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Journal of Functional Analysis 217 (2004) 448–488
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

An asymptotic theory of higher-order operator differential equations with nonsmooth nonlinearities

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Received 2 January 2004; accepted 8 April 2004

Communicated by H. Brezis

Abstract

We develop an asymptotic theory of nonlinear operator differential equations of an arbitrary order in Banach spaces. The nonlinear part of the equation is written in a divergent form. It is shown that the main term in an asymptotic representation of solutions at infinity satisfies a finite-dimensional dynamical system perturbed by a small nonlocal operator.

1. Introduction

The asymptotic theory of infinite-dimensional differential equations ([AN, Bre,DK,KM1,P] et al.) got a new impetus during the last three decades in connection with important applications to nonlinear problems of hydrodynamics, structural mechanics, chemistry, biology, etc. (see, in particular, the books [BV,ChV,EFNT,T,V] and the extensive bibliography cited there). In the present

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[1]Supported by the Swedish Research Council (VR).

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.04.004

paper, we study solutions to the equation

$$\mathscr{A}(D_t)u(t) = \sum_{j=0}^{n} D_t^{n-j} \mathscr{N}_j(t; u(t), \ldots, D_t^{\ell-n} u(t)) \tag{1}$$

on the real axis $\mathbb{R}$, where $\ell > n$, $D_t = -id/dt$, $\mathscr{A}(D_t)$ is an ordinary differential operator of order ℓ with operator coefficients and with nonlinear operators $\mathscr{N}_0, \ldots, \mathscr{N}_n$ defined on functions in a locally convex Sobolev space $W_{p,\mathrm{loc}}^{\ell-n}(\mathbb{R}; \{B_k\}_{k=0}^{\ell-n})$, where $B_0, \ldots, B_{\ell-n}$ are Banach spaces (see Section 2.1). We write the right-hand side in the divergence form in order to cover a wider class of equations. The operators $\mathscr{N}_j$, described in Section 2.5, are not assumed to be differentiable with respect to their arguments.

Our main concern is with the asymptotic behavior of solutions as $t \to +\infty$. We show that for a certain class of equations (1) the question of asymptotics can be reduced to that for a finite-dimensional dynamical system perturbed by a nonlocal operator. Our result is new also for the linear case and, apparently, even for linear ordinary differential equations with scalar variable coefficients.

Our conditions for the operators $\mathscr{A}$ and $\mathscr{N}_j$ include elliptic and parabolic partial differential operators, so that the theory developed in this paper can be applied directly to the study of local and boundary singularities of solutions to nonlinear elliptic and parabolic equations in the spirit of our works [KM2,KM3,KM4,KM5,KM6]. Note that our assumptions about $\mathscr{N}_j$ do not exclude fully nonlinear equations.

Let us describe a corollary of our main result concerning a solution $u(t)$ to (1) (see Theorem 2). Let u be subject to the growth requirement

$$||u||_{W_p^{\ell-n}(t,t+1;\{B_k\}_{k=0}^{\ell-n})} \leqslant \mathscr{M}(t), \tag{2}$$

where the majorant $\mathscr{M}$ is subject to conditions stated in Section 2.5. Estimates of this type are usually available in applications to partial differential equations, where one can use monotonicity properties of differential operators, the maximum modulus principle, differential inequalities, etc.

Under some natural assumptions on the nonlinearity we obtain the representation

$$\mathrm{col}(u(t), \ldots, D_t^{\ell-n-1} u(t)) = \sum_{s=1}^{\kappa} h_s(t)\, \mathrm{col}(U_s(t), \ldots, D_t^{\ell-n-1} U_s(t)) + \mathbf{w}(t), \tag{3}$$

which is a far-reaching generalization of the variation of arbitrary constants formula in the elementary theory of ordinary differential equations. The vector functions U_s are solutions to $\mathscr{A}(D_t)U(t) = 0$ of the form $\exp(i\lambda_v t)\varphi$ with λ_v being eigenvalues of the operator pencil $\mathscr{A}(\lambda)$ on a certain line $\Im\lambda = k_0$ and with φ denoting eigenvalues corresponding to λ_v. We assume here that there are no generalized eigenvectors corresponding to λ_v. The vector $\vec{\boldsymbol{h}} = (h_1, \ldots, h_\kappa)$ satisfies a finite-dimensional perturbed dynamical system of the form

$$\frac{d}{dt}\mathbf{h}(t) + \mathbb{N}(t;\mathbf{h}(t)) + \mathbb{K}[\mathbf{h}](t) = 0, \tag{4}$$

where $\mathbb{K}$ is a nonlocal nonlinear operator and the components of $\mathbb{N}$ are given by

$$\mathbb{N}_k(t;\mathbf{h}(t)) = \sum_{j=0}^{n}\left(\mathcal{N}_j\left(t;\sum_{s=1}^{\varkappa} h_s(t)U_s(t),\ldots,\sum_{s=1}^{\varkappa} h_s(t)D_t^{\ell-n}U_s(t)\right)\,\middle|\, D_t^{n-j}V_k(t)\right).$$

Here V_k are exponential solutions of the adjoint equation $\mathscr{A}^*(D_t)V(t) = 0$ which are connected with U_s by a biorthogonality condition and $(\cdot\,|\,\cdot)$ is the inner product.

The vector function $\mathbf{w}$ in (3) can be regarded as a remainder term. We give estimates which show that $\mathbb{K}$ and $\mathbf{w}$ are weak in a certain sense. In fact, Section 5.7 contains a general result with generalized eigenvectors permitted, where the vector functions U_s and V_s are polynomial exponential solutions of equations $\mathscr{A}(D_t)U = 0$ and $\mathscr{A}^*(D_t)V = 0$, respectively.

System (4) is the corner stone of our asymptotic theory. On one hand, it can be applied to construct solutions of (1) with the vector $\mathbf{h}(t)$ asymptotically close to a solution of the dynamical system

$$\frac{d}{dt}\vec{\chi}(t) + \mathbb{N}(t;\vec{\chi}(t)) = 0. \tag{5}$$

On the other hand, one can try to show that solutions of (1) subject to the growth restriction (2) have the asymptotic representation (3), where the vector $\mathbf{h}$ is asymptotically equivalent to a solution of (5).

In general, the dynamical system (5) is a very complicated object containing an arbitrary nonlinear term, so that even the appearance of a chaotic attractor is not excluded. Therefore, in order to deduce explicit asymptotic formulae for solutions of (1), one must restrict classes of nonlinearities.

If the line $\Im\lambda = k_0$ contains a single simple eigenvalue λ_0 of the pencil $\mathscr{A}(\lambda)$, then $\mathbf{h}$ becomes a scalar function h and system (5) turns into one equation with

$$\mathbb{N}(t;h(t)) = -ie^{-i\lambda_0 t}\sum_{s=0}^{n}\lambda_0^{n-s}(\mathcal{N}_s(t;h(t)e^{i\lambda_0 t}\varphi,\ldots,h(t)\lambda_0^{\ell-n}e^{i\lambda_0 t}\varphi)\,|\,\psi),$$

where φ and ψ are the eigenvectors corresponding to the eigenvalues λ_0 and $\bar{\lambda}_0$ of $\mathscr{A}(\lambda)$ and $\mathscr{A}^*(\lambda)$ respectively, subject to

$$\left(\frac{d\mathscr{A}}{d\lambda}(\lambda_0)\varphi\,\middle|\,\psi\right) = 1.$$

This situation appears, for example, in the problem of asymptotic behavior of solutions to the nonlinear elliptic differential equation of order $2m$

$$\mathbf{L}(D_x)u(x) = \sum_{|\alpha|\leqslant m} D_x^{\alpha}(\mathbf{N}_\alpha(x;\{D_x^{\gamma}u(x)\}_{|\gamma|\leqslant m}), \tag{6}$$

in a neighborhood of the origin of $\mathbb{R}^d$. Here $\mathbf{L}$ is homogeneous polynomial of degree $2m$ with constant coefficients. In the case $2m<d$ the dynamical system (5), which describes the behavior of solutions to Eq. (6) with a finite Dirichlet integral, reduces to one equation

$$\frac{d\chi}{dt}+e^{-td}\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}\int_{|\theta|=1}\mathbf{N}_\alpha(e^{-t}\theta;\chi(t),0)(D_x^\alpha G)(e^{-t}\theta)\,dS_\theta=0,$$

where G is the fundamental solution of $\mathbf{L}(D_x)$ in $\mathbb{R}^d$ and zero value of an argument of $\mathbf{N}_\alpha$ stands in place of derivatives of the orders $1,\dots,m$.

This and other applications of our asymptotic theory to partial differential equations will be discussed elsewhere.

Here is a plan of the paper. Definitions of function spaces and conditions for the operators involved are collected in Section 2, where basic properties of $\mathscr{A}(D_t)$ are reviewed as well. In Section 3 we use a new construction to reduce (1) to the equivalent first-order evolution system

$$(\mathscr{I}D_t+\mathfrak{A})\mathscr{U}(t)-\mathfrak{N}(t;\hat{\mathscr{U}}(t))=0\quad\text{on }\mathbb{R},\tag{7}$$

where $\mathscr{U}=(\mathscr{U}_1,\dots,\mathscr{U}_\ell)$, $\hat{\mathscr{U}}=(\mathscr{U}_1,\dots,\mathscr{U}_{\ell-n+1})$, $\mathfrak{A}$ is a linear operator independent of t, and $\mathfrak{N}$ is, in general, a nonlinear operator. The next Section 4 is devoted to the spectral analysis of the operator $\mathfrak{A}$. The results obtained are used in the last Section 5 to split system (7) into a finite- and infinite-dimensional parts. Resolving the infinite-dimensional part, we arrive at system (4). The paper is finished with derivation of asymptotic representations of solutions to (7) and (1).

2. Preliminaries on function spaces and operators

2.1. Local vector Sobolev spaces

Let ℓ and n be integers, $0\leqslant n<\ell$, and let $\{B_j\}_{j=-n}^{j=\ell-n}$ be a collection of reflexive Banach spaces such that B_j is densely imbedded in B_{j-1} and

$$||u||_{B_{j-1}}\leqslant c_j||u||_{B_j}\quad\text{for } j=1-n,\dots,\ell-n.$$

Let $B_{-j}{}'$ be the dual space of B_j. We denote by $(\cdot\,|\,\cdot)$ the duality between B_0 and $B_0{}'$ which can be extended by continuity onto $B_j\times B_{-j}{}'$ for $j=-n,\dots,\ell-n$.

We introduce the space $L_p(a,b;B)$ endowed with the norm

$$||u||_{L_p(a,b;B)}=\left(\int_a^b||u(t)||_B^p\,dt\right)^{1/p},$$

where B is a Banach space and $1<p<\infty$. We need the Sobolev space $W_p^j(a,b;\{B_k\}_{k=0}^j)$ with the norm

$$||u||_{W_p^j(a,b;\{B_k\}_{k=0}^j)} = \left(\int_a^b \sum_{0\leqslant k\leqslant j} ||D_t^k u(t)||_{B_{j-k}}^p \, dt\right)^{1/p}.$$

Let $L_{p,\mathrm{loc}}(\mathbb{R};B)$ and $W_{p,\mathrm{loc}}^j(\mathbb{R};\{B_k\}_{k=0}^j)$ be the space of functions belonging to $L_p(t,t+1;B)$ and $W_p^j(t,t+1;\{B_k\}_{k=0}^j)$ for all real t. We equip these spaces with the seminorms

$$||u||_{L_p(t,t+1;B)} \quad \text{and} \quad ||u||_{W_p^j(t,t+1;\{B_k\}_{k=0}^j)},$$

respectively.

We define $W_{p',\mathrm{comp}}^j(\mathbb{R};\{B_k{}'\}_{k=0}^j)$ to be the set of all $u\in W_{p'}^j(\mathbb{R};\{B_k{}'\}_{k=0}^j)$ with compact supports, $p'=p/(p-1)$. A sequence $\{u_k\}_{k\geqslant 1}$ of elements in $W_{p',\mathrm{comp}}^j(\mathbb{R};\{B_k{}'\}_{k=0}^j)$ converges to $u\in W_{p',\mathrm{comp}}^j(\mathbb{R};\{B_k{}'\}_{k=0}^j)$ if the supports of u_k are uniformly bounded and $u_k\to u$ in $W_{p'}^j(\mathbb{R};\{B_k{}'\}_{k=0}^j)$. By $W_{p,\mathrm{loc}}^{-j}(\mathbb{R};\{B_{-k}\}_{k=0}^j)$ we denote the spaces of sesquilinear bounded functionals on $W_{p',\mathrm{comp}}^j(\mathbb{R};\{B_k{}'\}_{k=0}^j)$ endowed with the seminorms

$$||f||_{W_p^j(t,t+1;\{B_{-k}\}_{k=0}^j)} = \sup_v \left|\int_{\mathbb{R}} (f(t)\mid v(t))\, dt\right|,$$

where supremum is taken over all $v\in W_{p',\mathrm{comp}}^j(\mathbb{R};\{B_k{}'\}_{k=0}^j)$ such that

$$\operatorname{supp} v\subset[t,t+1], ||v||_{W_{p'}^j(\mathbb{R};\{B_k{}'\}_{k=0}^j)} = 1. \tag{8}$$

If $f_k\in L_{p,\mathrm{loc}}(\mathbb{R};B_{-k})$, $k=0,\dots,j$, then the sum

$$f(t) = \sum_{k=0}^j D_t^{j-k} f_k(t), \tag{9}$$

where $D_t^{j-k}f_k$ is the distribution derivative, belongs to $W_{p,\mathrm{loc}}^{-j}(\mathbb{R};\{B_{-k}\}_{k=0}^j)$ and

$$||f||_{W_p^{-j}(t,t+1;\{B_{-k}\}_{k=0}^j)} \leqslant \left(\sum_{k=0}^j ||f_k||_{L_p(t,t+1;B_{-k})}^p\right)^{1/p}.$$

One can show that every element of $W_{p,\mathrm{loc}}^{-j}(\mathbb{R};\{B_{-k}\}_{k=0}^j)$ can be represented in form (9) so that

$$\left(\sum_{k=0}^j ||f_k||_{L_p(t,t+1;B_{-k})}^p\right)^{1/p} \leqslant c||f||_{W_p^{-j}(t-1,t+2;\{B_{-k}\}_{k=0}^j)},$$

where c does not depend on f and t. The proof is analogous to that of Lemma 10.8.2 in [KM1].

2.2. Conditions on the operator $\mathscr{A}(D_t)$

Here and elsewhere we use terminology of the spectral theory of operator pencils following [KM1].

By A_{jk} we denote a linear continuous operator which maps B_k into B_{-j} for $k = 0, \dots, \ell - n$, $j = 0, \dots, n$. We introduce the operator pencil

$$\mathscr{A}(\lambda) = \sum_{j=0}^{n} \sum_{k=0}^{\ell-n} A_{jk}\lambda^{\ell-j-k} : B_{\ell-n} \to B_{-n} \tag{10}$$

and assume that $\mathscr{A}(\lambda)$ is Fredholm for all $\lambda \in \mathbb{C}$ and invertible at least for one value of λ. Then the spectrum of $\mathscr{A}(\lambda)$ consists of isolated eigenvalues of finite algebraic multiplicity.

The operator pencil

$$\mathscr{A}^*(\lambda) : B_n{}' \to B_{n-\ell}{}' \tag{11}$$

is defined by $(x \mid \mathscr{A}^*(\lambda)y) = (\mathscr{A}(\bar{\lambda})x \mid y)$.

