Schauder estimates for solutions to a mixed boundary value problems for Stokes system in polyhedral domains

by V. Mazya and J. Rossmann

Abstract

A mixed boundary value problems for the Stokes system in a polyhedral domain is considered. The authors prove the existence of solutions in weighted and non-weighted Hölder spaces and obtain regularity assertions for the solutions. The results are essentially based on estimates of the Green's matrix.

Keywords: Stokes system, nonsmooth domains

MSC (1991): 35J25, 35J55, 35C15, 35Q30

0 Introduction

Schauder estimates, i.e. coercive estimates of Hölder norms, for solutions to linear elliptic equations and systems in domains with smooth boundaries have important applications to linear and especially nonlinear boundary value problems (see, e.g., Agmon, Douglis, Nirenberg [1] and Gilbarg, Trudinger [8]). In the present paper, we consider a mixed boundary value problem for the linear Stokes system

$$-\Delta u + \nabla p = f, \qquad -\nabla \cdot u = g \tag{0.1}$$

in a three-dimensional domain of polyhedral type, where components of the velocity and/or the friction are given on the boundary. To be more precise, we have one of the following boundary conditions on each side Γ_j:

(i) $u = h$,

(ii) $u_\tau = h, \quad -p + 2\varepsilon_{n,n}(u) = \phi$,

(iii) $u_n = h, \quad \varepsilon_{n,\tau}(u) = \phi$,

(iv) $-pn + 2\varepsilon_n(u) = \phi$,

where $u_n = u \cdot n$ denotes the normal and $u_\tau = u - u_n n$ the tangential component of u, $\varepsilon_n(u)$ is the vector $\varepsilon(u)\, n$, $\varepsilon_{n,n}(u)$ is the normal component and $\varepsilon_{n,\tau}(u)$ the tangential component of $\varepsilon_n(u)$. In our previous papers [22, 23] we proved estimates for the Green's matrix and estimates of solutions in weighted (and nonweighted) Sobolev spaces. The goal of the present paper is to prove the existence of solutions in weighted Hölder spaces. Furthermore, we will obtain regularity results for the solutions.

There is an extensive bibliography concerning elliptic boundary value problems in domains with edges (see e.g. the references in the books of Grisvard [10], Dauge [3], Nazarov and Plamenevskiĭ [25]). Moreover, many works deal with boundary value problems in Lipschitz domains. We mention here the papers of Jerison and Kenig [11], Kenig [12] and for the Stokes system the papers of Fabes, Kenig and Verchota [7], Brown and Shen [2], Deuring and von Wahl [5], Dindos and Mitrea [6]. However, most of the works in this field deal with solutions in Sobolev spaces with or without weight. Concerning Schauder estimates for solutions of boundary value problems in domains with edges, we mention the papers by Maz'ya and Plamenevskiĭ [16, 17], where boundary value problems for elliptic differential equations of arbitrary order were studied. The results obtained in [16, 17] are applicable, e.g., to the Dirichlet problem but not to the Stokes system with boundary conditions (i)–(iv). In [19], weighted L_p and Schauder estimates were obtained for solutions of the Stokes system with Dirichlet condition (i) and free surface condition (iii) on parts of the boundary. Weighted L_p and Schauder estimates for solutions to the Neumann problem in a domain with nonintersecting edges were proved in the preprint [26] by Solonnikov. For the Neumann problem to second order systems

$$\sum_{i,j=1}^{3} A_{i,j}\partial_{x_i}\partial_{x_j}u = f$$

we refer to our paper [21]. The boundary value problems considered in [16, 17, 19] allow to use weighted Hölder spaces $\mathcal{N}^{l,\sigma}_{\beta,\delta}$ with "homogeneous" norms. However, the Neumann problem or in general the mixed problem with boundary conditions (i)–(iv) requires the use of weighted spaces $C^{l,\sigma}_{\beta,\delta}$ with "inhomogeneous" norms. This makes the consideration of the boundary value problem more difficult. On the other hand, in some cases (e.g. the Dirichlet problem in convex polyhedral domains), the results can be improved when considering solutions in weighted spaces with inhomogeneous norms. We also note that on the set of functions vanishing in a neighborhood of the corners of the domain, the $C^{l,\sigma}_{\beta,\delta}$ norm with $\delta = 0$ is equivalent to the norm in the non-weighted Hölder space $C^{l,\sigma}$.

The largest part of the paper (Sections 2–4) concerns the boundary value problem for the Stokes system in a dihedron $\mathcal{D}$ and in polyhedral cone $\mathcal{K}$ with sides $\Gamma_1,\dots,\Gamma_N$ and edges $M_1,\dots,M_N$. We do not a priori suppose in this paper that the cone $\mathcal{K}$ is Lipschitz. Section 3 deals with the existence of solutions $(u,p)\in C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3\times C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$ of the boundary value problem if $f\in C^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $g\in C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$, and the boundary data h_j, ϕ_j are from the corresponding trace spaces. Here, $C^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ is a weighted Hölder space with weight parameters $\beta\in\mathbb{R}$, $\delta=(\delta_1,\dots,\delta_N)\in[0,\infty)^N$. In the special case $\delta_1=\cdots=\delta_N=\delta$, the norm in this space is given by

$$\|u\|_{C^{l,\sigma}_{\beta,\delta}(\mathcal{K})} = \sum_{|\alpha|\le l}\sup_{x\in\mathcal{K}}|x|^{\beta-l-\sigma+|\alpha|}\Big(\frac{r(x)}{|x|}\Big)^{\max(0,\delta-l-\sigma+|\alpha|)}|\partial_x^\alpha u(x)|$$
$$+\sum_{|\alpha|=l-k}\sup_{|x-y|<|x|/2}|x|^{\beta-\delta}\frac{\big|\partial_x^\alpha u(x)-\partial_y^\alpha u(y)\big|}{|x-y|^{k+\sigma-\delta}}+\sum_{|\alpha|=l}\sup_{|x-y|<r(x)/2}|x|^{\beta-\delta}\,r(x)^\delta\,\frac{\big|\partial_x^\alpha u(x)-\partial_y^\alpha u(y)\big|}{|x-y|^\sigma},$$

where $r(x)$ denotes the distance of x to the set $\mathcal{S}=\{0\}\cup M_1\cup\cdots\cup M_N$ and k is the integral part of $\delta-\sigma+1$. It is shown in Section 3 that there is a uniquely determined solution if g and the boundary data satisfy certain compatibility conditions on the edges, the line $\operatorname{Re}\lambda=2+\sigma-\beta$ is free of eigenvalues of a certain operator pencil $\mathfrak{A}(\lambda)$, and the components δ_k of δ are such that $\delta_k\ge 0$, $\delta_k-\sigma$ not integer, and $2-\mu_k<\delta_k-\sigma<2$ for $k=1,\dots,N$. Here μ_k are certain positive numbers depending on the angle θ_k at the edge M_k. For example, in the case of the Dirichlet problem, we have $\mu_k=\pi/\theta_k$ if $\theta_k<\pi$, while μ_k is the smallest positive solution of the equation $\sin(\mu\theta_k)+\mu\sin\theta_k=0$ if $\theta_k>\pi$. Estimates for the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ can be found e.g. in [4, 13, 14, 18]. It is further shown that the solution $(u,p)\in C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3\times C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$ belongs to $C^{l,\sigma'}_{\beta',\delta'}(\mathcal{K})^3\times C^{l-1,\sigma'}_{\beta',\delta'}(\mathcal{K})$ if $f\in C^{l-2,\sigma'}_{\beta',\delta'}(\mathcal{K})^3$, $g\in C^{1,\sigma'}_{\beta',\delta'}(\mathcal{K})$, the boundary data are from the corresponding trace spaces, the closed strip between the lines $\operatorname{Re}\lambda=2+\sigma-\beta$ and $\operatorname{Re}\lambda=l+\sigma'-\beta'$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$, and the components of δ' are such that $\delta'_k\ge 0$, $\delta_k-\sigma$ not integer, and $l-\mu_k<\delta'_k-\sigma'<l$. In Section 4, we deal with weak solutions of the boundary value problem. We prove that under conditions analogous to those in Section 3, there exists a unique weak solution $(u,p)\in C^{1,\sigma}_{\beta,\delta}(\mathcal{K})^3\times C^{0,\sigma}_{\beta,\delta}(\mathcal{K})$.

The boundary value problem for the Stokes system in a bounded polyhedral domain $\mathcal{G}$ is studied in the last Section 5. As an example, we consider the weak solution $(u,p)\in W^1(\mathcal{G})^3\times L_2(\mathcal{G})$ of the Dirichlet problem

$$-\Delta u+\nabla p=f,\quad -\nabla\cdot u=g\ \text{ in }\mathcal{G},\quad u=0\ \text{ on }\Gamma_j,\ j=1,\dots,N,$$

in a polyhedron $\mathcal{G}$ with sides Γ_j and edges M_k. We denote by θ_k the angle at the edge M_k. As a consequence of our results, we obtain for example the following regularity assertion in a neighborhood of an edge point $\xi\in M_k$.

> If $f\in C^{l-2,\sigma}$, $g\in C^{l-1,\sigma}$ in a neighborhood of ξ, $g|_{M_k}=0$, and $l+\sigma<\pi/\theta_k$, then $u\in C^{l,\sigma}$ and $p\in C^{l-1,\sigma}$ in a neighborhood of ξ.

This result is also true for $l = 1$. This means that in the case $\theta_k < \pi$, we obtain $u \in C^{1,\sigma}$, $p \in C^{0,\sigma}$ in a neighborhood of ξ provided $\sigma < (\pi - \theta_k)/\theta_k$ and f, g satisfy the above conditions. Another result is the following.

> If $f \in C^{l-2,\sigma}_{l,l}(\mathcal{G})^3$ and $g \in C^{l-1,\sigma}_{l,l}(\mathcal{G})$, where $0 < \sigma < \min(\mathrm{Re}\,\Lambda_j, \mu_k)$, then $(u,p) \in C^{l,\sigma}_{l,l}(\mathcal{G})^3$ $\times C^{l-1,\sigma}_{l,l}(\mathcal{G})$. In particular, we have $u \in C^{0,\sigma}(\mathcal{G})^3$. If the polyhedron $\mathcal{G}$ is convex, then this result is true for arbitrary $\sigma \in (0,1)$.

Here Λ_j denotes the eigenvalue of the pencil $\mathfrak{A}_j(\lambda)$ with smallest positive real part. For convex polyhedrons, we have $\Lambda_j = 1$. Then there is also the following result.

> Let $f \in C^{-1,\sigma}(\mathcal{G})^3$, $g \in C^{0,\sigma}(\mathcal{G})$, where ε is a sufficiently small positive number, and let $g|_{M_k=0}$ for all k. Then the solution (u,p) belongs to $C^{1,\sigma}(\mathcal{G})^3 \times C^{0,\sigma}(\mathcal{G})$.

Other examples are given at the end of Section 5. In a forthcoming paper, we will apply the results to the nonlinear Navier-Stokes system.

1 Weighted Hölder spaces

1.1 Weighted Hölder spaces in an angle an in a dihedron

Let K be the angle $\{(x_1,x_2) \in \mathbb{R}^2 :\ 0 < r < \infty,\ -\theta/2 < \varphi < \theta/2\}$, where r, φ are the polar coordinates of $x' = (x_1,x_2)$, and let $\gamma^\pm : \varphi = \pm\theta/2$ be the sides of K. Furthermore, let

$$\mathcal{D} = K \times \mathbb{R} = \{x = (x', x_3) : x' = (x_1,x_2) \in K,\ x_3 \in \mathbb{R}\}$$

be a dihedron with sides $\Gamma^\pm = \gamma^\pm \times \mathbb{R}$ and edge M. For arbitrary integer $l \ge 0$ and real δ, σ, $0 < \sigma < 1$, we define $\mathcal{N}^{l,\sigma}_\delta(K)$ as the space of all functions with continuous derivatives up to order l on $\bar{K}\backslash\{0\}$ such that

$$\|u\|_{\mathcal{N}^{l,\sigma}_\delta(K)} = \sum_{|\alpha|\le l} \sup_{x'\in K} |x'|^{\delta-l-\sigma+|\alpha|} |\partial^\alpha_{x'} u(x')| + \sum_{|\alpha|=l} \sup_{\substack{x',y'\in K \\ |x'-y'|<|x'|/2}} |x'|^\delta \frac{|\partial^\alpha_{x'} u(x') - \partial^\alpha_{y'} u(y')|}{|x'-y'|^\sigma} < \infty.$$

Analogously, we define the weighted Hölder space $\mathcal{N}^{l,\sigma}_\delta(\mathcal{D})$ as the set of all functions with continuous derivatives up to order l on $\bar{\mathcal{D}}\backslash M$ such that

$$\|u\|_{\mathcal{N}^{l,\sigma}_\delta(\mathcal{D})} = \sum_{|\alpha|\le l} \sup_{x\in\mathcal{D}} |x'|^{\delta-l-\sigma+|\alpha|} |\partial^\alpha_x u(x)| + \langle u\rangle_{l,\sigma,\delta;\mathcal{D}} < \infty, \tag{1.1}$$

where

$$\langle u\rangle_{l,\sigma,\delta;\mathcal{D}} = \sum_{|\alpha|=l} \sup_{\substack{x,y\in\mathcal{D} \\ |x-y|<|x'|/2}} |x'|^\delta \frac{|\partial^\alpha_x u(x) - \partial^\alpha_y u(y)|}{|x-y|^\sigma}.$$

Note that $\mathcal{N}^{l,\sigma}_\delta(\mathcal{D})$ is continuously imbedded into $C^{l-k,k-\delta+\sigma}(\mathcal{D})$ if $k-1 \le \delta-\sigma < k$, where k is a nonnegative integer, $k \le l$ (see [21]). Here $C^{l,\sigma}(\mathcal{D})$ denotes the nonweighted Hölder space with the norm

$$\|u\|_{C^{l,\sigma}(\mathcal{D})} = \sum_{|\alpha|\le l} \sup_{x\in\mathcal{D}} |\partial^\alpha u(x)| + \sum_{|\alpha|=l} \sup_{\substack{x,y\in\mathcal{D} \\ |x-y|<1}} \frac{\big|\partial^\alpha u(x) - \partial^\alpha u(y)\big|}{|x-y|^\sigma}$$

An equivalent norm is given in the following lemma.

Lemma 1.1 *The norm in $C^{l,\sigma}(\mathcal{D})$ is equivalent to*

$$\begin{aligned} \|u\| &= \sum_{|\alpha|\le l} \sup_{x\in\mathcal{D}} |\partial^\alpha u(x)| + \sum_{|\alpha|=l} \Big(\sup_{x'} \sup_{|x_3-y_3|<1} \frac{\big|\partial^\alpha u(x',x_3) - \partial^\alpha u(x',y_3)\big|}{|x_3-y_3|^\sigma} \\ &\qquad + \sup_{x_3} \sup_{|x'-y'|<|x'|/2} \frac{\big|\partial^\alpha u(x',x_3) - \partial^\alpha u(y',x_3)\big|}{|x'-y'|^\sigma} \Big). \end{aligned}$$

P r o o f. It suffices to prove the lemma for $l = 0$. Obviously, the norm in $C^{0,\sigma}(\mathcal{D})$ is equivalent to

$$\|u\| \;=\; \sup_{x\in\mathcal{D}} |u(x)| + \sup_{x'} \sup_{|x_3-y_3|<1} \frac{\big|u(x',x_3)-u(x',y_3)\big|}{|x_3-y_3|^\sigma} + \sup_{x_3} \sup_{|x'-y'|<1} \frac{\big|u(x',x_3)-u(y',x_3)\big|}{|x'-y'|^\sigma}\Big).$$

We show, that there exists a constant c independent of u such that

$$\sup_{x_3} \sup_{x',y'\in K} \frac{|u(x',x_3)-u(y',x_3)|}{|x'-y'|^\sigma} \le c \sup_{x_3} \sup_{\substack{x',y'\in K\\ |x'-y'|<|x'|/2}} \frac{|u(x',x_3)-u(y',x_3)|}{|x'-y'|^\sigma} \tag{1.2}$$

We denote the right-hand side of (1.2) by $A_\sigma(u)$. Let x', y' be arbitrary points in K such that $|x'-y'| > |x'|/2$, and let $x_3 \in \mathbb{R}$. We put $\xi_n = 2^{-n}x'$. Then

$$\big|u(\xi_n,x_3)-u(\xi_{n+1},x_3)\big| \le c_0\, |\xi_n-\xi_{n+1}|^\sigma = c_0\, |x'|^\sigma\, 2^{-(n+1)\sigma}, \quad \text{where } c_0 = A_\sigma(u).$$

Consequently,

$$\big|u(x',x_3)-u(0,x_3)\big| \le \sum_{n=0}^\infty \big|u(\xi_n,x_3)-u(\xi_{n+1},x_3)\big| \le c_0\, |x'|^\sigma \sum_{n=0}^\infty 2^{-(n+1)\sigma} = \frac{c_0}{2^\sigma-1}\, |x'|^\sigma$$

and, analogously,

$$\big|u(y',x_3)-u(0,x_3)\big| \le \frac{c_0}{2^\sigma-1}\, |y'|^\sigma$$

Since $|x'| < 2|x'-y'|$ and $|y'| < 3|x'-y'|$, it follows that

$$\big|u(x',x_3)-u(y',x_3)\big| \le \frac{c_0}{2^\sigma-1}\, \big(|x'|^\sigma + |y'|^\sigma\big) \le \frac{2^\sigma+3^\sigma}{2^\sigma-1}\, A_\sigma(u)\, |x'-y'|^\sigma$$

what proves (1.2). The result follows. □

Let $0 \le \delta < l+\sigma$ and $0 < \sigma \le 1$. Then by $C^{l,\sigma}_\delta(\mathcal{D})$, we denote the weighted Hölder space with the norm

$$\|u\|_{C^{l,\sigma}_\delta(\mathcal{D})} = \|u\|_{C^{l-k,k-\delta+\sigma}(\mathcal{D})} + \sum_{|\alpha|=l-k+1}^{l} \sup_{x\in\mathcal{D}} |x'|^{\delta-l-\sigma+|\alpha|} |\partial_x^\alpha u(x)| + \langle u\rangle_{l,\sigma,\delta;\mathcal{D}}\,,$$

where $k = [\delta-\sigma]+1$, $[s]$ denotes the greatest integer less or equal to s. In the case $\delta \ge l+\sigma$, we set $C^{l,\sigma}_\delta(\mathcal{D}) = \mathcal{N}^{l,\sigma}_\delta(\mathcal{D})$.

