n!R}{(R-r)^{n+1}}. \tag{5.4.42}$$

Let $z = re^{it}$ be a fixed point with $r < R$, and let $\xi = \rho e^{it}$. It follows from (5.4.38) that

$$|h_\xi^{(n)}(z)| = \frac{n!\rho}{(\rho-r)^{n+1}}. \tag{5.4.43}$$

Taking into account (5.4.39), (5.4.40), (5.4.41), and (5.4.43) we find

$$\mathcal{H}_n(z) \ge \frac{|h_\xi^{(n)}(z)|}{\Re h_\xi(0)} = \frac{n!\rho(R+\rho)}{(\rho-r)^{n+1}R}.$$

Passing to the limit as $\rho \downarrow R$ in the last inequality, we obtain

$$\mathcal{H}_n(z) \ge \frac{2n!R}{(R-r)^{n+1}},$$

which, along with (5.4.42), proves sharpness of the constant in inequality (5.4.36). □

5.5 The case $p = 2$

The next assertion is a particular case of Proposition 5.1 for $p = 2$.

Corollary 5.7. *Let f be analytic on D_R with $\Re f \in h_2(D_R)$. Further, let $n \ge 1$, and let $\mathcal{P}_m$ be a polynomial of degree $m \le n-1$. Then for any fixed point $z, |z| = r < R$, there holds*

$$|f^{(n)}(z)| \le \mathcal{H}_{n,2}(z) ||\Re\{f - \mathcal{P}_m\}||_2 \tag{5.5.1}$$

with the sharp constant

$$\mathcal{H}_{n,2}(z) = \frac{1}{R^{(2n+1)/2}} H_{n,2}\left(\frac{r}{R}\right), \tag{5.5.2}$$

where

$$H_{n,2}(\gamma) = \frac{n!}{\sqrt{\pi}(1-\gamma^2)^{(2n+1)/2}} \left\{ \sum_{k=0}^{n} \binom{n}{k}^2 \gamma^{2k} \right\}^{1/2}. \tag{5.5.3}$$

In particular,

$$|f^{(n)}(z)| \leq \mathcal{H}_{n,2}(z) E_{n-1,2}(\Re f), \tag{5.5.4}$$

where

$$E_{n,2}(\Re f) = \left|\left| \Re\left\{ f - \sum_{k=0}^{n} \frac{f^{(k)}(0)\zeta^k}{k!} \right\} \right|\right|_2$$

$$= \left\{ ||\Re f - \Re f(0)||_2^2 - \pi R \sum_{k=1}^{n} \frac{|f^{(k)}(0)|^2 R^{2k}}{(k!)^2} \right\}^{1/2}. \tag{5.5.5}$$

Here the sum in k from 1 to n is assumed to vanish for $n = 0$.

Proof. 1. Sharp constant in inequalities (5.5.1) *and* (5.5.4). Inequalities (5.5.1), (5.5.4) follow from Proposition 5.1. Consider the integral in (5.2.8) for $p = 2$. Putting $\zeta = 1/\xi$, we find

$$\int_{|\zeta|=1} \left| \Re\left\{ \frac{\zeta e^{i\alpha}}{(\zeta-\gamma)^{n+1}} \right\} \right|^2 |d\zeta| = \int_{|\xi|=1} \left| \Re\left\{ \frac{\xi^n e^{i\alpha}}{(1-\gamma\xi)^{n+1}} \right\} \right|^2 |d\xi|, \tag{5.5.6}$$

where $\gamma = r/R < 1$ and α is a real parameter. Similarly,

$$\int_{|\zeta|=1} \left| \Im\left\{ \frac{\zeta e^{i\alpha}}{(\zeta-\gamma)^{n+1}} \right\} \right|^2 |d\zeta| = \int_{|\xi|=1} \left| \Im\left\{ \frac{\xi^n e^{i\alpha}}{(1-\gamma\xi)^{n+1}} \right\} \right|^2 |d\xi|. \tag{5.5.7}$$

We use the following property (see, e.g. Polya and Szegö [75], Ex. 234): if the function

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

is regular in the disk $|z| < R$, and the equality

$$\int_0^{2\pi} [\Re f(\rho e^{i\vartheta})]^2 d\vartheta = \int_0^{2\pi} [\Im f(\rho e^{i\vartheta})]^2 d\vartheta \tag{5.5.8}$$

holds for $\rho = 0$, then it holds for any $\rho \in (0, R)$.

The function

$$g(\xi) = \frac{\xi^n e^{i\alpha}}{(1-\gamma\xi)^{n+1}}, \qquad 0 \leq \gamma < 1,$$

analytic in the disk $|\xi| < \gamma^{-1}$, satisfies (5.5.8) with $\rho = 0$. Therefore, the equality

$$\int_0^{2\pi} [\Re g(\rho e^{i\vartheta})]^2 d\vartheta = \int_0^{2\pi} [\Im g(\rho e^{i\vartheta})]^2 d\vartheta$$

is valid for $\rho = 1$, which together with (5.5.6) and (5.5.7) implies

$$\int_{|\zeta|=1} \left| \Re \left\{ \frac{\zeta e^{i\alpha}}{(\zeta - \gamma)^{n+1}} \right\} \right|^2 |d\zeta| = \int_{|\zeta|=1} \left| \Im \left\{ \frac{\zeta e^{i\alpha}}{(\zeta - \gamma)^{n+1}} \right\} \right|^2 |d\zeta|.$$

Using this identity and (5.2.8) with $p = 2$ we obtain

$$H_{n,2}(\gamma) = \frac{n!}{\pi} \left\{ \frac{1}{2} \int_{|\zeta|=1} \frac{|d\zeta|}{|\zeta - \gamma|^{2(n+1)}} \right\}^{1/2}. \tag{5.5.9}$$

Putting here $\zeta = (w + \gamma)(1 + \gamma w)^{-1}$, we find (see, e.g. Gradshtein and Ryzhik [41], **3.616**)

$$\begin{aligned}
\int_{|\zeta|=1} \frac{|d\zeta|}{|\zeta - \gamma|^{2(n+1)}} &= \frac{1}{(1-\gamma^2)^{2n+1}} \int_{|w|=1} |1 + \gamma w|^{2n} |dw| \\
&= \frac{1}{(1-\gamma^2)^{2n+1}} \int_0^{2\pi} (1 + 2\gamma \cos\varphi + \gamma^2)^n d\varphi \\
&= \frac{2\pi}{(1-\gamma^2)^{2n+1}} \sum_{k=0}^{n} \binom{n}{k}^2 \gamma^{2k},
\end{aligned} \tag{5.5.10}$$

which together with (5.5.9) leads to

$$H_{n,2}(\gamma) = \frac{n!}{\sqrt{\pi}(1-\gamma^2)^{(2n+1)/2}} \left\{ \sum_{k=0}^{n} \binom{n}{k}^2 \gamma^{2k} \right\}^{1/2}.$$

This and (5.2.7) with $p = 2$ imply (5.5.2), (5.5.3).

2. Proof of relations (5.5.5). We prove the first equality in (5.5.5). Let $z = Re^{i\varphi}$. We write the real part of the algebraic polynomial

$$\mathcal{P}_n(z) = \sum_{k=0}^{n} c_k z^k, \tag{5.5.11}$$

where $c_k = a_k + ib_k$, as a trigonometric polynomial $T_n(\varphi)$

$$\begin{aligned}
\Re \mathcal{P}_n(Re^{i\varphi}) &= \Re \left\{ \sum_{k=0}^{n} (a_k + ib_k) R^k e^{ik\varphi} \right\} \\
&= \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos k\varphi + \beta_k \sin k\varphi) = T_n(\varphi),
\end{aligned}$$

$\alpha_0 = a_0, \alpha_k = R^k a_k, \beta_k = -R^k b_k, 1 \le k \le n.$

Conversely, given a trigonometric polynomial

$$T_n(\varphi) = \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos k\varphi + \beta_k \sin k\varphi)$$

and introducing the coefficients $c_0 = \alpha_0 + i\beta_0, c_k = R^{-k}(\alpha_k - i\beta_k)$, where β_0 is an arbitrary real number, one restores the algebraic polynomial (5.5.11) (up to an imaginary constant) such that $\Re\mathcal{P}_n(Re^{i\varphi}) = T_n(\varphi)$.

Let $\{\mathfrak{T}_n\}$ be the set of trigonometric polynomials of degree at most n. Using the above relation between algebraic and trigonometric polynomials and the minimizing property of Fourier coefficients, we obtain

$$\begin{aligned} E_{n,2}(\Re f) &= \inf_{\mathcal{P}\in\{\mathfrak{P}_n\}} ||\Re\{f - \mathcal{P}\}||_2 \\ &= \inf_{\mathcal{P}\in\{\mathfrak{P}_n\}} \left\{ \int_{|\zeta|=R} [\Re f(\zeta) - \Re\mathcal{P}(\zeta)]^2 |d\zeta| \right\}^{1/2} \\ &= \sqrt{R} \inf_{\mathcal{T}\in\{\mathfrak{T}_n\}} \left\{ \int_{-\pi}^{\pi} [\Re f(Re^{i\varphi}) - \mathcal{T}(\varphi)]^2 d\varphi \right\}^{1/2} \\ &= \sqrt{R} \left\{ \int_{-\pi}^{\pi} [\Re f(Re^{i\varphi}) - \mathcal{F}_n(\varphi)]^2 d\varphi \right\}^{1/2}, \end{aligned} \tag{5.5.12}$$

where

$$\mathcal{F}_n(\varphi) = \sum_{k=-n}^{n} A_k e^{ik\varphi} \tag{5.5.13}$$

and

$$A_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re f(Re^{i\psi}) e^{-ik\psi} d\psi. \tag{5.5.14}$$