We deal with a variational form of the equation

$$\mathscr{A}(D_t)u = f \quad \text{in } \mathbb{R}, \tag{12}$$

where $f \in W^{-n}_{p,\mathrm{loc}}(\mathbb{R}; \{B_{-k}\}_{k=0}^{n})$. By a solution of this equation we mean a function $u \in W^{\ell-n}_{p,\mathrm{loc}}(\mathbb{R}; \{B_k\}_{k=0}^{\ell-n})$ satisfying

$$\sum_{j=0}^{n} \sum_{k=0}^{\ell-n} \int_{\mathbb{R}} (A_{jk} D_t^{\ell-n-k} u(t) \mid D_t^{n-j} v(t))\, dt = \int_{\mathbb{R}} (f(t) \mid v(t))\, dt \tag{13}$$

for all $v \in W^{n}_{p',\mathrm{comp}}(\mathbb{R}; \{B_k{}'\}_{k=0}^{n})$. We suppose that the operator A_{00} has a bounded inverse, which maps B_0 into B_0.

The left-hand side of (13) generates a bounded linear operator

$$\mathscr{A}(D_t) : W^{\ell-n}_{p,\mathrm{loc}}(\mathbb{R}; \{B_k\}_{k=0}^{\ell-n}) \to W^{-n}_{p,\mathrm{loc}}(\mathbb{R}; \{B_{-k}\}_{k=0}^{n}).$$

We assume that the local estimate

$$\begin{aligned} &||u||_{W_p^{\ell-n}(0,1;\{B_k\}_{k=0}^{\ell-n})} \\ &\quad \leqslant C(||\mathscr{A}(D_t)u||_{W_p^{-n}(-1,2;\{B_{-k}\}_{k=0}^{n})} + ||u||_{W_p^{\ell-n-1}(-1,2;\{B_k\}_{k=0}^{\ell-n-1})}) \end{aligned} \tag{14}$$

holds for every $u \in W_p^{\ell-n}(-1,2;\{B_k\}_{k=0}^{\ell-n})$. Substituting $u(t) = e^{i\lambda t}\varphi$, where $\lambda \in \mathbb{R}$ and $\varphi \in B_{\ell-n}$, we conclude that for sufficiently large $|\lambda|$

$$\sum_{k=0}^{\ell-n} |\lambda|^k ||\varphi||_{B_{\ell-n-k}} \leqslant C \sup |(\mathscr{A}(\lambda)\varphi \mid v)|, \tag{15}$$

where the supremum is taken over all $v \in B_n{}'$ subject to

$$\sum_{k=0}^{n} |\lambda|^{n-k} ||v||_{B_k{}'} = 1.$$

It is straightforward that estimate (15) holds for all complex λ with $|\arg(\pm\lambda)| \leqslant \theta$, $|\Re\,\lambda| \geqslant r$, where θ is a positive number and r is sufficiently large. Hence, an arbitrary strip $k_- < \Re\,\lambda < k_+$ contains at most a finite number of eigenvalues of the pencil $\mathscr{A}(\lambda)$, which have finite algebraic multiplicities.

2.3. *Basic properties of $\mathscr{A}(D_t)$*

We state two propositions concerning existence, uniqueness and asymptotics of a solution to Eq. (12). Their proofs are literally the same as those of Theorem 10.8.11 and Proposition 3.8.1 in [KM1].

Proposition 1. *Let $k_- < k_+$ and let the strip $k_- < \Im\lambda < k_+$ be free of eigenvalues of $\mathscr{A}(\lambda)$. We introduce the function*

$$\mu(t) = \begin{cases} e^{k_- t}(1+t)^{m_- - 1} & \textit{for } t \geqslant 0, \\ e^{k_+ t}(1+|t|)^{m_+ - 1} & \textit{for } t < 0, \end{cases} \tag{16}$$

where $m_\pm$ is the maximal partial multiplicity of all eigenvalues of $\mathscr{A}(\lambda)$ situated on the line $\Im\lambda = k_\pm$. In the case when there are no eigenvalues on the line $\Im\lambda = k_\pm$ we set $m_\pm = 1$. Suppose that f belongs to $W_{p,\mathrm{loc}}^{-n}(\mathbb{R};\{B_{-k}\}_{k=0}^{n})$ and satisfies

$$\int_{\mathbb{R}} \mu(\tau) ||f||_{W_p^{-n}(\tau,\tau+1;\{B_{-k}\}_{k=0}^n))}\, d\tau < \infty. \tag{17}$$

Then Eq. (12) *has a solution $u \in W_{p,\mathrm{loc}}^{\ell-n}(\mathbb{R};\{B_k\}_{k=0}^{\ell-n})$ subject to*

$$||u||_{W_p^{\ell-n}(t,t+1;\{B_k\}_{k=0}^{\ell-n})} \leqslant \mathscr{C} \int_{\mathbb{R}} \mu(\tau-t) ||f||_{W_p^{-n}(\tau,\tau+1;\{B_{-k}\}_{k=0}^n)}\, d\tau, \tag{18}$$

which implies

$$||u||_{W_p^{\ell-n}(t,t+1;\{B_k\}_{k=0}^{\ell-n})} = o(e^{-k_\mp t}) \quad as\ t \to \pm\infty. \tag{19}$$

The solution $u \in W_{p,\mathrm{loc}}^{\ell-n}(\mathbb{R};\{B_k\}_{k=0}^{\ell-n})$ *of* (12) *satisfying* (19) *is unique.*

Let the eigenvalues of the operator pencil $\mathscr{A}(\lambda)$ be indexed by integers. We denote by J_ν the geometric multiplicity of λ_ν and by $m_{\nu k}$, $k = 1, \dots, J_\nu$ its partial multiplicities. Then the algebraic multiplicity of λ_ν is equal to $m_{\nu 1} + \cdots + m_{\nu J_\nu}$. The number $\bar{\lambda}_\nu$ is an eigenvalue of $\mathscr{A}^*(\lambda)$ if and only if λ_ν is an eigenvalue of $\mathscr{A}(\lambda)$, and its geometric, partial and algebraic multiplicities coincide with those of λ_ν. We fix canonical sets of Jordan chains

$$\{\varphi_{kj}^{(\nu)}\} \quad \text{and} \quad \{\psi_{kj}^{(\nu)}\},$$

$k = 1, \dots, J_\nu$, $j = 0, \dots, m_{\nu k} - 1$, corresponding to the eigenvalues λ_ν and $\bar{\lambda}_\nu$ of the operator pencils (10) and (11). We suppose that the vectors are subject to the biorthogonality condition

$$\sum_{s=0}^{q} \sum_{\sigma=s+1}^{m_{\nu k}+s} \frac{1}{\sigma!}(\mathscr{A}^{(\sigma)}(\lambda_\nu)\varphi_{k,m_{\nu k}+s-\sigma}^{(\nu)} \mid \psi_{j,q-s}^{(\nu)}) = \delta_k^j \delta_q^0 \tag{20}$$

for $k,j = 1, \dots, J_\nu$, $q = 0, \dots, m_{\nu j} - 1$. By $\mathscr{A}^{(\sigma)}(\lambda)$ we mean the derivative of $\mathscr{A}(\lambda)$ of order σ.

We denote by $Z(\mathscr{A}, \lambda_\nu)$ the set of solutions of $\mathscr{A}(D_t)U = 0$ which have the form

$$U(t) = e^{it\lambda_\nu} \sum_{\sigma=0}^{m} \frac{(it)^\sigma}{\sigma!} u_{m-\sigma}, \quad u_{m-\sigma} \in B_{\ell-n}. \tag{21}$$

Here λ_ν is an eigenvalue of the pencil $\mathscr{A}(\lambda)$, u_0 is an eigenvector corresponding to λ_ν and $u_1, \dots, u_m$ are generalized eigenvectors. The dimension of $Z(\mathscr{A}, \lambda_\nu)$ is equal to the algebraic multiplicity $\varkappa_\nu$ of λ_ν. We put

$$\Phi_k^{(\nu)}(z) = \sum_{\sigma=0}^{m_{\nu k}-1} \frac{z^\sigma}{\sigma!} \varphi_{k,m_{\nu k}-1-\sigma}^{(\nu)}. \tag{22}$$

Then the vector functions

$$U_{ks}^{(\nu)}(t) = e^{it\lambda_\nu} D_t^s \Phi_k^{(\nu)}(it), \quad k = 1, \dots, J_\nu, \ \ s = 0, \dots, m_{\nu k} - 1, \tag{23}$$

form a basis in $Z(\mathscr{A}, \lambda_\nu)$.

Analogously, let $Z(\mathscr{A}^*, \bar{\lambda}_\nu)$ be the set of all solutions of $\mathscr{A}^*(D_t)V = 0$ which have the form

$$V(t) = e^{it\bar{\lambda}_\nu} \sum_{\sigma=0}^{m} \frac{(it)^\sigma}{\sigma!} v_{m-\sigma}, \quad v_{m-\sigma} \in B_n{}'.$$

Then the collection of the vector functions

$$V_{ks}^{(\nu)}(t) = ie^{it\bar{\lambda}_\nu} D_t^{m_{\nu k}-1-s} \Psi_k^{(\nu)}(it), \quad k = 1, \dots, J_\nu, \;\; s = 0, \dots, m_{\nu k} - 1, \tag{24}$$

where

$$\Psi_k^{(\nu)}(z) = \sum_{\sigma=0}^{m_{\nu k}-1} \frac{z^\sigma}{\sigma!} \psi_{k, m_{\nu k}-1-\sigma}^{(\nu)}, \tag{25}$$

is a basis in $Z(\mathscr{A}^*, \bar{\lambda}_\nu)$. The equivalent formulation of the biorthogonality condition (20) is

$$\int_{\mathbb{R}} \sum_{j=0}^{n} \sum_{s=0}^{\ell-n} (A_{js} D_t^{\ell-s-j}(\eta(t) U_{kp}^{(\nu)}(t)) \mid V_{mq}^{(\mu)}(t))\, dt = \delta_\mu^\nu \delta_m^k \delta_p^q, \tag{26}$$

where η is a smooth function equal to 1 in a neighborhood of $-\infty$ and to zero in a neighborhood of $+\infty$ (see [KM1, Section 1]).

Proposition 2. *Let $k_- < k_0 < k_+$ and let the strips $k_- < \Im\lambda < k_0$ and $k_0 < \Im\lambda < k_+$ be free of eigenvalues of $\mathscr{A}(\lambda)$. Suppose that f belongs to $W_{p,\mathrm{loc}}^{-n}(\mathbb{R}; \{B_{-k}\}_{k=0}^n)$ and satisfies*

$$\int_{-\infty}^{\infty} e^{k_0\tau}(1+|\tau|)^{m_0-1} ||f||_{W_p^{-n}(\tau,\tau+1;\{B_{-k}\}_{k=0}^n)}\, d\tau < \infty, \tag{27}$$

where m_0 is the maximal partial multiplicity of all eigenvalues of $\mathscr{A}(\lambda)$ situated on the line $\Im\lambda = k_0$. Then the solutions u_+ and u_- of Eq. (12) *constructed in Proposition* 1 *for the strips $k_- < \Im\lambda < k_0$ and $k_0 < \Im\lambda < k_+$ satisfy*

$$u_+(t) - u_-(t) = \sum_{\Im\lambda_\nu = k_0} \sum_{k=1}^{J_\nu} \sum_{p=0}^{m_{\nu k}-1} c_{kp}^{(\nu)} U_{kp}^{(\nu)}(t),$$

where

$$c_{kp}^{(\nu)} = \int_{\mathbb{R}} (f(\tau) \mid V_{kp}^{(\nu)}(\tau))\, d\tau.$$

2.4. Vector function spaces $\mathbb{T}'$ and $\hat{\mathbb{T}}$

By $\mathbb{T}'(a,b)$ we denote the space of vector-functions

$$\mathscr{U}' = (\mathscr{U}_1, \dots, \mathscr{U}_{\ell-n}),$$

with values in $B_{\ell-n}\times\cdots\times B_1$ which is endowed with the norm

$$||\mathscr{U}'||_{\mathbb{T}'(a,b)}=\left(\sum_{j=1}^{\ell-n}\int_a^b(||\mathscr{U}_j(\tau)||^p_{B_{\ell-n-j+1}}+||D_\tau\mathscr{U}_j(\tau)||^p_{B_{\ell-n-j}})\,d\tau\right)^{1/p}.$$

Also let $\hat{\mathbb{T}}(a,b)$ be the product $\mathbb{T}'(a,b)\times L_p(a,b;B_0)$ which consists of the vector functions

$$\hat{\mathscr{U}}=(\mathscr{U}',\mathscr{U}_{\ell-n+1})$$

with the norm

$$||\hat{\mathscr{U}}||_{\hat{\mathbb{T}}(a,b)}=(||\mathscr{U}'||^p_{\mathbb{T}'(a,b)}+||\mathscr{U}_{\ell-n+1}||^p_{L_p(a,b;B_0)})^{1/p}.$$

Furthermore, we use the spaces ${\mathbb{T}_{\mathrm{loc}}}'(\mathbb{R})$ and $\hat{\mathbb{T}}_{\mathrm{loc}}(\mathbb{R})$ endowed with the seminorms $||\mathscr{U}'||_{\mathbb{T}'(t,t+1)}$ and $||\hat{\mathscr{U}}||_{\hat{\mathbb{T}}(t,t+1)}$, $t\in\mathbb{R}$.

If $u\in W^{\ell-n}_{p,\mathrm{loc}}(\mathbb{R};\{B_k\}_{k=0}^{\ell-n})$ then by setting $\mathscr{U}_j=D_t^{j-1}u$ we see that

$$\mathscr{U}'\in{\mathbb{T}_{\mathrm{loc}}}'(\mathbb{R})\quad\text{and}\quad\hat{\mathscr{U}}\in\hat{\mathbb{T}}_{\mathrm{loc}}(\mathbb{R})$$

and

$$2^{-1/p}||\hat{\mathscr{U}}||_{\hat{\mathbb{T}}(t,t+1)}\leqslant||u||_{W_p^{\ell-n}(t,t+1;\{B_k\}_{k=0}^{\ell-n})}\leqslant||\mathscr{U}'||_{\mathbb{T}'(t,t+1)}. \tag{28}$$

2.5. *Conditions for the operators $\mathscr{N}_j$ in Eq.* (1) *and the majorant $\mathscr{M}(t)$*

Let us fix two real numbers $k_\pm$, $k_-<k_+$. By $m_\pm$ we denote the maximal partial multiplicities of the eigenvalues of the pencil $\mathscr{A}(\lambda)$ situated on the line $\Im\lambda=k_\pm$. In the case when there are no eigenvalues on the line $\Im\lambda=k_\pm$ we set $m_\pm=1$.

In what follows, by c and $\mathscr{C}$, sometimes with indices, we mean possibly different constants which depend on $k_\pm$, p and the unperturbed operator $\mathscr{A}(D_t)$.

In this section and elsewhere, $\mathscr{M}$ is a locally bounded, positive and measurable function on $\mathbb{R}$, which has appeared already in Introduction. Let us consider the set of the vector functions $\hat{\mathscr{U}}$ from $\hat{\mathbb{T}}_{\mathrm{loc}}(\mathbb{R})$ which satisfy

$$||\hat{\mathscr{U}}||_{\hat{\mathbb{T}}(t,t+1)}\leqslant 2^{1/p}\mathscr{M}(t)\quad\text{on }\mathbb{R}. \tag{29}$$

We require that the operator $\mathscr{N}_j$ appearing in (1) maps the space $\mathbb{R}\times B_{\ell-n}\times\cdots\times B_0$ into B_{-j}. We assume additionally that $\mathscr{N}_j$ maps the vector functions subject to (29) into $L_{p,\mathrm{loc}}(\mathbb{R};B_{-j})$. Also let the following Lipschitz-type condition hold for all vector functions $\hat{\mathscr{U}}$ and $\hat{\mathscr{V}}$ from $\hat{\mathbb{T}}_{\mathrm{loc}}(\mathbb{R})$ subject to (29):

$$\begin{aligned}&\left(\sum_{j=0}^{n}||\mathscr{N}_j(t;\hat{\mathscr{U}})-\mathscr{N}_j(t;\hat{\mathscr{V}})||^p_{L_p(t,t+1;B_{-j})}\right)^{1/p}\\&\qquad\leqslant\rho(t)||\hat{\mathscr{U}}-\hat{\mathscr{V}}||_{\hat{\mathbb{T}}(t,t+1)},\end{aligned} \tag{30}$$

where ρ is a measurable, bounded function, which may depend on the majorant $\mathscr{M}$. We put

$$\rho_0 = \sup \rho(t)$$

and assume that ρ_0 does not exceed a certain constant c°.