Every function $u \in C^{l,\sigma}_\delta(\mathcal{D})$, $0 \le \delta < l+\sigma$, is continuous in $\bar{\mathcal{D}}$. The restriction of u to M belongs to the Hölder space $C^{l-k,k-\delta+\sigma}(M)$, where $k = [\delta-\sigma]+1$. Conversely, every function $f \in C^{l-k,k-\delta+\sigma}(M)$ can be extended to a function $u \in C^{l,\sigma}_\delta(\mathcal{D})$. For this, we consider the operator

$$(Ef)(x',x_3) \stackrel{def}{=} \int_0^1 f(x_3+tr)\, \psi(t)\, dt, \tag{1.3}$$

where $\psi \in C_0^\infty(\mathbb{R})$ is a given function with support in $(0,1)$ satisfying the condition

$$\int_0^1 \psi(t)\, dt = 1, \qquad \int_0^1 t^j\, \psi(t)\, dt = 0 \;\text{ for } j = 1,2,\ldots,l-k.$$

In [21, Le.2.7] it is shown that E realizes a continuous mapping $C^{l-k,k-\delta+\sigma}(M) \to C^{l,\sigma}_\delta(\mathcal{D})$,

$$(Ef)|_M = f \;\text{ for arbitrary } f \in C^{l-k,k-\delta+\sigma}(M),\; k = [\delta-\sigma]+1 \le l$$

and

$$\partial_{x_i} Ef \in \mathcal{N}^{l,\sigma}_{\delta+1}(\mathcal{D}), \quad \|\partial_{x_i} Ef\|_{\mathcal{N}^{l,\sigma}_{\delta+1}(\mathcal{D})} \le c\, \|f\|_{C^{l-k,k-\delta+\sigma}(M)} \;\text{ for } i = 1,2.$$

For the following lemma we refer to [21, Le.2.8,Le.2.9].

Lemma 1.2 *Let $u \in C_\delta^{l,\sigma}(\mathcal{D})$, where $\delta \ge 0$, $k-1 < \delta - \sigma < k$, $k \in \{0,1,\dots,l\}$.*

1) *Furthermore, let $f_{i,j} = \partial_{x_1}^i \partial_{x_2}^j u|_M$ for $i+j \le l-k$ and $u_{i,j}(x) = \chi(|x'|)\,(Ef_{i,j})(x)$, where χ is a smooth cut-off function on $[0,\infty)$, $\operatorname{supp}\chi \subset [0,2)$, $\chi = 1$ on $[0,1]$. Then u admits the decomposition*

$$u = v + w, \quad \text{where } v \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D}),\ w = \sum_{i+j\le l-k} \frac{1}{i!\,j!}\, u_{i,j}\, x_1^i x_2^j \in C_{\delta+m}^{l+m,\sigma}(\mathcal{D}),\ m = 0,1,2,\dots.$$

2) *For the inclusion $u \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})$ it is necessary and sufficient that $\partial^\alpha u = 0$ on M for $|\alpha| \le l-k$.*

1.2 Weighted Hölder spaces in a polyhedral cone

Let

$$\mathcal{K} = \{x \in \mathbb{R}^3 :\ x/|x| \in \Omega\} \tag{1.4}$$

be a polyhedral cone in $\mathbb{R}^3$ whose boundary consists of plane sides Γ_k and edges M_k, $j = 1,\dots,N$. We denote by $r_k(x)$ the distance to the edge M_k and by $r(x)$ the distance to the set $\mathcal{S} = M_1 \cup \dots \cup M_N \cup \{0\}$. The subset $\{x \in \mathcal{K} :\ r_j(x) < 3r(x)/2\}$ is denoted by $\mathcal{K}_j$. Note that there are the inequalities

$$c_1\,|x| \prod_{k=1}^N \frac{r_k(x)}{|x|} \le r(x) \le c_2\,|x| \prod_{k=1}^N \frac{r_k(x)}{|x|} \tag{1.5}$$

with positive constants c_1, c_2 independent of $x \in \mathcal{K}$.

Let l be a nonnegative integer, $\beta \in \mathbb{R}$, and $\delta = (\delta_1,\dots,\delta_N) \in \mathbb{R}^N$. We define the space $\mathcal{N}_{\beta,\delta}^l(\mathcal{K})$ as the set of all l times continuously differentiable functions in $\bar{\mathcal{K}}\backslash\mathcal{S}$ such that

$$\|u\|_{\mathcal{N}_{\beta,\delta}^l(\mathcal{K})} = \sum_{|\alpha|\le l} \sup_{x\in\mathcal{K}} |x|^{\beta-l+|\alpha|} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k - l + |\alpha|} \big|\partial_x^\alpha u(x)\big|\, dx < \infty.$$

The weighted Hölder space $\mathcal{N}_{\beta,\delta}^{l,\sigma}(\mathcal{K})$ is defined as the set of all l times continuously differentiable functions on $\overline{\mathcal{K}}\backslash\mathcal{S}$ with finite norm

$$\begin{aligned}
\|u\|_{\mathcal{N}_{\beta,\delta}^{l,\sigma}(\mathcal{K})} &= \sum_{|\alpha|\le l} \sup_{x\in\mathcal{K}} |x|^{\beta-l-\sigma+|\alpha|} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k-l-\sigma+|\alpha|} \big|\partial_x^\alpha u(x)\big| \\
&\quad + \sum_{|\alpha|=l} \sup_{|x-y|<r(x)/2} |x|^\beta \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k} \frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^\sigma}\,.
\end{aligned} \tag{1.6}$$

Furthermore, the space $C_{\beta,\delta}^{l,\sigma}(\mathcal{K})$ is defined for nonnnegative δ_k, $k = 1,\dots,N$, as the set of all l times continuously differentiable functions on $\bar{\mathcal{K}}\backslash\mathcal{S}$ with finite norm

$$\begin{aligned}
\|u\|_{C_{\beta,\delta}^{l,\sigma}(\mathcal{K})} &= \sum_{|\alpha|\le l} \sup_{x\in\mathcal{K}} |x|^{\beta-l-\sigma+|\alpha|} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-l-\sigma+|\alpha|)} \big|\partial_x^\alpha u(x)\big| \\
&\quad + \sum_{j:\,k_j\le l}\ \sum_{|\alpha|=l-k_j}\ \sup_{\substack{x,y\in\mathcal{K}_j \\ |x-y|<|x|/2}} |x|^{\beta-\delta_j} \frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^{k_j+\sigma-\delta_j}} \\
&\quad + \sum_{|\alpha|=l} \sup_{\substack{x,y\in\mathcal{K} \\ |x-y|<r(x)/2}} |x|^\beta \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k} \frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^\sigma}\,,
\end{aligned} \tag{1.7}$$

where $k_j = [\delta_j - \sigma] + 1$, $[s]$ denotes the greatest integer less or equal to s. Replacing

$$\sum_{j=1}^N \sum_{|\alpha|=l-k_j} \sup_{\substack{x,y\in\mathcal{K}_j \\ |x-y|<|x|/2}} |x|^{\beta-\delta_j} \frac{\big|\partial^\alpha u(x) - \partial^\alpha u(y)\big|}{|x-y|^{k_j+\sigma-\delta_j}}$$

in (1.6) by

$$\sum_{j=1}^{N}\sum_{|\alpha|=l-k_j}\Big(\sup_{\substack{x,y\in\mathcal{K}_j\\ |x-y|<r(x)/2}}|x|^{\beta-\delta_j}\frac{\big|\partial^\alpha u(x)-\partial^\alpha u(y)\big|}{|x-y|^{k_j+\sigma-\delta_j}}+\sup_{x\in\mathcal{K}_j}\sup_{1/2<t<3/2}|x|^{\beta-\delta_j}\frac{\big|\partial^\alpha u(x)-\partial^\alpha u(tx)\big|}{|x-tx|^{k_j+\sigma-\delta_j}}\Big),$$

we obtain an equivalent norm in $C^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ (cf. Lemma 1.1). Obviously, $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ is a subset of $C^{l,\sigma}_{\beta,\delta}(\mathcal{K})$. If $\delta_k \ge l+\sigma$ for $k=1,\dots,N$, then both spaces coincide. Furthermore, there are the following continuous imbeddings:

$$\mathcal{N}^{l+1}_{\beta-\sigma+1,\delta-\sigma+1}(\mathcal{K}) \subset \mathcal{N}^{l,\sigma}_{\beta,\vec{\delta}}(\mathcal{K}) \subset \mathcal{N}^{l}_{\beta-\sigma,\delta-\sigma}(\mathcal{K}),$$
$$C^{l,\sigma}_{\beta,\delta}(\mathcal{K}) \subset C^{l',\sigma'}_{\beta',\delta'}(\mathcal{K}) \ \text{ if } l+\sigma \ge l'+\sigma',\ \beta-l-\sigma=\beta'-l'-\sigma',\ \delta_k-l-\sigma\le\delta'_k-l'-\sigma'.$$

The trace spaces for $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ and $C^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ on Γ_j are denoted by $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\Gamma_j)$ and $C^{l,\sigma}_{\beta,\delta}(\Gamma_j)$, respectively.

Finally, we introduce the following notation. If $\delta\in\mathbb{R}^N$ and $s\in\mathbb{R}$, then by $\mathcal{N}^{l,\sigma}_{\beta,\delta+s}(\mathcal{K})$ and $C^{l,\sigma}_{\beta,\delta+s}(\mathcal{K})$ we mean the spaces $\mathcal{N}^{l,\sigma}_{\beta,\delta'}(\mathcal{K})$ and $C^{l,\sigma}_{\beta,\delta'}(\mathcal{K})$ with $\delta'=(\delta_1+s,\dots,\delta_N+s)$.

2 The problem in a dihedron

Let $\mathcal{D}$ be the dihedron introduced in the previous section. We consider a boundary value problem for the Stokes system, where on each of the sides $\Gamma^\pm$ one of the boundary conditions (i)–(iv) is given. Let $n^\pm=(n_1^\pm,n_2^\pm,0)$ be the exterior normal to $\Gamma^\pm$, $\varepsilon_n^\pm(u)=\varepsilon(u)\,n^\pm$ and $\varepsilon_{nn}^\pm(u)=\varepsilon_n^\pm(u)\cdot n^\pm$. Furthermore, let $d^\pm\in\{0,1,2,3\}$ be integer numbers characterizing the boundary conditions on Γ^+ and Γ^-, respectively. We put

- $S^\pm u=u \quad \text{for } d^\pm=0,$
- $S^\pm u=u-(u\cdot n^\pm)n^\pm, \quad N^\pm(u,p)=-p+2\varepsilon^\pm_{nn}(u) \quad \text{for } d^\pm=1,$
- $S^\pm u=u\cdot n^\pm, \quad N^\pm(u,p)=\varepsilon^\pm_n(u)-\varepsilon^\pm_{nn}(u)\,n^\pm \quad \text{for } d^\pm=2$
- $N^\pm(u,p)=-pn^\pm+2\varepsilon^\pm_n(u) \quad \text{for } d^\pm=3$

and consider the boundary value problem

$$-\Delta u+\nabla p=f,\quad -\nabla\cdot u=g \quad \text{in } \mathcal{D}, \tag{2.8}$$
$$S^\pm u=h^\pm,\quad N^\pm(u,p)=\phi^\pm \quad \text{on } \Gamma^\pm. \tag{2.9}$$

Here the condition $N^\pm(u,p)=\phi^\pm$ is absent in the case $d^\pm=0$, while the condition $S^\pm u=h^\pm$ is absent in the case $d^\pm=3$.

2.1 Reduction to homogeneous boundary conditions

Lemma 2.1 *Let $h^\pm\in\mathcal{N}^{l,\sigma}_\delta(\Gamma^\pm)^{3-d^\pm}$, $\phi^\pm\in\mathcal{N}^{l-1,\sigma}_\delta(\Gamma^\pm)^{d^\pm}$, $l\ge1$, be given. Then there exists a vector function $u\in\mathcal{N}^{l,\sigma}_\delta(\mathcal{D})^3$ such that $S^\pm u=h^\pm$ and $N^\pm(u,0)=\phi^\pm$ on $\Gamma^\pm$. The norm of u can be estimated by the norms of $h^\pm$ and $\phi^\pm$.*

The analogous result in the space $C^{l,\sigma}_\delta$ holds only under additional assumptions on the boundary data. If $u\in C^{l,\sigma}_\delta(\mathcal{D})^3$, $\delta<l+\sigma$, then there exists the trace $u|_M\in C^{l-k,k-\delta+\sigma}(M)^3$, $k=[\delta-\sigma]+1$, and from the boundary conditions (2.9) it follows that $S^\pm u|_M=h^\pm|_M$. Here S^+ and S^- are considered as operators on $C^{l-k,k-\delta+\sigma}(M)^3$. Consequently, the boundary data h^+ and h^- must satisfy the compatibility condition $\big(h^+|_M\,,h^-|_M\big)\in R(T)$, where $R(T)$ is the range of the operator $T=(S^+,S^-)$. This condition can be also written in the form

$$A^+\,h^+|_M=A^-\,h^-|_M\,, \tag{2.10}$$

where A^+,A^- are certain constant matrices. For example in the case of the Dirichlet problem, A^+ and A^- are the identity matrices.

Lemma 2.2 *Let $h^\pm \in C_\delta^{l,\sigma}(\Gamma^\pm)^{3-d^\pm}$ and $\phi^\pm \in C_\delta^{l-1,\sigma}(\Gamma^\pm)^{d^\pm}$, $l \ge 1$, $l+\sigma-1 < \delta < l+\sigma$. Suppose that h^+ and h^- satisfy the compatibility condition* (2.10) *on M. Then there exists a vector function $u \in C_\delta^{l,\sigma}(\mathcal{D})^3$ such that $S^\pm u = h^\pm$, $N^\pm(u,0) = \phi^\pm$ on $\Gamma^\pm$. The norm of u can be estimated by the norms of $h^\pm$ and $\phi^\pm$.*

P r o o f. By (2.10), there exists a vector function $\psi \in C^{0,l+\sigma-\delta}(M)^3$ such that $S^\pm\psi = h^\pm|_M$. Let $v \in C_\delta^{l,\sigma}(\mathcal{D})^3$ be an extension of ψ. Then the trace of $h^\pm - S^\pm v|_{\Gamma^\pm}$ on M is equal to zero and, consequently, $h^\pm - S^\pm v|_{\Gamma^\pm} \in \mathcal{N}_\delta^{l,\sigma}(\Gamma^\pm)^{3-d^\pm}$ (cf. Lemma 1.1). Furthermore, $\phi^\pm - N^\pm(v,0)|_{\Gamma^\pm} \in C_\delta^{l-1,\sigma}(\Gamma^\pm)^{d^\pm} \subset \mathcal{N}_\delta^{l-1,\sigma}(\Gamma^\pm)^{d^\pm}$. Thus, according to Lemma 2.1, there exists a function $w \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})^3$ such that $S^\pm w = h^\pm - S^\pm v$ and $N^\pm(w,0) = \phi^\pm - N^\pm(v,0)$ on $\Gamma^\pm$. Then $u = v + w$ has the desired properties. □

Now let $g \in C_\delta^{l-1,\sigma}(\mathcal{D})$, $h^\pm \in C_\delta^{l,\sigma}(\Gamma^\pm)^{3-d^\pm}$, and $\phi \in C_\delta^{l,\sigma}(\Gamma^\pm)^{3-d^\pm}$, $l+\sigma-2 < \delta < l+\sigma-1$. Then the traces of $g, h^\pm, \partial_r h^\pm$ and $\phi^\pm$ on M exist. We suppose that there is a pair $(u,p) \in C_\delta^{l,\sigma}(\mathcal{D})^3 \times C_\delta^{l-1,\sigma}(\mathcal{D})$ such that

$$S^\pm u = h^\pm, \quad N^\pm(u,p) = \phi^\pm \ \text{ on } \Gamma^\pm \quad \text{and} \quad -\nabla\cdot u = g \ \text{ on } M \tag{2.11}$$

We put

$$b = u|_M\,, \quad c = (\partial_{x_1}u)|_M\,, \quad d = (\partial_{x_2}u)|_M \quad \text{and} \quad q = p|_M\,.$$

Then from the equations $S^\pm u = h^\pm$ on $\Gamma^\pm$ it follows that $S^\pm\partial_r u = \partial_r h^\pm$ on $\Gamma^\pm$, and therefore,

$$S^\pm b = h^\pm|_M\,, \tag{2.12}$$

$$S^\pm\big(c\cos\tfrac{\theta}{2} \pm d\sin\tfrac{\theta}{2}\big) = (\partial_r h^\pm)|_M\,. \tag{2.13}$$

Moreover $-\nabla\cdot u = g$ on M if and only if

$$c_1 + d_2 + \partial_{x_3}b_3 = -g|_M\,. \tag{2.14}$$

Obviously, the trace of $N^\pm(u,p)$ on M can be written as a linear form $M^\pm(c,d,\partial_{x_3}b,q)$. Thus, from $N^\pm(u,p) = \phi^\pm$ on $\Gamma^\pm$ it follows that

$$M^\pm(c,d,\partial_{x_3}b,q) = \phi^\pm|_M\,. \tag{2.15}$$

Lemma 2.3 *Suppose that $g \in C_\delta^{l-1,\sigma}(\mathcal{D})$, $h^\pm \in C_\delta^{l,\sigma}(\Gamma^\pm)^{3-d^\pm}$, and $\phi \in C_\delta^{l-1,\sigma}(\Gamma^\pm)^{3-d^\pm}$, $l \ge 1$, $l+\sigma-2 < \delta < l+\sigma-1$, are such that the system* (2.12)–(2.15) *with the unknowns b,c,d,q is solvable. Then there exists a pair $(u,p) \in C_\delta^{l,\sigma}(\mathcal{D})^3 \times C_\delta^{l-1,\sigma}(\mathcal{D})$ satisfying* (2.11).