We rewrite coefficients A_k using the Schwarz formula (1.3.1) and its corollary (5.2.4) for $z = 0$:

$$f(0) = i\,\Im f(0) + \frac{1}{2\pi R} \int_{|\zeta|=R} \Re f(\zeta) |d\zeta|, \tag{5.5.15}$$

$$f^{(k)}(0) = \frac{k!}{\pi R} \int_{|\zeta|=R} \frac{\Re f(\zeta)}{\zeta^k} |d\zeta|, \tag{5.5.16}$$

where $k \ge 1$. By (5.5.14) and (5.5.15) we have

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Re f(Re^{i\psi})d\psi$$

$$= \frac{1}{2\pi R}\int_{|\zeta|=R} \Re f(\zeta)|d\zeta| = f(0) - i\,\Im f(0) = \Re f(0). \quad (5.5.17)$$

In view of (5.5.14) and (5.5.16), we have for $1 \le k \le n$

$$A_k = \frac{R^k}{2\pi R}\int_{-\pi}^{\pi} \frac{\Re f(Re^{i\psi})}{R^k e^{ik\psi}} R d\psi$$

$$= \frac{R^{k-1}}{2\pi}\int_{|\zeta|=R} \frac{\Re f(\zeta)}{\zeta^k}|d\zeta| = \frac{R^k}{2k!} f^{(k)}(0). \quad (5.5.18)$$

Similarly, by (5.5.14) and (5.5.16), for $-n \le k \le -1$ there holds

$$A_k = \frac{R^{|k|}}{2\pi R}\overline{\int_{-\pi}^{\pi} \frac{\Re f(Re^{i\psi})}{R^{|k|} e^{i|k|\psi}} R d\psi}$$

$$= \frac{R^{|k|-1}}{2\pi}\overline{\int_{|\zeta|=R} \frac{\Re f(\zeta)}{\zeta^{|k|}}|d\zeta|} = \frac{R^{|k|}}{2|k|!}\overline{f^{(|k|)}(0)}. \quad (5.5.19)$$

Using (5.5.17)-(5.5.19) in (5.5.13), we find

$$\mathcal{F}_n(\varphi) = \Re f(0) + \sum_{k=1}^{n} \frac{R^k}{2k!}\left\{ f^{(k)}(0)e^{ik\varphi} + \overline{f^{(k)}(0)e^{ik\varphi}} \right\}$$

$$= \sum_{k=0}^{n} \frac{1}{k!}\Re\left\{ f^{(k)}(0)R^k e^{ik\varphi} \right\},$$

which implies that (5.5.12) can be written as

$$E_{n,2}(\Re f) = \sqrt{R}\left\{ \int_{-\pi}^{\pi}\left[\Re f(Re^{i\varphi}) - \sum_{k=0}^{n}\frac{1}{k!}\Re\left\{ f^{(k)}(0)R^k e^{ik\varphi} \right\}\right]^2 d\varphi \right\}^{1/2},$$

and, equivalently,

$$E_{n,2}(\Re f) = \left|\left| \Re\left\{ f - \sum_{k=0}^{n}\frac{f^{(k)}(0)\zeta^k}{k!} \right\} \right|\right|_2 .$$

This proves the first equality in (5.5.5).

Taking into account (5.5.13), (5.5.17), we write (5.5.12) as

$$E_{n,2}(\Re f) = \left\{ ||\Re f - \Re f(0)||_2^2 - 2\pi R\sum_{k=1}^{n}|A_k|^2 - 2\pi R\sum_{k=1}^{n}|A_{-k}|^2 \right\}^{1/2}.$$

Hence, by (5.5.18) and (5.5.19) we arrive at the second representation in (5.5.5). ∎

5.6 The case $p = \infty$

For $p = \infty$ inequality (5.2.9) can be written in terms of the infimum of the oscillation of $\Re f(\zeta) - \Re \mathcal{P}(\zeta)$ on the circle, where the infimum is taken over the set $\{\mathfrak{P}_{n-1}\}$ of polynomials $\mathcal{P}$ of degree $n-1$.

We introduce the notation

$$\mathcal{O}_{n,\Re f}(D_R) = \inf_{\mathcal{P}\in\{\mathfrak{P}_n\}} \mathcal{O}_{\Re\{f-\mathcal{P}\}}(D_R), \tag{5.6.1}$$

where $\mathcal{O}_{\Re f}(D_R)$ is the oscillation of $\Re f$ on the disk D_R. For $n = 0$ we use the notation $\mathcal{O}_{\Re f}(D_R)$ introduced earlier.

Corollary 5.13. *Let f be analytic on D_R with bounded $\Re f$, and let $n \geq 1$. Then for any fixed point $z, |z| = r < R$, there holds*

$$|f^{(n)}(z)| \leq \frac{1}{2}\mathcal{H}_{n,\infty}(z)\mathcal{O}_{n-1,\Re f}(D_R) \tag{5.6.2}$$

with the sharp constant

$$\mathcal{H}_{n,\infty}(z) = \frac{1}{R^n} H_{n,\infty}\left(\frac{r}{R}\right), \tag{5.6.3}$$

where

$$H_{n,\infty}(\gamma) = \frac{n!}{\pi}\sup_{\alpha}\int_{|\zeta|=1}\left|\Re\left\{\frac{\zeta e^{i\alpha}}{(\zeta-\gamma)^{n+1}}\right\}\right||d\zeta|. \tag{5.6.4}$$

In particular,

$$|f^{(n)}(0)| \leq \frac{2n!}{\pi R^n}\mathcal{O}_{n-1,\Re f}(D_R). \tag{5.6.5}$$

Proof. Suppose $\omega \in \mathbb{R}$. Then

$$\begin{aligned} E_{n,\infty}(\Re f) &= \inf_{\mathcal{P}\in\{\mathfrak{P}_n\}} ||\Re\{f-\mathcal{P}\}||_\infty = \inf_{\mathcal{P}\in\{\mathfrak{P}_n\}} ||\Re\{f-\mathcal{P}\}-\omega||_\infty \\ &= \inf_{\omega\in\mathbb{R}}\inf_{\mathcal{P}\in\{\mathfrak{P}_n\}} ||\Re\{f-\mathcal{P}\}-\omega||_\infty. \end{aligned}$$

Hence, permutating the infima and taking into account (3.4.1) and (5.6.1), we obtain

$$\begin{aligned} E_{n,\infty}(\Re f) &= \inf_{\mathcal{P}\in\{\mathfrak{P}_n\}}\inf_{\omega\in\mathbb{R}} ||\Re\{f-\mathcal{P}\}-\omega||_\infty = \inf_{\mathcal{P}\in\{\mathfrak{P}_n\}} E_{0,\infty}(\Re\{f-\mathcal{P}\}) \\ &= \inf_{\mathcal{P}\in\{\mathfrak{P}_n\}}\frac{1}{2}\mathcal{O}_{\Re\{f-\mathcal{P}\}}(D_R) = \frac{1}{2}\mathcal{O}_{n,\Re f}(D_R), \end{aligned}$$

which together with (5.2.9) and (5.2.7), (5.2.8) proves (5.6.2)-(5.6.4). Inequality (5.6.5) follows from (5.6.2) and Corollary 5.1. □

Note that the sharp inequality (3.7.13)

$$|f'(z)| \leq \frac{2R}{\pi(R^2 - |z|^2)} \mathcal{O}_{\Re f}(D_R)$$

is a particular case of (5.6.2) with $n = 1$.

The last estimate is somewhat similar to the Carathéodory inequality

$$|f'(z)| \leq \frac{R}{R^2 - |z|^2} \sup_{|\zeta|<R} |f(\zeta)|.$$

As a corollary of (3.7.13) we obtain

$$|f'(0)| \leq \frac{2}{\pi R} \mathcal{O}_{\Re f}(D_R)$$

(see Polya and Szegö [75], III, Ch. 5, § 2 and references there). Inequality (5.6.5) can be viewed as generalization of the last estimate to derivatives of arbitrary order.

6

Bohr's type real part estimates and theorems

6.1 Introduction

This chapter is connected with two classical assertions of the analytic functions theory, namely, with Hadamard-Borel-Carathéodory inequality

$$|f(z) - f(0)| \leq \frac{2r}{R-r} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\}, \tag{6.1.1}$$

and with Bohr's inequality

$$\sum_{n=0}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<R} |f(\zeta)| \tag{6.1.2}$$

for the majorant of the Taylor's series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{6.1.3}$$

where $|z| \leq R/3$ in (6.1.2) and the value $R/3$ cannot be improved.

In the chapter we deal, similarly to Aizenberg, Grossman and Korobeinik [6], Bénéteau, Dahlner and Khavinson [12], Djakov and Ramanujan [34], with the value of l_q-norm (quasi-norm, for $0 < q < 1$) of the remainder of the Taylor series (6.1.3).

In Section 1, we prove the inequality

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}} \, ||\Re f||_1 \tag{6.1.4}$$

with the sharp constant, where $r = |z| < R$, $m \geq 1$, $0 < q \leq \infty$.

Section 2 contains corollaries of (6.1.4) for analytic functions f in D_R with bounded $\Re f$, with $\Re f$ bounded from above, with $\Re f > 0$, as well as for bounded analytic functions. In particular, we obtain the estimate

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \le \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\},$$

with the best possible constant. This estimate, taken with $q = 1$, $m = 1$, is an improvement of (6.1.1). Other inequalities, which follow from (6.1.4), contain the supremum of $|\Re f(\zeta)| - |\Re f(0)|$ or $|f(\zeta)| - |f(0)|$ in D_R, as well as $\Re f(0)$ in the case $\Re f > 0$ on D_R. Each of these estimates specified for $q = 1$ and $m = 1$ improves a certain sharp Hadamard-Borel-Carathéodory type inequality.

Note that a sharp estimate of the full majorant series by the supremum modulus of f was obtained by Bombieri [18] for $r \in [R/3, R/\sqrt{2}]$.