From (30), we obtain

$$\left(\sum_{j=0}^{n} ||\mathscr{N}_j(\cdot; \hat{\mathscr{U}})||^p_{L_p(t,t+1;B_{-j})} \right)^{1/p} \leqslant \rho(t) ||\hat{\mathscr{U}}||_{\hat{\mathbb{T}}(t,t+1)} + \sigma(t), \tag{31}$$

where

$$\sigma(t) = \left(\sum_{j=0}^{n} ||\mathscr{N}_j(\cdot; 0)||^p_{L_p(t,t+1;B_{-j})} \right)^{1/p}. \tag{32}$$

We suppose that

$$\int_{\mathbb{R}} \mu(\tau - t)\sigma(\tau)\, d\tau \leqslant c^\diamond \mathscr{M}(t). \tag{33}$$

Note that (33) implies

$$\sigma(t) \leqslant C c^\diamond \mathscr{M}(t), \tag{34}$$

where C depends only on $k_\pm$ and $m_\pm$. In fact, for any $f \in L_{p,\mathrm{loc}}(\mathbb{R})$

$$\begin{aligned} ||f||_{L_p(t,t+1)} \leqslant & \int_{|\tau-t|<1} ||f||_{L_p(\tau,\tau+2)}\, d\tau \leqslant \int_{|\tau-t|<1} ||f||_{L_p(\tau,\tau+1)}\, d\tau \\ & + \int_{|\tau-t-1|<1} ||f||_{L_p(\tau,\tau+1)}\, d\tau \leqslant C \int_{\mathbb{R}} \mu(\tau - t) ||f||_{L_p(\tau,\tau+1)}\, d\tau, \end{aligned} \tag{35}$$

which implies (34).

Throughout the paper the majorant $\mathscr{M}$ fulfils the condition

$$e^{(k_- + c_- \rho_0^{1/m_-})t} \mathscr{M}(t) \text{ decreases} \quad \text{and} \quad e^{(k_+ - c_+ \rho_0^{1/m_+})t} \mathscr{M}(t) \text{ increases}. \tag{36}$$

Note that (36) leads to

$$c_1 \leqslant \frac{\mathscr{M}(t)}{\mathscr{M}(\tau)} \leqslant c_2 \quad \text{for } |t - \tau| \leqslant 1.$$

Remark. In what follows we require that the constants c° and $c^\diamond$ are sufficiently small whereas $c_\pm$ are large enough. Their values depend on $k_\pm$, p and the unperturbed operator $\mathscr{A}(D_t)$.

3. First-order system corresponding to Eq. (1)

3.1. *Reduction of Eq.* (1) *to a first-order system*

Let u be an arbitrary function from $W^{\ell-n}_{p,\mathrm{loc}}(\mathbb{R};\{B_k\}_{k=0}^{\ell-n})$ subject to estimate (2), which implies (29) owing to (28). By conditions for $\mathcal{N}_j$ imposed in Section 2.5, the operators

$$u \to \mathcal{N}_j(t; u, \dots, D_t^{\ell-n}u), \quad j = 0, \dots, n,$$

map functions from $W^{\ell-n}_{p,\mathrm{loc}}(\mathbb{R};\{B_k\}_{k=0}^{\ell-n})$ satisfying (2) into $L_{p,\mathrm{loc}}(\mathbb{R}; B_{-j})$. We say that u is a solution of Eq. (1) if

$$\sum_{j=0}^{n}\sum_{k=0}^{\ell-n}\int_{\mathbb{R}}(A_{jk}D_t^{\ell-n-k}u(t)\mid D_t^{n-j}v(t))\,dt$$
$$= \sum_{j=0}^{n}\int_{\mathbb{R}}(\mathcal{N}_j(t;u(t),\dots,D_t^{\ell-n}u(t))\mid D_t^{n-j}v(t))\,dt \tag{37}$$

for all $v \in W^n_{p',\mathrm{comp}}(\mathbb{R};\{B_k{}'\}_{k=0}^n)$. We write Eq. (1) in the form

$$\sum_{j=0}^{n} D_t^{n-j}\mathscr{A}_j(D_t)u(t) = \sum_{j=0}^{n} D_t^{n-j}\mathcal{N}_j(t,u(t),\dots,D_t^{\ell-n}u(t)) \quad \text{on } \mathbb{R}, \tag{38}$$

where

$$\mathscr{A}_j(D_t) = \sum_{k=0}^{\ell-n} A_{jk}D_t^{\ell-n-k}.$$

Let $\mathscr{U} = \mathrm{col}(\mathscr{U}_1, \dots, \mathscr{U}_\ell)$, where

$$\mathscr{U}_k = D_t^{k-1}u, \quad k = 1, \dots, \ell - n, \tag{39}$$

$$\mathscr{U}_{\ell-n+1} = \mathscr{A}_0(D_t)u - \mathcal{N}_0(t, u, \dots, D_t^{\ell-n}u) \tag{40}$$

and

$$\mathscr{U}_{\ell-n+j} = D_t\mathscr{U}_{\ell-n+j-1} + \mathscr{A}_{j-1}(D_t)u - \mathcal{N}_{j-1}(t, u, \dots, D_t^{\ell-n}u) \tag{41}$$

for $j = 2, \dots, n$. With this notation, (38) takes the form

$$D_t \mathscr{U}_\ell + \mathscr{A}_n(D_t)u - \mathscr{N}_n(t, u, \dots, D_t^{\ell-n}u) = 0. \tag{42}$$

Using (39) and the notation $\mathscr{Z} = D_t^{\ell-n}u$ we write (40) as

$$A_{00}\mathscr{Z} = \mathscr{N}_0(t; \mathscr{U}', \mathscr{Z}) + \mathscr{U}_{\ell-n+1} - \sum_{k=0}^{\ell-n-1} A_{0,\ell-n-k}\mathscr{U}_{k+1}, \tag{43}$$

where as before $\mathscr{U}' = \mathrm{col}(\mathscr{U}_1, \dots, \mathscr{U}_{\ell-n})$.

Lemma 1. *There exist constants C_1, C_2 and C_3 such that the following assertions hold*:

(i) *Let $\hat{\mathscr{U}} = (\mathscr{U}', \mathscr{U}_{\ell-n+1})$ belong to $\hat{\mathbb{T}}_{\mathrm{loc}}(\mathbb{R})$ and satisfy*

$$||\hat{\mathscr{U}}||_{\hat{\mathbb{T}}(t,t+1)} \leqslant C_1 \mathscr{M}(t) \quad \text{on } \mathbb{R}, \tag{44}$$

where $C_1 < 1$. Then Eq. (43) *has a unique solution*

$$\mathscr{Z} = \mathscr{S}(t; \hat{\mathscr{U}})$$

in $L_{p,\mathrm{loc}}(\mathbb{R}; B_0)$ subject to

$$||\mathscr{Z}||_{L_p(t,t+1;B_0)} \leqslant C_2(||\hat{\mathscr{U}}||_{\hat{\mathbb{T}}(t,t+1)} + \sigma(t)) \tag{45}$$

with σ given by (32).

(ii) *For all $\hat{\mathscr{U}}$ satisfying* (44)

$$||\mathscr{S}(t; \hat{\mathscr{U}}) - \mathscr{S}_0\hat{\mathscr{U}}||_{L_p(t,t+1;B_0)} \leqslant C_3(\rho(t)||\hat{\mathscr{U}}||_{\hat{\mathbb{T}}(t,t+1)} + \sigma(t)), \tag{46}$$

where $\mathscr{S}_0$ is a linear operator defined by

$$\mathscr{S}_0\hat{\mathscr{U}} = A_{00}^{-1}\left(\mathscr{U}_{\ell-n+1} - \sum_{k=0}^{\ell-n-1} A_{0,\ell-n-k}\mathscr{U}_{k+1}\right). \tag{47}$$

(iii) *For all $\hat{\mathscr{U}}_1, \hat{\mathscr{U}}_2 \in \hat{T}_{\mathrm{loc}}(\mathbb{R})$ subject to* (44)

$$||\mathscr{S}(t; \hat{\mathscr{U}}_1) - \mathscr{S}(t; \hat{\mathscr{U}}_2)||_{L_p(t,t+1;B_0)} \leqslant c||\hat{\mathscr{U}}_1 - \hat{\mathscr{U}}_2||_{\hat{\mathbb{T}}(t,t+1)}. \tag{48}$$

Proof. (i) We write (43) as

$$\mathscr{Z} = \mathscr{F}(t, \hat{\mathscr{U}}, \mathscr{Z}), \tag{49}$$

where

$$\mathscr{F}(t,\hat{\mathscr{U}},\mathscr{Z}) = A_{00}^{-1}\left(\mathscr{N}_0(t;\mathscr{U}',\mathscr{Z}) + \mathscr{U}_{\ell-n+1} - \sum_{k=0}^{\ell-n-1} A_{0,\ell-n-k}\mathscr{U}_{k+1}\right).$$

We introduce the Banach space $\mathfrak{B}$ of functions in $L_{p,\mathrm{loc}}(\mathbb{R};B_0)$ with the norm

$$\sup_t \frac{||u||_{L_p(t,t+1;B_0)}}{\mathscr{M}(t)}.$$

By $\mathfrak{B}(r)$ we denote the ball in $\mathfrak{B}$ with radius r centered at the origin. Let $\mathscr{Z}\in\mathfrak{B}(C_2)$. By (31) we have

$$\begin{aligned}&||\mathscr{F}(\cdot;\hat{\mathscr{U}},\mathscr{Z})||_{L_p(t,t+1;B_0)}\\ &\quad\leqslant c(\sigma(t)+\rho(t)(||\mathscr{U}'||_{\mathbb{T}'(t,t+1)}+||\mathscr{Z}||_{L_p(t,t+1;B_0)})+||\hat{\mathscr{U}}||_{\hat{\mathbb{T}}(t,t+1)}).\end{aligned} \tag{50}$$

Using (34) and (44), we obtain

$$||\mathscr{F}(\cdot;\hat{\mathscr{U}},\mathscr{Z})||_{L_p(t,t+1;B_0)}\leqslant c(Cc^{\diamond}+\rho_0(C_1+q)+C_1)\mathscr{M}(t).$$

We can choose C_1, C_2, $c^{\diamond}$ and the upper bound c° for ρ_0 so that

$$c(Cc^{\diamond}+c^{\circ}(C_1+C_2)+C_1)\leqslant C_2,$$

which means that $\mathscr{F}$ maps $\mathfrak{B}(C_2)$ into itself. The Lipschitz condition (30) leads to

$$||\mathscr{F}(\cdot;\hat{\mathscr{U}},\mathscr{Z}_1)-\mathscr{F}(\cdot;\hat{\mathscr{U}},\mathscr{Z}_2)||_{L_p(t,t+1;B_0)}\leqslant c\rho_0||\mathscr{Z}_1-\mathscr{Z}_2||_{L_p(t,t+1;B_0)}$$

for all $\mathscr{Z}_1,\mathscr{Z}_2\in\mathfrak{B}(C_2)$. Hence there exists a unique solution of (49) from $\mathfrak{B}(C_2)$.

Estimate (45) follows directly from (49) and (50).

(ii) Since

$$\mathscr{S}(t;\hat{\mathscr{U}})-\mathscr{S}_0\hat{\mathscr{U}} = A_{00}^{-1}\mathscr{N}_0(t;\mathscr{U}',\mathscr{S}(t;\hat{\mathscr{U}})),$$

we deduce (46) from (31) and (45).

(iii) Replacing the pair $(\hat{\mathscr{U}},\mathscr{Z})$ in (43) by $(\hat{\mathscr{U}}_1,\mathscr{Z}_1)$ and $(\hat{\mathscr{U}}_2,\mathscr{Z}_2)$, we obtain from (30) that

$$\begin{aligned}||\mathscr{Z}_1-\mathscr{Z}_2||_{L_p(t,t+1;B_0)}\leqslant c\rho_0(||\mathscr{Z}_1-\mathscr{Z}_2||_{L_p(t,t+1;B_0)}+||\hat{\mathscr{U}}_1-\hat{\mathscr{U}}_2||_{\hat{\mathbb{T}}(t,t+1)})\\ +\ c||\hat{\mathscr{U}}_1-\hat{\mathscr{U}}_2||_{\hat{\mathbb{T}}(t,t+1)},\end{aligned}$$

which implies (48). $\square$

By (39) and (40)

$$D_t^{\ell-n} u = \mathscr{S}(t; \hat{\mathscr{U}}), \tag{51}$$

It follows from (39) that

$$D_t \mathscr{U}_k = \mathscr{U}_{k+1} \quad \text{for } k = 1, \dots, \ell - n - 1. \tag{52}$$

Hence

$$D_t \mathscr{U}_{\ell-n} = \mathscr{S}(t; \hat{\mathscr{U}}). \tag{53}$$

Using (51), we write (41) as

$$\begin{aligned} D_t \mathscr{U}_{\ell-n+j} = {} & \mathscr{U}_{\ell-n+j+1} - \sum_{k=0}^{\ell-n-1} A_{j,\ell-n-k} \mathscr{U}_{k+1} - A_{j0} \mathscr{S}(t; \hat{\mathscr{U}}) \\ & + \mathscr{N}_j(t; \mathscr{U}', \mathscr{S}(t; \hat{\mathscr{U}})) \end{aligned} \tag{54}$$

for $j = 1, \dots, n-1$. Then (42) takes the form

$$D_t \mathscr{U}_\ell + \sum_{k=0}^{\ell-n-1} A_{n,\ell-n-k} \mathscr{U}_{k+1} + A_{n0} \mathscr{S}(t; \hat{\mathscr{U}}) - \mathscr{N}_n(t; \mathscr{U}', \mathscr{S}(t; \hat{\mathscr{U}})) = 0. \tag{55}$$

Relations (52)–(55) can be written as the evolution system

$$(\mathscr{I} D_t + \mathfrak{A}) \mathscr{U}(t) - \mathfrak{N}(t; \hat{\mathscr{U}}(t)) = 0 \quad \text{on } \mathbb{R}, \tag{56}$$

where the operator $\mathfrak{N}$ is given by

$$\mathfrak{N}(t; \hat{\mathscr{U}}) = \operatorname{col}(0, \dots, 0, \mathfrak{N}_{\ell-n}(t; \hat{\mathscr{U}}), \mathfrak{N}_{\ell-n+1}(t; \hat{\mathscr{U}}), \dots, \mathfrak{N}_\ell(t; \hat{\mathscr{U}})), \tag{57}$$

with

$$\mathfrak{N}_{\ell-n}(t; \hat{\mathscr{U}}) = \mathscr{S}(t; \hat{\mathscr{U}}) - A_{00}^{-1} \left(\mathscr{U}_{\ell-n+1} - \sum_{k=0}^{\ell-n-1} A_{0,\ell-n-k} \mathscr{U}_{k+1} \right)$$

and

$$\begin{aligned} & \mathfrak{N}_{\ell-n+j}(t; \hat{\mathscr{U}}) \\ & \quad = \mathscr{N}_j(t; \mathscr{U}', \mathscr{S}(t; \hat{\mathscr{U}})) - A_{j0} \left(\mathscr{S}(t; \hat{\mathscr{U}}) - A_{00}^{-1} \left(\mathscr{U}_{\ell-n+1} - \sum_{k=0}^{\ell-n-1} A_{0,\ell-n-k} \mathscr{U}_{k+1} \right) \right) \end{aligned}$$

for $j = 1, \dots, n$. By (43) these relations can be written as

$$\mathfrak{N}_{\ell-n}(t; \hat{\mathscr{U}}) = A_{00}^{-1} \mathscr{N}_0(t; \mathscr{U}', \mathscr{S}(t; \hat{\mathscr{U}})) \tag{58}$$

and

$$\mathfrak{N}_{\ell-n+j}(t;\hat{\mathscr{U}}) = \mathscr{N}_j(t;\mathscr{U}',\mathscr{S}(t;\hat{\mathscr{U}})) - A_{j0}A_{00}^{-1}\mathscr{N}_0(t;\mathscr{U}',\mathscr{S}(t;\hat{\mathscr{U}})) \tag{59}$$

for $j = 1, \dots, n$.