P r o o f. Let $b \in C^{1,l-1-\delta+\sigma}(M)^3$, $c,d \in C^{0,l-1-\delta+\sigma}(M)^3$, $q \in C^{0,l-1-\delta+\sigma}(M)$ solve the linear system (2.12)–(2.15). We set

$$v = Eb + x_1\,Ec + x_2\,Ed, \quad p = Eq,$$

where E is the extension operator (1.3). Then $v \in C_\delta^{l,\sigma}(\mathcal{D})^3$, $p \in C_\delta^{l-1,\sigma}(\mathcal{D})$, $v|_M = b$, $\partial_{x_1}v|_M = c$, $\partial x_2 v|_M = d$, and $p|_M = q$. Consequently,

$$S^\pm v|_M = h^\pm|_M, \quad \partial_r S^\pm v|_M = \partial_r h^\pm|_M, \quad -\nabla\cdot v|_M = g|_M, \quad N^\pm(v,p)|_M = \phi^\pm|_M.$$

From this and from Lemma 1.1 we conclude that $S^\pm v - h^\pm \in \mathcal{N}_\delta^{l,\sigma}(\Gamma^\pm)^{3-d^\pm}$ and $N^\pm(v,p) - \phi^\pm \in \mathcal{N}_\delta^{l-1,\sigma}(\Gamma^\pm)^{d^\pm}$. Thus, by Lemma 2.1, there exists a vector function $w \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})^3$ such that

$$S^\pm w = h^\pm - S^\pm v, \quad N^\pm(w,0) = \phi^\pm \ \text{ on } \Gamma^\pm.$$

Then $(u,p) = (v+w,p)$ satisfies (2.11). □

Remark 2.1 The condition of Lemma 2.3 is always satisfied if $d^+ + d^-$ is odd, h^+ and h^- satisfy the compatibility condition (2.10) and $\sin 2\theta \ne 0$ for $d^+ + d^- = 3$, $\cos\theta\cos 2\theta \ne 0$ for $d^+ + d^- \in \{1,5\}$. If $d^+ + d^-$ is even, then additionally to (2.10) another compatibility condition must be satisfied. For example, in the case of the Dirichlet problem, $\theta \neq \pi$, $\theta \neq 2\pi$, for the validity of Lemma 2.3 it is necessary and sufficient that

$$h^+|_M = h^-|_M, \quad n^- \cdot \partial_r h^+|_M + n^+ \cdot \partial_r h^-|_M = (g|_M + \partial_{x_3} h_3^+|_M) \sin\theta.$$

In the case $d^- = 0$, $d^+ = 2$, $\theta \neq \pi/2$, $\theta \neq 3\pi/2$, the data h^+, h^-, ϕ^+ and g must satisfy the compatibility conditions $h^- \cdot n^+ = h^+$ and

$$\partial_r h^+ \cos 2\theta - (2n^+ \cos\theta + n^-)\, \partial_r h^- + 2\sin^2\theta\, (\phi_1^+ \cos\theta/2 + \phi_2^+ \sin\theta/2) + \frac{1}{2}(g + \partial_{x_3} h_3^-) \sin 2\theta = 0,$$

and in the case of the Neumann problem, $\theta \neq \pi$, $\theta \neq 2\pi$, the compatibility condition $\phi^+ \cdot n^- = \phi^- \cdot n^+$ on M is necessary and sufficient for the validity of Lemma 2.3, see [22].

Lemma 2.4 *Let $f \in C_\delta^{l-2,\sigma}(\mathcal{D})^3$, $g \in C_\delta^{l-1,\sigma}(\mathcal{D})$, $h^\pm \in C_\delta^{l,\sigma}(\Gamma^\pm)^{3-d^\pm}$, and $\phi \in C_\delta^{l-1,\sigma}(\Gamma^\pm)^{3-d^\pm}$, $l \geq 2$, $0 \leq \delta < l + \sigma - 1$, $\delta - \sigma$ not integer. Suppose that g, $h^\pm$ and $\phi^\pm$ are such that the system (2.12)–(2.15) with the unknowns b, c, d, q is solvable. In the case $l + \sigma - \delta > 2$ we assume furthermore that for $k = 2, 3, \dots, [l + \sigma - \delta]$ there do not exist homogeneous polynomials*

$$u = \sum_{i+j=k} c_{i,j}\, x_1^i\, x_2^j, \quad p = \sum_{i+j=k-1} d_{i,j}\, x_1^i\, x_2^j$$

satisfying $-\Delta u + \nabla p = 0$, $-\nabla \cdot u = 0$ in $\mathcal{D}$ and the homogeneous boundary conditions $S^\pm u = 0$, $N^\pm(u,p) = 0$ on $\Gamma^\pm$. Then there exists a pair $(u,p) \in C_\delta^{l,\sigma}(\mathcal{D})^3 \times C_\delta^{l-1,\sigma}(\mathcal{D})$ such that

$$\Delta u - \nabla p + f \in \mathcal{N}_{\beta,\delta}^{l-2,\sigma}(\mathcal{D})^3, \quad \nabla \cdot u + g \in \mathcal{N}_{\beta,\delta}^{l-1,\sigma}(\mathcal{D}), \quad S^\pm u|_{\Gamma^\pm} = h^\pm, \quad N^\pm(u,p)|_{\Gamma^\pm} = \phi^\pm.$$

P r o o f. Let $s < l + \sigma - \delta < s + 1$, where $s \in \{1, \dots, l\}$, and let k be an integer, $1 \leq k \leq s$. We show (by induction in k) that there exists a pair $(u,p) \in C_\delta^{l,\sigma}(\mathcal{D})^3 \times C_\delta^{l-1,\sigma}(\mathcal{D})$ satisfying the following condition (C_k):

$$\big(\partial_{x_1}^i \partial_{x_2}^j (\Delta u - \nabla p + f)\big)\big|_M = 0 \text{ for } i + j \leq k - 2, \quad \big(\partial_{x_1}^i \partial_{x_2}^j (\nabla \cdot u + g)\big)\big|_M = 0 \text{ for } i + j \leq k - 1,$$
$$\big(\partial_r^j (S^\pm u|_{\Gamma^\pm} - h^\pm)\big)\big|_M = 0 \text{ for } j \leq k, \quad \big(\partial_r^j (N^\pm(u,p)|_{\Gamma^\pm} - \phi^\pm)\big)\big|_M = 0 \text{ for } j \leq k - 1.$$

For $k = 1$ condition (C_k) means that

$$-\nabla \cdot u = g, \quad S^\pm u = h^\pm, \quad \partial_r (S^\pm u|_{\Gamma^\pm}) = \partial_r h^\pm, \quad N^\pm(u,p) = \phi^\pm \quad \text{on } M.$$

Suppose that $2 \leq k \leq s$ and the assertion is true for $k - 1$, i.e. there exists a pair $(u,p) \in C_\delta^{l,\sigma}(\mathcal{D})^3 \times C_\delta^{l-1,\sigma}(\mathcal{D})$ satisfying condition (C_{k-1}). We put

$$v_\nu = \sum_{|\alpha|=k} (E v_\alpha^{(\nu)}) \frac{(x')^\alpha}{\alpha!}, \quad \nu = 1, 2, 3, \quad q = \sum_{|\gamma|=k-1} (E q_\gamma) \frac{(x')^\gamma}{\gamma!}$$

and show that the functions $v_\alpha^{(\nu)}, q_\gamma \in C^{s-k, l-s+\sigma-\delta}(M)$ can be chosen such that

$$\partial_{x_1}^i \partial_{x_2}^j (\Delta v - \nabla q) = -\partial_{x_1}^i \partial_{x_2}^j (\Delta u - \nabla p + f) \quad \text{on } M \quad \text{for } i + j = k - 2, \tag{2.16}$$
$$\partial_{x_1}^i \partial_{x_2}^j \nabla \cdot v = -\partial_{x_1}^i \partial_{x_2}^j (\nabla \cdot u + g) \quad \text{on } M \quad \text{for } i + j = k - 1, \tag{2.17}$$
$$\partial_r^k (S^\pm v|_{\Gamma^\pm}) = \partial_r^k (h^\pm - S^\pm u|_{\Gamma^\pm}), \quad \partial_r^{k-1}\big(N^\pm(v,q)|_{\Gamma^\pm}\big) = \partial_r^{k-1}\big(\phi^\pm - N^\pm(v,q)|_{\Gamma^\pm}\big) \quad \text{on } M. \tag{2.18}$$

Equation (2.16) is equivalent to

$$v_{i+2,j}^{(\nu)} + v_{i,j+2}^{(\nu)} + \delta_{\nu,1}\, q_{i+1,j} + \delta_{\nu,2}\, q_{i,j+1} = -\big(\partial_{x_1}^i \partial_{x_2}^j (\Delta u - \nabla p + f)\big)\big|_M \quad \text{for } i + j = k - 2, \ \nu = 1, 2, 3,$$

while equation (2.16) is equivalent to

$$v_{i+1,j}^{(1)} + v_{i,j+1}^{(2)} = -\big(\partial_{x_1}^i \partial_{x_2}^j (\nabla \cdot u + g)\big)\big|_M \quad \text{for } i + j = k - 1.$$

Analogously, (2.18) can be written in the form

$$\mathfrak{S}^\pm(\{v_\alpha^{(\nu)}\},\{q_\gamma\}) = \partial_r^k (h^\pm - S^\pm u|_{\Gamma^\pm})|_M, \quad \mathfrak{N}^\pm(\{v_\alpha^{(\nu)}\},\{q_\gamma\}) = \partial_r^{k-1}\big(\phi^\pm - N^\pm(v,q)|_{\Gamma^\pm}\big)\big|_M\,,$$

where $\mathfrak{S}^\pm$, $\mathfrak{N}^\pm$ are linear forms. Thus, (2.16)–(2.18) is equivalent to a system of $4k+3$ linear equations with constant coefficients and $4k+3$ unknowns $v_\alpha^{(\nu)}$, q_γ, $\nu = 1,2,3$, $|\alpha| = k$, $|\gamma| = k-1$. This system is uniquely solvable. Otherwise, the corresponding homogeneous system has a nontrivial solution $\big(\{c_\alpha^{(\nu)}\},\{d_\gamma\}\big)$, and the functions

$$U_\nu = \sum_{|\alpha|=k} c_\alpha^{(\nu)} \frac{(x')^\alpha}{\alpha!}, \quad \nu = 1,2,3, \quad P = \sum_{|\gamma|=k-1} d_\gamma \frac{(x')^\gamma}{\gamma!}$$

satisfy the homogeneous equations (2.8), (2.9) what contradicts the assumptions of the lemma.

This proves that $v_\alpha^{(\nu)}$ and q_γ can be chosen such that (v,q) satisfies (2.16)–(2.18). Obviously,

$$\partial_{x_1}^i \partial_{x_2}^j (\Delta v - \nabla q) = 0 \ \text{ on } M \ \text{ for } i+j \le k-3, \quad \partial_{x_1}^i \partial_{x_2}^j \nabla\cdot v = 0 \ \text{ on } M \ \text{ for } i+j \le k-2,$$
$$\partial_r^j (S^\pm v|_{\Gamma^\pm}) = 0, \quad \partial_r^{k-1}\big(N^\pm(v,q)|_{\Gamma^\pm}\big) \ \text{ on } M \ \text{ for } j \le k-1.$$

Consequently, the pair $(u+v, p+q)$ satisfy condition (C$_k$).

In particular, it follows that there exists a pair $(u,p) \in C_\delta^{l,\sigma}(\mathcal{D})^3 \times C_\delta^{l-1,\sigma}(\mathcal{D})$ satisfying condition (C$_k$) for $k = s$. This means that $\Delta u - \nabla p + f \in \mathcal{N}_{\beta,\delta}^{l-2,\sigma}(\mathcal{D})^3$, $\nabla\cdot u + g \in \mathcal{N}_{\beta,\delta}^{l-1,\sigma}(\mathcal{D})$, $S^\pm u - h^\pm \in \mathcal{N}_{\beta,\delta}^{l,\sigma}(\Gamma^\pm)^{3-d^\pm}$, and $N^\pm(u,p) - \phi^\pm \in \mathcal{N}_{\beta,\delta}^{l-1,\sigma}(\Gamma^\pm)^{d^\pm}$ (see Lemma 1.2). Applying Lemma 2.1, we obtain the assertion of the lemma. □

Remark 2.2 The last assumptions in Lemma 2.4 (the nonexistence of homogeneous polynomials of degrees k and $k-1$, respectively, satisfying the homogeneous equations (2.8) and (2.9)) is satisfied, e.g., if $\lambda = k$ is not an eigenvalue of the pencil $A(\lambda)$ introduced below.

2.2 Regularity assertions for the solution

The next lemma follows from [24, Th.6.3.7] and [1, Th.9.3].

Lemma 2.5 *Let G_1, G_2 be bounded subdomains of $\mathbb{R}^3$ such that $\overline{G}_1 \subset G_2$, $G_1 \cap \mathcal{D} \neq \emptyset$ and $\overline{G}_1 \cap M = \emptyset$. If (u,p) is a solution of* (2.8), (2.9), *$u \in W^{2,s}(\mathcal{D}\cap G_2)^3$, $p \in W^{1,s}(\mathcal{D}\cap G_2)$, $f \in C^{l-2,\sigma}(\mathcal{D}\cap G_2)^3$, $g \in C^{l-2,\sigma}(\mathcal{D}\cap G_2)$, $h^\pm \in C^{l,\sigma}(\Gamma^\pm\cap G_2)^{3-d^\pm}$, $\phi^\pm \in C^{l-1,\sigma}(\Gamma^\pm \cap G_2)^{d^\pm}$, $l \ge 2$, $0 < \sigma < 1$, then $u \in C^{l,\sigma}(\mathcal{D}\cap G_1)^3 \times C^{l-1,\sigma}(\mathcal{D}\cap G_2)$, and*

$$\begin{aligned}\|u\|_{C^{l,\sigma}(\mathcal{D}\cap G_1)^\ell} + \|p\|_{C^{l-1,\sigma}(\mathcal{D}\cap G_2)} \ \le\ & c\Big(\|f\|_{C^{l-2,\sigma}(\mathcal{D}\cap G_2)^\ell} + \|g\|_{C^{l-2,\sigma}(\mathcal{D}\cap G_2)} + \sum_\pm \|h^\pm\|_{C^{l,\sigma}(\Gamma^\pm\cap G_2)} \\ & + \sum_\pm \|\phi^\pm\|_{C^{l-1,\sigma}(\Gamma^\pm\cap G_2)} + \|u\|_{C(\mathcal{D}\cap G_2)^3} + \|p\|_{C(\mathcal{D}\cap G_2)}\Big)\end{aligned}$$

with a constant c independent of u and p.

Let $W_{loc}^{l,s}(\bar{\mathcal{D}}\backslash M)$ be the set of all functions u such that $\zeta u \in W^{l,s}(\mathcal{D})$ for all $\zeta \in C_0^\infty(\bar{\mathcal{D}}\backslash M)$.

Lemma 2.6 *Let $(u,p) \in W_{loc}^{2,s}(\bar{\mathcal{D}}\backslash M)^3 \times W_{loc}^{1,s}(\bar{\mathcal{D}}\backslash M)$ be a solution of problem* (2.8), (2.9) *such that*

$$\sup |x'|^{\delta-l-\sigma}\, |u(x)| + \sup |x'|^{\delta-l-\sigma+1}\, |p(x)| < \infty.$$

If $f \in \mathcal{N}_\delta^{l-2,\sigma}(\mathcal{D})^3$, $l \ge 2$, $g \in \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})$, $h^\pm \in \mathcal{N}_\delta^{l,\sigma}(\Gamma^\pm)^{3-d^\pm}$, $\phi^\pm \in \mathcal{N}_\delta^{l-1,\sigma}(\Gamma^\pm)^{d^\pm}$, then $u \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})^3$, $p \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})$, and

$$\begin{aligned}\|u\|_{\mathcal{N}_\delta^{l,\sigma}(\mathcal{D})^3} + \|p\|_{\mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})} \ \le\ & c\Big(\|f\|_{\mathcal{N}_\delta^{l-2,\sigma}(\mathcal{D})^3} + \|g\|_{\mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})} + \sum_\pm \|h^\pm\|_{\mathcal{N}_\delta^{l,\sigma}(\Gamma^\pm)} \\ & + \sum_\pm \|\phi^\pm\|_{\mathcal{N}_\delta^{l-1,\sigma}(\Gamma^\pm)} + \|u\|_{\mathcal{N}^0_{\delta-l-\sigma}(\mathcal{D})^3} + \|p\|_{\mathcal{N}^0_{\delta-l+1-\sigma}(\mathcal{D})}\Big). \quad (2.19)\end{aligned}$$

P r o o f. Due to Lemma 2.1, we may restrict ourselves to the case $h^\pm = 0$, $\phi^\pm = 0$. For an arbitrary point $y \in \mathcal{D}$ we denote by B_y the ball with center y and radius $|y'|/2$ and by B'_y the ball with center y and radius $3|y'|/4$. For an arbitrary subdomain $\mathcal{U} \subset \mathcal{D}$ let the norm in $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{U})$ be defined by (1.1), where $\mathcal{D}$ is replaced by $\mathcal{U}$. For $|y'| = 1$ this norm is equivalent to the $C^{l,\sigma}$ norm, and Lemma 2.5 implies

$$\begin{aligned} \|u\|_{\mathcal{N}^{l,\sigma}_\delta(B_y\cap\mathcal{D})^3} + \|p\|_{\mathcal{N}^{l-1,\sigma}_\delta(B_y\cap\mathcal{D})} \ \le\ & c\Big(\|f\|_{\mathcal{N}^{l-2,\sigma}_\delta(B'_y\cap\mathcal{D})^3} + \|g\|_{\mathcal{N}^{l-1,\sigma}_\delta(B_y\cap\mathcal{D})} + \|u\|_{\mathcal{N}^0_{\delta-l-\sigma}(B'_y\cap\mathcal{D})^3} \\ & + \|p\|_{\mathcal{N}^0_{\delta-l+1-\sigma}(B'_y\cap\mathcal{D})}\Big) \end{aligned} \tag{2.20}$$

with a constant c independent of y. Let $|y'| \neq 1$ and $z = |y'|^{-1}\, y$. We introduce the functions $\tilde{u}(\xi) = u(|y'|\xi)$, $\tilde{p}(\xi) = |y'|\, p(|y'|\xi)$, $\tilde{f}(\xi) = |y'|^2\, f(|y'|\xi)$, and $\tilde{g}(\xi) = |y'|\, g(|y'|\xi)$. Then

$$-\Delta\tilde{u} + \nabla\tilde{p} = \tilde{f},\quad -\nabla\cdot\tilde{u} = \tilde{g} \ \text{ in } \mathcal{D},\qquad S^\pm\tilde{u} = 0,\quad N^\pm(\tilde{u},\tilde{p}) = 0 \ \text{ on } \Gamma^\pm.$$

Therefore, by (2.20), we have

$$\begin{aligned} \|\tilde{u}\|_{\mathcal{N}^{l,\sigma}_\delta(B_z\cap\mathcal{D})^3} + \|\tilde{p}\|_{\mathcal{N}^{l-1,\sigma}_\delta(B_z\cap\mathcal{D})} \ \le\ & c\Big(\|\tilde{f}\|_{\mathcal{N}^{l-2,\sigma}_\delta(B'_z\cap\mathcal{D})^3} + \|\tilde{g}\|_{\mathcal{N}^{l-1,\sigma}_\delta(B_z\cap\mathcal{D})} + \|\tilde{u}\|_{\mathcal{N}^0_{\delta-l-\sigma}(B'_z\cap\mathcal{D})^3} \\ & + \|\tilde{p}\|_{\mathcal{N}^0_{\delta-l+1-\sigma}(B'_z\cap\mathcal{D})}\Big). \end{aligned}$$