In Section 3 we give modifications of Bohr's theorem as consequences of our inequalities with sharp constants derived in Section 2. For example, if the function (6.1.3) is analytic on D_R, then for any $q \in (0, \infty]$, integer $m \ge 1$ and $|z| \le R_{m,q}$ the inequality

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \le \sup_{|\zeta|<R} \Re\{e^{-i \arg f(0)} f(\zeta)\} - |f(0)| \tag{6.1.5}$$

holds, where $R_{m,q} = r_{m,q} R$, and $r_{m,q}$ is the root of the equation

$$2^q r^{mq} + r^q - 1 = 0$$

in the interval $(0, 1)$, and $R_{m,q}$ is the radius of the largest disk centered at $z = 0$ in which (6.1.5) takes place.

In particular,

$$r_{1,q} = (1 + 2^q)^{-1/q} \quad \text{and} \quad r_{2,q} = 2^{1/q}\big(1 + \sqrt{1 + 2^{q+2}}\,\big)^{-1/q}. \tag{6.1.6}$$

Some of the inequalities presented in Section 3 contain known analogues of Bohr's theorem with $\Re f$ in the right-hand side (see Aizenberg, Aytuna and Djakov [3], Paulsen, Popescu and Singh [72], Sidon [84], Tomić [87]).

6.2 Estimate for the l_q -norm of the Taylor series remainder by $||\Re f||_1$

In the sequel, we use the notation $r = |z|$ and $D_\varrho = \{z \in \mathbb{C} : |z| < \varrho\}$.

We start with a sharp inequality for an analytic function f. The right-hand side of the inequality contains the norm in the space $L_1(\partial D_R)$.

Proposition 6.1. *Let the function* (6.1.3) *be analytic on* D_R *with* $\Re f \in h_1(D_R)$*, and let* $q > 0$*,* $m \geq R$*,* $|z| = r < R$*. Then the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}} \, ||\Re f||_1 \tag{6.2.1}$$

holds with the sharp constant.

Proof. 1. Proof of inequality (6.2.1). Let a function f, analytic in D_R with $\Re f \in h_1(D_R)$ be given by (6.1.3). By Corollary 5.1

$$|c_n| \leq \frac{1}{\pi R^{n+1}} \, ||\Re f||_1 \tag{6.2.2}$$

for any $n \geq 1$.

Using (6.2.2), we find

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{1}{\pi R} \left\{ \sum_{n=m}^{\infty} \left(\frac{r}{R} \right)^{nq} \right\}^{1/q} ||\Re f||_1$$

$$= \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}} \, ||\Re f||_1$$

for any z with $|z| = r < R$.

2. Sharpness of the constant in (6.2.1). By (6.2.1), proved above, the sharp constant $C(r)$ in

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq C(r) \, ||\Re f||_1 \tag{6.2.3}$$

satisfies

$$C(r) \leq \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}}. \tag{6.2.4}$$

We show that the converse inequality for $C(r)$ holds as well.

Let $\rho > R$. Consider the families of analytic functions in $\overline{D}_R$

$$f_\rho(z) = \frac{z}{z - \rho}, \qquad w_\rho(z) = f_\rho(z) - \beta_\rho, \tag{6.2.5}$$

depending on the parameter ρ, with the real constant β_ρ defined by

$$||\Re f_\rho - \beta_\rho||_1 = \min_{c \in \mathbb{R}} ||\Re f_\rho - c||_1.$$

Then, for any real constant c

$$||\Re w_\rho - c||_1 \geq ||\Re w_\rho||_1.$$

Setting here

$$c = A_\rho = \max_{|\zeta|=R} \Re w_\rho(\zeta)$$

and taking into account

$$||\Re w_\rho - A_\rho||_1 = \int_{|\zeta|=R} \big[A_\rho - \Re w_\rho(\zeta)\big]|d\zeta|$$

$$= 2\pi R\{A_\rho - \Re w_\rho(0)\} = 2\pi R \max_{|\zeta|=R} \Re\{w_\rho(\zeta) - w_\rho(0)\},$$

we arrive at

$$2\pi R \max_{|\zeta|=R} \Re\{w_\rho(\zeta) - w_\rho(0)\} \geq ||\Re w_\rho||_1. \tag{6.2.6}$$

In view of

$$c_n(\rho) = \frac{w_\rho^{(n)}(0)}{n!} = -\frac{1}{\rho^n} \quad \text{for} \quad n \geq 1,$$

we find

$$\sum_{n=m}^{\infty} |c_n(\rho) z^n|^q = \sum_{n=m}^{\infty} \left(\frac{r}{\rho}\right)^{nq} = \frac{r^{mq}}{\rho^{(m-1)q}(\rho^q - r^q)}. \tag{6.2.7}$$

By (6.2.5), (1.4.6) and (1.4.7) we have

$$\max_{|\zeta|=R} \Re\big\{w_\rho(\zeta) - w_\rho(0)\big\} = \max_{|\zeta|=R} \Re\big\{f_\rho(\zeta) - f_\rho(0)\big\} = \frac{R}{\rho + R}. \tag{6.2.8}$$

It follows from (6.2.3), (6.2.6), (6.2.7) and (6.2.8) that

$$C(r) \geq \frac{(\rho + R) r^m}{2\pi R^2 \rho^{m-1}(\rho^q - r^q)^{1/q}}. \tag{6.2.9}$$

Passing to the limit as $\rho \downarrow R$ in the last inequality, we obtain

$$C(r) \geq \frac{r^m}{\pi R^m (R^q - r^q)^{1/q}}, \tag{6.2.10}$$

which together with (6.2.4) proves the sharpness of the constant in (6.2.1). $\square$

6.3 Others estimates for the l_q-norm of the Taylor series remainder

In this section we obtain estimates with sharp constants for the l_q -norm (quasi-norm for $0 < q < 1$) of the Taylor series remainder for bounded analytic functions and analytic functions whose real part is bounded or on-side bounded.

We start with a theorem concerning analytic functions with real part bounded from above which refines Hadamard-Borel-Carathéodory inequality (6.1.1).

Theorem 6.1. *Let the function* (6.1.3) *be analytic on* D_R *with* $\Re f$ *bounded from above, and let* $q > 0$, $m \geq 1$, $|z| = r < R$. *Then the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\} \qquad (6.3.1)$$

holds with the sharp constant.

Proof. We write (6.2.1) for the disk D_ϱ, $\varrho \in (r, R)$, with f replaced by $f - \omega$, where ω is an arbitrary real constant. Then

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{r^m}{\pi \varrho^m (\varrho^q - r^q)^{1/q}} \, ||\Re f - \omega||_{L_1(\partial D_\varrho)}. \qquad (6.3.2)$$

Putting here

$$\omega = \mathcal{A}_f(R) = \sup_{|\zeta|<R} \Re f(\zeta)$$

and taking into account that

$$||\Re f - \mathcal{A}_f(R)||_{L_1(\partial D_\varrho)} = 2\pi\rho\{\mathcal{A}_f(R) - \Re f(0)\} = 2\pi\rho \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\},$$

we find

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\},$$

which implies (6.3.1) after the passage to the limit as $\varrho \uparrow R$.

Hence, the sharp constant $C(r)$ in

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq C(r) \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\} \qquad (6.3.3)$$

obeys

$$C(r) \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}}. \qquad (6.3.4)$$

To get the lower estimate for $C(r)$, we shall use functions f_ρ given by (6.2.5). Taking into account equality

$$f_\rho^{(n)}(0) = w_\rho^{(n)}(0),$$

as well as (6.3.3), (6.2.7) and (6.2.8), we arrive at

$$C(r) \geq \frac{(\rho+R)r^m}{R\rho^{m-1}(\rho^q - r^q)^{1/q}}. \tag{6.3.5}$$

Passing to the limit as $\rho \downarrow R$ in the last inequality, we obtain

$$C(r) \geq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}}, \tag{6.3.6}$$

which together with (6.3.4) proves the sharpness of the constant in (6.3.1). □

Remark 6.2. Inequality (6.3.1) for $q = m = 1$ is well known (see, e.g. Polya and Szegö [75], III, Ch. 5, § 2). Adding $|c_0|$ and $|f(0)|$ to the left- and right-hand sides of (6.3.1) with $q = m = 1$, respectively, and replacing $-\Re f(0)$ by $|f(0)|$ in the resulting relation, we arrive at

$$\sum_{n=0}^{\infty} |c_n z|^n \leq \frac{R+r}{R-r}|f(0)| + \frac{2r}{R-r} \sup_{|\zeta|<R} \Re f(\zeta),$$

which is a refinement of the Hadamard-Borel-Carathéodory inequality

$$|f(z)| \leq \frac{R+r}{R-r}|f(0)| + \frac{2r}{R-r} \sup_{|\zeta|<R} \Re f(\zeta)$$

(see, e.g., Burckel [22], Ch. 6 and references there, Titchmarsh [86], Ch. 5).

The next assertion contains an sharp estimate for analytic functions on D_R with bounded real part. It is a refinement of the inequality

$$|f(z) - f(0)| \leq \frac{2r}{R-r} \sup_{|\zeta|<R} \{|\Re f(\zeta)| - |\Re f(0)|\}$$

which follows from (6.1.1).

Theorem 6.2. *Let the function* (6.1.3) *be analytic on* D_R *with bounded real part, and let* $q > 0$, $m \geq 1$, $|z| = r < 1$. *Then the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \{|\Re f(\zeta)| - |\Re f(0)|\} \tag{6.3.7}$$

holds with the sharp constant.

Proof. Setting

$$\omega = \mathcal{R}_f(R) = \sup_{|\zeta|<R} |\Re f(\zeta)|$$

in (6.3.2) and making use of the equalities

$$||\Re f - \mathcal{R}_f(R)||_{L_1(\partial D_\varrho)} = 2\pi\rho\{\mathcal{R}_f(R) - \Re f(0)\} = 2\pi\rho \sup_{|\zeta|<R}\{|\Re f(\zeta)| - \Re f(0)\},$$

we arrive at

$$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \le \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<R}\{|\Re f(\zeta)| - \Re f(0)\}.$$

This estimate leads to

$$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \le \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R}\{|\Re f(\zeta)| - \Re f(0)\} \qquad (6.3.8)$$

after the passage to the limit as $\varrho \uparrow R$. Replacing f by $-f$ in the last inequality, we obtain

$$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \le \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R}\{|\Re f(\zeta)| + \Re f(0)\},$$

which together with (6.3.8) results at (6.3.7).