In (56), we denote the identity operator by $\mathscr{I}$. We also use the operator matrix

$$\mathfrak{A} = -\mathscr{J} + \mathfrak{L} \tag{60}$$

with $\mathscr{J} = \{\mathscr{J}_{jk}\}_{j,k=1}^{\ell}$ given by

$$\begin{array}{cc} & (\ell-n+1) \\ (\ell-n) & \begin{pmatrix} 0 & I & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & I & \cdots & \vdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & A_{00}^{-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & I \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \end{array}$$

and $\mathfrak{L} = \{\mathfrak{L}_{jk}\}_{j,k=1}^{\ell}$ equal to

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ A_{00}^{-1}A_{0,\ell-n} & \cdots & A_{00}^{-1}A_{0,1} & 0 & \cdots & 0 \\ A_{1,\ell-n} - A_{1,0}A_{00}^{-1}A_{0,\ell-n} & \cdots & A_{1,1} - A_{1,0}A_{00}^{-1}A_{0,1} & A_{10}A_{00}^{-1} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \ddots & \vdots \\ A_{n,\ell-n} - A_{n,0}A_{00}^{-1}A_{0,\ell-n} & \cdots & A_{n,1} - A_{n,0}A_{00}^{-1}A_{0,1} & A_{n0}A_{00}^{-1} & \cdots & 0 \end{pmatrix}$$

with zeros in the last $n-1$ columns.

We put

$$\mathscr{T} = B_{\ell-n} \times \cdots \times B_1 \times B_0 \times (B_{-n})^{n-1} \tag{61}$$

and

$$\mathscr{R} = B_{\ell-n-1} \times \cdots \times B_1 \times B_0 \times (B_{-n})^{n}. \tag{62}$$

By the continuity of $A_{jk} : B_k \to B_{-j}$, the operator $\mathfrak{A} : \mathscr{T} \to \mathscr{R}$ is continuous.

3.2. Vector function spaces $\mathbb{T}$ and $\mathbf{S}$

Here we collect definitions of some spaces which are used in the sequel. Let $\mathcal{T}$ and $\mathcal{R}$ be the spaces defined by (61) and (62). We introduce the space $\mathbb{T}(a,b)$ of vector functions $\mathcal{U} = \mathrm{col}(\mathcal{U}_j)_{j=1}^{\ell}$ defined on (a,b), taking values in $\mathcal{T}$ and supplied with the norm

$$||\mathcal{U}||_{\mathbb{T}(a,b)} = \left(\int_a^b (||\mathcal{U}(\tau)||_{\mathcal{T}}^p + ||D_\tau \mathcal{U}(\tau)||_{\mathcal{R}}^p)\, d\tau\right)^{1/p}.$$

This definition is equivalent to

$$\mathbb{T}(a,b) = \{\mathcal{U} : \mathcal{U} \in L_p(a,b;\mathcal{T}), D_t\mathcal{U} \in L_p(a,b;\mathcal{R})\}. \tag{63}$$

Note that $\mathcal{U} \in \mathbb{T}(a,b)$ implies $\mathcal{U}' = (\mathcal{U}_1, \dots, \mathcal{U}_{\ell-n}) \in \mathbb{T}'(a,b)$ and $\hat{\mathcal{U}} = (\mathcal{U}', \mathcal{U}_{\ell-n+1}) \in \hat{\mathbb{T}}(a,b)$ (see Section 2.4).

By $\mathbf{S}(a,b)$ we denote the space of all vector functions $\mathcal{U}(t)$ represented in the form

$$\mathcal{U}(t) = \mathrm{col}(u(t), \dots, D_t^{\ell-n-1}u(t), u_{\ell-n+1}(t), \dots, u_\ell(t)), \tag{64}$$

where $u \in W_p^{\ell-n}(a,b;\{B_j\}_{j=0}^{\ell-n})$,

$$u_{\ell-n+1} \in L_p(a,b;B_0), \quad D_t u_{\ell-n+1} \in L_p(a,b;B_{-n}) \tag{65}$$

and

$$u_{\ell-n+j}, D_t u_{\ell-n+j} \in L_p(a,b;B_{-n}), \quad j = 2, \dots, n. \tag{66}$$

We equip the space $\mathbf{S}(a,b)$ with the norm

$$||\mathcal{U}||_{\mathbf{S}(a,b)} = \Bigg(||u||^p_{W_p^{\ell-n}(a,b;\{B_j\}_{j=0}^{\ell-n})} + ||u_{\ell-n+1}||^p_{L_p(a,b;B_0)}$$
$$+ ||D_t u_{\ell-n+1}||^p_{L_p(a,b;B_{-n})} + \sum_{j=2}^{n}(||u_{\ell-n+j}||^p_{L_p(a,b;B_{-n})} + ||D_t u_{\ell-n+j}||^p_{L_p(a,b;B_{-n})})\Bigg)^{1/p}.$$

Clearly, $\mathbf{S}(a,b) \subset \mathbb{T}(a,b)$ and

$$||\mathcal{U}||_{\mathbb{T}(a,b)} \leqslant c||\mathcal{U}||_{\mathbf{S}(a,b)} \tag{67}$$

for all $\mathcal{U} \in \mathbf{S}(a,b)$.

We use the notation $\mathbb{T}_{\mathrm{loc}}(\mathbb{R})$ for the space of vector functions defined on $\mathbb{R}$ whose restrictions to an arbitrary finite interval (a,b) belong to $\mathbb{T}(a,b)$. In the same way, the space $\mathbf{S}_{\mathrm{loc}}(\mathbb{R})$ is defined.

3.3. *Equivalence of Eq.* (1) *and system* (56)

Lemma 2. (i) *If* $u \in W^{\ell-n}_{p,\mathrm{loc}}(\mathbb{R};\{B\}^{\ell-n}_{k=0})$ *is a solution of* (1) *subject to* (2), *then the vector function* $\mathcal{U}$ *given by* (39)–(41) *belongs to* $\mathbf{S}_{\mathrm{loc}}(\mathbb{R})$ *and satisfies* (56) *as well as*

$$||\mathcal{U}||_{\mathbf{S}(t,t+1)} \leqslant c\mathcal{M}(t) \quad \textit{for } t \in \mathbb{R}. \tag{68}$$

(ii) *If* $\mathcal{U} \in \mathbb{T}_{\mathrm{loc}}(\mathbb{R})$ *is a solution of* (56) *subject to*

$$||\mathcal{U}||_{\mathbb{T}(t,t+1)} \leqslant c\mathcal{M}(t) \quad \textit{for } t \in \mathbb{R}, \tag{69}$$

then $\mathcal{U} \in \mathbf{S}_{\mathrm{loc}}(\mathbb{R})$ *and the function* $u = \mathcal{U}_1$ *is a solution of* (1) *satisfying* (2).

Proof. (i) In order to verify the inclusion $\mathcal{U} \in \mathbf{S}_{\mathrm{loc}}(\mathbb{R})$ and estimate (68), it suffices to show that

$$\mathcal{U}_{\ell-n+1} \in L_{p,\mathrm{loc}}(\mathbb{R};B_0), \quad D_t\mathcal{U}_{\ell-n+1} \in L_{p,\mathrm{loc}}(\mathbb{R};B_{-n}), \tag{70}$$

$$\mathcal{U}_{\ell-n+j}, D_t u_{\ell-n+j} \in L_{p,\mathrm{loc}}(\mathbb{R};B_{-n}), \quad j=2,\dots,n, \tag{71}$$

and that

$$\Bigg(||\mathcal{U}_{\ell-n+1}||^p_{L_p(t,t+1;B_0)} + ||D_t\mathcal{U}_{\ell-n+1}||^p_{L_p(t,t+1;B_{-n})} + \sum_{j=2}^{n} (||\mathcal{U}_{\ell-n+j}||^p_{L_p(t,t+1;B_{-n})} + ||D_t\mathcal{U}_{\ell-n+j}||^p_{L_p(t,t+1;B_{-n})}) \Bigg)^{1/p} \leqslant c\mathcal{M}(t). \tag{72}$$

The first inclusion in (70) and estimate (72) for the norm of $\mathcal{U}_{\ell-n+1}$ follow from (40) and (34).

Let $\eta \in C_0^\infty(0,1)$ and $\eta_t(\tau) = \eta(\tau - t)$. From (40) and (41) combined with (2) and (34) we obtain

$$\left| \int_t^{t+1} \mathcal{U}_{\ell-n+j}(\tau)\eta(\tau-t)\,d\tau \right| \leqslant c\mathcal{M}(t), \tag{73}$$

where $j = 1, \ldots, n$. Let the mean value of η on $(0,1)$ is equal to 1. By the identity

$$\mathscr{U}_{\ell-n+j}(\tau) - \int_\tau^{\tau+1} \mathscr{U}_{\ell-n+j}(s)\eta(s-\tau)\,ds$$
$$= -i\int_\tau^{\tau+1} D_\sigma \mathscr{U}_{\ell-n-j}(\sigma)\,d\sigma \int_\sigma^{\tau+1} \eta(s-\tau)\,ds$$

and by (73), we arrive at

$$||\mathscr{U}_{\ell-n+j}||_{L_p(t,t+1;B_{-n})} \leqslant c(\mathscr{M}(t) + ||D_t\mathscr{U}_{\ell-n+j}||_{L_p(t,t+1;B_{-n})}). \tag{74}$$

This inequality with $j = 0$, along with (42), yield (71) with $j = n$ and the estimate

$$(||\mathscr{U}_\ell||^p_{L_p(t,t+1;B_{-n})} + ||D_t\mathscr{U}_\ell||^p_{L_p(t,t+1;B_{-n})})^{1/p} \leqslant c\mathscr{M}(t).$$

Estimates for the terms in (72) with $j = n-1, \ldots, 1$ together with the remaining inclusions in (70) and (71) result from (41) and (74).

It follows from the reduction of (1) to the first-order system (56) performed in Section 3.1 that $\mathscr{U}$ satisfies (56). The result follows.

(ii) By (52) and (53) we obtain $\mathscr{U}_k = D_t^{k-1}\mathscr{U}_1$ for $k = 1, \ldots, \ell - n$ and $\mathscr{S}(t;\hat{\mathscr{U}}) = D_t^{\ell-n}\mathscr{U}_1$. These equalities along with (69) imply (2) for $u = \mathscr{U}_1$. Equality (43) becomes

$$\mathscr{U}_{\ell-n+1} = \mathscr{A}_0(D_t)\mathscr{U}_1 - \mathscr{N}_0(t;\mathscr{U}_1, \ldots, D_t^{\ell-n}\mathscr{U}_1),$$

whereas (54) takes the form

$$D_t\mathscr{U}_{\ell-n+j} = \mathscr{U}_{\ell-n+j+1} - \mathscr{A}_j(D_t)\mathscr{U}_1 + \mathscr{N}_j(t;\mathscr{U}_1, \ldots, D_t^{\ell-n}\mathscr{U}_1)$$

for $j = 1, \ldots, n-1$, and (55) can be written as

$$D_t\mathscr{U}_\ell + \mathscr{A}_n(D_t)\mathscr{U}_1 - \mathscr{N}_n(t;\mathscr{U}_1, \ldots, D_t^{\ell-n}\mathscr{U}_1) = 0.$$

The last three equations imply (38) for $u = \mathscr{U}_1$. $\square$

4. Properties of the pencil $(\lambda\mathscr{I} + \mathfrak{A})$

4.1. *Correspondence between $\mathscr{A}(\lambda)$ and the linear pencil $\lambda\mathscr{I} + \mathfrak{A}$*

Lemma 3. *Let the row vector $e(\lambda) = (e_1(\lambda), \ldots, e_\ell(\lambda))$ be given by*

$$e_{\ell-j}(\lambda) = \lambda^j, \quad j = 0, \ldots, n-1, \tag{75}$$

$$e_{\ell-n}(\lambda) = \sum_{j=0}^{n} \lambda^{n-j} A_{j0}, \tag{76}$$

$$e_{\ell-n-k}(\lambda) = \sum_{s=0}^{k} \sum_{j=0}^{n} \lambda^{k+n-s-j} A_{js}, \quad k = 1, \dots, \ell - n - 1. \tag{77}$$

Then for all $\lambda \in \mathbb{C}$ the equality

$$e(\lambda)(\lambda \mathscr{I} + \mathfrak{A}) = (\mathscr{A}(\lambda), 0, \dots, 0) \tag{78}$$

holds.

Proof. It follows from (78) that

$$\lambda e_{\ell-j} - e_{\ell-j-1} = 0 \tag{79}$$

for $j = 0, \dots, n-2$,

$$\lambda e_{\ell-n+1} - e_{\ell-n} A_{00}^{-1} + \sum_{j=1}^{n} e_{\ell-n+j} A_{j0} A_{00}^{-1} = 0, \tag{80}$$

$$\lambda e_{\ell-n-k+1} - e_{\ell-n-k} + e_{\ell-n} A_{00}^{-1} A_{0k} + \sum_{j=1}^{n} e_{\ell-n+j} (A_{jk} - A_{j0} A_{00}^{-1} A_{0k}) = 0 \tag{81}$$

for $k = 1, \dots, \ell - n - 1$ and

$$\lambda e_1 + e_{\ell-n} A_{00}^{-1} A_{0,\ell-n} + \sum_{j=1}^{n} e_{\ell-n+j} (A_{j,\ell-n} - A_{j0} A_{00}^{-1} A_{0,\ell-n}) = \mathscr{A}(\lambda). \tag{82}$$

Putting $e_\ell = 1$ and using (79) we obtain (75). Equality (76) results from (80) and (75). Now, using (81) together with (75) and (76), we arrive at

$$e_{\ell-n-k} = \lambda e_{\ell-n-k+1} + \sum_{j=0}^{n} \lambda^{n-j} A_{jk} \tag{83}$$

for $k = 1, \dots, \ell - n - 1$ and (77) with $k = 1, \dots, \ell - n - 2$ follows from (83) by induction. Finally, using (75) and (76), we write (82) as

$$\lambda e_1 + \sum_{j=0}^{n} \lambda^{n-j} A_{j,\ell-n} = \mathscr{A}(\lambda),$$

which leads to (77) with $k = \ell - n - 1$. The proof is complete. $\square$

We introduce the operator matrix $\mathscr{E}(\lambda) = \{\mathscr{E}_{pq}(\lambda)\}_{p,q=1}^{\ell}$ as

$$\begin{array}{cc} & (\ell - n) \\ (\ell - n + 1) & \begin{pmatrix} e_1(\lambda) & e_2(\lambda) & \cdots & e_{\ell-n}(\lambda) & \cdots & e_{\ell-1}(\lambda) & e_\ell(\lambda) \\ -I & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & -I & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -A_{00} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & -I & 0 \end{pmatrix} \end{array}. \tag{84}$$

One can check directly that $\mathscr{E}^{-1}(\lambda)$ is given by

$$\begin{array}{cc} & (\ell - n + 1) \\ (\ell - n) & \begin{pmatrix} 0 & -I & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -A_{00}^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & -I \\ I & e_1(\lambda) & \cdots & e_{\ell-n}(\lambda) & \cdots & e_{\ell-2}(\lambda) & e_{\ell-1}(\lambda) \end{pmatrix} \end{array}.$$

We obtain the key result of the present section which describes an explicit correspondence between the pencils $\mathscr{A}(\lambda)$ and $\lambda\mathscr{I} + \mathfrak{A}$.