Using the inequalities

$$c_1\, |y'|^{l+\sigma-\delta}\|u\|_{\mathcal{N}^{l,\sigma}_\delta(B_y\cap\mathcal{D})^3} \le \|\tilde{u}\|_{\mathcal{N}^{l,\sigma}_\delta(B_z\cap\mathcal{D})^3} \le c_2\, |y'|^{l+\sigma-\delta}\|u\|_{\mathcal{N}^{l,\sigma}_\delta(B_y\cap\mathcal{D})^3}$$

and the analogous inequalities for the norms of $\tilde{p}$, $\tilde{f}$ and $\tilde{g}$, we obtain estimate (2.20) for arbitrary $y \in \mathcal{D}$. This proves the lemma. $\square$

We further need the following modification of Lemma 2.6

Lemma 2.7 *Let ζ, η be smooth functions with compact supports, $\eta = 1$ in a neighborhood of $\operatorname{supp}\zeta$. Furthermore, let $(u,p) \in W^{2,s}_{loc}(\bar{\mathcal{D}}\backslash M)^3 \times W^{1,s}_{loc}(\bar{\mathcal{D}}\backslash M)$ be a solution of problem* (2.8), (2.9) *such that*

$$\sup |x'|^{\delta-l-\sigma}\, \big|\eta(x)\, u(x)\big| + \sup |x'|^{\delta-l-\sigma+1}\, \big|\eta(x)\, p(x)\big| < \infty.$$

If $\eta f \in \mathcal{N}^{l-2,\sigma}_\delta(\mathcal{D})^3$, $l \ge 2$, $\eta g \in \mathcal{N}^{l-1,\sigma}_\delta(\mathcal{D})$, $\eta h^\pm \in \mathcal{N}^{l,\sigma}_\delta(\Gamma^\pm)^{3-d^\pm}$, $\eta\phi^\pm \in \mathcal{N}^{l-1,\sigma}_\delta(\Gamma^\pm)^{d^\pm}$, then $\zeta u \in \mathcal{N}^{l,\sigma}_\delta(\mathcal{D})^3$, $\zeta p \in \mathcal{N}^{l,\sigma}_\delta(\mathcal{K})$, and

$$\begin{aligned} \|\zeta u\|_{\mathcal{N}^{l,\sigma}_\delta(\mathcal{D})^3} + \|\zeta p\|_{\mathcal{N}^{l-1,\sigma}_\delta(\mathcal{D})} \ \le\ & c\Big(\|\eta f\|_{\mathcal{N}^{l-2,\sigma}_\delta(\mathcal{D})^3} + \|\eta g\|_{\mathcal{N}^{l-1,\sigma}_\delta(\mathcal{D})} + \sum_\pm \|\eta h^\pm\|_{\mathcal{N}^{l,\sigma}_\delta(\Gamma^\pm)} \\ & + \sum_\pm \|\eta\phi^\pm\|_{\mathcal{N}^{l-1,\sigma}_\delta(\Gamma^\pm)} + \|\eta u\|_{\mathcal{N}^0_{\delta-l-\sigma}(\mathcal{D})^3} + \|\eta p\|_{\mathcal{N}^0_{\delta-l+1-\sigma}(\mathcal{D})}\Big). \end{aligned}$$

P r o o f. We may again restrict ourselves in the proof to the case $h^\pm = 0$, $\phi^\pm = 0$. Let $\mathcal{U}$ be a neighborhood of $\operatorname{supp}\zeta$ such that $\eta = 1$ in a neighborhood of $\mathcal{U}$. Obviously, we obtain an equivalent norm in $\mathcal{N}^{l,\sigma}_\delta(\mathcal{D})$ if we replace the expression $\langle u\rangle_{l,\sigma,\delta;\mathcal{D}}$ in (1.1) by

$$\langle u\rangle'_{l,\sigma,\delta;\mathcal{D}} = \sum_{|\alpha|=l} \sup_{\substack{x,y\in\mathcal{D}\\ |x-y|<\varepsilon|x'|}} |x'|^\delta\, \frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|}{|x-y|^\sigma}\,,$$

where ε is an arbitrarily small positive number. Using this norm with sufficiently small ε, then we have

$$\|\zeta u\|_{\mathcal{N}^{l,\sigma}_\delta(\mathcal{D})^3} + \|\zeta p\|_{\mathcal{N}^{l-1,\sigma}_\delta(\mathcal{D})} \le c\Big(\|u\|_{\mathcal{N}^{l,\sigma}_\delta(\mathcal{U}\cap\mathcal{D})^3} + \|p\|_{\mathcal{N}^{l-1,\sigma}_\delta(\mathcal{U}\cap\mathcal{D})}\Big).$$

Here, we used the same notation as in the proof of Lemma 2.6. Furthermore, estimate (2.20) is also valid if we denote by B_y and B'_y the balls centered at y with radii $\varepsilon|y'|$ and $2\varepsilon|y'|$, respectively. From this it follows (if ε is sufficiently small) that

$$\begin{aligned}\|u\|_{\mathcal{N}^{l,\sigma}_\delta(\mathcal{U}\cap\mathcal{D})^3} + \|p\|_{\mathcal{N}^{l-1,\sigma}_\delta(\mathcal{U}\cap\mathcal{D})} \;\le\; c\,\Big(&\|f\|_{\mathcal{N}^{l-2,\sigma}_\delta(\mathcal{U}'\cap\mathcal{D})^3} + \|g\|_{\mathcal{N}^{l-1,\sigma}_\delta(\mathcal{U}'\cap\mathcal{D})}\\ &+ \|u\|_{\mathcal{N}^{0}_{\delta-l-\sigma}(\mathcal{U}'\cap\mathcal{D})^3} + \|p\|_{\mathcal{N}^{0}_{\delta-l+1-\sigma}(\mathcal{U}'\cap\mathcal{D})}\Big),\end{aligned}$$

where $\mathcal{U}' = \{x\in\mathcal{D}:\ \eta(x)=1\}$. This proves the lemma. □

Next, we prove a regularity assertion for the solution in the class of the spaces $C^{l,\sigma}_\delta$,

Lemma 2.8 *Let $(u,p)\in W^{2,s}_{loc}(\bar{\mathcal{D}}\backslash M)^3\times W^{1,s}_{loc}(\bar{\mathcal{D}}\backslash M)$ be a solution of problem* (2.8), (2.9), *and let ζ, η be smooth functions with compact supports, $\eta=1$ in a neighborhood of $\operatorname{supp}\zeta$. Suppose that $\eta u\in C^{l-1,\sigma}_{\delta-1}(\mathcal{D})^3$, $\eta p\in C^{l-2,\sigma}_{\delta-1}(\mathcal{D})$, $\eta f\in C^{l-2,\sigma}_\delta(\mathcal{D})^3$, $\eta g\in C^{l-1,\sigma}_\delta(\mathcal{D})$, $\eta h^\pm\in C^{l,\sigma}_\delta(\Gamma^\pm)^{3-d^\pm}$, $\eta\phi^\pm\in C^{l-1,\sigma}_\delta(\Gamma^\pm)^{d^\pm}$, where $l\ge 2$, $\delta\ge 1$, $0<\sigma<1$. Then $\zeta(u,p)\in C^{l,\sigma}_\delta(\mathcal{D})^3\times C^{l-1,\sigma}_\delta(\mathcal{D})$.*

P r o o f. By Lemma 1.2, there are the representations

$$\eta u = u' + u'',\quad \eta p = p' + p'',$$

where $u'\in\mathcal{N}^{l-1,\sigma}_{\delta-1}(\mathcal{D})^3$, $p'\in\mathcal{N}^{l-2,\sigma}_{\delta-1}(\mathcal{D})$, $u''\in C^{l,\sigma}_\delta(\mathcal{D})^3$, and $p''\in C^{l-1,\sigma}_\delta(\mathcal{D})$. Let χ be a smooth cut-off function equal to one in a neighborhood of $\operatorname{supp}\zeta$ such that $\eta=1$ in a neighborhood of $\operatorname{supp}\chi$. Then

$$\chi(-\Delta u'+\nabla p') = \chi f + \chi(\Delta u''-\nabla p'')\in C^{l-2,\sigma}_\delta(\mathcal{D})^3,\quad -\chi\nabla\cdot u' = \chi g + \chi\nabla\cdot u''\in C^{l-1}_\delta(\mathcal{D}).$$

In the case $\delta\ge l-2+\sigma$, the space $C^{l-2,\sigma}_\delta(\mathcal{D})$ coincides with $\mathcal{N}^{l-2,\sigma}_\delta(\mathcal{D})$, and in the case $1\le\delta<l-2+\sigma$ we have $l\ge 3$ and $\chi(-\Delta u'+\nabla p')\in\mathcal{N}^{l-3,\sigma}_{\delta-1}(\mathcal{D})^3$. From Lemma 1.2 it follows that $\mathcal{N}^{l-3,\sigma}_{\delta-1}(\mathcal{D})\cap C^{l-2,\sigma}_\delta(\mathcal{D})\subset\mathcal{N}^{l-2,\sigma}_\delta(\mathcal{D})$. Therefore in both cases, we obtain $\chi(-\Delta u'+\nabla p')\in\mathcal{N}^{l-2,\sigma}_\delta(\mathcal{D})^3$. Analogously, $\chi\nabla\cdot u'\in\mathcal{N}^{l-1,\sigma}_\delta(\mathcal{D})$, $\chi S^\pm u'\in\mathcal{N}^{l,\sigma}_\delta(\Gamma^\pm)^{3-d^\pm}$, and $\chi N^\pm(u',p')\in\mathcal{N}^{l-1,\sigma}_\delta(\Gamma^\pm)^{d^\pm}$. This together with Lemma 2.7 implies $\zeta u'\in\mathcal{N}^{l,\sigma}_\delta(\mathcal{D})^3$, $\zeta p'\in\mathcal{N}^{l-1,\sigma}_\delta(\mathcal{D})$. The result follows. □

We define the operator $A(\lambda)$ as follows

$$A(\lambda)\,\big(U(\varphi),P(\varphi)\big) = \big(r^{2-\lambda}(-\Delta u+\nabla p)\,,\ -r^{1-\lambda}\nabla\cdot u\,,\ r^{-\lambda}S^\pm u|_{\varphi=\pm\theta/2}\,,\ r^{1-\lambda}N^\pm(u,p)|_{\varphi=\pm\theta/2}\big),$$

where $u=r^\lambda U(\varphi)$, $p=r^{\lambda-1}P(\varphi)$, $\lambda\in\mathbb{C}$, r,φ are the polar coordinates of the point $x'=(x_1,x_2)$. The operator $A(\lambda)$ depends quadratically on the parameter λ and realizes a continuous mapping

$$W^{2,s}((-\tfrac{\theta}{2},+\tfrac{\theta}{2}))^3\times W^{1,s}((-\tfrac{\theta}{2},+\tfrac{\theta}{2}))\to W^{1,s}((-\tfrac{\theta}{2},+\tfrac{\theta}{2}))^3\times L^s((-\tfrac{\theta}{2},\tfrac{\theta}{2}))\times\mathbb{C}^3\times\mathbb{C}^3$$

for every $\lambda\in\mathbb{C}$. In [22] a description of the spectrum of the pencil $A(\lambda)$ is given for different d^- and d^+. For example, in the cases of the Dirichlet problem ($d^+=d^-=0$) and Neumann problem ($d^+=d^-=3$), the spectrum of $A(\lambda)$ consists of the solutions of the equation

$$\sin(\lambda\theta)\,\big(\lambda^2\sin^2\theta-\sin^2(\lambda\theta)\big)=0,$$

$\lambda\ne 0$ for $d^+=d^-=0$. In the case $d^-=0,\ d^+=1$, the eigenvalues of $A(\lambda)$ are the nonzero solutions of the equation

$$\sin(\lambda\theta)\,\big(\lambda\sin(2\theta)+\sin(2\lambda\theta)\big)=0.$$

If $d^-=0,\ d^+=2$, then the eigenvalues are the nonzero solutions of the equation

$$\sin(2\lambda\theta)\,\big(\lambda\sin(2\theta)-\sin(2\lambda\theta)\big)=0,$$

while the nonzero solutions of the equation

$$\sin(2\lambda\theta)\,\big(\lambda^2\sin^2\theta-\cos^2(\lambda\theta)\big)=0$$

are eigenvalues of $A(\lambda)$ if $d^-=0$ and $d^+=3$.

Lemma 2.9 *Let ζ,η be the same functions as in Lemma* 2.8*, and let (u,p) be a solution of problem* (2.8), (2.9) *such that $\eta\partial_{x_3}^j(u,p) \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})^3 \times \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})$ for $j=0$ and $j=1$, where $l \ge 2$, $0<\sigma<1$. If*

$$\eta f \in \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})^3, \quad \eta g \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D}), \quad \eta h^\pm \in \mathcal{N}_\delta^{l+1,\sigma}(\Gamma^\pm)^{3-d^\pm}, \quad \eta\phi^\pm \in \mathcal{N}_\delta^{l,\sigma}(\Gamma^\pm)^{d^\pm},$$

and the strip $l+\sigma-\delta \le \operatorname{Re}\lambda \le l+1+\sigma-\delta$ does not contain eigenvalues of the pencil $A(\lambda)$, then $\zeta(u,p) \in \mathcal{N}_\delta^{l+1,\sigma}(\mathcal{D})^3 \times \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})$.

P r o o f. Let χ be a smooth cut-off function such that $\chi=1$ in a neighborhood of $\operatorname{supp}\zeta$ and $\eta=1$ in a neighborhood of $\operatorname{supp}\chi$. We denote by $\Delta_{x'}$, $\nabla_{x'}$ the Laplace and Nabla operators in the coordinates $x'=(x_1,x_2)$. Then

$$-\Delta_{x'}(\chi u_3) = F_3 = \chi\big(f_3 + \partial_{x_3}^2 u_3 - \partial_{x_3} p\big) - 2\nabla_{x'}\chi\cdot\nabla_{x'}u_3 - u_3\,\Delta_{x'}\chi \in \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D}).$$

Furthermore, χu_3 satisfies the boundary conditions

$$\chi u_3\big|_{\Gamma^\pm} = H_3^\pm \in \mathcal{N}_\delta^{l+1,\sigma}(\Gamma^\pm) \ \text{ for } d^\pm \le 1, \qquad \frac{\partial(\chi u_3)}{\partial n^\pm}\Big|_{\Gamma^\pm} = \Phi_3^\pm \in \mathcal{N}_\delta^{l,\sigma}(\Gamma^\pm) \ \text{ for } d^\pm \ge 2$$

on $\Gamma^\pm$, where $H_3^\pm = \chi h_3^\pm$ and $\Phi_3^\pm = \chi\,\big((4-d^\pm)\phi_3^\pm - n^\pm\cdot\partial_{x_3}u\big) + u_3\,\partial\chi/\partial_{n^\pm}$. Analogously for $u'=(u_1,u_2)$ and p, we obtain the equations

$$-\Delta_{x'}(\chi u') + \nabla_{x'}(\chi p) = F', \quad -\nabla_{x'}\cdot(\chi u') = G, \quad \tilde S^\pm(\chi u')|_{\Gamma^\pm} = H'^\pm, \quad \tilde N^\pm(\chi u',\chi p)|_{\Gamma^\pm} = \Phi'^\pm$$

with certain functions $F' \in \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})^2$, $G \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})$, $H'^\pm \in \mathcal{N}_\delta^{l+1,\sigma}(\Gamma^\pm)$, $\Phi'^\pm \in \mathcal{N}_\delta^{l,\sigma}(\Gamma^\pm)$, where $\tilde S^\pm u' = S^\pm(u',0)$ and $\tilde N^\pm(u',p) = N^\pm(u',0,p)$. Consequently, by [**?**, Th.8.4], we obtain $(\chi u)(\cdot,x_3) \in \mathcal{N}_\delta^{l+1,\sigma}(K)^3$, $(\chi p)(\cdot,x_3) \in \mathcal{N}_\delta^{l,\sigma}(K)$, and

$$\begin{aligned} \|(\chi u)(\cdot,x_3)\|_{\mathcal{N}_\delta^{l+1,\sigma}(K)^3} + \|(\chi p)(\cdot,x_3)\|_{\mathcal{N}_\delta^{l+1,\sigma}(K)} \ \le\ & c\,\Big(\|F(\cdot,x_3)\|_{\mathcal{N}_\delta^{l-1,\sigma}(K)^3} + \|G(\cdot,x_3)\|_{\mathcal{N}_\delta^{l,\sigma}(K)} \\ & + \sum_\pm \|H\|_{\mathcal{N}_\delta^{l+1,\sigma}(\gamma^\pm)} + \sum_\pm \|\Phi^\pm\|_{\mathcal{N}_\delta^{l,\sigma}(\gamma^\pm)}\Big) \end{aligned}$$

with a constant c independent of x_3. From this and from the inclusions $\eta\partial_{x_3}u \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})^3$, $\eta\partial_{x_3}p \in \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})$ we conclude that $\zeta(u,p) \in \mathcal{N}_\delta^{l+1,\sigma}(\mathcal{D})^3 \times \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})$. □

We prove the analogous result for the spaces $C_\delta^{l,\sigma}$.

Lemma 2.10 *Let ζ,η be the same functions as in Lemma* 2.8*, and let (u,p) be a solution of problem* (2.8), (2.9) *such that $\eta\partial_{x_3}^j(u,p) \in C_\delta^{l,\sigma}(\mathcal{D})^3 \times C_\delta^{l-1,\sigma}(\mathcal{D})$ for $j=0$ and $j=1$, where $l\ge 2$, $0<\sigma<1$, $\delta-\sigma$ is not integer. If*

$$\eta f \in C_\delta^{l-1,\sigma}(\mathcal{D})^3, \quad \eta g \in C_\delta^{l,\sigma}(\mathcal{D}), \quad \eta h^\pm \in C_\delta^{l+1,\sigma}(\Gamma^\pm)^{3-d^\pm}, \quad \eta\phi^\pm \in C_\delta^{l,\sigma}(\Gamma^\pm)^{d^\pm},$$

and the strip $l+\sigma-\delta \le \operatorname{Re}\lambda \le l+1+\sigma-\delta$ does not contain eigenvalues of the pencil $A(\lambda)$, then $\zeta(u,p) \in C_\delta^{l+1,\sigma}(\mathcal{D})^3 \times C_\delta^{l,\sigma}(\mathcal{D})$.