Let us show that the constant in (6.3.7) is sharp. By $C(r)$ we denote the best constant in

$$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \le C(r) \sup_{|\zeta|<R}\{|\Re f(\zeta)| - |\Re f(0)|\}. \qquad (6.3.9)$$

As shown above, $C(r)$ obeys (6.3.4).

We introduce the family of analytic functions in $\overline{D}_R$

$$g_\rho(z) = \frac{\rho}{z-\rho} + \frac{\rho^2}{\rho^2 - R^2}, \qquad (6.3.10)$$

depending on a parameter $\rho > R$. By (5.4.21) and (5.4.25) we have

$$\sup_{|\zeta|<R}\{|\Re g_\rho(\zeta)| - |\Re g_\rho(0)|\} = \frac{R}{\rho + R}. \qquad (6.3.11)$$

Taking into account that the functions (6.2.5) and (6.3.10) differ by a constant, and using (6.3.9), (6.2.7) and (6.3.11), we arrive at (6.3.5). Passing there to the limit as $\rho \downarrow R$, we conclude that (6.3.6) holds, which together with (6.3.4) proves the sharpness of the constant in (6.3.7). $\square$

The following assertion contains an estimate with the sharp constant for bounded analytic functions in D_R. It gives a refinement of the estimate

$$|f(z) - f(0)| \leq \frac{2r}{R-r} \sup_{|\zeta|<R} \{|f(\zeta)| - |f(0)|\}$$

which follows from (6.1.1).

Theorem 6.3. *Let the function* (6.1.3) *be analytic and bounded on* D_R*, and let* $q > 0$*,* $m \geq 1$*,* $|z| = r < 1$*. Then the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \{|f(\zeta)| - |f(0)|\} \qquad (6.3.12)$$

holds with the sharp constant.

Proof. Setting

$$\omega = \mathcal{M}_f(R) = \sup_{|\zeta|<R} |f(\zeta)|$$

in (6.3.2) and using the equalities

$$||\Re f - \mathcal{M}_f(R)||_{L_1(\partial D_\varrho)} = 2\pi\rho\{\mathcal{M}_f(R) - \Re f(0)\} = 2\pi\rho \sup_{|\zeta|<R} \{|f(\zeta)| - \Re f(0)\},$$

we obtain

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<1} \{|f(\zeta)| - \Re f(0)\}.$$

Passing here to the limit as $\varrho \uparrow R$, we find

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \{|f(\zeta)| - \Re f(0)\}.$$

Replacing f by $fe^{i\alpha}$, we arrive at

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \sup_{|\zeta|<R} \{|f(\zeta)| - \Re\big(f(0)e^{i\alpha}\big)\},$$

which implies (6.3.12) by the arbitrariness of α.

Let us show that the constant in (6.3.12) is sharp. By $C(r)$ we denote the best constant in

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq C(r) \sup_{|\zeta|<R} \{|f(\zeta)| - |f(0)|\}. \qquad (6.3.13)$$

As shown above, $C(r)$ obeys (6.3.4).

We consider the family h_ρ of analytic functions in $\overline{D}$, defined by (6.3.10). By (5.4.33) we have

$$\sup_{|\zeta|<R} \{|g_\rho(\zeta)| - |g_\rho(0)|\} = \frac{R}{\rho + R}. \tag{6.3.14}$$

Taking into account that the functions (6.2.5) and (6.3.10) differ by a constant, and using (6.3.13), (6.2.7) and (6.3.14), we arrive at (6.2.9). Passing there to the limit as $\rho \downarrow R$, we obtain (6.2.10), which together with (6.2.4) proves the sharpness of the constant in (6.3.12). □

Remark 6.3. We note that a consequence of (5.1.6) is the inequality

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \le \frac{r^m}{R^{m-1}(R^q - r^q)^{1/q}} \, \frac{[\mathcal{M}_f(R)]^2 - |f(0)|^2}{\mathcal{M}_f(R)} \tag{6.3.15}$$

with the constant factor in the right-hand side twice as small as in (6.3.12) and sharp, which can be checked using the sequence of functions given by (6.3.10) and the limit passage as $\rho \downarrow R$. Inequality (6.3.15) for $q = 1, m = 1$ with $\mathcal{M}_f(R) \le 1$ was derived by Paulsen, Popescu and Singh [72].

The next assertion refines the inequality

$$|f(z) - f(0)| \le \frac{2r}{R - r} \, \Re f(0)$$

resulting from (6.1.1) for analytic functions in D_R with $\Re f > 0$.

Theorem 6.4. *Let the function* (6.1.3) *be analytic with positive* $\Re f$ *on* D_R, *and let* $q > 0$, $m \ge 1$, $|z| = r < 1$. *Then the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \le \frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \, \Re f(0) \tag{6.3.16}$$

holds with the sharp constant.

Proof. Setting $\omega = 0$ in (6.3.2), with f such that $\Re f > 0$ in D_R, we obtain

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \le \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \, \Re f(0),$$

which leads to (6.3.16) as $\varrho \uparrow R$.

Thus, the sharp constant $C(r)$ in

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq C(r)\, \Re f(0) \tag{6.3.17}$$

obeys the estimate (6.3.4).

To show the sharpness of the constant in (6.3.16), consider the family of analytic functions in $\overline{D}_R$

$$h_\rho(z) = \frac{\rho}{\rho - z} - \frac{\rho}{\rho + R}, \tag{6.3.18}$$

depending on the parameter $\rho > R$. By (5.4.39), the real part of h_ρ is positive in D_R. Taking into account that the functions (6.2.5) and (6.3.18) differ by a constant and using (6.3.17), (6.2.7) and $\Re h_\rho(0) = R(\rho+1)^{-1}$, we arrive at (6.3.5). Passing there to the limit as $\rho \downarrow R$, we obtain (6.3.6), which together with (6.3.4) proves the sharpness of the constant in (6.3.16). □

6.4 Bohr's type modulus and real part theorems

In this section we collect some corollaries of the theorems in Sect. 2.

Corollary 6.1. *Let the function* (6.1.3) *be analytic on* D_R*, and let*

$$\sup_{|\zeta|<R} \Re\{e^{-i \arg f(0)} f(\zeta)\} < \infty,$$

where $\arg f(0)$ *is replaced by zero if* $f(0) = c_0 = 0$.

Then for any $q \in (0, \infty]$*, integer* $m \geq 1$ *and* $|z| \leq R_{m,q}$ *the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \sup_{|\zeta|<R} \Re\{e^{-i \arg f(0)} f(\zeta)\} - |f(0)| \tag{6.4.1}$$

holds, where $R_{m,q} = r_{m,q} R$*, and* $r_{m,q}$ *is the root of the equation* $2^q r^{mq} + r^q - 1 = 0$ *in the interval* $(0, 1)$*. Here* $R_{m,q}$ *is the radius of the largest disk centered at* $z = 0$ *in which* (6.4.1) *takes place for all* f*. In particular,* (6.1.6) *holds.*

Proof. The condition

$$\frac{2r^m}{R^{m-1}(R^q - r^q)^{1/q}} \leq 1$$

ensuring the sharpness of the constant in (6.3.1) holds if $|z| \leq R_{m,q}$, where $R_{m,q} = r_{m,q} R$, and $r_{m,q}$ is the root of the equation $2^q r^{mq} + r^q - 1 = 0$ in the interval $(0, 1)$.

The disk of radius $R_{m,q}$ centered at $z = 0$ is the largest disk, where the inequality

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \sup_{|\zeta|<R} \Re f(\zeta) - \Re f(0) \tag{6.4.2}$$

holds for all f. The last inequality coincides with (6.4.1) for $f(0) = c_0 = 0$.

Suppose now that $f(0) \neq 0$. Setting $e^{-i \arg f(0)} f$ in place of f in (6.4.2) and noting that the coefficients $|c_n|$ in the left-hand side of (6.4.2) do not change, when $\Re f(0)$ is replaced by $|f(0)| = |c_0|$, we arrive at (6.4.1). □

Inequality (6.4.1) with $q = 1, m = 1$ becomes

$$\sum_{n=1}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<R} \Re\{e^{-i \arg f(0)} f(\zeta)\} - |f(0)| \tag{6.4.3}$$

with $|z| \leq R/3$, where $R/3$ is the radius of the largest disk centered at $z = 0$ in which (6.4.3) takes place. Note that (6.4.3) is equivalent to a sharp inequality obtained by Sidon [84] in his proof of Bohr's theorem and to the inequality derived by Paulsen, Popescu and Singh [72].

For $q = 1, m = 2$ inequality (6.4.1) is

$$\sum_{n=2}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<R} \Re\{e^{-i \arg f(0)} f(\zeta)\} - |f(0)|, \tag{6.4.4}$$

where $|z| \leq R/2$ and $R/2$ is the radius of the largest disk about $z = 0$ in which (6.4.4) takes place.

The next assertion follows from Theorem 6.3. For $q = 1, m = 1$ it contains Bohr's inequality (6.1.2).