Proposition 3. *For all* $\lambda \in \mathbb{C}$

$$\mathscr{E}(\lambda)(\lambda\mathscr{I} + \mathfrak{A}) = \operatorname{diag}(\mathscr{A}(\lambda), I, \dots, I)\begin{pmatrix} J_{\ell-n}(\lambda) & 0 \\ -\mathscr{B}(\lambda) & J_n(\lambda) - M \end{pmatrix}, \tag{85}$$

where the $d \times d$ *matrix* $J_d(\lambda)$ *is defined by*

$$J_d(\lambda) = \begin{pmatrix} I & 0 & \dots & 0 & 0 \\ -\lambda & I & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \\ 0 & 0 & \dots & -\lambda & I \end{pmatrix}.$$

Here the $n \times n$ matrix M is defined by

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ A_{10}A_{00}^{-1} & 0 & \dots & 0 & 0 \\ A_{20}A_{00}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ A_{n-1,0}A_{00}^{-1} & 0 & \dots & 0 & 0 \end{pmatrix}$$

and the $n \times (\ell - n)$ matrix $\mathscr{B}(\lambda)$ is given by

$$\mathscr{B}(\lambda) = \begin{pmatrix} A_{0,\ell-n} & \dots & A_{02} & A_{01} \\ A_{1,\ell-n} & \dots & A_{12} & A_{11} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n-1,\ell-n} & \dots & A_{n-1,2} & A_{n-1,1} \end{pmatrix}$$

$$- \begin{pmatrix} 0 & \dots & 0 & -\lambda A_{00} \\ A_{10}A_{00}^{-1}A_{0,\ell-n} & \dots & A_{10}A_{00}^{-1}A_{02} & A_{10}A_{00}^{-1}A_{01} \\ \vdots & \dots & \vdots & \vdots \\ A_{n-1,0}A_{00}^{-1}A_{0,\ell-n} & \dots & A_{n-1,0}A_{00}^{-1}A_{02} & A_{n-1,0}A_{00}^{-1}A_{01} \end{pmatrix}.$$

Proof. By Lemma 3, the left-hand side in (85) is a triangular matrix with the diagonal $\mathscr{A}(\lambda), I, \dots, I$. One can verify directly that this matrix is equal to the right-hand side in (85).

Clearly, $J_d(\lambda)$ has the inverse

$$J_d(\lambda)^{-1} = \begin{pmatrix} I & 0 & 0 & \dots & 0 & 0 \\ \lambda & I & 0 & \dots & \vdots & \vdots \\ \lambda^2 & \lambda & I & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & I & 0 \\ \lambda^{d-1} & \lambda^{d-2} & \lambda^{d-3} & \dots & \lambda & I \end{pmatrix}. \tag{86}$$

We recall that the matrix $\mathscr{E}^{-1}(\lambda)$ is polynomial. In the next lemma we prove the same for the last matrix in (85).

Lemma 4. *The following formula is valid*:

$$\begin{pmatrix} J_{\ell-n}(\lambda) & 0 \\ -\mathscr{B}(\lambda) & J_n(\lambda) - M \end{pmatrix}^{-1} = \begin{pmatrix} J_{\ell-n}^{-1}(\lambda) & 0 \\ Q(\lambda) & J_n^{-1}(\lambda)(I+M) \end{pmatrix}, \tag{87}$$

where the elements of the matrix $Q(\lambda) = \{Q_{jk}(\lambda)\}$, $j = 1, \dots, n$, $k = 1, \dots, \ell - n$, *are given by*

$$Q_{jk}(\lambda) = \sum_{s=0}^{j-1} \sum_{q=k-1}^{\ell-n} \lambda^{q+s+1-k} A_{j-s-1,\ell-n-q}. \tag{88}$$

Proof. Let us look for $(J_n(\lambda) - M)^{-1}$ in the form $J_n^{-1}(\lambda) + S(\lambda)$, where $S(\lambda)$ has nonzero elements only in the first column and $S_{11}(\lambda) = 0$. We have

$$(J_n(\lambda) - M)(J_n^{-1}(\lambda) + S(\lambda)) = I + J_n(\lambda)S(\lambda) - MJ_n^{-1}(\lambda).$$

Hence,

$$S(\lambda) = J_n^{-1}(\lambda)M.$$

Therefore, we arrive at (87) with

$$Q(\lambda) = (J_n^{-1}(\lambda) + S(\lambda))\mathscr{B}(\lambda)J_n^{-1}(\lambda).$$

One can check that the last equality gives (88). $\square$

The next proposition is a straightforward application of (85).

Proposition 4. (i) *The operator*

$$\lambda\mathscr{I} + \mathfrak{A} : \mathscr{T} \to \mathscr{R} \tag{89}$$

is Fredholm for all $\lambda \in \mathbb{C}$.

(ii) *The spectra of the operator* $-\mathfrak{A}$ *and the pencil* $\mathscr{A}(\lambda)$ *coincide and consist of eigenvalues of the same geometric, algebraic and partial multiplicities.*

Proof. Let

$$\mathfrak{B} = B_{-n} \times B_{\ell-n-1} \times \cdots \times B_1 \times B_0 \times (B_{-n})^{n-1}.$$

The operator

$$\mathscr{E}(\lambda) : \mathscr{R} \to \mathfrak{B}$$

is an isomorphism for all $\lambda \in \mathbb{C}$. Analogously, one verifies that the operator

$$\begin{Bmatrix} J(\lambda) & 0 \\ -\mathscr{B}(\lambda) & J(\lambda) - M \end{Bmatrix} : \mathscr{T} \to \mathscr{T}$$

is isomorphic for all $\lambda \in \mathbb{C}$. Hence and by (85) the polynomial operator functions

$$\lambda\mathscr{I} + \mathfrak{A} : \mathscr{T} \to \mathscr{R}$$

and

$$\mathrm{diag}(\mathscr{A}(\lambda), I, \dots, I) : \mathscr{T} \to \mathfrak{B}$$

are equivalent and therefore these functions have the same spectrum and the geometric, partial and algebraic multiplicities of their eigenvalues coincide (see, for example, [KM1, Appendix]). $\square$

4.2. *Spectral properties of the pencil $\lambda\mathscr{I} + \mathfrak{A}$*

We introduce the vector functions

$$\mathscr{U}^{(v)}_{kj}(t) = \mathrm{col}(\mathscr{U}^{(v)}_{kj,s}(t))^{\ell}_{s=1}$$

by the equality

$$\mathscr{U}^{(v)}_{kj}(t) = \begin{pmatrix} J^{-1}_{\ell-n}(D_t) & 0 \\ Q(D_t) & J^{-1}_n(D_t)(I+M) \end{pmatrix} \mathrm{col}(U^{(v)}_{kj}(t), 0, \dots, 0).$$

Using the description of the operators given in the previous section, we get

$$\mathscr{U}^{(v)}_{kj,s}(t) = D^{s-1}_t U^{(v)}_{kj}(t), \quad s = 1, \dots, \ell - n, \tag{90}$$

and

$$\mathscr{U}^{(v)}_{kj,\ell-n+s}(t) = \sum_{p=0}^{s-1} \sum_{q=0}^{\ell-n} A_{s-p-1,\ell-n-q} D^{q+p}_t U^{(v)}_{kj}(t) \tag{91}$$

for $s = 1, \dots, n$.

We introduce the vector functions

$$\mathscr{V}^{(v)}_{kj}(t) = \mathrm{col}(\mathscr{V}^{(v)}_{kj,s}(t))^{\ell}_{s=1},$$

which are given by

$$\mathscr{V}^{(v)}_{kj}(t) = \mathscr{E}^*(D_t)\,\mathrm{col}(V^{(v)}_{kj}(t), \dots, 0).$$

This is equivalent to the equalities

$$\mathscr{V}^{(v)}_{kj,s}(t) = \sum_{p=0}^{\ell-n-s} \sum_{q=0}^{n} A^*_{qp} D^{\ell-s-q-p}_t V^{(v)}_{kj}(t)$$

for $s = 1, \dots, \ell - n - 1$,

$$\mathscr{V}^{(v)}_{kj,\ell-n}(t) = \sum_{q=0}^{n} A^*_{q0} D_t^{n-q} V^{(v)}_{kj}(t) \tag{92}$$

and

$$\mathscr{V}^{(v)}_{kj,\ell-n+s}(t) = D_t^{n-s} V^{(v)}_{kj}(t) \tag{93}$$

for $s = 1, \dots, n$. By the definitions of $\mathscr{U}^{(v)}_{kj}$ and $\mathscr{V}^{(v)}_{kj}$ for every $t \in \mathbb{R}$

$$\mathscr{U}^{(v)}_{kj}(t) \in \mathscr{T} \quad \text{and} \quad \mathscr{V}^{(v)}_{kj}(t) \in \mathscr{R}^* = (B_n{}')^n \times B_0{}' \times B_{-1}{}' \times \cdots \times B_{n-\ell+1}{}'. \tag{94}$$

Using (85) and (87), we obtain

$$(\mathscr{I} D_t + \mathfrak{A}) \mathscr{U}^{(v)}_{kj}(t) = 0,$$

$$(\mathscr{I} D_t + \mathfrak{A}^*) \mathscr{V}^{(v)}_{kj}(t) = 0. \tag{95}$$

The next assertion contains analogs of Lemma 12.4.1, Proposition 12.4.2 and Corollary 12.4.3 in [KM1]. Its proof follows the same lines as that in Section 12.4 [KM1].

Proposition 5. (i) *Let η be the same function as in* (26). *Then*

$$\int_{\mathbb{R}} \langle (\mathscr{I} D_t + \mathfrak{A})(\eta(t) \mathscr{U}^{(v)}_{kj}(t)) \mid \mathscr{V}^{(\mu)}_{pq}(t) \rangle \, dt = \delta^v_\mu \delta^k_p \delta^j_q, \tag{96}$$

where we use the duality

$$\langle \mathscr{U} \mid \mathscr{V} \rangle = \sum_{s=1}^{\ell} (\mathscr{U}_s \mid \mathscr{V}_s) \tag{97}$$

for all $\mathscr{U} = (\mathscr{U}_1, \dots, \mathscr{U}_\ell)$ and $\mathscr{V} = (\mathscr{V}_1, \dots, \mathscr{V}_\ell)$.

(ii) *For every $\tau \in \mathbb{R}$, the systems*

$$\{\mathscr{U}^{(v)}_{kj}(\tau)\}, \{\mathscr{V}^{(v)}_{k, m_{\mu k} - 1 - j}(\tau)\},$$

where $k = 1, \dots, J_v$, $j = 0, \dots, m_{vk} - 1$, form canonical sets of Jordan chains corresponding to the eigenvalues λ_v and $\bar{\lambda}_v$ of the operator pencils $\lambda \mathscr{I} + \mathfrak{A}$ and $\lambda \mathscr{I} + \mathfrak{A}^*$. The biorthogonality relation holds:

$$\langle \mathscr{U}^{(\mu)}_{kj}(\tau) \mid \mathscr{V}^{(v)}_{sq}(\tau) \rangle = -i \delta^\mu_v \delta^k_s \delta^j_q. \tag{98}$$

(iii) *The equalities*

$$(\mathscr{I} \lambda_v + \mathfrak{A}) \mathscr{U}^{(v)}_{k0}(\tau) = 0, \tag{99}$$

$$(\mathscr{I}\lambda_\nu + \mathfrak{A})\mathscr{U}^{(\nu)}_{kh}(\tau) + \mathscr{U}^{(\nu)}_{k,h-1}(\tau) = 0 \tag{100}$$

hold for $h = 1, \dots, m_{\nu k} - 1$.

(iv) *The vectors* $\mathscr{V}^{(\nu)}_{kh}$ *satisfy*

$$(\mathscr{I}\bar{\lambda}_\nu + \mathfrak{A}^*)\mathscr{V}^{(\nu)}_{k,m_{\nu k}-1}(\tau) = 0, \tag{101}$$

$$(\mathscr{I}\bar{\lambda}_\nu + \mathfrak{A}^*)\mathscr{V}^{(\nu)}_{kh}(\tau) + \mathscr{V}^{(\nu)}_{k,h+1}(\tau) = 0 \tag{102}$$

for $h = 0, 1, \dots, m_{\nu k} - 2$.

Let $k_\pm$ and k_0 be real numbers such that $k_- < k_0 < k_+$ and all eigenvalues of $\mathscr{A}(\lambda)$ in the strip $k_- < \Im\lambda < k_+$ are situated on the line $\Im\lambda = k_0$. We introduce the Riesz projector $\mathscr{P}$ by

$$\mathscr{P}F = \frac{1}{2\pi i}\int_\Gamma (\lambda\mathscr{I} + \mathfrak{A})^{-1} F\, d\lambda, \tag{103}$$

where Γ is a smooth simple contour consisting of regular points of $\lambda I + \mathfrak{A}$ and having the eigenvalues λ_ν with $\Im\lambda_\nu = k_0$ inside, F is an element of $\mathscr{R}$. In the next proposition we collect properties of the operator $\mathscr{P}$ which are proved in the same way as Propositions 12.5.1–12.5.3 in [KM1].

Proposition 6. (i) *For every* $\tau \in \mathbb{R}$ *the formula*

$$\mathscr{P} = i \sum_{\Im\lambda_\nu = k_0} \sum_{k=1}^{J_\nu} \sum_{j=0}^{m_{\nu k}-1} \langle \cdot \mid \mathscr{V}^{(\nu)}_{kj}(\tau) \rangle \mathscr{U}^{(\nu)}_{kj}(\tau) \tag{104}$$

holds. (*In particular, the right-hand side in* (104) *does not depend on* τ).

(ii) *The operator* $\mathscr{P}$ *maps* $\mathscr{R}$ *into* $\mathscr{D}$ *and* $\mathscr{P}^2 = \mathscr{P}$.

(iii) *For an arbitrary* $\mathscr{U} \in \mathscr{T}$,

$$\mathfrak{A}\mathscr{P}\mathscr{U} = P\mathfrak{A}\mathscr{U}. \tag{105}$$

(iv) *The equalities*

$$\mathscr{P}\mathscr{U}^{(\nu)}_{kj}(\tau) = \mathscr{U}^{(\nu)}_{kj}(\tau), \tag{106}$$

$$\mathscr{P}^*\mathscr{V}^{(\nu)}_{kj}(\tau) = \mathscr{V}^{(\nu)}_{kj}(\tau) \tag{107}$$

hold.

(v) *The operator function*

$$(t,\tau)\to\sum_{k=1}^{J_v}\sum_{j=0}^{m_{vk}-1}\langle\,\cdot\mid\mathscr{V}_{kj}^{(v)}(\tau)\rangle\,\mathscr{U}_{kj}^{(v)}(t) \tag{108}$$

depends only on $t-\tau$.

In the following statement we use the space $\mathbb{T}(a,b)$ defined by (63).