P r o o f. Suppose that $k-1<\delta-\sigma<k$, where k is an integer, $k\le l$. Then both ηu and $\partial_{x_3}(\eta u)$ belong to $C^{l-k,k-\delta+\sigma}(\mathcal{D})^3$. Consequently, the traces $u^{(i,j)}$ of $\partial_{x_1}^i\partial_{x_2}^j(\eta u)$ on M are from $C^{l-k-i-j+1,k-\delta+\sigma}(M)^3$ for $i+j\le l-k$. Analogously, the traces $p^{(i,j)}$ of $\partial_{x_1}^i\partial_{x_2}^j(\eta p)$ on M are from $C^{l-k-i-j,k-\delta+\sigma}(M)$ for $i+j\le l-k-1$. By Lemma 1.2, there are the representations

$$\zeta u = \zeta \sum_{i+j\le l-k} \frac{Eu^{(i,j)}}{i!j!}\,x_1^i\,x_2^j + v, \qquad \zeta p = \zeta \sum_{i+j\le l-k-1} \frac{Ep^{(i,j)}}{i!j!}\,x_1^i\,x_2^j + q,$$

where E is the extension operator (1.3), $\partial_{x_3}^j v \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})^3$, and $\partial_{x_3}^j q \in \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})$ for $j=0,1$. From the properties of the extension operator E it follows that

$$\zeta u - v \in C_\delta^{l+1,\sigma}(\mathcal{D})^3, \quad \zeta p - q \in C_\delta^{l,\sigma}(\mathcal{D}),$$

Therefore, $-\Delta v + \nabla q \in C_\delta^{l-1,\sigma}(\mathcal{D})^3$, $-\nabla\cdot v \in C_\delta^{l-1,\sigma}(\mathcal{D})$, $S^\pm v \in C_\delta^{l+1,\sigma}(\Gamma^\pm)^{3-d^\pm}$, and $N^\pm(v,q) \in C_\delta^{l+1,\sigma}(\Gamma^\pm)^{d^\pm}$. Since $v \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})^3$ and $q \in \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})$, we have $\partial_x^\alpha(\nabla q - \Delta v) = 0$ on M for $|\alpha| \le l-k-2$ and $\partial_x^\alpha \nabla\cdot v = 0$ on M for $|\alpha| \le l-k-1$. Furthermore, $\partial_r^j S^\pm v = 0$ on M for $j \le l-k$ and $\partial_r^j N^\pm(v,q) = 0$ on M for $j \le l-k-1$. We put

$$f^{(i,j)} = \partial_{x_1}^i \partial_{x_2}^j (\nabla q - \Delta v)\big|_M \quad \text{for } i+j = l-k-1, \qquad g^{(i,j)} = -\partial_{x_1}^i \partial_{x_2}^j \nabla\cdot v\big|_M \quad \text{for } i+j = l-k,$$

$H^\pm = \partial_r^{l-k+1} S^\pm v|_M$, and $\Phi^\pm = \partial_r^{l-k} N^\pm(v,q)|_M$. Obviously $f^{(i,j)}, g^{(i,j)}, H^\pm, \Phi^\pm$ belong to the space $C^{0,k+\sigma-\delta}(M)$. By virtue of Lemma 1.2, we have

$$\eta\Big(\nabla q - \Delta v - \sum_{i+j=l-k-1} \frac{Ef^{(i,j)}}{i!\,j!}\, x_1^i\, x_2^j\Big) \in \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})^3, \quad \eta\big(\nabla\cdot v + \sum_{i+j=l-k} \frac{Eg^{(i,j)}}{i!\,j!}\, x_1^i\, x_2^j\big) \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D}),$$

$$\eta\Big(S^\pm v - \frac{EH^\pm}{(l-k+1)!}\, r^{l-k+1}\Big) \in \mathcal{N}_\delta^{l+1,\sigma}(\Gamma^\pm)^{3-d^\pm}, \qquad \eta\Big(N^\pm(v,q) - \frac{E\Phi^\pm}{(l-k)!}\, r^{l-k}\Big) \in \mathcal{N}_\delta^{l,\sigma}(\Gamma^\pm)^{d^\pm}.$$

Since $\lambda = l-k+1$ is not an eigenvalue of the pencil $A(\lambda)$, there exist homogeneous vector-valued polynomials $U^{(i,j,\mu)}(x_1,x_2)$ of degree $l-k+1$ and homogeneous polynomials $P^{(i,j,\mu)}(x_1,x_2)$ of degree $l-k$, $\mu = 1,2,3,4$, $i+j = l-k-1-\delta_{\mu,4}$, satisfying the equations

$$-\Delta U_\nu^{(i,j,\mu)} + \partial_{x_\nu} P^{(i,j,\mu)} = \delta_{\mu,\nu}\, \frac{x_1^i\, x_2^j}{i!\, j!} \quad \text{for } \nu = 1,2,2, \qquad -\nabla\cdot U^{(i,j,\mu)} = \delta_{\mu,4}\, \frac{x_1^i\, x_2^j}{i!\, j!}$$

and the boundary conditions $S^\pm U^{(i,j,\mu)} = 0$, $N^\pm(U^{(i,j,\mu)}, P^{(i,j,\mu)}) = 0$ on $\Gamma^\pm$ (see Remark 2.2). We define

$$\begin{aligned} U(x) &= \sum_{\mu=1}^{3} \sum_{i+j=l-k-1} U^{(i,j,\mu)}(x')\,(Ef_\mu^{(i,j)})(x) + \sum_{i+j=l-k-2} U^{(i,j,4)}(x')\,(Eg^{(i,j)})(x) \\ P(x) &= \sum_{\mu=1}^{3} \sum_{i+j=l-k-1} P^{(i,j,\mu)}(x')\,(Ef_\mu^{(i,j)})(x) + \sum_{i+j=l-k-2} P^{(i,j,4)}(x')\,(Eg^{(i,j)})(x) \end{aligned}$$

Then $\eta(U,P) \in C_\delta^{l+1,\sigma}(\mathcal{D})^3 \times C_\delta^{l,\sigma}(\mathcal{D})$, $\eta\,\partial_{x_3}^j(U,P) \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})^3 \times \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})$ for $j = 0,1$, $S^\pm U = 0$ on $\Gamma^\pm$

$$\eta\,\big(\Delta(U-v) - \nabla(P-q)\big) \in \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})^3, \quad \eta\,\nabla\cdot(U-v) \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D}), \quad \eta\, N^\pm(U,P) \in \mathcal{N}_\delta^{l,\sigma}(\Gamma^\pm).$$

For the last, we used the fact that $\eta\partial_{x_\nu} Ef^{(i,j)} \in \mathcal{N}_{\delta+l-k+1}^{l,\sigma}(\mathcal{D})^3$ and $\eta\partial_{x_\nu} Eg^{(i,j)} \in \mathcal{N}_{\delta+l-k+1}^{l,\sigma}(\mathcal{D})$ for $\nu = 1,2,3$.

Analogously there exist functions V and Q such that $\eta(V,Q) \in C_\delta^{l+1,\sigma}(\mathcal{D})^3 \times C_\delta^{l,\sigma}(\mathcal{D})$, $\eta\partial_{x_3}^j(V,Q) \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})^3 \times \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})$ for $j = 0,1$,

$$\begin{gathered} \eta\,(\Delta V - \nabla Q) \in \mathcal{N}_\delta^{l-1,\sigma}(\mathcal{D})^3, \quad \eta\,\nabla\cdot V \in \mathcal{N}_\delta^{l,\sigma}(\mathcal{D}) \\ \eta\, S^\pm(V-v) \in \mathcal{N}_\delta^{l+1,\sigma}(\Gamma^\pm), \quad \eta\, N^\pm(V-v, Q-q) \in \mathcal{N}_\delta^{l,\sigma}(\Gamma^\pm). \end{gathered}$$

Applying Lemma 2.9 to the vector-function $(U+V-v, P+Q-q)$, we obtain $\chi(U+V-v, P+Q-q) \in \mathcal{N}_\delta^{l+1,\sigma}(\mathcal{D})^3 \times \mathcal{N}_\delta^{l,\sigma}(\mathcal{D})$, where χ is the same cut-off function as in the proof of Lemma 2.9. The result follows. □

3 Solvability of the problem in a cone

Let $d_j \in \{0,1,2,3\}$ for $j = 1,\dots,N$. We consider the boundary value problem

$$-\Delta u + \nabla p = f \qquad -\nabla\cdot u = g \quad \text{in } \mathcal{K}, \tag{3.1}$$

$$S_j u = h_j, \quad N_j(u,p) = \phi_j \quad \text{on } \Gamma_j,\ j = 1,\dots,N, \tag{3.2}$$

where

$$S_j u = \begin{cases} u & \text{for } d_j = 0, \\ u - u_n\, n & \text{for } d_j = 1, \\ u_n & \text{for } d_j = 2, \end{cases} \qquad N_j(u,p) = \begin{cases} -p + 2\varepsilon_{nn}(u) & \text{for } d_j = 1, \\ \varepsilon_n(u) - \varepsilon_{nn}(u)\, n & \text{for } d_j = 2, \\ -pn + 2\varepsilon_n(u) & \text{for } d_j = 3. \end{cases}$$

We will prove that this problem is uniquely solvable in $C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$ under certain conditions on the data g, h_j, ϕ_j and on β and δ.

3.1 Operator pencils

1) Let $\Gamma_{k_\pm}$ be the sides of $\mathcal{K}$ adjacent to the edge M_k, and let θ_k be the angle at the edge M_k. We consider the Stokes system in the dihedron $\mathcal{D}_k$ bounded by the half-planes $\Gamma^\circ_{k_\pm} \supset \Gamma_{k_\pm}$ with the boundary conditions

$$S_{k_\pm} u = h^\pm, \quad N_{k_\pm}(u,p) = \phi^\pm \ \text{ on } \Gamma^\circ_{k_\pm}.$$

By $A_k(\lambda)$ we denote the operator pencil introduced before Lemma 2.9 for this problem. Furthermore, let $\lambda_1^{(k)}$ denote the eigenvalue with smallest positive real part of this pencil, while $\lambda_2^{(k)}$ is the eigenvalue with smallest real part greater than 1. Finally, we define

$$\mu_k = \begin{cases} \operatorname{Re}\lambda_1^{(k)} & \text{if } d_{k_+} + d_{k_-} \text{ is odd} \quad \text{or} \quad d_{k_+} + d_{k_-} \text{ is even and } \alpha_k \ge \pi/m_k, \\ \operatorname{Re}\lambda_2^{(k)} & \text{if } d_{k_+} + d_{k_-} \text{ is even and } \theta_k < \pi/m_k, \end{cases} \tag{3.3}$$

where $m_k = 1$ if $d_{k_+} = d_{k_-}$, $m_k = 2$ if $d_{k_+} \ne d_{k_-}$.

2) Let $\rho = |x|$, $\omega = x/|x|$, $V_\Omega = \{u \in W^1(\Omega)^3 : S_j u = 0 \text{ on } \gamma_j \text{ for } j = 1, \dots, n\}$, and

$$a\Big(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix}; \lambda \Big) = \frac{1}{\log 2} \int\limits_{\substack{\mathcal{K} \\ 1<|x|<2}} \Big(2 \sum_{i,j=1}^3 \varepsilon_{i,j}(U) \cdot \varepsilon_{i,j}(V) - P \nabla \cdot V - (\nabla \cdot U)\, Q \Big)\, dx,$$

where $U = \rho^\lambda u(\omega)$, $V = \rho^{-1-\lambda} v(\omega)$, $P = \rho^{\lambda-1} p(\omega)$, $Q = \rho^{-2-\lambda} q(\omega)$, $u, v \in V_\Omega$, $p, q \in L_2(\Omega)$, and $\lambda \in \mathbb{C}$, $\varepsilon_{i,j}(U) = \frac{1}{2}\big(\partial_{x_i} U_j + \partial_{x_j} U_i\big)$. The bilinear form $a(\cdot,\cdot;\lambda)$ generates the linear and continuous operator

$$\mathfrak{A}(\lambda) : V_\Omega \times L_2(\Omega) \to V^*_\Omega \times L_2(\Omega)$$

by

$$\int_\Omega \mathfrak{A}(\lambda) \begin{pmatrix} u \\ p \end{pmatrix} \cdot \begin{pmatrix} v \\ q \end{pmatrix} d\omega = a\Big(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix}; \lambda \Big), \quad u, v \in V_\Omega, \ p, q \in L_2(\Omega).$$

3.2 Reduction to homogeneous boundary conditions

Lemma 3.1 *Let $h_j \in \mathcal{N}^{l,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in \mathcal{N}^{l-1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}$, $j = 1, \dots, N$, $l \ge 1$, be given. Then there exists a vector function $u \in \mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3$ such that $S_j u = h_j$ and $N_j(u,0) = \phi_j$ on Γ_j. The norm of u can be estimated by the norms of h_j and ϕ_j.*

P r o o f. Let ζ_k be smooth functions on $(0,\infty)$ such that

$$\operatorname{supp} \zeta_k \subset (2^{k-1}, 2^{k+1}), \quad \big|\partial_\rho^j \zeta_k(\rho)\big| \le c\, 2^{-kj}, \quad \text{and} \quad \sum_{k=-\infty}^{+\infty} \zeta_k = 1. \tag{3.4}$$

We set

$$h_{k,j}(x) = \zeta_k(2^k|x|)\, h_j(2^k x), \qquad \phi_{k,j}(x) = 2^k\, \zeta_k(2^k|x|)\, \phi_j(2^k x).$$

The supports of $h_{k,j}$ and $\phi_{k,j}$ are contained in $\{x : 1/2 < |x| < 2\}$. Consequently, by Lemma 2.1, there exists a vector function $v_k \in \mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3$ such that $v_k(x) = 0$ for $|x| < 1/4$ and $|x| > 4$, $S_j v_k = h_{k,j}$ on $N_j(v_k, 0) = h_{k,j}$ on Γ_j, $j = 1, \dots, N$,

$$\|v_k\|_{\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3} \le c \sum_{j=1}^N \Big(\|h_{k,j}\|_{\mathcal{N}^{l,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}} + \|\phi_{k,j}\|_{\mathcal{N}^{l-1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}} \Big) \tag{3.5}$$

where c is independent of k. From this we conclude that the functions $u_k(x) = v_k(2^{-k}x)$ satisfy $S_j u_k(x) = \zeta_k(|x|)\, g_j(x)$, $N_j(u_k,0) = \zeta_k(|x|)\, \phi_j(x)$ on Γ_j, and the estimate (3.5) with $\zeta_k h_j$, $\zeta_k\phi_j$ instead of $h_{k,j}$ and $\phi_{k,j}$, respectively. Thus, $u = \sum u_k$ has the desired properties. $\square$

An analogous result in $C^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ is only valid under additional compatibility conditions on the boundary data. Denote by Γ_{k_+} and Γ_{k_-} the sides of the cone $\mathcal{K}$ adjacent to the edge M_k and by θ_k the inner angle at M_k. If $u \in C^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ and $\delta_k < l+\sigma$, then the trace of u on M_k exists and from the equations $S_j u = h_j$ on Γ_j it follows that the pair $\big(h_{k_+}|_{M_j}, h_{k_-}|_{M_j}\big)$ belongs to the range of the matrix operator (S_{k_+}, S_{k_-}). This condition can be also written in the form

$$A_k^+ h_{k_+}|_{M_k} = A_k^- h_{k_-}|_{M_k}\,, \tag{3.6}$$

where A_k^+, A_k^- are certain constant matrices (see Section 2.1).

Using Lemma 2.4, one can prove the following result analogously to Lemma 3.1.

Lemma 3.2 *Let $h_j \in C^{l,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in C^{l-1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}$, $f \in C^{l-2,\sigma}_{\beta,\delta}(\mathcal{K})^3$, and $g \in C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$, where $l \ge 1$, $0 \le \delta_k < l+\sigma$, $\delta_k - \sigma$ not integer for $k = 1,\dots,N$ (in the case $l = 1$ the condition on f can be omitted). Suppose that the boundary data h_j satisfy the compatibility condition (3.6) and that in the case $\delta_k < l-1+\sigma$ the functions $g, h_{k_\pm}, \phi_{k_\pm}$ satisfy the compatibility conditions given in Lemma 2.3. Furthermore, we assume that the numbers $2,3,\dots,[l+\sigma-\delta_k]$ do not belong to the spectrum of the pencil $A_k(\lambda)$ if $\delta_k < l-2+\sigma$. Then there exists a vector function $(u,p) \in C^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$ satisfying*

$$S_j u = h_j,\;\; N_j(u,p) = \phi_j \text{ on } \Gamma_j,\; j = 1,\dots,n,\;\; \Delta u - \nabla p + f \in \mathcal{N}^{l-2,\sigma}_{\beta,\delta}(\mathcal{K})^3,\;\; \nabla\cdot u + g \in \mathcal{N}^{l-1,\sigma}_{\beta,\delta}(\mathcal{K}) \tag{3.7}$$

The norms of u and p can be estimated by the norms of f, g, h_j and ϕ_j.

Note again that the condition of Lemma 2.3 is always satisfied if $d_{k_+} + d_{k_-} = 3$ and $\sin(2\theta_k) \ne 0$ or $d_{k_+} + d_{k_-} \in \{1,5\}$ and $\cos\theta_k \cos(2\theta_k) \ne 0$. For even $d_{k_+} + d_{k_-}$, one can find explicit conditions on $g|_{M_k}$, $h_{k_\pm}|_{M_k}$, and $\phi_{k_\pm}|_{M_k}$ for different combinations of boundary conditions on Γ_{k_+} and Γ_{k_-}, see Remark 2.1.

3.3 Regularity result for solutions of the boundary value problem

Lemma 3.3 *Let $(u,p) \in W^{2,s}_{loc}(\bar{\mathcal{K}}\backslash\mathcal{S})^3 \times W^{1,s}_{loc}(\bar{\mathcal{K}}\backslash\mathcal{S})$ be a solution of problem (3.1), (3.2) such that*

$$\sup |x|^{\delta-l-\sigma} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k-l-\sigma} |u(x)| + \sup |x|^{\delta-l+1-\sigma} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k-l+1-\sigma} |p(x)| < \infty.$$

If $f \in \mathcal{N}^{l-2,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $l \ge 2$, $g \in \mathcal{N}^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$, $h_j \in \mathcal{N}^{l,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in \mathcal{N}^{l-1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}$, $j = 1,\dots,N$, then $u \in \mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3$ and $p \in \mathcal{N}^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$.

P r o o f. Due to Lemma 3.1, we may assume without loss of generality that $h_j = 0$ and $\phi_j = 0$. From Lemma 2.7 it follows that $\zeta u \in \mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3$ and $\zeta p \in \mathcal{N}^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$ for every smooth function ζ with compact support vanishing in a neighborhood of the origin.

