DOI 10.1007/s10958-017-3444-y
Journal of Mathematical Sciences, Vol. 224, No. 5, August, 2017

NOTE ON A NONSTANDARD EIGENFUNCTION OF THE PLANAR FOURIER TRANSFORM

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UDC 517.95

We consider a nontrivial example of distributional eigenfunction of the planar Fourier transform. This eigenfunction is not a tensor product of univariate eigenfunctions. As a consequence, we obtain a formula for multi-dimensional eigenfunctions in dimension $2N$. *Bibliography*: 6 *titles.*

The Fourier transform of a function $f \in L^2(\mathbb{R}^N)$ is defined by

$$\mathscr{F}(f)(\xi) = \frac{1}{(2\pi)^{N/2}} \int\limits_{\mathbb{R}^N} f(\mathbf{x}) \mathrm{e}^{-i\,\mathbf{x}\cdot\xi} d\mathbf{x}\,,$$

where $\mathbf{x}\cdot\xi = x_1\xi_1 + \ldots + x_N\xi_N$. Since the Fourier transform has period 4, i.e., applying the Fourier transform four times, we get the identity operator, we see that if f is an eigenfunction, $\mathscr{F}(f) = \lambda f$, then λ satisfies $\lambda^4 = 1$. Hence $\lambda = \pm 1, \pm i$ are the only possible eigenvalues of the Fourier transform. The exponential $\mathrm{e}^{-|\mathbf{x}|^2/2}$ is an eigenfunction associated to the eigenvalue 1.

In dimension $N = 1$, the Hermite functions

$$\Phi_n(x) = \frac{1}{(\sqrt{\pi}2^n n!)^{1/2}} H_n(x)\,\mathrm{e}^{-x^2/2},$$

where

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{d^n}{dx^n} \mathrm{e}^{-x^2}$$

are the Hermite polynomials, give the remaining eigenfunctions. They satisfy $\mathscr{F}(\Phi_n) = (-i)^n \Phi_n$ and form an orthonormal basis for the space $L^2(\mathbb{R})$ (cf. [1]).

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Translated from *Problemy Matematicheskogo Analiza* **88**, March 2017, pp. 83-86.

In higher dimensions, the eigenfunctions of the Fourier transform can be obtained by taking the tensor products of Hermite functions, one in each coordinate variable. A list of eigenfunctions of the cosine or sine Fourier transforms can be found in [1].

In dimension $N = 2$, the separable Hermite–Gaussian functions $\Phi_m(x_1)\Phi_n(x_2)$ are eigenfunctions corresponding to the eigenvalues $(-i)^{m+n}$. Denote by

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

a rotation matrix where α is the rotation angle in the counterclockwise direction. Since A is an orthogonal matrix, we have

$$(\mathscr{F} f(A\cdot))(\xi) = (\mathscr{F} f(\cdot))(A\xi)\,.$$

Hence the rotated Hermite–Gaussian functions

$$h^{(\alpha)}_{m,n}(x_1,x_2) = \Phi_m(x_1\cos\alpha + x_2\sin\alpha)\,\Phi_n(-x_1\sin\alpha + x_2\cos\alpha)$$

are eigenfunctions with the same eigenvalues as those of $\Phi_m(x_1)\Phi_n(x_2)$'s.

The functions $r^\nu L^{(\nu)}_p(r^2)\mathrm{e}^{-r^2/2}$, with the generalized Laguerre polynomials $L^{(\nu)}_p$, are eigenfunctions of the Hankel transform

$$\mathscr{H}_\nu(f)(r) = \int\limits_0^\infty f(\rho)J_\nu(r\rho)\rho d\rho$$

corresponding to the eigenvalues $(-1)^p$ (cf. [2, formula (2.5)]). Here, $J_\nu(\rho)$ is the Bessel function of the first kind of order ν and argument ρ. Because of the integral representation (cf. [3, p. 20])

$$J_\nu(\rho) = \frac{i^\nu}{2\pi}\int\limits_0^{2\pi} \mathrm{e}^{-ir\rho\cos\theta}\mathrm{e}^{-i\,\nu\,\theta}d\theta,$$

the Laguerre–Gaussian functions

$$l_{m,n}(r,\theta) = N_{p,\nu}r^\nu L^{(\nu)}_p(r^2)\mathrm{e}^{-r^2/2}\mathrm{e}^{-i\nu\theta}$$

are eigenfunctions in the polar coordinates (r,θ) of the planar Fourier transform corresponding to the eigenvalues $(-i)^{m+n}$. Here, $p = \min\{m,n\}$, $\nu = |m-n|$, $N_{p,\nu}$ is the normalization factor.

The above sets of eigenfunctions form a complete orthonormal basis for $L^2(\mathbb{R}^2)$. In [4], the three sets considered above are all obtained as special cases of a general form.

In dimension $N > 2$, the separable Hermite–Gaussian functions

$$\Phi_{\mathbf{m}}(\mathbf{x}) = \prod_{j=1}^N \Phi_{m_j}(x_j)$$

are eigenfunctions corresponding to the eigenvalues $(-i)^{m_1+\ldots+m_N}$. Moreover, if A is an orthogonal matrix of order N, then $\Phi_{\mathbf{m}}(A\mathbf{x})$ are eigenfunctions with the same eigenvalues of $\Phi_{\mathbf{m}}$.