Proposition 7. *Let* $\mathscr{P}$ *be the Riesz projector* (104). *If* $\mathscr{U}\in\mathbb{T}(a,b)$ *then* $\mathscr{P}\mathscr{U}\in\mathbb{T}(a,b)$ *and*

$$\|\mathscr{P}\mathscr{U}\|_{\mathbb{T}(a,b)}\leqslant c\|\mathscr{U}\|_{\mathbb{T}(a,b)}, \tag{109}$$

where c *does not depend on* a,b.

Proof. By (104) with $\tau=0$,

$$\mathscr{P}\mathscr{U}(t)=i\sum_{\Im\lambda_v=k_0}\sum_{k=1}^{J_v}\sum_{j=0}^{m_{vk}-1}\langle\mathscr{U}(t)\mid\mathscr{V}_{kj}^{(v)}(0)\rangle\,\mathscr{U}_{kj}^{(v)}(0).$$

It suffices to estimate

$$\|\mathscr{P}\mathscr{U}\|_{L_p(a,b;\mathscr{T})}\quad\text{and}\quad\|D_t\mathscr{P}\mathscr{U}\|_{L_p(a,b;\mathscr{R})},$$

which is achieved by using (94) with $t=0$. $\square$

5. Asymptotic representation of solutions to Eq. (1)

5.1. Spaces $\mathbb{X}$ and $\mathbb{Y}$

Here we add some new function spaces to spaces $\mathbb{T}$ and $\mathbf{S}$ defined in Section 3.2. By $\mathbb{X}(a,b)$ we denote the space of all vector functions

$$\mathscr{U}(t)=(\mathscr{I}-\mathscr{P})\mathscr{V}(t) \tag{110}$$

with $\mathscr{V}\in\mathbf{S}(a,b)$. We equip the space $\mathbb{X}(a,b)$ with the norm

$$\|\mathscr{U}\|_{\mathbb{X}(a,b)}=\inf\|\mathscr{V}\|_{\mathbf{S}(a,b)},$$

where the infimum is taken over all $\mathscr{V}$ in (110).

Lemma 5. *The space $\mathbb{X}(a,b)$ is continuously embedded into $\mathbb{T}(a,b)$ and the estimate*

$$||\mathscr{U}||_{\mathbb{T}(a,b)} \leqslant c||\mathscr{U}||_{\mathbb{X}(a,b)} \tag{111}$$

holds, where c does not depend on a, b and $\mathscr{U}$.

Proof. Let $\mathscr{U}$ be given by (110) where

$$\mathscr{V}(t) = \mathrm{col}(u(t), \dots, D_t^{\ell-n-1}u(t), u_{\ell-n+1}, \dots, u_\ell)$$

with $u \in W_p^{\ell-n}(a,b;\{B_j\}_{j=0}^{\ell-n})$ and other components subject to (65) and (66). Then

$$||\mathscr{U}||_{\mathbb{T}(a,b)} \leqslant C||\mathscr{V}||_{\mathbb{T}(a,b)}$$

by (109). Using (67), we arrive at

$$||\mathscr{U}||_{\mathbb{T}(a,b)} \leqslant C||\mathscr{V}||_{\mathbf{S}(a,b)}.$$

Since this is valid for all $\mathscr{V} \in \mathbf{S}(a,b)$ satisfying (110), we obtain (111). $\square$

We define the space $\mathbb{X}_{\mathrm{loc}}(\mathbb{R})$ of all vector functions on $\mathbb{R}$ which are represented in form (110) with a certain $\mathscr{V} \in \mathbf{S}_{\mathrm{loc}}(\mathbb{R})$.

We shall also use the space $\mathbb{Y}_{\mathrm{loc}}(\mathbb{R})$ of vector functions $\mathscr{F}(t) = \mathrm{col}(\mathscr{F}_j(t))_{j=1}^{\ell-n}$ given as

$$\mathscr{F}(t) = \mathrm{col}(0, \dots, 0, f_{\ell-n}(t), \dots, f_\ell) \tag{112}$$

with some $f_{\ell-n+j} \in L_{p,\mathrm{loc}}(\mathbb{R}; B_{-j})$, $j = 0, \dots, n$. We equip this space with the seminorms

$$||\mathscr{F}||_{\mathbb{Y}(t,t+1)} = \left(\sum_{j=0}^{n} ||f_{\ell-n+j}||^p_{L_p(t,t+1;B_{-j})}\right)^{1/p}.$$

5.2. *Spectral splitting of the nonlinear system* (56)

Applying $\mathscr{P}$ and $\mathscr{I} - \mathscr{P}$ to system (56) we arrive at

$$(\mathscr{I}D_t + \mathfrak{A})\mathbf{u} - \mathscr{P}\mathfrak{N}(t; \hat{\mathbf{u}} + \hat{\mathbf{v}}) = 0 \quad \text{on } \mathbb{R} \tag{113}$$

and

$$(\mathscr{I}D_t + \mathfrak{A})\mathbf{v} - (\mathscr{I} - \mathscr{P})\mathfrak{N}(t; \hat{\mathbf{u}} + \hat{\mathbf{v}}) = 0 \quad \text{on } \mathbb{R}, \tag{114}$$

where $\hat{\mathbf{u}} = \mathrm{col}(\mathbf{u}_1, \dots, \mathbf{u}_{\ell-n+1})$, $\hat{\mathbf{v}} = \mathrm{col}(\mathbf{v}_1, \dots, \mathbf{v}_{\ell-n+1})$ and

$$\mathbf{u}(t) = \mathscr{P}\mathscr{U}(t), \quad \mathbf{v}(t) = (\mathscr{I} - \mathscr{P})\mathscr{U}(t). \tag{115}$$

Thus we have split system (56) into the finite-dimensional system (113) and the infinite-dimensional system (114). Clearly, $\mathscr{U} \in \mathbb{T}_{\mathrm{loc}}(\mathbb{R})$ implies that $\mathbf{u}$ and $D_t\mathbf{u}$ belong to $\in L_{p,\mathrm{loc}}(\mathbb{R}; \mathscr{T})$.

The next proposition shows the equivalence of (56) and the split system (113), (114).

Proposition 8. (i) *Let* $\mathscr{U} \in \mathbb{T}_{\mathrm{loc}}(\mathbb{R})$ *be a solution of* (56) *subject to* (68). *Then the pair* $\mathbf{u}$, $\mathbf{v}$ *given by* (115) *satisfies the inequalities*

$$||\mathbf{u}||_{\mathbb{T}(t,t+1)} \leqslant c\mathscr{M}(t), \tag{116}$$

$$||\mathbf{v}||_{\mathbb{T}(t,t+1)} \leqslant c\mathscr{M}(t) \tag{117}$$

and systems (113), (114)

(ii) *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *belong to* $\mathbb{T}_{\mathrm{loc}}(\mathbb{R})$ *and be subject to* (116) *and* (117). *Also let* $\mathbf{u}(t) = \mathscr{P}\mathbf{u}(t)$ *and* $\mathbf{v}(t) = (\mathscr{I} - \mathscr{P})\mathbf{v}(t)$ *on* $\mathbb{R}$. *Then* $\mathscr{U} = \mathbf{u} + \mathbf{v}$ *satisfies* (69) *and system* (56).

Proof. The proof is obvious. □

This proposition, combined with Lemma 2, ensures the equivalence of equation (1) and the split system (113), (114).

5.3. Solvability of the unperturbed infinite-dimensional part of the split system

In what follows, we fix numbers k_-, k_+ and k_0 so that $k_- < k_0 < k_+$ and the eigenvalues of $\mathscr{A}(\lambda)$ in the strip $k_- < \Im\lambda < k_+$ are situated on the line $\Im\lambda = k_0$. By $m_\pm$ we denote the maximal partial multiplicity of all eigenvalues of $\mathscr{A}(\lambda)$ situated on the line $\Im\lambda = k_\pm$. In the case when there are no eigenvalues on the line $\Im\lambda = k_\pm$, we set $m_\pm = 1$. As before, by c we mean possibly different positive constants depending on the operator $\mathscr{A}(D_t)$ and the numbers $k_\pm$, k_0.

Here, we deal with the system

$$(\mathscr{I}D_t + \mathfrak{A})\mathbf{w} = (\mathscr{I} - \mathscr{P})\mathscr{F} \quad \text{on } \mathbb{R}. \tag{118}$$

Proposition 9. (i) (Existence) *Let* $\mathscr{F} \in \mathbb{Y}_{\mathrm{loc}}(\mathbb{R})$. *Suppose that*

$$\int_{\mathbb{R}} \mu(\tau)||\mathscr{F}||_{\mathbb{Y}(\tau,\tau+1)}\, d\tau < \infty, \tag{119}$$

where μ *is defined by* (16). *Then Eq.* (118) *has a solution* $\mathbf{w} \in \mathbb{X}_{\mathrm{loc}}(\mathbb{R})$ *satisfying*

$$||\mathbf{w}||_{\mathbb{X}(t,t+1)} \leqslant c\int_{\mathbb{R}} \mu(\tau - t)||\mathscr{F}||_{\mathbb{Y}(\tau,\tau+1)}\, d\tau. \tag{120}$$

(ii) (Uniqueness) *Let* $\mathbf{w} \in \mathbb{X}_{\mathrm{loc}}(\mathbb{R})$ *be a solution to* (118) *with* $\mathscr{F} = 0$. *Also let*

$$||\mathbf{w}||_{\mathbb{T}(t,t+1)} = \begin{cases} o(e^{-k_- t}) & \text{if } t \to +\infty, \\ o(e^{-k_+ t}) & \text{if } t \to -\infty \end{cases} \tag{121}$$

be valid. Then $\mathbf{w} = 0$.

Proof. (i) It is the same as that of Theorem 12.7.1 in [KM1], and the proof of (ii) follows the same lines as that of Lemma 12.8.1 in [KM1]. $\square$

Remark. Obviously, (121) follows from (120).

5.4. The infinite-dimensional part of the perturbed split system

We start with a unique solvability result for system (114), where $\mathfrak{N}$ is given by (57)–(59). First, we obtain some properties of the operators $\mathfrak{N}$ to be used in this section and in the sequel. Combining (30), (46) and (48), we conclude that for all $\hat{\mathscr{U}}_1$ and $\hat{\mathscr{U}}_2$ subject to (44)

$$||\mathfrak{N}(\cdot;\hat{\mathscr{U}}_1) - \mathfrak{N}(\cdot;\hat{\mathscr{U}}_2)||_{\mathbb{Y}(t,t+1)} \leqslant c\rho(t)||\hat{\mathscr{U}}_1 - \hat{\mathscr{U}}_2||_{\hat{\mathbb{T}}(t,t+1)}. \tag{122}$$

This together with (31) implies

$$||\mathfrak{N}(\cdot;\hat{\mathscr{U}})||_{\mathbb{Y}(t,t+1)} \leqslant c(\rho(t)||\hat{\mathscr{U}}||_{\hat{\mathbb{T}}(t,t+1)} + \sigma(t)). \tag{123}$$

Proposition 10. *Let* c_* *be a sufficiently small constant. For any* $\hat{\mathbf{u}} \in \hat{\mathbb{T}}_{\mathrm{loc}}(\mathbb{R})$ *such that*

$$||\hat{\mathbf{u}}||_{\hat{\mathbb{T}}(t,t+1)} \leqslant c_* \mathscr{M}(t), \tag{124}$$

system (114) *has a unique solution* $\mathbf{v} \in \mathbb{T}_{\mathrm{loc}}(\mathbb{R})$ *subject to*

$$||\mathbf{v}||_{\mathbb{T}(t,t+1)} \leqslant c_* \mathscr{M}(t). \tag{125}$$

Proof. We need the Banach space $\mathfrak{W}(\mathscr{M})$ of vector functions from $\mathbb{T}_{\mathrm{loc}}(\mathbb{R})$ such that

$$||w||_{\mathfrak{W}(\mathscr{M})} = \sup_{t \in \mathbb{R}} \frac{||w||_{\mathbb{T}(t,t+1)}}{\mathscr{M}(t)} < \infty.$$

The notation $\mathfrak{W}_r(\mathscr{M})$ will be used for the ball of radius r in $\mathfrak{W}(\mathscr{M})$ centered at the origin.

We introduce the inverse operator $\mathfrak{S} : (\mathscr{I} - \mathscr{P})\mathscr{F} \to \mathbf{w}$, where $\mathbf{w} \in \mathbb{X}_{\text{loc}}(\mathbb{R})$ satisfies (118). The domain $\mathscr{D}(\mathfrak{S})$ consists of all vector functions $(\mathscr{I} - \mathscr{P})\mathscr{F}$ with $\mathscr{F} \in \mathbb{Y}_{\text{loc}}(\mathbb{R})$ subject to (119). By Proposition 9, the solution $\mathbf{w}$ exists, is unique and satisfies (120).

We look for a solution $\mathbf{v}$ to (114) subject to (125). Let us set $\mathscr{F} = \mathfrak{N}(t; \hat{\mathbf{u}} + \hat{\mathbf{v}})$. By (123),

$$||\mathscr{F}||_{\mathbb{Y}(t,t+1)} \leqslant c(\rho(t)||\hat{\mathbf{u}} + \hat{\mathbf{v}}||_{\hat{\mathbb{T}}(t,t+1)} + \sigma(t)).$$

Making use of (34) combined with (124) and (125), we see that

$$||\mathscr{F}||_{\mathbb{Y}(t,t+1)} \leqslant c(\rho_0 c_* + c^\diamond)\mathscr{M}(t).$$

Therefore, $(\mathscr{I} - \mathscr{P})\mathscr{F}$ belongs to $\mathscr{D}(\mathfrak{S})$ and we can define the operator

$$T(\hat{\mathbf{v}}) = \mathfrak{S}(\mathscr{I} - \mathscr{P})\mathfrak{N}(t; \hat{\mathbf{u}} + \hat{\mathbf{v}}).$$

We need to prove the existence and uniqueness of a fixed point of T. It follows from (120) and (123) that

$$||T(\hat{\mathbf{v}})||_{\mathbb{X}(t,t+1)} \leqslant \mathscr{C} \int_{\mathbb{R}} \mu(\tau - t)(\rho(\tau)||\hat{\mathbf{v}}||_{\hat{\mathbb{T}}(\tau,\tau+1)} + \sigma_1(\tau))\, d\tau, \tag{126}$$

where

$$\sigma_1(\tau) = \rho(\tau)||\hat{\mathbf{u}}||_{\hat{\mathbb{T}}(\tau,\tau+1)} + \sigma(\tau).$$

Inequality (67) shows that T is a continuous map of $\mathfrak{W}_{c_*}(\mathscr{M})$ into $\mathfrak{W}(\mathscr{M})$. Continuing estimate (126) with help of (33), (124) and (125) we arrive at

$$||T(\hat{\mathbf{v}})||_{\mathbb{X}(t,t+1)} \leqslant c_2 \left(\rho_0 c_* \int_{\mathbb{R}} \mu(\tau - t)\mathscr{M}(\tau)\, d\tau + c^\diamond \mathscr{M}(t) \right).$$

Using definition (16) and (36), we obtain

$$\begin{aligned} &\rho_0 \int_{\mathbb{R}} \mu(\tau - t)\mathscr{M}(\tau)\, d\tau \\ &\leqslant \rho_0 \mathscr{M}(t) \left(\int_t^\infty e^{-c_- \rho_0^{1/m_-}(\tau - t)} (1 + \tau - t)^{m_- - 1}\, d\tau \right. \\ &\quad \left. + \int_{-\infty}^t e^{c_+ \rho_0^{1/m_+}(\tau - t)} (1 + t - \tau)^{m_+ - 1}\, d\tau \right) \leqslant c(c_+^{-m_+} + c_-^{-m_-})\mathscr{M}(t). \end{aligned} \tag{127}$$

We may assume that the constants $c^\diamond$ and $c_\pm$ satisfy the inequality

$$c(c_*(c_+^{-m_+} + c_-^{-m_-}) + c^\diamond) < 1. \tag{128}$$

Hence, the operator T maps $\mathfrak{W}_{c_*}(\mathscr{M})$ into itself.