Let ρ be a positive integer, $\mathcal{K}_\rho = \{x \in \mathcal{K} :\ \rho/2 < |x| < 2\rho\}$, and $\mathcal{K}'_\rho = \{x \in \mathcal{K} :\ \rho/4 < |x| < 4\rho\}$. Furthermore, let $\tilde{u}(x) = u(\rho x)$, $\tilde{p}(x) = \rho\, p(\rho x)$, $\tilde{f}(x) = \rho^2 f(\rho x)$, and $\tilde{g}(x) = \rho\, g(\rho x)$. Then $-\Delta\tilde{u} + \nabla\tilde{p} = \tilde{f}$ and $-\nabla\cdot\tilde{u} = \tilde{g}$ in $\mathcal{K}$. Moreover $\tilde{u}$ and $\tilde{p}$ satisfy the homogeneous boundary conditions (3.2). Consequently, by Lemma 2.7, we have

$$\begin{aligned} \|\tilde{u}\|_{\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K}_1)^3} + \|\tilde{p}\|_{\mathcal{N}^{l-1,\sigma}_{\beta,\delta}(\mathcal{K}_1)} \;\le\; & c\Big(\|\tilde{f}\|_{\mathcal{N}^{l-2,\sigma}_{\beta,\delta}(\mathcal{K}'_1)^3} + \|\tilde{g}\|_{\mathcal{N}^{l-1,\sigma}_{\beta,\delta}(\mathcal{K}'_1)} \\ & + \|\tilde{u}\|_{\mathcal{N}^0_{\beta-l-\sigma,\delta-l-\sigma}(\mathcal{K}'_1)^3} + \|\tilde{p}\|_{\mathcal{N}^0_{\beta-l+1-\sigma,\delta-l+1-\sigma}(\mathcal{K}'_1)}\Big) \end{aligned} \tag{3.8}$$

with a constant c independent of u, p and ρ. Here the norm in $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K}_\rho)$ is defined by (1.6), where $\mathcal{K}$ has to be replaced by $\mathcal{K}_\rho$. Since

$$\|\tilde{u}\|_{\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K}_1)^3} = \rho^{l+\sigma-\beta} \|u\|_{\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K}_\rho)^3},$$

we obtain an analogous estimate for the norms of u and p in $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K}_\rho)^3$ and $\mathcal{N}^{l-1,\sigma}_{\beta,\delta}(\mathcal{K}_\rho)$, respectively. Here, the constant c is the same as in (3.8). The result follows. $\square$

In the same way, the following two lemmas can be proved using Lemmas 2.8 and 2.9.

Lemma 3.4 *Let $(u,p) \in W^{2,s}_{loc}(\bar{\mathcal{K}}\backslash\mathcal{S})^3 \times W^{1,s}_{loc}(\bar{\mathcal{K}}\backslash\mathcal{S})$ be a solution of problem* (3.1), (3.2). *Suppose that $u \in C^{l-1,\sigma}_{\beta-1,\delta-1}(\mathcal{K})^3$, $p \in C^{l-2,\sigma}_{\beta-1,\delta-1}(\mathcal{K})$, $f \in C^{l-2,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $g \in C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$, $h_j \in C^{l,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in C^{l-1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}$, where $l \geq 2$, $\delta_k \geq 1$ for $k = 1,\dots,N$, $0 < \sigma < 1$. Then $(u,p) \in C^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$.*

Lemma 3.5 *Let (u,p) be a solution of problem* (3.1), (3.2) *such that $(\rho\partial_\rho)^j(u,p) \in C^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$ for $j = 0,1$, where $l \geq 2$, $0 < \sigma < 1$. If*

$$f \in C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})^3, \quad g \in C^{l,\sigma}_{\beta,\delta}(\mathcal{K}), \quad h_j \in C^{l+1,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}, \quad \phi_j \in C^{l,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j},$$

$j = 1,\dots,N$, and the strip $l + \sigma - \delta_k \leq \operatorname{Re}\lambda \leq l + 1 + \sigma - \delta_k$ does not contain eigenvalues of the pencil $A_k(\lambda)$, $k = 1,\dots,n$, then $(u,p) \in C^{l+1,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{l,\sigma}_{\beta,\delta}(\mathcal{K})$.

3.4 Estimates of Green's matrix

We denote by $V^l_\kappa(K)$ the weighted Sobolev space wit the norm

$$\|u\|_{V^l_\kappa(\mathcal{K})} = \Big(\int_{\mathcal{K}} \sum_{|\alpha|\leq l} |x|^{2(\kappa-l+|\alpha|)}\, |\partial_x^\alpha u(x)|^2\, dx\Big)^{1/2}.$$

A matrix $G(x,\xi) = \big(G_{i,j}(x,\xi)\big)_{i,j=1}^4$ is called Green's matrix for problem (3.1), (3.2) if

$$-\Delta_x \vec{G}_j(x,\xi) + \nabla_x G_{4,j}(x,\xi) = \delta(x-\xi)\,(\delta_{1,j},\delta_{2,j},\delta_{3,j})^t \quad \text{for } x,\xi \in \mathcal{K}, \tag{3.9}$$

$$-\nabla_x \cdot \vec{G}_j(x,\xi) = \delta_{4,j}\,\delta(x-\xi) \quad \text{for } x,\xi \in \mathcal{K}, \tag{3.10}$$

$$S_k\vec{G}_j(x,\xi) = 0, \quad N_k(\partial_x)\big(\vec{G}_j(x,\xi), G_{4,j}(x,\xi)\big) = 0 \quad \text{for } x \in \Gamma_k,\ \xi \in \mathcal{K},\quad k = 1,\dots,N. \tag{3.11}$$

Here $\vec{G}_j$ denotes the vector with the components $G_{1,j}, G_{2,j}, G_{3,j}$.

Suppose that the line $\operatorname{Re}\lambda = -\kappa - 1/2$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$. Then, by [22, Th.4.5], there exists a unique Green matrix $G(x,\xi)$ such that the function $x \to \zeta(|x-\xi|/r(\xi))\,G_{i,j}(x,\xi)$ belongs to $V^1_\kappa(\mathcal{K})$ for $i = 1,2,3$ and to $V^0_\kappa(\mathcal{K})$ for $i = 4$, where ζ is an arbitrary smooth function on $(0,\infty)$ equal to one in $(1,\infty)$ and to zero in $(0,\frac{1}{2})$.

We denote by $\Lambda_- < \operatorname{Re}\lambda < \Lambda_+$ the widest strip in the complex plane which is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and which contains the line $\operatorname{Re}\lambda = -\kappa - 1/2$. Furthermore, we introduce the following notation.

$$\sigma_{k,i,\alpha} = \min(0, \mu_k - |\alpha| - \delta_{i,4} - \varepsilon), \quad \sigma_{i,\alpha}(x) = \min(0, \mu_x - |\alpha| - \delta_{i,4} - \varepsilon)$$

Here ε is an arbitrarily small positive real number, $\mu_x = \mu_{k(x)}$, and $k(x)$ is the smallest integer k such that $r(x) = r_k(x)$. For the following theorem we refer to [22, Th.4.5,Th.4.6].

Theorem 3.1 *Let $G(x,\xi)$ be the above introduced Green matrix. Then for $|x| > 2|\xi|$ there is the estimate*

$$\big|\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x,\xi)\big| \leq c\,|x|^{\Lambda_- - \delta_{i,4} - |\alpha| + \varepsilon}\ |\xi|^{-\Lambda_- - 1 - \delta_{j,4} - |\gamma| - \varepsilon} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma_{k,i,\alpha}} \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\sigma_{k,j,\gamma}},$$

For $|\xi| > 2|x|$ we have

$$\big|\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x,\xi)\big| \leq c\,|x|^{\Lambda_+ - \delta_{i,4} - |\alpha| - \varepsilon}\ |\xi|^{-\Lambda_+ - 1 - \delta_{j,4} - |\gamma| + \varepsilon} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma_{k,i,\alpha}} \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\sigma_{k,j,\gamma}},$$

while for $|x|/2 < |\xi| < 2|x|$ *the estimates*

$$\big|\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x,\xi)\big| \le c\, |x-\xi|^{-T-|\alpha|-|\gamma|} \quad \text{if } |x-\xi| < \min(r(x), r(\xi)),$$
$$\big|\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x,\xi)\big| \le c\, |x-\xi|^{-T-|\alpha|-|\gamma|} \Big(\frac{r(x)}{|x-\xi|}\Big)^{\sigma_{i,\alpha}(x)} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\sigma_{j,\gamma}(\xi)} \quad \text{if } |x-\xi| > \min(r(x), r(\xi))$$

are valid, where $T = 1 + \delta_{i,4} + \delta_{j,4}$. *Furthermore, for* $i = 1, \dots, 4$ *there is the representation* $G_{i,4}(x,\xi) = -\nabla_\xi \cdot \vec{\mathcal{P}}_i(x,\xi) + \mathcal{Q}_i(x,\xi)$, *where* $\vec{\mathcal{P}}_i(x,\xi) \cdot n = 0$ *for* $\xi \in \Gamma_k$, $x \in \mathcal{D}$, *and* $\vec{\mathcal{P}}_i$ *and* $\mathcal{Q}_i$ *satisfy the estimates*

$$|\partial_x^\alpha \partial_\xi^\gamma \vec{\mathcal{P}}_i(x,\xi)| \le c_{\alpha,\gamma}\, |x-\xi|^{-1-\delta_{i,4}-|\alpha|-|\gamma|}, \quad |\partial_x^\alpha \partial_\xi^\gamma \mathcal{Q}_i(x,\xi)| \le c_{\alpha,\gamma}\, r(\xi)^{-2-\delta_{i,4}-|\alpha|-|\gamma|}$$

for $|x-\xi| < \min(r(x), r(\xi))$.

Remark 3.1 For derivatives with respect to $\rho = |x|$ there are the sharper estimates

$$\big|\partial_\rho \partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x,\xi)\big| \le c\, |x|^{\Lambda_- -1-\delta_{i,4}-|\alpha|+\varepsilon}\, |\xi|^{-\Lambda_- -1-\delta_{j,4}-|\gamma|-\varepsilon} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma_{k,i,\alpha}} \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\sigma_{k,j,\gamma}},$$
$$\big|\partial_\rho \partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x,\xi)\big| \le c\, |x|^{\Lambda_+ -\delta_{i,4}-|\alpha|-\varepsilon}\, |\xi|^{-\Lambda_+ -1-\delta_{j,4}-|\gamma|+\varepsilon} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma_{k,i,\alpha}} \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\sigma_{k,j,\gamma}}$$

if $|\xi| < |x|/2$ and $|\xi| > 2|x|$, respectively. For $|x|/2 < |\xi| < 2|x|$, $|x-\xi| > \min(r(x), r(\xi))$, the estimate

$$\big|\partial_\rho \partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x,\xi)\big| \le c\, |x-\xi|^{-T-1-|\alpha|-|\gamma|} \Big(\frac{r(x)}{|x-\xi|}\Big)^{\sigma_{i,\alpha}(x)} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\sigma_{j,\gamma}(\xi)}$$

is valid.

3.5 Representation of the solution by Green's matrix

Suppose that $f \in \mathcal{N}_{\beta,\delta}^{0,\sigma}(\mathcal{K})$, $g \in \mathcal{N}_{\beta,\delta}^{1,\sigma}(\mathcal{K})$, where $\beta \in \mathbb{R}$ and $\delta = (\delta_1, \dots, \delta_N) \in [0,\infty)^N$ are such that

$$2 - \mu_k < \delta_k - \sigma < 2, \quad \delta_k - \sigma \text{ not integer for } k = 1, \dots, N, \tag{3.12}$$
$$\Lambda_- < 2 + \sigma - \beta < \Lambda_+. \tag{3.13}$$

Here Λ_+, Λ_- are the same numbers as in Theorem 3.1. We consider the functions

$$u_i(x) = \sum_{j=1}^3 \int_{\mathcal{K}} \big(f_j(\xi) + \partial_{\xi_j} g(\xi)\big)\, G_{i,j}(x,\xi)\, d\xi + \int_{\mathcal{K}} g(\xi)\, G_{i,4}(x,\xi)\, d\xi, \quad i = 1,2,3, \tag{3.14}$$
$$p(x) = -g(x) + \sum_{j=1}^3 \int_{\mathcal{K}} \big(f_j(\xi) + \partial_{\xi_j} g(\xi)\big)\, G_{4,j}(x,\xi)\, d\xi + \int_{\mathcal{K}} g(\xi)\, G_{4,4}(x,\xi)\, d\xi. \tag{3.15}$$

The vector-function (u,p) is a solution of problem (3.1), (3.2) with $h_j = 0$, $\phi_j = 0$ (see [22, Th.4.5]). Let χ be an arbitrary smooth cut-off function on $[0,\infty)$, $\chi(t) = 1$ for $t < 1/4$, $\chi(t) = 0$ for $t > 1/2$. We put

$$\chi^+(x,\xi) = \chi\Big(\frac{|x-\xi|}{r(x)}\Big), \quad \chi^-(x,\xi) = 1 - \chi^+(x,\xi).$$

Then $\chi^+(x,\xi) = 0$ for $|x-\xi| > r(x)/2$, $\chi^-(x,\xi) = 0$ for $|x-\xi| < r(x)/4$, and

$$\big|\partial_x^\alpha \chi^\pm(x,\xi)\big| \le c\, r(x)^{-|\alpha|}$$

with a constant c independent of x and ξ. We write u nd p in the form

$$u = u^+ + u^-, \quad p = p^+ + p^-,$$

where

$$u_i^\pm(x) = \sum_{j=1}^3 \int_{\mathcal{K}} \big(f_j(\xi)+\partial_{\xi_j} g(\xi)\big)\, \chi^\pm(x,\xi)\, G_{i,j}(x,\xi)\, d\xi + \int_{\mathcal{K}} g(\xi)\, \chi^\pm(x,\xi)\, G_{i,4}(x,\xi)\, d\xi, \tag{3.16}$$

$$p^\pm(x) = -\frac{g(x)}{2} + \sum_{j=1}^3 \int_{\mathcal{K}} \big(f_j(\xi)+\partial_{\xi_j} g(\xi)\big)\, \chi^\pm(x,\xi)\, G_{4,j}(x,\xi)\, d\xi + \int_{\mathcal{K}} g(\xi)\, \chi^\pm(x,\xi)\, G_{4,4}(x,\xi)\, d\xi. \tag{3.17}$$

3.6 Weighted L_∞ estimates for u^+, p^+

We consider the functions u^+, p^+ defined by (3.16), (3.17), where $G(x,\xi)$ is the Green matrix introduced in Section 3.4.

Lemma 3.6 *Suppose conditions* (3.12), (3.13) *are satisfied. Then for arbitrary* $f \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3$ *and* $g \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})$*, there are the estimates*

$$|x|^{\beta-2-\sigma} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k-2-\sigma} \sum_{i=1}^3 |u_i^+(x)| \le c\,\Big(\|f\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})}\Big), \tag{3.18}$$

$$|x|^{\beta-1-\sigma} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k-1-\sigma} \Big(\sum_{i,\nu=1}^3 |\partial_{x_\nu} u_i^+(x)| + |p^+(x)|\Big) \le c\,\Big(\|f\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})}\Big). \tag{3.19}$$

P r o o f. We have $|\partial_{x_\nu} u_i^+(x)| \le A + B$, where

$$A = \sum_{j=1}^3 \Big| \int_{\mathcal{K}} \big(f_j(\xi)+\partial_{\xi_j} g(\xi)\big)\, \partial_{x_\nu}\big(\chi^+(x,\xi)\, G_{i,j}(x,\xi)\big)\, d\xi\Big| \quad \text{and}$$

$$B = \Big|\partial_{x_\nu} \int_{\mathcal{K}} g(\xi)\, \chi^+(x,\xi)\, G_{i,4}(x,\xi)\, d\xi\Big|$$

On the support of χ^+ there are the inequalities $|x|/2 \le |\xi| \le 3|x|/2$, $r_k(x)/2 \le r_k(\xi) \le 3r_k(x)/2$. This together with Theorem 3.1 implies

$$\begin{aligned} A &\le c\,|x|^{\sigma-\beta} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma-\delta_k} \Big(\|f\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})^3} + \|\nabla g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})}\Big) \int\limits_{|x-\xi|<r(x)/2} |x-\xi|^{-2}\, d\xi \\ &\le c\,|x|^{\sigma-\beta+1} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma-\delta_k+1} \Big(\|f\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})^3} + \|\nabla g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})}\Big) \end{aligned}$$

Using the representation $G_{i,4}(x,\xi) = -\nabla_\xi \cdot \vec{\mathcal{P}}_i(x,\xi) + \mathcal{Q}_i(x,\xi)$ in Theorem 3.1 and the properties of $\mathcal{P}_i$, $\mathcal{Q}_i$, we obtain

$$\begin{aligned} B &= \Big|\partial_{x_\nu} \int_{\mathcal{K}} \Big(g(\xi)\,\big(\nabla_\xi \chi^+(x,\xi)\cdot \vec{\mathcal{P}}_i(x,\xi) + \mathcal{Q}_i(x,\xi)\big) + \chi^+(x,\xi)\, \nabla_\xi g(\xi) \cdot \vec{\mathcal{P}}_i(x,\xi)\Big)\, d\xi\Big| \\ &\le c\,|x|^{\sigma-\beta+1} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma-\delta_k+1} \|g\|_{\mathcal{N}^0_{\beta-1-\sigma,\delta-1-\sigma}(\mathcal{K})} \int\limits_{|x-\xi|<r(x)/2} r(x)^{-3}\, d\xi \\ &\quad + c\,|x|^{\sigma-\beta} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma-\delta_k} \|\nabla g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})} \int\limits_{|x-\xi|<r(x)/2} |x-\xi|^{-2}\, d\xi \\ &\le c\,|x|^{\sigma-\beta+1} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma-\delta_k+1} \Big(\|g\|_{\mathcal{N}^0_{\beta-1-\sigma,\delta-1-\sigma}(\mathcal{K})^3} + \|\nabla g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})}\Big). \end{aligned}$$

This proves the desired estimate for ∇u. Analogously, the estimates for u and p hold. $\square$

3.7 Weighted L_∞ estimates for the derivatives of u^-, p^-

Next we show that

$$|x|^{\beta-2-\sigma+|\alpha|} \prod_{k=1}^{N} \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-2-\sigma+|\alpha|)} \big|\partial_x^\alpha u^-(x)\big| \le c\,\Big(\|f\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})}\Big) \tag{3.20}$$

and

$$|x|^{\beta-1-\sigma+|\alpha|} \prod_{k=1}^{N} \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-1-\sigma+|\alpha|)} \big|\partial_x^\alpha p^-(x)\big| \le c\,\Big(\|f\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})}\Big) \tag{3.21}$$

for an arbitrary multi-index α. For this we need the following lemmas.

Lemma 3.7 *Let $f \in \mathcal{N}^{t,\sigma}_{\beta,\delta}(\mathcal{K})$ and*

$$v(x) = \int\limits_{\substack{\mathcal{K} \\ |\xi|<|x|/2}} K(x,\xi)\, f(\xi)\, d\xi,$$

where the kernel K satisfies the estimate

$$|K(x,\xi)| \le c\,|x|^{\Lambda_- - s+\varepsilon}\,|\xi|^{-\Lambda_- -1-t-\varepsilon} \prod_{k=1}^{N} \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k-s-\varepsilon)} \prod_{k=1}^{N} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0,\mu_k-t-\varepsilon)}.$$

with nonnegative integers s,t. Suppose that β,δ satisfy conditions (3.12), (3.13)*. Then*

$$|x|^{\beta-2+s-\sigma} \prod_{k=1}^{N} \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-2+s-\sigma)} |v(x)| \le c\,\|f\|_{\mathcal{N}^{0}_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})} \tag{3.22}$$

with a constant c independent of x.