Corollary 6.2. *Let the function* (6.1.3) *be analytic and bounded on* D_R*. Then for any* $q \in (0, \infty]$*, integer* $m \geq 1$ *and* $|z| \leq R_{m,q}$ *the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \sup_{|\zeta|<R} |f(\zeta)| - |f(0)| \tag{6.4.5}$$

holds, where $R_{m,q} = r_{m,q} R$*, and* $r_{m,q}$ *is the root of the equation* $2^q r^{mq} + r^q - 1 = 0$ *in the interval* $(0, 1)$*. Here* $R_{m,q}$ *is the radius of the largest disk centered at* $z = 0$ *in which* (6.4.5) *takes place for all* f*. In particular,* (6.1.6) *holds.*

For $q = 1, m = 2$ inequality (6.4.5) takes the form

$$|c_0| + \sum_{n=2}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<R} |f(\zeta)|, \tag{6.4.6}$$

where $|z| \leq R/2$. The value $R/2$ of the radius of the disk where (6.4.6) is valid cannot be improved. Note that the inequality

$$|c_0|^2 + \sum_{n=1}^{\infty} |c_n z^n| \le 1, \tag{6.4.7}$$

was obtained by Paulsen, Popescu and Singh [72] for functions (6.1.3) satisfying the condition $|f(\zeta)| \le 1$ in D_R and is valid for $|z| \le R/2$. The value $R/2$ of the radius of the disk where (6.4.7) holds is largest. Comparison of (6.4.6) and (6.4.7) shows that none of these inequalities is a consequence of the other one.

We conclude this section by an assertion which follows from Theorem 6.4.

Corollary 6.3. *Let the function* (6.1.3) *be analytic, and* $\Re\{e^{-i\arg f(0)} f\} > 0$ *on* D_R. *Then for any* $q \in (0, \infty]$, *integer* $m \ge 1$ *and* $|z| \le R_{m,q}$ *the inequality*

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \le |f(0)| \tag{6.4.8}$$

holds, where $R_{m,q} = r_{m,q} R$, *and* $r_{m,q}$ *is the root of the equation* $2^q r^{mq} + r^q - 1 = 0$ *in the interval* $(0, 1)$. *Here* $R_{m,q}$ *is the radius of the largest disk centered at* $z = 0$ *in which* (6.4.5) *takes place for all* f. *In particular,* (6.1.6) *holds.*

Note that the inequality (6.4.8) for $q = 1, m = 1$ with $|z| \le R/3$ was obtained by Aizenberg, Aytuna and Djakov [3] (see also Aizenberg, Grossman and Korobeinik [6]).

7

Estimates for the increment of derivatives of analytic functions

7.1 Introduction

The present chapter is connected with the contents of Chapters 1 and 3. One inequality, proved in Chapter 3, intimately relates the questions we address in this chapter. We mean the inequality which stems from Proposition 3.1:

$$|f(z) - f(0)| \leq \mathcal{K}_{0,p}(z)||\Re f - c||_p. \tag{7.1.1}$$

Here

$$\mathcal{K}_{0,p}(z) = \sup\{\mathcal{K}_p(z, \alpha) : 0 \leq \alpha \leq 2\pi\},$$

$z \in D_R$, c is an arbitrary real constant, $|| \cdot ||_p$ denote the L_p-norm of a real valued function on the circle $|\zeta| = R$, $1 \leq p \leq \infty$, and $\mathcal{K}_p(z, \alpha)$ is given by (3.2.2) and (3.2.3).

In this chapter, we generalize (7.1.1) to derivatives of an analytic function f in D_R. As a consequence, we derive a generalization of the Hadamard-Borel-Carathéodory inequality (1.1.2) for derivatives as well as analogues of the Carathéodory and Landau inequalities (5.1.1), (5.1.3) for the increment of derivatives at zero.

In Section 2 we find a representation for the best constant in the generalization of estimate (7.1.1)

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \mathcal{K}_{n,p}(z)||\Re\{f - \mathcal{P}_m\}||_p, \tag{7.1.2}$$

where $n \geq 0, \mathcal{P}_m$ is a polynomial of degree $m, m \leq n$. From (7.1.2) we obtain the following estimate with right-hand side containing the best polynomial approximation of $\Re f$ on the circle $|\zeta| = R$ in the $L_p(\partial D_R)$-norm:

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \mathcal{K}_{n,p}(z)E_{n,p}(\Re f), \tag{7.1.3}$$

with $n \geq 0$.

Section 3 of this chapter concerns corollaries of inequality (7.1.2) for $p = 1$. First, we show that

$$\mathcal{K}_{n,1}(z) = \frac{n![R^{n+1} - (R-r)^{n+1}]}{\pi(R-r)^{n+1}R^{n+1}},$$

where $|z| = r < R$. From inequality (7.1.2) with $p = 1$ and $m = 0$ we deduce estimates with sharp constants for the increment at zero of derivatives,

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \frac{2n![R^{n+1} - (R-r)^{n+1}]}{(R-r)^{n+1}R^n} \sup_{|\zeta|<R} \Re\{f(\zeta) - f(0)\}, \quad (7.1.4)$$

for $n \geq 0$, where z is a fixed point of the circle $|z| = r < R$. As a particular case, the estimate just mentioned contains the Hadamard-Borel-Carathéodory inequality (1.1.2). Similar sharp estimates are obtained when the right-hand side of (7.1.4) contains the expressions

$$\sup_{|\zeta|<R} \{|\Re f(\zeta)| - |\Re f(0)|\}, \qquad \sup_{|\zeta|<R} \{|f(\zeta)| - |f(0)|\}$$

in analogues of the Landau inequality, as well as $\Re f(0)$ provided that $\Re f(\zeta) \geq 0$ for $|\zeta| = R$ in the analogue of the Carathéodory inequality.

In Section 4 of this chapter, we deduce corollaries of (7.1.3) for $p = 2$ and $p = \infty$. In particular, we show that inequality (7.1.3) with $p = 2$ holds with the sharp constant

$$\mathcal{K}_{n,2}(z) = \frac{1}{R^{(2n+1)/2}} K_{n,2}\left(\frac{r}{R}\right),$$

where

$$K_{n,2}(\gamma) = \frac{n!}{\sqrt{\pi}(1-\gamma^2)^{(2n+1)/2}} \left\{ \sum_{k=0}^{n} \binom{n}{k}^2 \gamma^{2k} - (1-\gamma^2)^{2n+1} \right\}^{1/2},$$

and with $E_{n,2}(\Re f)$ given by (5.5.5).

We note, that a corollary of (7.1.3) for $p = \infty$ contains estimates for the modulus of the increment at zero of derivatives of f in terms of

$$\mathcal{O}_{n,\Re f}(\partial D_R) = \inf_{\mathcal{P} \in \{\mathfrak{P}_n\}} \mathcal{O}_{\Re\{f-\mathcal{P}\}}(D_R),$$

where $\mathcal{O}_{\Re f}(D_R)$ is the oscillation of $\Re f$ on the disk D_R.

7.2 Estimate for $|\Delta f^{(n)}(z)|$ by $||\Re\{f - \mathcal{P}_m\}||_p$. General case

The main assertion of this section is

Proposition 7.1. *Let f be analytic on D_R with $\Re f \in h_p(D_R)$, $1 \le p \le \infty$. Further, let $n \ge 0$, and let $\mathcal{P}_m$ be a polynomial of degree $m \le n$. Then for any fixed point $z, |z| = r < R$, there holds*

$$|f^{(n)}(z) - f^{(n)}(0)| \le \mathcal{K}_{n,p}(z)||\Re\{f - \mathcal{P}_m\}||_p \tag{7.2.1}$$

with the sharp constant

$$\mathcal{K}_{n,p}(z) = \frac{1}{R^{(np+1)/p}} K_{n,p}\left(\frac{r}{R}\right), \tag{7.2.2}$$

where

$$K_{n,p}(\gamma) = \frac{n!}{\pi} \sup_{\alpha} \left\{ \int_{|\zeta|=1} \left| \Re\left\{ \frac{\zeta^{n+1} - (\zeta - \gamma)^{n+1}}{(\zeta - \gamma)^{n+1}\zeta^n} e^{i\alpha} \right\} \right|^q |d\zeta| \right\}^{1/q}, \tag{7.2.3}$$

and $1/p + 1/q = 1$.

In particular,

$$|f^{(n)}(z) - f^{(n)}(0)| \le \mathcal{K}_{n,p}(z) E_{n,p}(\Re f). \tag{7.2.4}$$

Proof. By Lemma 5.1 and notation (5.2.5), we have

$$|\Delta f^{(n)}(z)| = \frac{n!}{\pi R} \sup_{\alpha} \int_{|\zeta|=R} \left\{ \mathcal{G}_{n,z,\alpha}(\zeta) - \mathcal{G}_{n,0,\alpha}(\zeta) \right\} \Re f(\zeta)\, |d\zeta|. \tag{7.2.5}$$

By (7.2.5) we arrive at the formula

$$\mathcal{K}_{n,p}(z) = \frac{n!}{\pi R} \sup_{\alpha} ||\mathcal{G}_{n,z,\alpha} - \mathcal{G}_{n,0,\alpha}||_q \tag{7.2.6}$$

for the sharp constant $\mathcal{K}_{n,p}(z)$ in

$$|f^{(n)}(z) - f^{(n)}(0)| \le \mathcal{K}_{n,p}(z)||\Re f||_p. \tag{7.2.7}$$

In view of (5.2.5), representation (7.2.6) can be written as

$$\mathcal{K}_{n,p}(z) = \frac{n!}{\pi R} \sup_{\beta} \left\{ \int_{|\xi|=R} \left| \Re\left\{ \frac{\xi^{n+1} - (\xi - z)^{n+1}}{(\xi - z)^{n+1}\xi^n} e^{i\beta} \right\} \right|^q |d\xi| \right\}^{1/q}, \tag{7.2.8}$$

where the case $p = 1$ ($q = \infty$) is understood in the sense of passage to the limit.