We deduce from (122) that for all $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathfrak{W}_{c_*}(\mathcal{M})$

$$||T(\hat{\mathbf{v}}_1) - T(\hat{\mathbf{v}}_2)||_{\mathbb{T}(t,t+1)} \leqslant C \int_{\mathbb{R}} \mu(\tau - t)\rho(\tau)||\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2||_{\hat{\mathbb{T}}(\tau,\tau+1)}\, d\tau. \tag{129}$$

Hence,

$$||T(\hat{\mathbf{v}}_1) - T(\hat{\mathbf{v}}_2)||_{\mathfrak{W}(\mathcal{M})} \leqslant C\rho_0 \sup_t \frac{\rho_0}{\mathcal{M}(t)} \int_{\mathbb{R}} \mu(\tau - t)\mathcal{M}(\tau)\, d\tau ||\mathbf{v}_1 - \mathbf{v}_2||_{\mathfrak{W}(\mathcal{M})}.$$

Making use of (127) and (128) we derive

$$||T(\hat{\mathbf{v}}_1) - T(\hat{\mathbf{v}}_2)||_{\mathfrak{W}(\mathcal{M})} \leqslant c||\mathbf{v}_1 - \mathbf{v}_2||_{\mathfrak{W}(\mathcal{M})}$$

with $c < 1$ for all $\mathbf{v}_1$, $\mathbf{v}_2$ from $\mathfrak{W}_{c_*}(\mathcal{M})$. The result follows from the Banach fixed point theorem. □

We shall use the function

$$\mu_0(t) = \begin{cases} e^{k_-(\rho_0)t} & \text{for } t \geqslant 0, \\ e^{k_+(\rho_0)t} & \text{for } t < 0, \end{cases} \tag{130}$$

where

$$k_{\mp}(x) = k_{\mp} \pm \frac{1}{2} c_{\mp} x^{1/m_{\mp}}, \quad x \geqslant 0 \tag{131}$$

with $c_{\pm}$ in (36).

In the following proposition, we show that estimate (125) can be improved.

Proposition 11. *Let* $\hat{\mathbf{u}}$ *satisfy* (124). *Assume that*

$$\int_{\mathbb{R}} \mu_0(\tau)\sigma(\tau)\, d\tau \leqslant c^{\diamond} \mathcal{M}(t), \tag{132}$$

where $c^{\diamond}$ *is the same as in* (33). *Then there exists a constant* C^* *such that*

$$||\mathbf{v}||_{\mathbb{T}(t,t+1)} \leqslant C^* \int_{\mathbb{R}} \mu_0(\tau - t)(\rho(\tau)||\hat{\mathbf{u}}||_{\hat{\mathbb{T}}(\tau,\tau+1)} + \sigma(\tau))\, d\tau. \tag{133}$$

Before proving this proposition, we note that (132) implies (33) by $\mu(t) \leqslant \mu_0(t)$.

Proof of Proposition. By (124) and (132), the integral on the right-hand side in (133) has the majorant

$$\rho_0 \int_{\mathbb{R}} \mu_0(\tau - t)c_* \mathcal{M}(\tau)\, d\tau + c^{\diamond} \mathcal{M}(t).$$

The first term does not exceed

$$cc_*\left(\frac{1}{c_-}+\frac{1}{c_+}\right)\mathcal{M}(t)$$

(compare with (127)), which shows that the right-hand side in (133) is less then

$$C^*\left(cc_*\left(\frac{1}{c_-}+\frac{1}{c_+}\right)+c^\diamond\right)\mathcal{M}(t). \tag{134}$$

Using the smallness of c_-^{-1}, c_+^{-1} and $c^\diamond$ we estimate (134) by $c_*\mathcal{M}(t)$. Hence, (125) follows from (133).

Let us prove (133). By $\mathcal{M}_1(t)$ we denote the integral on the right-hand side in (133) and we shall use the notation introduce in the proof of Proposition 10, now with $\mathcal{M}_1$ in place of $\mathcal{M}$. Let us verify the inequality

$$||T\mathbf{v}||_{\mathbb{T}(t,t+1)}\leqslant C^*\mathcal{M}_1(t)$$

for all $\mathbf{v}\in\mathfrak{W}_{C^*}(\mathcal{M}_1)$. Inserting estimate (133) into (126) we arrive at

$$||T\mathbf{v}||_{\mathbb{T}(t,t+1)}\leqslant C\bigg(\int_{\mathbb{R}}\mu(\tau-t)\sigma_1(\tau)\,d\tau + C^*\rho_0\int_{\mathbb{R}}\int_{\mathbb{R}}\mu(\tau-t)\mu_0(s-\tau)\sigma_1(s)\,ds\,d\tau\bigg). \tag{135}$$

Let us find a majorant for the double integral in the right-hand side. Recalling the definitions of μ and μ_0, we see that for $s\geqslant t$

$$\int_{\mathbb{R}}\mu(\tau-t)\mu_0(s-\tau)\,d\tau\leqslant e^{k_-(\rho_0)(s-t)}\left(\frac{1}{k_-(\rho_0)-k_-}+\frac{1}{k_+-k_-(\rho_0)}\right) + \frac{e^{k_-(s-t)}}{k_+(\rho_0)-k_-}\leqslant\left(c+\frac{2}{\rho_0c_-}\right)e^{k_-(\rho_0)(s-t)}.$$

Similarly, for $t\geqslant s$ we have

$$\int_{\mathbb{R}}\mu(\tau-t)\mu_0(s-\tau)\,d\tau\leqslant\left(c+\frac{2}{\rho_0c_+}\right)e^{k_+(\rho_0)(s-t)}.$$

Hence, the double integral in (135) does not exceed

$$\left(C+\left(\frac{1}{c_+}+\frac{1}{c_-}\right)\frac{2}{\rho_0}\right)\int_{\mathbb{R}}\mu_0(s-t)\sigma_1(s)\,ds. \tag{136}$$

Therefore,

$$||T\mathbf{v}||_{\mathbb{T}(t,t+1)}\leqslant C\left(1+C^*\left(cc^\circ+\frac{2}{c_-}+\frac{2}{c_+}\right)\right)\mathcal{M}_1(t),$$

where c° is the same as in Section 2.5. Since the constants $1/c^\circ$, c_- and c_+ are large enough and C^* can be chosen to satisfy $2C \leqslant C^*$, there holds

$$||T\mathbf{v}||_{\mathbb{T}(t,t+1)} \leqslant C^* \mathscr{M}_1(t).$$

Thus, T maps $\mathfrak{W}_{C^*}(\mathscr{M}_1)$.

We show that T is contractive. By (129)

$$||T(\hat{\mathbf{v}}_1) - T(\hat{\mathbf{v}}_2)||_{\mathfrak{W}(\mathscr{M}_1)} \leqslant C\rho_0 \int_{\mathbb{R}} \mu(\tau - t)\mathscr{M}_1(\tau)\, d\tau ||\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2||_{\mathfrak{W}(\mathscr{M}_1)}.$$

The last integral is equal to the double integral in (135), which has the majorant (136). Hence,

$$||T(\hat{\mathbf{v}}_1) - T(\hat{\mathbf{v}}_2)||_{\mathfrak{W}(\mathscr{M}_1)} \leqslant C\left(cc^\diamond + \left(\frac{2}{c_+} + \frac{2}{c_-}\right)\right)||\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2||_{\mathfrak{W}(\mathscr{M}_1)}.$$

By smallness of $c^\diamond$, $1/c_+$ and $1/c_-$ the constant factor on the right-hand side can be made less than 1. The result follows by the Banach fixed point theorem.

5.5. *The finite-dimensional system*

By Propositions 10–11, for any $\hat{\mathbf{u}} \in \hat{\mathbb{T}}_{\mathrm{loc}}(\mathbb{R})$ subject to (124), one can define the inverse operator $\mathfrak{M} : \hat{\mathbf{u}} \to \mathbf{v} \in \mathbb{X}_{\mathrm{loc}}(\mathbb{R})$ solving system (114). Proposition 11 guarantees the estimate

$$||\mathfrak{M}[\hat{\mathbf{u}}]||_{\mathbb{X}(t,t+1)} \leqslant C \int \mu_1(\tau - t)(\rho(\tau)||\hat{\mathbf{u}}||_{\hat{\mathbb{T}}(\tau,\tau+1)} + \sigma(\tau))\, d\tau. \tag{137}$$

Using the operator $\mathfrak{M}$, we write (113) in the form

$$(\mathscr{I} D_t + \mathfrak{A})\mathbf{u} - \mathscr{P}\mathfrak{N}(t; \hat{\mathbf{u}} + \widehat{\mathfrak{M}[\hat{\mathbf{u}}]}) = 0 \quad \text{on } \mathbb{R}. \tag{138}$$

We rewrite this system as

$$(\mathscr{I} D_t + \mathfrak{A})\mathbf{u} - \mathscr{P}\mathfrak{N}^{(0)}(t; \hat{\mathbf{u}}) - \mathscr{P}\mathscr{K}[\hat{\mathbf{u}}] = 0 \quad \text{on } \mathbb{R}, \tag{139}$$

where

$$\mathfrak{N}^{(0)}(t; \hat{\mathbf{u}}) = \mathrm{col}(0, \ldots, 0, \mathfrak{N}^{(0)}_{\ell-n}(t; \hat{\mathbf{u}}), \mathfrak{N}^{(0)}_{\ell-n+1}(t; \hat{\mathbf{u}}), \ldots, \mathfrak{N}^{(0)}_{\ell}(t; \hat{\mathbf{u}})), \tag{140}$$

with

$$\mathfrak{N}^{(0)}_{\ell-n}(t; \hat{\mathbf{u}}) = A_{00}^{-1} \mathscr{N}_0(t; \mathscr{U}', \mathscr{S}_0\hat{\mathbf{u}}) \tag{141}$$

and

$$\mathfrak{N}^{(0)}_{\ell-n+j}(t;\hat{\mathbf{u}}) = \mathcal{N}_j(t;\mathbf{u}',\mathcal{S}_0\hat{\mathbf{u}}) - A_{j0}A_{00}^{-1}\mathcal{N}_0(t;\mathbf{u}',\mathcal{S}_0\hat{\mathbf{u}}) \tag{142}$$

for $j = 1, \dots, n$. Here $\mathcal{S}_0$ is given by (47). The operator $\mathcal{K}$ in (139) is defined as

$$\mathcal{K}[\hat{\mathbf{u}}](t) = \mathfrak{N}(t;\hat{\mathbf{u}} + \widehat{\mathfrak{M}[\hat{\mathbf{u}}]}) - \mathfrak{N}^{(0)}(t;\hat{\mathbf{u}}). \tag{143}$$

We wish to obtain estimates for $\mathcal{K}$ to show that it plays the role of a weak perturbation at infinity of the differential dynamical system

$$(\mathcal{I}D_t + \mathfrak{A})\mathbf{u} - \mathcal{P}\mathfrak{N}^{(0)}(t;\hat{\mathbf{u}}) = 0 \quad \text{on } \mathbb{R}.$$

Lemma 6. (i) *For all* $\hat{\mathbf{u}} \in \hat{\mathbb{T}}_{\mathrm{loc}}(\mathbb{R})$ *subject to* (124)

$$||\mathcal{K}[\hat{\mathbf{u}}]||_{\mathbb{Y}(t,t+1)} \leqslant c\rho(t)\int_{\mathbb{R}} \mu_0(\tau - t)(\rho(\tau)||\hat{\mathbf{u}}||_{\hat{\mathbb{T}}(\tau,\tau+1)} + \sigma(\tau))\, d\tau. \tag{144}$$

(*See* (130) *for the definition of* μ_0).

Proof. We represent $\mathcal{K}$ as the sum $\mathcal{K}_1 + \mathcal{K}_2$, where

$$\mathcal{K}_1[\hat{\mathbf{u}}] = \mathfrak{N}(t;\hat{\mathbf{u}} + \widehat{\mathfrak{M}[\hat{\mathbf{u}}]}) - \mathfrak{N}(t;\hat{\mathbf{u}})$$

and

$$\mathcal{K}_2[\hat{\mathbf{u}}] = \mathfrak{N}(t;\hat{\mathbf{u}}) - \mathfrak{N}^{(0)}(t;\hat{\mathbf{u}}).$$

Estimate (144) for $\mathcal{K}_1$ follows from (122) and (137).

By (122)

$$\left(\sum_{j=0}^{n} ||\mathcal{N}_j(\cdot;\mathbf{u}',\mathcal{S}(\cdot;\hat{\mathbf{u}}) - \mathcal{N}_j(\cdot;\mathbf{u}',\mathcal{S}_0\hat{\mathbf{u}}||^p_{L_p(t,t+1;B_{-j})}\right)^{1/p}$$
$$\leqslant c\rho(t)||\mathcal{S}(\cdot;\hat{\mathbf{u}}) - \mathcal{S}_0\hat{\mathbf{u}}||_{L_p(t,t+1;B_0)}.$$

Combining this with (58), (59) and (141), (142), we arrive at

$$||\mathcal{K}_2[\hat{\mathbf{u}}]||_{\mathbb{Y}(t,t+1)} \leqslant c\rho(t)||\mathcal{S}(\cdot;\hat{\mathbf{u}}) - \mathcal{S}_0(\hat{\mathbf{u}})||_{L_p(t,t+1;B_0)}.$$

We note that by (35) with μ replaced by μ_0 the right-hand side does not exceed

$$c\rho(t)\int_{\mathbb{R}} \mu_0(\tau - t)||\mathcal{S}(\cdot;\hat{\mathbf{u}}) - \mathcal{S}_0\hat{\mathbf{u}}||_{L_p(t,t+1;B_0)}\, d\tau.$$

Using (46), we obtain (144) for $\mathcal{K}_2$. The proof is complete. □

5.6. Asymptotic representation of solutions to Eq. (1)

We are in a position to formulate an asymptotic representation for solutions to Eq. (1).

Theorem 1. *Let as before $k_\pm$ and k_0 be real numbers such that $k_- < k_0 < k_+$ and let all eigenvalues of the operator pencil $\mathscr{A}(\lambda)$ in the strip $k_- < \Im\lambda < k_+$ be situated on the line $\Im\lambda = k_0$. Suppose that the nonlinear operators $\mathscr{N}_j$ are subject to the conditions in Section* 2.5 *and* (132) *is fulfilled.*

(i) *Denote by u a function in $W^{\ell-n}_{p,\mathrm{loc}}(\mathbb{R};\{B_k\}_{k=0}^{\ell-n})$ satisfying Eq.* (1) *and estimate* (2). *Then the vector $\mathscr{U}$ defined by* (39)–(41) *can be written as*

$$\mathscr{U} = \mathbf{u} + \mathbf{v}, \tag{145}$$

where $\mathbf{u} = \mathscr{P}\mathbf{u}$ belongs to $\mathbb{T}_{\mathrm{loc}}(\mathbb{R})$, and satisfies (124) *and* (139). *The vector $\mathbf{v} \in \mathbb{X}_{\mathrm{loc}}(\mathbb{R})$ admits estimate* (133).