P r o o f. Obviously

$$\begin{aligned} |v(x)| \;\; &\le \;\; c\,|x|^{\Lambda_- - s+\varepsilon} \prod_{k=1}^{N} \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k-s-\varepsilon)} \|f\|_{\mathcal{N}^{0}_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})} \\ &\quad \times \int\limits_{\substack{\mathcal{K} \\ |\xi|<|x|/2}} |\xi|^{-\Lambda_- -1-\beta+\sigma-\varepsilon} \prod_{k=1}^{N} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0,\mu_k-t-\varepsilon)-\delta_k+t+\sigma} d\xi \end{aligned} \tag{3.23}$$

From the conditions on β and δ it follows that $-\Lambda_- -1-\beta+\sigma > -3$ and $\min(0,\mu_k-t-\varepsilon)-\delta_k+t+\sigma > -2$. Hence the integral on the right-hand side of (3.23) is equal to $c\,|x|^{-\Lambda_- +2-\beta+\sigma-\varepsilon}$. Therefore,

$$|v(x)| \le c\,|x|^{-\beta+2-s+\sigma} \prod_{k=1}^{N} \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k-s-\varepsilon)} \|f\|_{\mathcal{N}^{0}_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})}$$

Using the inequality $\min(0,\mu_k-s-\varepsilon) \ge \min(0, 2-\delta_k+\sigma-s) = -\max(0,\delta_k-2+s-\sigma)$, we obtain the desired estimate for v. □

Analogously, the following lemma can be proved.

Lemma 3.8 *Let $f \in \mathcal{N}^{t,\sigma}_{\beta,\delta}(\mathcal{K})$ and*

$$v(x) = \int\limits_{\substack{\mathcal{K} \\ |\xi|>2|x|}} K(x,\xi)\, f(\xi)\, d\xi,$$

where the kernel K satisfies the estimate

$$|K(x,\xi)| \le c\,|x|^{\Lambda_+ - s + \varepsilon}\,|\xi|^{-\Lambda_+ - 1 - t - \varepsilon} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k - s - \varepsilon)} \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0,\mu_k - t - \varepsilon)}.$$

with nonnegative integers s, t. Suppose that β, δ satisfy conditions (3.12), (3.13). *Then* (3.22) *is valid with a constant c independent of x.*

Note that the functions $\partial_x^\alpha G_{i,j}(x,\xi)$ satisfy the assumption on the kernel K in Lemmas 3.7 and 3.8 with $s = |\alpha| + \delta_{i,4}$, $t = \delta_{j,4}$. For the proof of an analogous estimate for the integral over the set $\{x \in \mathcal{K} : |x|/2 < |\xi| < 2|x|\}$, we need the following lemma.

Lemma 3.9 *Let $\mathcal{D}$ be a dihedron, $x \in \mathcal{D}$ and $R \ge r(x)/4$. Then*

$$\int\limits_{\substack{\mathcal{D} \\ r(x)/4 < |x-\xi| < R}} |x-\xi|^{-\alpha}\, r(\xi)^{-\delta}\, d\xi \le c\, R^{3-\alpha-\delta} \quad \textit{if } \alpha+\delta < 3,\ \delta < 2, \tag{3.24}$$

$$\int\limits_{\substack{\mathcal{D} \\ |x-\xi| > R}} |x-\xi|^{-\alpha}\, r(\xi)^{-\delta}\, d\xi \le c\, R^{3-\alpha-\delta} \quad \textit{if } \alpha+\delta > 3,\ \delta < 2. \tag{3.25}$$

Here the constant c is independent of x and R.

P r o o f. 1) We denote the left-hand side of (3.24) by A. Substituting $x/r(x) = y$ and $\xi/r(x) = \eta$, we obtain

$$A = r(x)^{3-\alpha-\delta} \int\limits_{\substack{\mathcal{D} \\ 1/4 < |y-\eta| < R/r(x)}} |y-\eta|^{-\alpha}\, r(\eta)^{-\delta}\, d\eta,$$

where $r(y) = 1$. Obviously the integral

$$\int\limits_{\substack{\mathcal{D} \\ 1/4 < |y-\eta| < 2}} |y-\eta|^{-\alpha}\, r(\eta)^{-\delta}\, d\eta$$

is finite and can be estimated by a constant independent of y. We denote by $y^* = (0,0,y_3)$ the orthogonal projection of y onto the edge M. Then, for $2 < |y-\eta| < R/r(x)$, we have $1 < |\eta - y^*| < 1 + R/r(x)$, $2|\eta - y^*|/3 < |y-\eta| < 2|\eta - y^*|$ and, therefore,

$$\int\limits_{2 < |y-\eta| < R/r(x)} |y-\eta|^{-\alpha}\, r(\eta)^{-\beta}\, d\eta \le c \int\limits_{1 < |\eta - y^*| < 1 + R/r(x)} |\eta - y^*|^{-\alpha}\, r(\eta - y^*)^{-\delta}\, d\eta \le c\Big(\frac{R}{r(x)}\Big)^{3-\alpha-\delta.}$$

This proves (3.24).

2) If $\delta \le 0$, then $\alpha > 3$ and $r(\xi)^{-\delta} \le c\big(r(x)^{-\delta} + |x-\xi|^{-\delta}\big) \le c\big(R^{-\delta} + |x-\xi|^{-\delta}\big)$, and

$$\int\limits_{|x-\xi| > R} |x-\xi|^{-\alpha}\, r(\xi)^{-\delta}\, d\xi \le c \int\limits_{|x-\xi| > R} \big(|x-\xi|^{-\alpha} R^{-\delta} + |x-\xi|^{-\alpha-\delta}\big)\, d\xi \le c\, R^{3-\alpha-\delta}.$$

Let $\delta \ge 0$. Then

$$\int\limits_{\substack{\mathcal{D} \\ r(\xi) > |x-\xi| > R}} |x-\xi|^{-\alpha}\, r(\xi)^{-\delta}\, d\xi \le c \int\limits_{\substack{\mathcal{D} \\ |x-\xi| > R}} |x-\xi|^{-\alpha-\delta}\, d\xi \le c\, R^{3-\alpha-\delta}.$$

If $|x-\xi| > r(\xi) > R$, then $|\xi - x^*| \ge r(\xi) > R$ and $|\xi - x^*| \le |\xi-\xi^*| + |\xi^* - x^*| \le r(\xi) + |\xi - x| < 2|\xi - x|$. Here again $x^* = (0,0,x_3)$, $\xi^* = (0,0,\xi_3)$ denote the nearest points to x and ξ on M. Therefore,

$$\int\limits_{\substack{\mathcal{D}\\ |x-\xi|>r(\xi)>R}} |x-\xi|^{-\alpha}\, r(\xi)^{-\delta}\, d\xi \le c \int\limits_{\substack{\mathcal{D}\\ |\xi-x^*|>R}} |\xi - x^*|^{-\alpha}\, r(\xi-x^*)^{-\delta}\, d\xi \le c\, R^{3-\alpha-\delta}.$$

Finally, since $2|x-\xi| > R + |x_3 - \xi_3|$ for $|x-\xi| > R$, we obtain

$$\int\limits_{\substack{\mathcal{D}\\ |x-\xi|>R>r(\xi)}} |x-\xi|^{-\alpha}\, r(\xi)^{-\delta}\, d\xi \le c \int_{-\infty}^{+\infty} \big(R + |x_3-\xi_3|\big)^{-\alpha}\, d\xi_3 \int_0^R r^{1-\delta}\, dr = c\, R^{3-\alpha-\delta}.$$

This proves (3.25). □

Lemma 3.10 *Let $f \in \mathcal{N}^{t,\sigma}_{\beta,\delta}(\mathcal{K})$ and*

$$v(x) = \int\limits_{\substack{\mathcal{K}\\ |x|/2<|\xi|<2|x|}} K(x,\xi)\, f(\xi)\, d\xi,$$

where $K(x,\xi)$ vanishes for $|x-\xi| < r(x)/4$ and satisfies the estimate

$$|K(x,\xi)| \le c\, |x-\xi|^{-1-s-t} \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-s-\varepsilon)} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-t-\varepsilon)}.$$

with nonnegative integers s,t. Suppose that δ satisfy condition (3.12)*. Then* (3.22) *is valid with a constant c independent of x.*

P r o o f. Obviously,

$$|v(x)| \le c\, |x|^{\sigma+t-\beta}\, \|f\| \int\limits_{\substack{\mathcal{K}\\ |x|/2<|\xi|<2|x|}} |K(x,\xi)| \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\sigma+t-\delta_k} d\xi$$

(here by $\|f\|$, we mean the $\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}$-norm of f). If $r(x) > |x|/2$, then $|x|/2 < r_k(x) < |x|$ for $k = 1,\dots,N$ and $|x|/8 < |x-\xi| < 3|x|$ for all $\xi \in \mathcal{K}$ satisfying the inequalities $|x|/2 < |\xi| < 2|x|$, $|x-\xi| > r(x)/4$. This implies

$$|v(x)| \le c\, |x|^{-1-\beta-s+\sigma}\, \|f\| \int\limits_{\substack{\mathcal{K}\\ |x|/2<|\xi|<2|x|}} \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\sigma+t-\delta_k+\min(0,\mu_k-t-\varepsilon)} d\xi$$

Since $\sigma + t - \delta_k + \min(0,\mu_k - t - \varepsilon) \ge \sigma - \delta_k > -2$, it follows that

$$|v(x)| \le c\, |x|^{2-\beta-s+\sigma}\, \|f\| \le c\, |x|^{2-\beta-s+\sigma}\, \|f\| \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{-\max(0,\delta_k-2+s-\sigma)}.$$

This proves (3.22) for the case $r(x) > |x|/2$.

Suppose now that $r(x) = r_1(x) \le |x|/2$ and $f(\xi) = 0$ for $r_1(\xi) < 2r(\xi)$. Then there exist positive constants c_1, c_2 such that

$$c_1\, |x| < |x-\xi| < 3|x|, \quad \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-s-\varepsilon)} \le c_2 \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k-s-\varepsilon)}$$

for $\xi \in \operatorname{supp} K(x,\cdot)\, f$, $|x|/2 < |\xi| < 2|x|$. From this and from the inequality $\mu_k > 2 - \delta_k + \sigma$ we conclude that

$$\begin{aligned} |v(x)| &\le c\, |x|^{-1-\beta-s+\sigma}\, \|f\| \prod_{k=1}^{N} \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k-s-\varepsilon)} \\ &\quad \times \int\limits_{\substack{\mathcal{K} \\ |x|/2<|\xi|<2|x|}} \prod_{k=1}^{N} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\sigma+t-\delta_k+\min(0,\mu_k-t-\varepsilon)} d\xi \\ &\le c\, |x|^{2-\beta-s+\sigma}\, \|f\| \prod_{k=1}^{N} \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,2-s-\delta_k+\sigma)} \end{aligned}$$

what implies (3.22).

Next we suppose that $r(x) = r_1(x) \le |x|/2$ and $f(\xi) = 0$ for $r_1(\xi) > 3r(\xi)$. Then there exist positive constants c_1, c_2 such that

$$\Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-t-\varepsilon)} \le c_1 \Big(\frac{r_1(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_1-t-\varepsilon)}, \quad \prod_{k=1}^{N} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\sigma+t-\delta_k} \le c\Big(\frac{r_1(\xi)}{|\xi|}\Big)^{\sigma+t-\delta_1}$$

for $\xi \in \operatorname{supp} K(x,\cdot)\, f$, $|x|/2 < |\xi| < 2|x|$. Thus,

$$\begin{aligned} |v(x)| &\le c\, |x|^{\delta_1-\beta}\, r_1(x)^{\min(0,\mu_1-s-\varepsilon)}\, \|f\| \\ &\quad \times \int_{\mathcal{K}_x} |x-\xi|^{-1-s-t-\min(0,\mu_1-s-\varepsilon)-\min(0,\mu_1-t-\varepsilon)}\, r_1(\xi)^{\sigma+t-\delta_1+\min(0,\mu_1-t-\varepsilon)}\, d\xi, \end{aligned} \tag{3.26}$$

where $\mathcal{K}_x$ denotes the domain $\{\xi \in \mathcal{K} :\ |x|/2 < |\xi| < 2|x|,\ r_1(\xi) < 3r(\xi),\ |x-\xi| > r(x)/4\}$. Let $\delta_1 - \sigma > 2 - s$. Then according to (3.12), we have

$$-1 - s - \min(0, \mu_1 - s - \varepsilon) + \sigma - \delta_1 = -3 + \max(2 - s, 2 - \mu_1 + \varepsilon) + \sigma - \delta_1 < -3.$$

Hence, using (3.25) with $R = r_1(x)/4$, we can estimate the integral on the right of (3.26) by

$$c\, r_1(x)^{2-s-\min(0,\mu_1-s-\varepsilon)+\sigma-\delta_1}$$

what implies

$$|v(x)| \le c\, |x|^{\delta_1-\beta}\, r_1(x)^{2-\delta_1-s+\sigma}\, \|f\| = c\, |x|^{2-\beta-s+\sigma} \Big(\frac{r_1(x)}{|x|}\Big)^{2-\delta_1-s+\sigma} \|f\|.$$

If $2 - \mu_1 < \delta_1 - \sigma < 2 - s$, then $-1 - s - \min(0, \mu_1 - s - \varepsilon) + \sigma - \delta_1 = -1 - s + \sigma - \delta_1 > -3$. Consequently, using (3.24) with $R = 3|x|$, the integral on the right of (3.26) can be estimated by $c\, |x|^{2-\delta_1-s+\sigma}$ from what we obtain

$$|v(x)| \le c\, |x|^{2-\beta-s+\sigma}\, \|f\| = c\, |x|^{2-\beta-s+\sigma} \Big(\frac{r_1(x)}{|x|}\Big)^{\min(0,2-\delta_1-s+\sigma)} \|f\|$$

Thus, both in the cases $\delta_1 - \sigma > 2 - s$ and $\delta_1 - \sigma < 2 - s$, the estimate (3.22) follows.

It remains to note that every $f \in \mathcal{N}^{t,\sigma}_{\beta,\delta}(\mathcal{K})$ can be written as a sum $f = f_1 + f_2$ such that $f_1(\xi) = 0$ for $r_1(\xi) < 2r(\xi)$, $f_2(\xi) = 0$ for $r_1(\xi) > 3r(\xi)$ and

$$\|f_1\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})} + \|f_2\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})} \le c\, \|f\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})}$$

This completes the proof. □

Theorem 3.2 *Let $f \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $g \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})$, and let (u^-, p^-) be defined by* (3.16), (3.17), *where $G(x,\xi)$ is the Green matrix introduced in Section* 3.4. *Suppose that β, δ satisfy conditions* (3.12) *and* (3.13). *Then u^-, p^- satisfy* (3.20), (3.21).

P r o o f. It is sufficient to note that $K(x,\xi) = \partial_x^\alpha\big(\chi^-(x,\xi) G_{i,j}(x,\xi)\big)$ satisfies the conditions of Lemmas 3.7, 3.8 and 3.10 with $s = |\alpha| + \delta_{i,4}$, $t = \delta_{j,4}$ for $|\xi| < |x|/2$, $|\xi| > 2|x|$ and $|x|/2 < |\xi| < 2|x|$, respectively. Hence, the result follows immediately from the representation of u^-, p^- and from Lemmas 3.7, 3.8 and 3.10. □

3.8 Hölder estimates for u^-, p^-

Let $\mathcal{K}_\nu = \{x \in \mathcal{K} : r_\nu(x) < 3r(x)/2\}$ for $\nu = 1, \dots, N$.

Lemma 3.11 *Let $f \in \mathcal{N}^{t,\sigma}_{\beta,\delta}(\mathcal{K})$ and*

$$v(x) = \int_{\mathcal{K}} K(x,\xi)\, f(\xi)\, d\xi.$$

We assume that β, δ satisfy conditions (3.12), (3.13), $K(x,\xi)$ vanishes for $|x-\xi| > r(x)/4$ and that the estimates

$$\begin{aligned}
|\partial_\rho^s K(x,\xi)| &\le c\, |x-\xi|^{-3-s-t+k_\nu} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi - t-\varepsilon)} && \text{for } x \in \mathcal{K}_\nu,\ |x|/4 < |\xi| < 2|x|,\ s=0,1,\\
|\partial_\rho K(x,\xi)| &\le c\, |x|^{\Lambda_- -3+k_\nu+\varepsilon}\, |\xi|^{-\Lambda_- -1-t-\varepsilon} \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0,\mu_k-t-\varepsilon)} && \text{for } x \in \mathcal{K}_\nu,\ |\xi| < 3|x|/4,\\
|\partial_\rho K(x,\xi)| &\le c\, |x|^{\Lambda_+ -3+k_\nu-\varepsilon}\, |\xi|^{-\Lambda_+ -1-t+\varepsilon} \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0,\mu_k-t-\varepsilon)} && \text{for } x \in \mathcal{K}_\nu,\ |\xi| > 3|x|/2
\end{aligned}$$

are valid, where $k_\nu = 1 + [\delta_\nu - \sigma]$ and $\rho = |x|$. Then there ist the estimate

$$|x|^{\beta-\delta_\nu} \frac{|v(x)-v(y)|}{|x-y|^{\sigma-\delta_\nu+k_\nu}} \le c\, \|f\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})} \tag{3.27}$$

for $x \in \mathcal{K}_\nu$, $y = \tau x$, $4/5 < \tau < 5/4$, $|x-y| > r(x)/4$, where c is independent of f, x, τ.