Suppose $1 < p \le \infty$. Setting $z = re^{i\tau}, \xi = Re^{it}, \gamma = r/R$ and $\varphi = t - \tau$ in (7.2.8), we obtain

$$\mathcal{K}_{n,p}(z) = \frac{n!}{\pi R} \sup_{\beta} \left\{ \int_{\tau}^{2\pi+\tau} \left| \Re\left\{ \frac{e^{i(n+1)t} - (e^{it} - \gamma e^{i\tau})^{n+1}}{(e^{it} - \gamma e^{i\tau})^{n+1} R^n e^{int}} e^{i\beta} \right\} \right|^q R dt \right\}^{1/q}$$

$$= \frac{n!}{\pi R^{(np+1)/p}} \sup_{\beta} \left\{ \int_{0}^{2\pi} \left| \Re\left\{ \frac{e^{i(n+1)\varphi} - (e^{i\varphi} - \gamma)^{n+1}}{(e^{i\varphi} - \gamma)^{n+1} e^{in\varphi}} e^{i(\beta - n\tau)} \right\} \right|^q d\varphi \right\}^{1/q}.$$

Putting here $\alpha = \beta - n\tau$ and using 2π-periodicity of the resulting function in α, we find

$$\mathcal{K}_{n,p}(z) = \frac{1}{R^{(np+1)/p}} K_{n,p}\left(\frac{r}{R}\right),$$

where

$$K_{n,p}(\gamma) = \frac{n!}{\pi} \sup_{\alpha} \left\{ \int_{|\zeta|=1} \left| \Re\left\{ \frac{\zeta^{n+1} - (\zeta - \gamma)^{n+1}}{(\zeta - \gamma)^{n+1} \zeta^n} e^{i\alpha} \right\} \right|^q |d\zeta| \right\}^{1/q},$$

which proves (7.2.2) and (7.2.3).

Replacing f by $f - \mathcal{P}_m$ with $m \le n$ in (7.2.7), we arrive at inequality (7.2.1), which leads to (7.2.4). □

7.3 The case $p = 1$ and its corollaries

7.3.1 Explicit estimate in the case $p = 1$

In this section we obtain an explicit representation for the sharp constant in (7.2.1) for $p = 1$ and derive some corollaries. In particular, the next assertion contains an explicit formula for $\mathcal{K}_{n,1}(z)$.

Corollary 7.1. *Let f be analytic on D_R with $\Re f \in h_1(D_R)$. Further, let $n \ge 0$, and let $\mathcal{P}_m$ be a polynomial of degree $m \le n$. Then for any fixed point $z, |z| = r < R$, there holds*

$$|f^{(n)}(z) - f^{(n)}(0)| \le \mathcal{K}_{n,1}(z) ||\Re\{f - \mathcal{P}_m\}||_1 \tag{7.3.1}$$

with the sharp constant

$$\mathcal{K}_{n,1}(z) = \frac{n![R^{n+1} - (R-r)^{n+1}]}{\pi (R-r)^{n+1} R^{n+1}}. \tag{7.3.2}$$

In particular,

$$|f^{(n)}(z) - f^{(n)}(0)| \le \mathcal{K}_{n,1}(z) E_{n,1}(\Re f). \tag{7.3.3}$$

Proof. Inequalities (7.3.1) and (7.3.3) follow as particular cases from Proposition 7.1. Representation (7.2.6) for $p = 1$ can be written as

$$\mathcal{K}_{n,1}(z) = \frac{n!}{\pi R} \sup_{\alpha} \sup_{|\zeta|=R} \left| \Re \left\{ \frac{\zeta^{n+1} - (\zeta - z)^{n+1}}{(\zeta - z)^{n+1}\zeta^n} e^{i\alpha}. \right\} \right| \tag{7.3.4}$$

Permutating the suprema in (7.3.4), we obtain the equality

$$\mathcal{K}_{n,1}(z) = \frac{n!}{\pi R} \sup_{|\zeta|=R} \sup_{\alpha} \left| \Re \left\{ \frac{\zeta^{n+1} - (\zeta - z)^{n+1}}{(\zeta - z)^{n+1}\zeta^n} e^{i\alpha} \right\} \right|$$

$$= \frac{n!}{\pi R} \sup_{|\zeta|=R} \left| \frac{\zeta^{n+1} - (\zeta - z)^{n+1}}{(\zeta - z)^{n+1}\zeta^n} \right|. \tag{7.3.5}$$

On one hand, (7.3.5) implies

$$\mathcal{K}_{n,1}(z) \geq \frac{n!}{\pi R} \sup_{|\zeta|=R} \frac{|\zeta|^{n+1} - |\zeta - z|^{n+1}}{|\zeta - z|^{n+1}|\xi|^n} \geq \frac{n![R^{n+1} - (R-r)^{n+1}]}{\pi(R-r)^{n+1}R^{n+1}}. \tag{7.3.6}$$

On the other hand, (7.3.5) gives

$$\mathcal{K}_{n,1}(z) = \frac{n!}{\pi R^{n+1}} \sup_{|\zeta|=R} \left| 1 - \left(\frac{\zeta}{\zeta - z} \right)^{n+1} \right|$$

$$= \frac{n!}{\pi R^{n+1}} \sup_{|\zeta|=R} \left| 1 - \frac{\zeta}{\zeta - z} \right| \left| \sum_{k=0}^{n} \left(\frac{\zeta}{\zeta - z} \right)^k \right|$$

$$\leq \frac{n!}{\pi R^{n+1}} \sup_{|\zeta|=R} \frac{|z|}{|\zeta - z|} \sum_{k=0}^{n} \frac{|\zeta|^k}{|\zeta - z|^k}.$$

From the last inequality it follows

$$\mathcal{K}_{n,1}(z) \leq \frac{n!r}{\pi R^{n+1}(R-r)} \sum_{k=0}^{n} \frac{R^k}{(R-r)^k} = \frac{n![R^{n+1} - (R-r)^{n+1}]}{\pi(R-r)^{n+1}R^{n+1}},$$

which together with (7.3.6) proves the equality (7.3.2). □

7.3.2 Hadamard-Borel-Carathéodory type inequality for derivatives

The estimate for $|f^{(n)}(z) - f^{(n)}(0)|$ with $n \geq 0$ below contains the value

$$\sup_{|\zeta|<R} \Re f(\zeta) - \Re f(0)$$

in the right-hand side and generalizes the Hadamard-Borel-Carathéodory inequality (1.1.2) for derivatives. In particular, for $n = 0$ that inequality coincides with (1.1.2).

Corollary 7.2. *Let f be analytic on D_R with $\Re f$ bounded from above. Then for any fixed $z, |z| = r < R$, the inequality*

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n} \sup_{|\zeta|<R} \Re\Delta f(\zeta) \tag{7.3.7}$$

holds with the best constant, where $n \geq 0$.

Proof. By Corollary 7.1,

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \frac{n![\rho^{n+1} - (\rho-r)^{n+1}]}{\pi(\rho-r)^{n+1}\rho^{n+1}} ||\Re f - \omega||_{L_1(\partial D_\rho)}, \tag{7.3.8}$$

where $\rho \in (r, R)$, ω is a real constant and $n \geq 0$.

We set

$$\omega = \mathcal{A}_f(R) = \sup_{|\zeta|<R} \Re f(\zeta)$$

in (7.3.8). Taking into account (5.4.8), we arrive at

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \frac{2n![\rho^{n+1} - (\rho-r)^{n+1}]}{(\rho-r)^{n+1}\rho^n} \sup_{|\zeta|<R} \Re\Delta f(\zeta).$$

Passing to the limit as $\rho \uparrow R$ in the last inequality, we obtain (7.3.7).

Now, we show that the constant in (7.3.7) is sharp. Consider the family of analytic functions on $\overline{D}_R$

$$f_\xi(z) = \frac{\xi}{z - \xi}, \tag{7.3.9}$$

where ξ is a complex parameter, $|\xi| > R$.

Let n be a nonnegative integer. The function $f_\xi(z)$ defined by (7.3.9) satisfies

$$|f_\xi^{(n)}(z) - f_\xi^{(n)}(0)| = \frac{n!|\xi^{n+1} - (\xi - z)^{n+1}|}{|\xi - z|^{n+1}|\xi|^n}. \tag{7.3.10}$$

Fixing $z = re^{it}$ in D_R and putting $\xi = \rho e^{it}$, by (7.3.10) we obtain

$$|f_\xi^{(n)}(z) - f_\xi^{(n)}(0)| = \frac{n![\rho^{n+1} - (\rho-r)^{n+1}]}{(\rho-r)^{n+1}\rho^n}. \tag{7.3.11}$$

Let $\mathcal{K}_n(z)$ be the best constant in

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \mathcal{K}_n(z) \sup_{|\zeta|<R} \Re\Delta f(\zeta). \tag{7.3.12}$$

As shown above,

$$\mathcal{K}_n(z) \leq \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n}. \tag{7.3.13}$$

By (7.3.12),

$$|f_\xi^{(n)}(z) - f_\xi^{(n)}(0)| \leq \mathcal{K}_n(z) \max_{|\zeta|=R} \Re\Delta f_\xi(\zeta),$$

which together with (5.4.11), (7.3.11) implies

$$\mathcal{K}_n(z) \geq \frac{n![\rho^{n+1} - (\rho-r)^{n+1}](\rho+R)}{(\rho-r)^{n+1}\rho^n R}.$$

Passing here to the limit as $\rho \downarrow R$, we obtain

$$\mathcal{K}_n(z) \geq \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n},$$

which, along with (7.3.13), proves the sharpness of the constant in the inequality (7.3.7). □

Observe also that replacing f by $-f$ in (7.3.7), we deduce the inequality

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n} \sup_{|\zeta|<R} \Re\Delta\{-f(\zeta)\} \tag{7.3.14}$$

for analytic functions f on D_R with $\Re f$ bounded from below, where $n \geq 0$. Unlike (7.3.7) with

$$\mathbf{A}_f(R) = \sup_{|\zeta|<R} \Re\Delta f(\zeta) = \sup_{|\zeta|<R} \Re f(\zeta) - \Re f(0)$$

in the right-hand side, inequality (7.3.14) contains the expression

$$\mathbf{B}_f(R) = \sup_{|\zeta|<R} \Re\Delta\{-f(\zeta)\} = \Re f(0) - \inf_{|\zeta|<R} \Re f(\zeta).$$

Unifying (7.3.7) with (7.3.14), we arrive at the estimate

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n} \min\left\{\mathbf{A}_f(R), \mathbf{B}_f(R)\right\},$$

for analytic functions f on D_R with bounded $\Re f$, where $n \geq 0$.