(ii) *Let $\mathbf{u} = \mathscr{P}\mathbf{u}$ be a solution to* (139) *subject to* (116) *with a sufficiently small c and let $\mathbf{v} = \mathfrak{M}[\hat{\mathbf{u}}]$. Then $\mathscr{U}$ given by* (145) *satisfies* (69) *and system* (56) *and $u = \mathscr{U}_1$ belongs to $W^{\ell-n}_{p,\mathrm{loc}}(\mathbb{R};\{B_k\}_{k=0}^{\ell-n})$, is a solution of* (1) *and subject to* (2).

Proof. (i) Formula (145) follows from (115). Estimate (133) is established in Proposition 11. Eq. (139) for $\mathbf{u}$ is derived in Section 5.5.

(ii) The assertion follows directly from Proposition 8(ii). $\square$

Remark. Suppose that $\rho(t) \to 0$ as $t \to +\infty$ and that $\sigma(t) = 0$ for large positive t. Then (133) shows that for large positive t

$$\|\mathbf{v}\|_{\mathbb{X}(t,t+1)} \leqslant C\left(\int_{\mathbb{R}} \mu_0(\tau - t)\rho(\tau)\|\hat{\mathbf{u}}\|_{\hat{\mathbb{T}}(\tau,\tau+1)}\, d\tau + e^{-k_+(\rho_0)t}\right).$$

One can check that the integral on the right-hand side is $o(\|\hat{\mathbf{u}}\|_{\hat{\mathbb{T}}(t,t+1)})$ under mild assumptions about $\|\hat{\mathbf{u}}\|_{\hat{\mathbb{T}}(t,t+1)}$. This means that $\mathbf{v}$ plays the role of a remainder term in the asymptotics $\mathscr{U} \sim \mathbf{u}$.

We give another plausible argument in favor of the relation $\mathscr{U} \sim \mathbf{u}$. Under the assumptions $\rho(t) = o(1)$ and $\sigma(t) = o(\mathscr{M}(t))$ as $t \to +\infty$, estimate (133) implies

$$\|\mathbf{v}\|_{\mathbb{X}(t,t+1)} = o(\mathscr{M}(t)) \quad \text{as } t \to +\infty. \tag{146}$$

So, if $\mathscr{M}(t)$ is an asymptotic majorant for $\|u\|_{W_p^{\ell-n}(t,t+1;\{B_k\}_{k=0}^{\ell-n})}$, then $\mathbf{u}$ is the leading term in (145).

Let us justify (146). We have

$$\rho(t)\|\hat{\mathbf{u}}\|_{\hat{\mathbb{T}}(t,t+1)} + \sigma(t) \leqslant \varepsilon(t)\mathscr{M}(t),$$

where ε is a bounded nonincreasing function such that $\varepsilon(t)\to 0$ as $t\to+\infty$. Now we majorize the right-hand side in (133) by

$$C\bigg(\int_t^\infty e^{k_-(\rho_0)(\tau-t)}\varepsilon(\tau)\mathscr{M}(\tau)\,d\tau+\int_N^t e^{k_+(\rho_0)(\tau-t)}\varepsilon(\tau)\mathscr{M}(\tau)\,d\tau$$
$$+\int_{-N}^N e^{k_+(\rho_0)(\tau-t)}\varepsilon(\tau)\mathscr{M}(\tau)\,d\tau+\int_{-\infty}^{-N} e^{k_+(\rho_0)(\tau-t)}\varepsilon(\tau)\mathscr{M}(\tau)\,d\tau\bigg).$$

By the monotonicity condition (36), the last sum is dominated by

$$C(\varepsilon(t)\mathscr{M}(t)+\varepsilon(N)\mathscr{M}(N)e^{k_+(\rho_0)(N-t)}$$
$$+\,e^{-k_+(\rho_0)t}\int_{-N}^N\varepsilon(\tau)\mathscr{M}(\tau)\,d\tau+\varepsilon(-\infty)e^{-k_+(\rho_0)(t+N)}\mathscr{M}(-N)),$$

which together with (36) gives (146).

5.7. *Another form of system* (139)

In the present section we write system (139) in a form, which does not contain the operator $\mathfrak{A}$. This is achieved by the following change of the unknown vector $\mathbf{u}$.

Using $\mathscr{P}\mathbf{u}=\mathbf{u}$ and (104), we write

$$\mathbf{u}(t)=\sum_{\Im\lambda_\nu=k_0}\sum_{k=1}^{J_\nu}\sum_{j=0}^{m_{\nu k}-1}h_{kj}^{(\nu)}(t)\mathscr{U}_{kj}^{(\nu)}(t) \tag{147}$$

with the coefficients

$$h_{kj}^{(\nu)}(t)=i\langle\mathbf{u}(t)\mid\mathscr{V}_{kj}^{(\nu)}(t)\rangle \tag{148}$$

Clearly, $\mathbf{u}\in\mathbb{T}_{\mathrm{loc}}(\mathbb{R})$ implies $h_{kj}^{(\nu)}\in W_{p,\mathrm{loc}}^1(\mathbb{R})$.

For brevity, in the next formulae, we omit indices ν, k and j in $\mathscr{U}_{kj}^{(\nu)}$ and $h_{kj}^{(\nu)}$. According to (90) and (91),

$$\mathscr{U}_s(t)=D_t^{s-1}U(t),\quad s=1,\dots,\ell-n, \tag{149}$$

and

$$\mathscr{U}_{\ell-n+s}(t)=\sum_{p=0}^{s-1}\sum_{q=0}^{\ell-n}A_{s-p-1,\ell-n-q}D_t^{q+p}U(t)\quad\text{for } s=1,\dots,n,$$

where the functions U (with indices) were defined by (23). Therefore,

$$\mathbf{u}_s(t) = \sum h(t) D_t^{s-1} U(t) \quad \text{for } s = 1, \ldots, \ell - n.$$

Also note that

$$\mathscr{S}_0 \hat{\mathscr{U}} = D_t^{\ell-n} U, \tag{150}$$

where $\mathscr{S}_0$ is defined by (47).

Making use of (95), we rewrite system (138) as

$$\sum (D_t h)(t) \mathscr{U}(t) - \mathscr{P}\mathfrak{N}^{(0)}(t; \hat{\mathbf{u}}(t)) - \mathscr{P}\mathscr{K}[\hat{\mathbf{u}}](t) = 0. \tag{151}$$

Hence,

$$\sum (D_t h)(t) \langle \mathscr{U}(t) \mid \mathscr{V}(t) \rangle - \langle \mathscr{P}\mathfrak{N}^{(0)}(t; \hat{\mathbf{u}}(t)) \mid \mathscr{V}(t) \rangle - \langle \mathscr{P}\mathscr{K}[\hat{\mathbf{u}}](t) \mid \mathscr{V}(t) \rangle = 0,$$

where $\mathscr{V} = \mathscr{V}_{pq}^{(\mu)}$. By the biorthogonality condition (98) and by (107)

$$\frac{d}{dt} h(t) + \langle \mathfrak{N}^{(0)}(t; \hat{\mathbf{u}}(t)) \mid \mathscr{V}(t) \rangle + \langle \mathscr{K}[\hat{\mathbf{u}}](t) \mid \mathscr{V}(t) \rangle = 0. \tag{152}$$

Using the coordinate notation as well as (141) and (142), we see that the last functional is equal to

$$\begin{aligned} \sum_{q=0}^{n} (\mathfrak{N}^{(0)}_{\ell-n+q} \mid \mathscr{V}_{\ell-n+s}(t)) = {} & (A_{00}^{-1} \mathscr{N}_0(t; \mathbf{u}', \mathscr{S}_0 \hat{\mathbf{u}}) \mid \mathscr{V}_{\ell-n}) \\ & + \sum_{q=1}^{n} (\mathscr{N}_q(t; \mathbf{u}', \mathscr{S}_0 \hat{\mathbf{u}}) \\ & - A_{q0} A_{00}^{-1} \mathscr{N}_0(t; \mathbf{u}', \mathscr{S}_0 \hat{\mathbf{u}}) \mid \mathscr{V}_{\ell-n+q}). \end{aligned} \tag{153}$$

According to (92) and (93), we may write this in the form

$$\left(A_{00}^{-1} \mathscr{N}_0 \,\middle|\, \sum_{q=0}^{n} A_{q0}^{*} D_t^{n-q} V \right) + \sum_{q=1}^{n} (\mathscr{N}_q - A_{q0} A_{00}^{-1} \mathscr{N}_0 \mid D_t^{n-q} V).$$

Hence,

$$\sum_{q=0}^{n} (\mathfrak{N}^{(0)}_{\ell-n+q} \mid \mathscr{V}_{\ell-n+s}(t)) = \sum_{q=0}^{n} (\mathscr{N}_q(t; \mathbf{u}', \mathscr{S}_0 \hat{\mathbf{u}}) \mid D_t^{n-q} V(t)) \tag{154}$$

Let us introduce the $\varkappa$-dimensional vector

$$\mathbb{N}(t; \mathbf{h}(t)) = \sum_{q=0}^{n} \left(\mathscr{N}_q \left(t; \sum h(t) U(t), \ldots, \sum h(t) D_t^{\ell-n} U(t) \right) \,\middle|\, D_t^{n-q} V(t) \right), \tag{155}$$

where $\varkappa$ is the number of components in $\mathbf{h} = \{h_{kj}^{(v)}\}$, $\Im\lambda_v = k_0$, $k = 1, \ldots, J_v$, $j = 0, \ldots, m_{vk} - 1$. Combining (152), (154), (155) and (150) we arrive at the following equivalent form of the $\varkappa$-dimensional system (152):

$$\frac{d}{dt}\mathbf{h}(t) + \mathbb{N}(t; \mathbf{h}(t)) + \mathbb{K}[\mathbf{h}](t) = 0, \tag{156}$$

where

$$\mathbb{K}[\mathbf{h}](t) = \langle \mathscr{K}[\hat{\mathbf{u}}](t) \mid \mathscr{V}(t) \rangle. \tag{157}$$

We are interested in solutions of (156) subject to the estimate

$$\left\|\sum h\mathscr{U}\right\|_{\mathbb{T}(t,t+1)} \leqslant c\mathscr{M}(t). \tag{158}$$

where m_0 is the maximal partial multiplicity of the eigenvalues on the line $\Im\lambda = k_0$. By (147), this restriction enables one to obtain solutions $\mathbf{u}$ of (138) satisfying (124).

5.8. *The case of eigenvalues without generalized eigenfunctions*

Let us consider the particular case $m_{vk} = 1$ for all v and k in (147). In other words let there exist no generalized eigenfunctions corresponding to the eigenvalues on the line $\Im\lambda = k_0$. Let as before all eigenvalues of the operator pencil $\mathscr{A}(\lambda)$ in the strip $k_- < \Im\lambda < k_+$ be situated on the line $\Im\lambda = k_0$. Here we deal with the eigenvalues λ_v on the line $\Im\lambda = k_0$. Let us denote the corresponding eigenfunctions of the pencil $\mathscr{A}(\lambda)$ by $\varphi_k^{(v)}$, $k = 1, \ldots, J_v$. The eigenvectors of the pencil $\mathscr{A}^*(\lambda)$ corresponding to the eigenvalue $\bar{\lambda}_v$ will be denoted by $\psi_k^{(v)}$. The biorthogonality relation (20) becomes

$$\left(\frac{d\mathscr{A}}{d\lambda}(\lambda_v)\varphi_k^{(v)} \mid \psi_j^{(v)}\right) = \delta_k^j.$$

Theorem 2. *Suppose that the nonlinear operators $\mathscr{N}_j$ are subject to the conditions in Section* 2.5 *and* (132) *is fulfilled.*

(i) *Denote by u a function in $W_{p,\mathrm{loc}}^{\ell-n}(\mathbb{R}; \{B_k\}_{k=0}^{\ell-n})$ satisfying Eq.* (1) *and estimate* (2). *Then*

$$\begin{aligned}
&\mathrm{col}(u(t), \ldots, D_t^{\ell-n-1}u(t)) \\
&\quad = \sum_{\Im\lambda_v = k_0} \sum_{k=1}^{J_v} h_k^{(v)}(t) e^{i\lambda_v t} \varphi_k^{(v)} \,\mathrm{col}(1, \lambda_v, \ldots, \lambda_v^{\ell-n-1}) \\
&\qquad + \mathrm{col}(\mathbf{v}_1(t), \ldots, \mathbf{v}_{\ell-n}(t)),
\end{aligned} \tag{159}$$

where the vector function $\mathbf{h} = \{h_k^{(v)}\}$ *is subject to*

$$||\mathbf{h}||_{W_p^1(t,t+1)^\varkappa} \leqslant c e^{k_0 t} \mathscr{M}(t) \tag{160}$$

and satisfies the system

$$\frac{d}{dt} h_k^{(v)}(t) + \mathbb{N}_k^{(v)}(t; \mathbf{h}(t)) + \mathbb{K}_k^{(v)}[\mathbf{h}](t) = 0, \tag{161}$$

where

$$\mathbb{N}_k^{(v)}(t; \mathbf{h}(t)) \\ = e^{-i\lambda_v t} \sum_{j=0}^{n} \lambda_v^{n-j} \left(\mathscr{N}_j \left(t; \sum_{\Im \lambda_m u = k_0} \sum_{m=1}^{J_v} h_m^{(\mu)}(t) e^{i\lambda_\mu t} \varphi_m^{(\mu)} \operatorname{col}(1, \dots, \lambda_\mu^{\ell-n-1}) \right) \middle| \psi_k^{(v)} \right)$$

and the nonlocal operators $\mathbb{K}_k^{(v)}$ *satisfy the estimate*

$$||\mathbb{K}_k^{(v)}[\mathbf{h}]||_{L_p(t,t+1)} \leqslant C\rho(t) \int_{\mathbb{R}} \mu_0(\tau - t)(\rho(\tau)||\mathbf{h}||_{W_p^1(\tau,\tau+1)^\varkappa} + \sigma(\tau))\, d\tau.$$

The remainder $\mathbf{v}' = (\mathbf{v}_1, \dots, \mathbf{v}_{\ell-n})$ *satisfies*

$$||\mathbf{v}'||_{\mathbb{T}'(t,t+1)} \leqslant C \int_{\mathbb{R}} \mu_0(\tau - t)(\rho(\tau)||\mathbf{h}||_{W_p^1(\tau,\tau+1)^\varkappa} + \sigma(\tau))\, d\tau. \tag{162}$$

(ii) *Let* $\mathbf{h} \in W_{p,\mathrm{loc}}^1(\mathbb{R})^\varkappa$ *be a solution to* (161) *subject to* (160) *with a sufficiently small* c *and let* $\mathbf{v} = \mathfrak{M}[\hat{\mathbf{u}}]$. *Then*

$$u = \sum e^{i\lambda_v t} h_k^{(v)}(t) \varphi_k^{(v)} + \mathbf{v}_1$$

belongs to $W_{p,\mathrm{loc}}^{\ell-n}(\mathbb{R}; \{B_k\}_{k=0}^{\ell-n})$, *is a solution of* (1) *and is subject to* (2).

Proof. (i) In our case

$$\mathbf{u}'(t) = \sum_{\Im \lambda_v = k_0} \sum_{k=1}^{J_v} h_k^{(v)}(t) e^{i\lambda_v t} \varphi_k^{(v)} \operatorname{col}(1, \lambda_v, \dots, \lambda_v^{\ell-n-1}),$$

which makes (159) a direct consequence of (145). System (161) is another form of (139). Estimate (160) follows from (116) and the estimates

$$c_1 ||\hat{\mathbf{u}}||_{\hat{\mathbb{T}}(t,t+1)} \leqslant ||\mathbf{h}||_{W_p^1(t,t+1)^\varkappa} \leqslant c_2 ||\hat{\mathbf{u}}||_{\hat{\mathbb{T}}(t,t+1)}.$$

The proof of (i) is completed by reference to Theorem 1(i).

The part (ii) is a direct corollary of Theorem 1(ii). $\square$

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