Another example is provided by functions of the form

$$Y_k(\mathbf{x})\mathrm{e}^{-|\mathbf{x}|^2/2},$$

where Y_k is a homogeneous harmonic polynomial of degree k. They are eigenfunctions of the Fourier transform and satisfy the equality

$$\mathscr{F}(Y_k(\cdot)\mathrm{e}^{-|\cdot|^2/2})(\xi) = (-i)^k Y_k(\xi)\mathrm{e}^{-|\xi|^2/2}.$$

The above result is known as the Bochner–Hecke formula for the Fourier transform [5, p. 85].

More generally, one can consider eigenfunctions in the sense of distributions. Such eigenfunctions do not need to belong to $L^2(\mathbb{R}^N)$. It is known that $1/|\mathbf{x}|^{N/2}$ is an eigenfunction of $\mathscr{F}$ in the sense of distributions and corresponds to the eigenvalue 1 (cf. [5, p. 71]):

$$\mathscr{F}(|\cdot|^{-N/2})(\xi) = |\xi|^{-N/2}.$$

The goal of this note is to consider a nontrivial example of distributional eigenfunction of the planar Fourier transform. This eigenfunction is not a tensor product of univariate eigenfunctions. As a consequence, we obtain a formula for multi-dimensional eigenfunctions in dimension $2N$.

Theorem. *The function*

$$f(x,y) = \frac{\sqrt{x^2+y^2}}{x\,y}$$

is an eigenfunction of the Fourier transform

$$\mathscr{F}(f)(k,l) = \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} f(x,y)\mathrm{e}^{-i(k\,x+l\,y)}dx\,dy \tag{1}$$

corresponding to the eigenvalue -1. *The integral* (1) *is understood in the sense of Cauchy principal value.*

Proof. Let

$$F(k,l) = \iint\limits_{\mathbb{R}^2} \frac{\sqrt{x^2+y^2}}{xy}\mathrm{e}^{-i(k\,x+l\,y)}dx\,dy.$$

By the symmetry of the integrand,

$$F(k,l) = F(l,k), \quad F(-k,l) = -F(k,l), \quad F(k,l) = F(-k,-l)\,. \tag{2}$$

Taking the partial derivative of $F(k,l)$, we get

$$\frac{\partial F(k,l)}{\partial k} = -i\iint\limits_{\mathbb{R}^2} \frac{\sqrt{x^2+y^2}}{y}\mathrm{e}^{-i(k\,x+l\,y)}dx\,dy,$$

$$\frac{\partial^2 F(k,l)}{\partial k\partial l} = -\iint\limits_{\mathbb{R}^2} \sqrt{x^2+y^2}\mathrm{e}^{-i(k\,x+l\,y)}dx\,dy.$$

We introduce the polar coordinates (r,φ) so that $kx+ly = r\sqrt{k^2+l^2}\cos\varphi$. We write the last integral in the form

$$\frac{\partial^2 F(k,l)}{\partial k\partial l} = -\int\limits_0^{2\pi}\int\limits_0^{\infty} \mathrm{e}^{-ir\sqrt{k^2+l^2}\cos\varphi}r^2 dr\,d\varphi\,.$$

Making the change of variable $\rho = r\sqrt{k^2+l^2}$ and setting

$$a = \int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{-i\rho\cos\varphi}\rho^2 d\rho d\varphi,$$

we get

$$\frac{\partial^2 F(k,l)}{\partial k \partial l} = -\frac{a}{(k^2+l^2)^{3/2}}\,.$$

Integrating with respect to l, we find

$$\frac{\partial F(k,l)}{\partial k} = -\frac{a\,l}{k^2(k^2+l^2)^{1/2}} + A(k),$$

where $A(k)$ denotes an unknown function. Integrating with respect to k, we get the following formula for the original function:

$$F(k,l) = a\frac{\sqrt{k^2+l^2}}{kl} + A(k) + B(l),$$

where $A(k)$ and $B(l)$ are unknown functions. Since $F(k,l) = F(l,k)$, we deduce that $A \equiv B$ and $2\,A(l) = -A(k) - A(-k)$ in view of (2), which implies $A(l) = \text{const}$. Hence $A(l) = 0$. Thus, we have proved that $F(k,l) = af(k,l)$, i.e., $\lambda = a/(2\pi)$ is an eigenvalue for the Fourier transform and f is the corresponding eigenfunction. It remains to compute a.

We consider the Fourier transform of the radial function $\sqrt{x^2+y^2}$ (cf. [6, p. 194])

$$\frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} \sqrt{x^2+y^2}\mathrm{e}^{-i(kx+ly)}dx\,dy = 4\frac{\Gamma(\frac{3}{2})}{\Gamma(-\frac{1}{2})}(k^2+l^2)^{-3/2} = -\frac{1}{(k^2+l^2)^{3/2}}.$$

Hence, in the polar coordinates,

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \rho^2 \mathrm{e}^{-i\rho\sqrt{k^2+l^2}\cos\varphi} d\rho d\varphi = -\frac{1}{(k^2+l^2)^{3/2}}.$$

We infer that $a = -2\pi$. The theorem is proved. □

From this theorem it follows that $\prod_{j=1}^{N} f(x_{2j-1}, x_{2j})$ is an eigenfunction of the Fourier transform in dimension $2N$ corresponding to the eigenvalue $(-1)^N$.

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Submitted on December 20, 2016