7.3.3 Landau type inequalities

The following assertion contains a sharp estimate for $|f^{(n)}(z) - f^{(n)}(0)|$ with

$$\sup_{|\zeta|<R} |\Re f(\zeta)| - |\Re f(0)|$$

in the right-hand side. The estimate below is similar to the Landau inequality (5.1.3).

Corollary 7.3. *Let f be analytic on D_R with bounded $\Re f$. Then for any fixed $z, |z| = r < R$, the inequality*

$$|f^{(n)}(z) - f^{(n)}(0)| \le \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n} \sup_{|\zeta|<R} \Delta|\Re f(\zeta)| \qquad (7.3.15)$$

holds with the best constant, where $n \ge 0$.

Proof. Setting

$$\omega = \mathcal{R}_f(R) = \sup_{|\zeta|<R} |\Re f(\zeta)|$$

in (7.3.8) and taking into account (5.4.18), we arrive at

$$|f^{(n)}(z) - f^{(n)}(0)| \le \frac{2n!\left\{\rho^{n+1} - (\rho-r)^{n+1}\right\}}{(\rho-r)^{n+1}\rho^n} \left\{\mathcal{R}_f(R) - \Re f(0)\right\}.$$

Passing to the limit as $\rho \uparrow R$ in the last inequality, we obtain

$$|f^{(n)}(z) - f^{(n)}(0)| \le \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n} \left\{\mathcal{R}_f(R) - \Re f(0)\right\}. \quad (7.3.16)$$

Replacing f by $-f$ in (7.3.16), we have

$$|f^{(n)}(z) - f^{(n)}(0)| \le \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n} \left\{\mathcal{R}_f(R) + \Re f(0)\right\},$$

which together with (7.3.16) implies (7.3.15).

Let us show that the constant in (7.3.15) is sharp. We introduce the family of analytic functions in $\overline{D}_R$

$$g_\xi(z) = \frac{\xi}{z-\xi} + \frac{|\xi|^2}{|\xi|^2 - R^2}, \qquad (7.3.17)$$

depending on a complex parameter $\xi = \rho e^{i\tau},\ \rho > R$.

Let $z = re^{it}$ be a fixed point, $r < R$, and let $\xi = \rho e^{it}$. Taking into account that the functions (7.3.9) and (7.3.17) differ by a constant, for any nonnegative integer n by (7.3.11) we have

$$|g_\xi^{(n)}(z) - g_\xi^{(n)}(0)| = \frac{n![\rho^{n+1} - (\rho-r)^{n+1}]}{(\rho-r)^{n+1}\rho^n}. \qquad (7.3.18)$$

By $\mathcal{K}_n(z)$ we denote the best constant in

$$|f^{(n)}(z) - f^{(n)}(0)| \le \mathcal{K}_n(z)\{\mathcal{R}_f(R) - |\Re f(0)|\}. \qquad (7.3.19)$$

As shown above,

$$\mathcal{K}_n(z) \le \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n}. \tag{7.3.20}$$

By (7.3.19),

$$|g_\xi^{(n)}(z) - g_\xi^{(n)}(0)| \le \mathcal{K}_n(z)\{\mathcal{R}_{g_\xi}(R) - |\Re g_\xi(0)|\},$$

which together with (5.4.25), (7.3.18) implies

$$\mathcal{K}_n(z) \ge \frac{n![\rho^{n+1} - (\rho-r)^{n+1}](\rho+R)}{(\rho-r)^{n+1}\rho^n R}.$$

Passing to the limit as $\rho \downarrow R$ in the last inequality, we obtain

$$\mathcal{K}_n(z) \ge \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n},$$

which, along with (7.3.20), proves the sharpness of the constant in (7.3.15).□

The assertion below contains a sharp estimate of $|f^{(n)}(z) - f^{(n)}(0)|$ with

$$\sup_{|\zeta|<R} |f(\zeta)| - |f(0)|$$

in the right-hand side. This estimate is closely related to the Landau inequality (5.1.3).

Corollary 7.4. *Let f be analytic and bounded on D_R. Then for any fixed $z, |z| = r < R$, the sharp inequality*

$$|f^{(n)}(z) - f^{(n)}(0)| \le \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n} \sup_{|\zeta|<R} \Delta|f(\zeta)| \tag{7.3.21}$$

holds for $n \ge 0$.

Proof. Setting

$$\omega = \mathcal{M}_f(R) = \sup_{|\zeta|<R} |f(\zeta)|$$

in (7.3.8) and taking into account (5.4.29), we arrive at

$$|f^{(n)}(z) - f^{(n)}(0)| \le \frac{2n!\left\{\rho^{n+1} - (\rho-r)^{n+1}\right\}}{(\rho-r)^{n+1}\rho^n} \left\{\mathcal{M}_f(R) - \Re f(0)\right\}.$$

Passing to the limit as $\rho \uparrow R$ in the last inequality, we obtain

$$|f^{(n)}(z) - f^{(n)}(0)| \le \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n} \left\{\mathcal{M}_f(R) - \Re f(0)\right\}. \tag{7.3.22}$$

Replacing f by $fe^{i\alpha}$ in (7.3.22), we have

$$|f^{(n)}(z) - f^{(n)}(0)| \le \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n}\left\{\mathcal{M}_f(R) - \Re(f(0)e^{i\alpha})\right\},$$

which due to the arbitrariness of α implies (7.3.21).

We check that the constant in (7.3.21) is sharp. Let n be a nonnegative integer and let $\mathcal{K}_n(z)$ denote the sharp constant in

$$|f^{(n)}(z) - f^{(n)}(0)| \le \mathcal{K}_n(z)\left\{\mathcal{M}_f(R) - |f(0)|\right\}. \tag{7.3.23}$$

As shown above,

$$\mathcal{K}_n(z) \le \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n}. \tag{7.3.24}$$

A particular case of (7.3.23) is

$$|g_\xi^{(n)}(z) - g_\xi^{(n)}(0)| \le \mathcal{K}_n(z)\left\{\mathcal{M}_{g_\xi}(R) - |g_\xi(0)|\right\}, \tag{7.3.25}$$

where $g_\xi(z)$ is the analytic function defined by (7.3.17). For any fixed $z = re^{it}, r < R$, and $\xi = \rho e^{it}$ by (5.4.33), (7.3.18), and (7.3.25) we have

$$\mathcal{K}_n(z) \ge \frac{n!\left\{\rho^{n+1} - (\rho-r)^{n+1}\right\}(\rho+R)}{(\rho-r)^{n+1}\rho^n R}.$$

Passing to the limit as $\rho \downarrow R$ in the last inequality, we obtain

$$\mathcal{K}_m(z) \ge \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n},$$

which, along with (7.3.24), proves the sharpness of the constant in the inequality (7.3.21). □

7.3.4 Carathéodory type inequality

The following assertion contains estimate for $|f^{(n)}(z) - f^{(n)}(0)|$ in terms of $\Re f(0)$ under the assumption that $\Re f(\zeta) > 0$ for $|\zeta| < R$. This estimate is closely related to the Carathéodory inequality (5.1.1).

Corollary 7.5. *Let f be analytic with $\Re f(\zeta) > 0$ on the disk D_R. Then for any fixed $z, |z| = r < R$, the inequality*

$$|f^{(n)}(z) - f^{(n)}(0)| \le \frac{2n!\left\{R^{n+1} - (R-r)^{n+1}\right\}}{(R-r)^{n+1}R^n}\Re f(0) \tag{7.3.26}$$

holds with the best constant, where $n \ge 0$.

Proof. Suppose $\Re f(\zeta) > 0$ for $|\zeta| < R$. We put $\omega = 0$ in (7.3.8). Since

$$||\Re f||_{L_1(\partial D_\rho)} = 2\pi\rho\, \Re f(0),$$

it follows from (7.3.8) that

$$|f^{(n)}(z) - f^{(n)}(0)| \le \frac{2n!\left\{\rho^{n+1} - (\rho - r)^{n+1}\right\}}{(\rho - r)^{n+1}\rho^n}\, \Re f(0),$$

where $n \ge 0$. Passing to the limit as $\rho \uparrow R$ in the last inequality, we arrive at (7.3.26).

We prove that the constant in (7.3.26) is sharp. Introduce the family of analytic functions

$$h_\xi(z) = \frac{\xi}{\xi - z} - \frac{|\xi|}{|\xi| + R}, \tag{7.3.27}$$

which depend on the complex parameter $\xi = \rho e^{i\tau}$, $\rho > R$. By (5.4.39), $\Re h_\xi(\zeta) \ge 0$, for $|\zeta| = R$.

Let $z = re^{it}$ be a fixed point, $r < R$, and let $\xi = \rho e^{it}$. Taking into account that functions (7.3.9) and (7.3.27) differ by a constant, for any nonnegative integer n by (7.3.11) we have

$$|h_\xi^{(n)}(z) - h_\xi^{(n)}(0)| = \frac{n![\rho^{n+1} - (\rho - r)^{n+1}]}{(\rho - r)^{n+1}\rho^n}. \tag{7.3.28}$$

By $\mathcal{K}_n(z)$ we denote the sharp constant in

$$|f^{(n)}(z) - f^{(n)}(0)| \le \mathcal{K}_n(z)\Re f(0). \tag{7.3.29}$$

As shown above,

$$\mathcal{K}_n(z) \le \frac{2n!\left\{R^{n+1} - (R - r)^{n+1}\right\}}{(R - r)^{n+1}R^n}. \tag{7.3.30}$$

By (7.3.29),

$$|h_\xi^{(n)}(z) - h_\xi^{(n)}(0)| \le \mathcal{K}_n(z)\Re h_\xi(0),$$

Hence, by (5.4.40), (7.3.28)

$$\mathcal{K}_n(z) \ge \frac{n!\left\{\rho^{n+1} - (\rho - r)^{n+1}\right\}(\rho + R)}{(\rho - r)^{n+1}\rho^m R}.$$

Passing to the limit as $\rho \downarrow R$ in the last inequality, we obtain

$$\mathcal{K}_n(z) \ge \frac{2n!\left\{R^{n+1} - (R - r)^{n+1}\right\}}{(R - r)^{n+1}R^n},$$

which, along with (7.3.30), proves the sharpness of the constant in the inequality (7.3.26). □

7.4 The cases $p = 2$ and $p = \infty$

The next assertion is a particular case of Proposition 7.1 for $p = 2$.

Corollary 7.6. *Let f be analytic on D_R with $\Re f \in h_2(D_R)$. Further, let $n \geq 0$, and let $\mathcal{P}_m$ be a polynomial of degree $m \leq n$. Then for any fixed point $z, |z| = r < R$, there holds*

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \mathcal{K}_{n,2}(z)||\Re\{f - \mathcal{P}_m\}||_2, \tag{7.4.1}$$

with the sharp constant

$$\mathcal{K}_{n,2}(z) = \frac{1}{R^{(2n+1)/2}} K_{n,2}\left(\frac{r}{R}\right), \tag{7.4.2}$$

where

$$K_{n,2}(\gamma) = \frac{n!}{\sqrt{\pi}(1-\gamma^2)^{(2n+1)/2}} \left\{ \sum_{k=0}^{n} \binom{n}{k}^2 \gamma^{2k} - (1-\gamma^2)^{2n+1} \right\}^{1/2}. \tag{7.4.3}$$

In particular,

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \mathcal{K}_{n,2}(z) E_{n,2}(\Re f), \tag{7.4.4}$$

where

$$E_{n,2}(\Re f) = \left|\left| \Re\left\{ f - \sum_{k=0}^{n} \frac{f^{(k)}(0)\zeta^k}{k!} \right\} \right|\right|_2$$
$$= \left\{ ||\Re f - \Re f(0)||_2^2 - \pi R \sum_{k=1}^{n} \frac{|f^{(k)}(0)|^2 R^{2k}}{(k!)^2} \right\}^{1/2}.$$

Here the sum in k from 1 to n is assumed to vanish for $n = 0$.

Proof. Consider the integral in (7.2.3) for $p = 2$. Putting $\zeta = 1/\xi$ there, we obtain

$$\int_{|\zeta|=1} \left| \Re\left\{ \frac{\zeta^{n+1} - (\zeta-\gamma)^{n+1}}{(\zeta-\gamma)^{n+1}\zeta^n} e^{i\alpha} \right\} \right|^2 |d\zeta|$$
$$= \int_{|\xi|=1} \left| \Re\left\{ \frac{\xi^n[1-(1-\gamma\xi)^{n+1}]}{(1-\gamma\xi)^{n+1}} e^{i\alpha} \right\} \right|^2 |d\xi|, \tag{7.4.5}$$

where $\gamma = r/R < 1$ and α is a real parameter. Similarly,

$$\int_{|\zeta|=1} \left| \Im\left\{ \frac{\zeta^{n+1} - (\zeta-\gamma)^{n+1}}{(\zeta-\gamma)^{n+1}\zeta^n} e^{i\alpha} \right\} \right|^2 |d\zeta|$$
$$= \int_{|\xi|=1} \left| \Im\left\{ \frac{\xi^n[1-(1-\gamma\xi)^{n+1}]}{(1-\gamma\xi)^{n+1}} e^{i\alpha} \right\} \right|^2 |d\xi|. \tag{7.4.6}$$

The function

$$g(\xi) = \frac{\xi^n[1-(1-\gamma\xi)^{n+1}]}{(1-\gamma\xi)^{n+1}}e^{i\alpha}, \qquad 0 \le \gamma < 1,$$

analytic in the disk $|\xi| < \gamma^{-1}$, satisfies (5.5.8) for $\rho = 0$. Therefore, the equality

$$\int_0^{2\pi} [\Re g(\rho e^{i\vartheta})]^2 d\vartheta = \int_0^{2\pi} [\Im g(\rho e^{i\vartheta})]^2 d\vartheta$$

holds for $\rho = 1$, which together with (7.2.3) for $p = 2$ and (7.4.5), (7.4.6) leads to

$$K_{n,2}(\gamma) = \frac{n!}{\pi}\left\{\frac{1}{2}\int_{|\zeta|=1} \frac{|\zeta^{n+1}-(\zeta-\gamma)^{n+1}|^2}{|\zeta-\gamma|^{2(n+1)}}|d\zeta|\right\}^{1/2}.$$

By straightforward calculations we find

$$K_{n,2}(\gamma) = \frac{n!}{\pi}\left\{\frac{1}{2}\int_{|\zeta|=1}\left|1-\left(\frac{\zeta}{\zeta-\gamma}\right)^{n+1}\right|^2 |d\zeta|\right\}^{1/2}$$

$$= \frac{n!}{\pi}\left\{\frac{1}{2}\int_{|\zeta|=1}\left(1-2\Re\left(\frac{\zeta}{\zeta-\gamma}\right)^{n+1}+\frac{1}{|\zeta-\gamma|^{2(n+1)}}\right)|d\zeta|\right\}^{1/2}.$$

Now, taking into account the equality

$$\int_{|\zeta|=1}\Re\left(\frac{\zeta}{\zeta-\gamma}\right)^{n+1}|d\zeta| = \Re\left\{\frac{1}{i}\int_{|\zeta|=1}\frac{\zeta^n}{(\zeta-\gamma)^{n+1}}d\zeta\right\} = 2\pi$$

and (5.5.10), we arrive at

$$K_{n,2}(\gamma) = \frac{n!}{\sqrt{\pi}(1-\gamma^2)^{(2n+1)/2}}\left\{\sum_{k=0}^{n}\binom{n}{k}^2\gamma^{2k}-(1-\gamma^2)^{2n+1}\right\}^{1/2},$$

which together with (7.2.2) for $p = 2$ results at (7.4.2), (7.4.3).

The formula for $E_{n,2}(\Re f)$ stated in Corollary 7.6 was obtained in Corollary 5.7. □

Remark 7.1. For $n = 0$, inequalities (7.4.1) and (7.4.4) were obtained in previous chapters (see (3.3.5) and (2.3.5), respectively).

For $p = \infty$ inequality (7.2.4) can be written in terms of $\mathcal{O}_{n,\Re f}(D_R)$ which is the infimum of the oscillation of $\Re f(\zeta) - \Re\mathcal{P}(\zeta)$ on the disk D_R taken over the set $\{\mathfrak{P}_n\}$ of polynomials $\mathcal{P}$ of degree at most n. The next assertion follows from Proposition 7.1 for $p = \infty$ and can be proved in the same manner as Corollary 5.13.

Corollary 7.11. *Let f be analytic on D_R with bounded $\Re f$, and let $n \geq 0$. Then for any fixed point $z, |z| = r < R$, the inequality*

$$|f^{(n)}(z) - f^{(n)}(0)| \leq \frac{1}{2}\mathcal{K}_{n,\infty}(z)\mathcal{O}_{n,\Re f}(D_R) \tag{7.4.7}$$

holds with the sharp constant

$$\mathcal{K}_{n,\infty}(z) = \frac{1}{R^n} K_{n,\infty}\left(\frac{r}{R}\right),$$

where

$$K_{n,\infty}(\gamma) = \frac{n!}{\pi} \sup_{\alpha} \int_{|\zeta|=1} \left| \Re\left\{ \frac{\zeta^{n+1} - (\zeta - \gamma)^{n+1}}{(\zeta - \gamma)^{n+1}\zeta^n} e^{i\alpha} \right\} \right| |d\zeta|.$$

Remark 7.2. Note that the sharp inequality (3.4.23)

$$|\Delta f(z)| \leq \frac{1}{\pi} \log \frac{R+r}{R-r}\, \mathcal{O}_{\Re f}(D_R)$$

is a particular case of (7.4.7) for $n = 0$.

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Index

List of Symbols

Point Sets

D_R	disk $\|z\| < R$ in complex plane $\mathbb{C}$ and real plane $\mathbb{R}^2$	xi
G	a set in $\mathbb{C}$ or $\mathbb{R}^2$ with closure $\overline{G}$ and boundary ∂G	14
$\mathbb{C}_+$	upper half-plane of the complex plane	16
$\mathbb{R}^2_+$	upper half-plane of the real plane	57
z	$x + iy$ in $\mathbb{C}$, and (x, y) in $\mathbb{R}^2$	57

Vectors

ℓ_ϑ	unit vector at an angle ϑ with respect to the x-axis	61
l_ϑ	unit vector having an angle ϑ with the radial direction	63

Functions

f	analytic function in a domain of $\mathbb{C}$	xi
$\Re f$	real part of analytic function f	xii
$\Im f$	imaginary part of analytic function f	xii
$\|g\|$	modulus (absolute value) of complex (real) valued function g	xi
$\mathcal{P}_m$	polynomial of degree m	xiv
$\varDelta g(z)$	$= g(z) - g(0)$, increment of a function g at $z = 0$	6
$\arg f(z)$	argument of $f(z)$	13
$\varDelta_\xi g(z)$	$= g(z) - g(\xi)$, increment of a function g	15
$B(\alpha, \beta)$	Beta-function	18
$F(\alpha, \beta; \gamma, x)$	hypergeometric Gauss function	31
u	harmonic function in a domain of $\mathbb{R}^2$	57
∇u	gradient of u	57
$\|\nabla u\|$	Euclidean length of ∇u	57
$\partial u/\partial l$	directional derivative of u in the direction of a unit vector l	58
$\varGamma(\alpha)$	Gamma-function	75

Functional Spaces

$L_p(\partial D_R)$	space of real valued functions for which $\|f\|^p$ is integrable on ∂D_R	19
$h_p(D_R)$	Hardy space of harmonic functions in the disk D_R which are represented by the Poisson integral with a density in $L_p(\partial D_R)$	6
$C(G)$	space of real valued functions continuous on G	6
$C^1(G)$	space of real valued functions whose elements have continuous first derivatives on $G \subset \mathbb{R}^2$	63

Characteristics of Functions