P r o o f. Let $x \in \mathcal{K}_\nu$ and $y = \tau x$, $4/5 < \tau < 5/4$, $|x-y| > r(x)/4$. Then $4|x-y| < \min(|x|,|y|)$. We have $|v(x)-v(y)| \le A_1 + A_2 + A_3$, where

$$A_1 = \int\limits_{\substack{\mathcal{K}\\ |\xi-x|<2|x-y|}} |K(x,\xi)\, f(\xi)|\, d\xi, \quad A_2 = \int\limits_{\substack{\mathcal{K}\\ |\xi-x|<2|x-y|}} |K(y,\xi)\, f(\xi)|\, d\xi,$$
$$A_3 = \int\limits_{\substack{\mathcal{K}\\ |\xi-x|>2|x-y|}} \big|(K(x,\xi)-K(y,\xi))\, f(\xi)\big|\, d\xi,$$

From $|x-\xi| < 2|x-y| < |x|/2$ it follows that $|x|/2 < |\xi| < 3|x|/2$. Therefore,

$$A_1 \le c\, \|f\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})}\, |x|^{t+\sigma-\beta} \int |x-\xi|^{-3-t+k_\nu} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-t-\varepsilon)} \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{t+\sigma-\delta_k} d\xi,$$

where the domain of integration is contained in the set of all $\xi \in \mathcal{K}$ satisfying the inequalities $|x|/2 < |\xi| < 2|x|$, $r(x)/4 < |x-\xi| < 2|x-y|$. Since $k_\nu + \sigma - \delta_\nu > 0$ and $\min(0,\mu_k - t - \varepsilon) + t + \sigma - \delta_k > -2$, by virtue of (3.24), we obtain

$$A_1 \le c\, \|f\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})}\, |x|^{\delta_\nu-\beta}\, |x-y|^{\sigma-\delta_\nu+k_\nu}.$$

Analogously, this estimate holds for A_2. For the proof, one can use the fact that $|\xi-x| < 2|x-y|$ implies $|\xi-y| < 3|x-y|$ and $|y|/4 < |\xi| < 2|y|$.

We consider the expression A_3. By the mean value theorem, there is the inequality

$$\big|K(x,\xi) - K(y,\xi)\big| \le \big|\partial_\rho K(\tilde{x},\xi)\big| \cdot |x-y|,$$

where $\tilde{x}$ is a certain point on the line between x and y, i.e. $4|x|/5 < |\tilde{x}| < 5|x|/4$. Hence,

$$A_3 \le |x-y| \int\limits_{\substack{\mathcal{K}\\ |\xi-x|>2|x-y|}} \big|\partial_\rho K(\tilde{x},\xi)\, f(\xi)\big|\, d\xi.$$

Here, for the integral over the set of all $\xi \in \mathcal{K}$ such that $|\xi| < |x|/2$, we obtain

$$\begin{aligned}
&\int \big|\partial_\rho K(\tilde{x},\xi)\, f(\xi)\big|\, d\xi \\
&\le c\, \|f\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})}\ |x|^{\Lambda_- -3+k_\nu+\varepsilon} \int |\xi|^{\sigma-\beta-\Lambda_- -1-\varepsilon} \prod_{k=1}^n \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{t+\sigma-\delta_k+\min(0,\mu_k-t-\varepsilon)} d\xi \\
&\le c\, \|f\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})}\ |x|^{\sigma-\beta+k_\nu-1}.
\end{aligned}$$

The same estimate holds for the integral over the set of all $\xi \in \mathcal{K}$ such that $|\xi| > 2|x|$, while for the integral over the set of all $\xi \in \mathcal{K}$ satisfying $|x|/2 < |\xi| < 2|x|$ and $|\xi - x| > 2|x-y|$ we have

$$\begin{aligned}
&\int \big|\partial_\rho K(\tilde{x},\xi)\, f(\xi)\big|\, d\xi \\
&\le c\, \|f\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})}\ |x|^{t+\sigma-\beta} \int |x-\xi|^{-4-t+k_\nu} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-t-\varepsilon)} \prod_{k=1}^n \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{t+\sigma-\delta_k} d\xi \\
&\le c\, \|f\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})}\ |x|^{\delta_\nu-\beta}\, |x-y|^{\sigma-\delta_\nu+k_\nu-1}.
\end{aligned}$$

Here, we used the inequality $|x-\xi|/2 < |\tilde{x}-\xi| < 3|x-\xi|/2$ and the estimate (3.25) with $R = 2|x-y|$. Thus, we obtain

$$\begin{aligned}
A_3 &\le c\, \|f\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})}\ |x-y|\, \big(|x|^{\sigma-\beta+k_\nu-1} + |x|^{\delta_\nu-\beta}\, |x-y|^{\sigma-\delta_\nu+k_\nu-1}\big) \\
&\le c\, \|f\|_{\mathcal{N}^0_{\beta-t-\sigma,\delta-t-\sigma}(\mathcal{K})}\ |x|^{\delta_\nu-\beta}\, |x-y|^{\sigma-\delta_\nu+k_\nu}.
\end{aligned}$$

This proves the lemma. □

Theorem 3.3 *Let the condition of Theorem* 3.2 *be satisfied. Then* $u^- \in C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $p^- \in C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$, *and there is the inequality*

$$\|u^-\|_{C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|p^-\|_{C^{1,\sigma}_{\beta,\delta}(\mathcal{K})} \le c\Big(\|f\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})}\Big) \tag{3.28}$$

with a constant c independent of f and g.

P r o o f. According to Theorem 3.2, the vector function u^- satisfies (3.20). We show that

$$|x|^\beta \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k} \frac{\big|\partial_x^\alpha u^-(x) - \partial_y^\alpha u^-(y)\big|}{|x-y|^\sigma} \le c\Big(\|f\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})}\Big) \tag{3.29}$$

for $|\alpha| = 2$, $|x-y| < r(x)/2$. By the mean value theorem, we have

$$\partial^\alpha u(x) - \partial^\alpha u(y) = (x-y)\cdot \nabla\partial^\alpha u(\tilde{x}),$$

where $\tilde{x} = x + t(y-x)$, $t \in (0,1)$. Furthermore, for $|x-y| < r(x)/2$, there are the inequalities $|x|/2 < |\tilde{x}| < 3|x|/2$, $r_j(x)/2 < r_j(\tilde{x}) < 3r_j(x)/2$, $j = 1,\dots,N$. From this and from (1.5) it follows that

$$\begin{aligned}
|x|^\beta \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k} \frac{\big|\partial_x^\alpha u^-(x) - \partial_y^\alpha u^-(y)\big|}{|x-y|^\sigma} &\le |x|^\beta \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k} r(x)^{1-\sigma}\, |\nabla\partial^\alpha u(\tilde{x})| \\
&\le c\, |\tilde{x}|^{\beta+1-\sigma} \prod_{k=1}^N \Big(\frac{r_k(\tilde{x})}{|\tilde{x}|}\Big)^{\delta_k-1+\sigma} |\nabla\partial^\alpha u(\tilde{x})|
\end{aligned}$$

for $|\alpha| = 2$. This together with (3.20) implies (3.29). Analogously, the estimate

$$|x|^{\beta-\delta_\nu} \frac{\big|\partial_x^\alpha u^-(x) - \partial_y^\alpha u^-(y)\big|}{|x-y|^{k_\nu+\sigma-\delta_\nu}} \le c\Big(\|f\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})}\Big) \tag{3.30}$$

holds for $|\alpha| = 2 - k_\nu = 1 - [\delta_\nu - \sigma]$, $x, y \in \mathcal{K}_\nu$, $|x-y| < r(x)/2$, $\nu = 1, \dots, N$.

Next we prove that (3.30) is valid for $|\alpha| = 2-k_\nu$, $x \in \mathcal{K}_\nu$, $y = \tau|x|$, $1/2 < \tau < 3/2$. For $|x-y| < r(x)/2$ this is already shown. For $1/2 < \tau < 4/5$ or $5/4 < \tau < 3/2$, the left-hand side of (3.30) does not exceed

$$c \Big(|x|^{\beta-\sigma-k_\nu} \left|\partial_x^\alpha u^-(x)\right| + |y|^{\beta-\sigma-k_\nu} \left|\partial_y^\alpha u^-(y)\right| \Big)$$

and can estimated by means of (3.20). For $|x-y| > r(x)/2$, $\tau \in (4/5, 5/4)$, we may refer to Lemma 3.11, since $K(x,\xi) = \partial_x^\alpha \big(\chi^-(x,\xi)\, G_{i,j}(x,\xi)\big)$ satisfies the conditions of this lemma for $|\alpha| = 2 - \delta_{i,4} - k_\nu$, $t = \delta_{j,4}$, $k_\nu \le 2 - \delta_{i,4}$ (see Theorem 3.1, Remark 3.1).

Hence, the norm of u^- in $C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3$ is majorized by the right-hand side of (3.28). Analogously, the desired estimate for p^- can be proved by means of (3.21) and Lemma 3.11. The proof is complete. □

3.9 Solvability in $C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$

Theorem 3.4 *Let $f \in C^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $g \in C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$, $h_j \in C^{2,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in C^{1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}$, $j = 1, \dots, N$, where δ satisfies condition* (3.12) *and β is such that the line* $\operatorname{Re}\lambda = 2 + \sigma - \beta$ *is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$. Suppose that the boundary data h_j satisfy the compatibility condition* (3.6)*and that in the case $\delta_k < 1 + \sigma$ the functions $g, h_{k_\pm}, \phi_{k_\pm}$ satisfy the compatibility conditions given in Lemma* 2.3. *Then problem* (3.1), (3.2) *has a unique solution $(u, p) \in C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$.*

P r o o f. Due to condition (3.12), we have $\mu_k > 2$ if $\delta_k < \sigma$. This means, the number $\lambda = 2$ does not belong to the spectrum of the pencil $A_k(\lambda)$ if $\delta_k < \sigma$. Hence, Lemma 3.2 allows us to restrict ourselves in the proof to the case $f \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $g \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})$, $h_j = 0$, $\phi_j = 0$.

Let κ be an arbitrary real number such that the closed strip between the lines $\operatorname{Re}\lambda = 2 + \sigma - \beta$ and $\operatorname{Re}\lambda = -\kappa - 1/2$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$, and let $G(x,\xi)$ be the Green matrix introduced in Section 3.4. Then condition (3.13) is satisfied. We consider the vector-functions (u^+, p^+) and (u^-, p^-) defined by (3.16), (3.17).

By Theorem 3.3, the vector-function (u^-, p^-) belongs to the space $C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$, while (u^+, p^+) satisfies the estimates (3.18), (3.19). From the definition of u^-, p^- and from the equation

$$-\Delta_x G_{i,j}(x,\xi) + \partial_{x_i} G_{4,j}(x,\xi) = \delta_{i,j}\, \delta(x-\xi), \quad i = 1,2,3,$$

it follows that

$$-\Delta u_i^-(x) + \partial_{x_i} p^-(x) = -\frac{1}{2} \partial_{x_i} g(x) - \sum_{j=1}^{3} \int_{\mathcal{K}} \big(f_j(\xi) + \partial_{\xi_j} g(\xi)\big)\, K_{i,j}(x,\xi)\, d\xi - \int_{\mathcal{K}} g(\xi)\, K_{i,4}(x,\xi)\, d\xi,$$

where

$$K_{i,j}(x,\xi) = G_{i,j}(x,\xi)\, \Delta_x \chi^-(x,\xi) + 2 \nabla_x G_{i,j}(x,\xi) \cdot \nabla_x \chi^-(x,\xi) - G_{4,j}(x,\xi)\, \partial_{x_i} \chi^-(x,\xi).$$

The functions $K_{i,j}(x,\xi)$ vanish for $|x-\xi| < r(x)/4$ and for $|x-\xi| > r(x)/2$ and satisfy the estimate

$$\left|\partial_x^\alpha K_{i,j}(x,\xi)\right| \le c_\alpha\, r(x)^{-3-|\alpha|-\delta_{4,j}}$$

with constants c_α independent of x and ξ. Consequently,

$$\begin{aligned}
&\Big|\partial_x^\alpha \int_{\mathcal{K}} \big(f_j(\xi) + \partial_{\xi_j} g(\xi)\big)\, K_{i,j}(x,\xi)\, d\xi\Big| \\
&\le c\, \|f_j + \partial_{x_j} g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})}\ |x|^{\sigma-\beta} \prod_{k=1}^{N} \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma-\delta_k} \int_{\mathcal{K}} \left|\partial_x^\alpha K_{i,j}(x,\xi)\right| d\xi \\
&\le c\, \|f_j + \partial_{x_j} g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})}\ |x|^{\sigma-\beta-|\alpha|} \prod_{k=1}^{N} \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma-\delta_k-|\alpha|}
\end{aligned}$$

for arbitrary multi-indices α, $j = 1, 2, 3$, and analogously

$$\Big|\partial_x^\alpha \int_{\mathcal{K}} g(\xi)\, K_{i,4}(x,\xi)\, d\xi\Big| \le c\, \|g\|_{\mathcal{N}^0_{\beta-1-\sigma,\delta-1-\sigma}(\mathcal{K})}\ |x|^{\sigma-\beta-|\alpha|} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma-\delta_k-|\alpha|}.$$

Thus, $-\Delta u^- + \nabla p \in \mathcal{N}^1_{\beta+1-\sigma,\delta+1-\sigma}(\mathcal{K})^3 \subset \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3$ and

$$\| -\Delta u^- + \nabla p\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} \le c\,\big(\|f\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})}\big).$$

Analogously, we obtain $\nabla \cdot u^- \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})$, $S_j u^- = 0$ on Γ_j and $N_j(u^-,p^-) \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}$. Since $(u,p) = (u^+ + u^-, p^+ + p^-)$ is a solution of problem (3.1), (3.2), it follows that

$$\begin{aligned} &-\Delta u^+ + \nabla p^+ = f + \Delta u^- - \nabla p^- \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3, \quad -\nabla\cdot u^+ = g + \nabla \cdot u^- \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K}),\\ &S_j u^+|_{\Gamma_j} = 0, \quad N_j(u^+,p^+)|_{\Gamma_j} = -N_j(u^-,p^-)|_{\Gamma_j} \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\Gamma_j), \quad j = 1,\dots,N. \end{aligned}$$

Applying Lemma 3.3, we conclude that $(u^+,p^+) \in \mathcal{N}^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})$ and, therefore, $(u,p) \in C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$. This proves the existence of a solution.

We prove the uniqueness. Let $(u,p) \in C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$ be a solution of the homogeneous problem (3.1), (3.2), and let χ be a smooth cut-off function on $\bar{\mathcal{K}}$ equal to one for $|x| < 1$ and to zero for $|x| > 2$. Furthermore, let $\beta' = \beta - \sigma - 3/2$ and δ'_k be real numbers such that $\max(0, \delta_k - \sigma, 2 - \mu_k) - 1 < \delta'_k < 1$. We denote by $W^l_{\beta,\delta}(\mathcal{K})$ the weighted Sobolev space with the norm

$$\|u\|_{W^l_{\beta,\delta}(\mathcal{K})} = \Big(\int_{\mathcal{K}} \sum_{|\alpha|\le l} |x|^{2(\beta-l+|\alpha|)} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{2\delta_k}\ |\partial_x^\alpha u(x)|^2\, dx\Big)^{1/2}$$

and by $W^{l-1/2}_{\beta,\delta}(\Gamma_j)$ the corresponding trace space. Then $\chi(u,p) \in W^2_{\beta'+\varepsilon,\delta'}(\mathcal{K})^3 \times W^1_{\beta'+\varepsilon,\delta'}(\mathcal{K})$ and $(1-\chi)(u,p) \in W^2_{\beta'-\varepsilon,\delta'}(\mathcal{K})^3 \times W^1_{\beta'-\varepsilon,\delta'}(\mathcal{K})$, where ε is an arbitrary positive number. Consequently,

$$-\Delta(\chi u) + \nabla(\chi p) = \Delta(u - \chi u) - \nabla(p - \chi p) \in W^0_{\beta'-\varepsilon,\vec{\delta}'}(\mathcal{K})^3$$

and analogously, $\nabla\cdot(\chi u) \in W^1_{\beta'-\varepsilon,\delta'}(\mathcal{K})$, $S_j(\chi u) = 0$, $N_j(\chi u, \chi p) \in W^{1/2}_{\beta'-\varepsilon,\delta'}(\Gamma_j)$. Applying [23, Th.3.3], we obtain $\chi(u,p) \in W^2_{\beta'-\varepsilon,\delta'}(\mathcal{K})^3 \times W^1_{\beta'-\varepsilon,\delta'}(\mathcal{K})$ if ε is sufficiently small. Hence, $u \in W^2_{\beta'-\varepsilon,\vec{\delta}'}(\mathcal{K})^3 \times W^1_{\beta'-\varepsilon,\delta'}(\mathcal{K})$ and [23, Th.3.2] implies $u = 0$, $p = 0$. The proof of the theorem is complete. □

Theorem 3.5 *Let $(u,p) \in C^{2,\sigma'}_{\beta',\delta'}(\mathcal{K}) \times C^{1,\sigma'}_{\beta',\delta'}(\mathcal{K})$ be a solution of problem* (3.1), (3.2)*, where*

$$\begin{aligned} &f \in C^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3 \cap C^{0,\sigma'}_{\beta',\delta'}(\mathcal{K})^3, \quad g \in C^{1,\sigma}_{\beta,\delta}(\mathcal{K}) \cap C^{1,\sigma'}_{\beta',\delta'}(\mathcal{K}),\\ &h_j \in C^{2,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j} \cap C^{2,\sigma'}_{\beta',\delta'}(\Gamma_j)^{3-d_j}, \quad \phi_j \in C^{1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j} \cap C^{1,\sigma'}_{\beta',\delta'}(\Gamma_j)^{d_j}. \end{aligned}$$

We suppose that δ_k, δ'_k are nonnegative numbers such that $\delta_k - \sigma$ and $\delta'_k - \sigma'$ are not integer and

$$2 - \mu_k < \delta_k - \sigma < 2, \quad 2 - \mu_k < \delta'_k - \sigma' < 2$$

for $k = 1,\dots,N$. Furthermore, we assume that there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ in the closed strip between the lines $\operatorname{Re}\lambda = 2 + \sigma - \beta$ and $\operatorname{Re}\lambda = 2 + \sigma' - \beta'$ and that g, h_j, ϕ_j satisfy the same compatibility conditions as in Theorem 3.4*. Then $(u,p) \in C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$.*

P r o o f. 1) Let first $\delta = \delta'$ and $\sigma = \sigma'$. Then analogously to Lemma 3.2, there exist $v \in C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \cap C^{2,\sigma}_{\beta',\delta}(\mathcal{K})^3$ and $q \in C^{1,\sigma}_{\beta,\delta}(\mathcal{K}) \cap C^{1,\sigma}_{\beta',\delta}(\mathcal{K})$ such that

$$\Delta v - \nabla q + f \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3 \cap \mathcal{N}^{0,\sigma}_{\beta',\delta}(\mathcal{K})^3, \quad \nabla\cdot v + g \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K}) \cap \mathcal{N}^{1,\sigma}_{\beta',\delta}(\mathcal{K})$$

$S_jv = h_j$ and $N_j(v,q) = \phi_j$ on Γ_j. Therefore, we may assume without loss of generality that $f \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3 \cap \mathcal{N}^{0,\sigma}_{\beta',\delta}(\mathcal{K})^3$, $g \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K}) \cap \mathcal{N}^{1,\sigma}_{\beta',\delta}(\mathcal{K})$, $h_j = 0$, and $\phi_j = 0$. Then, as was shown in the proof Theorem 3.4, the solution $(u,p) \in C^{2,\sigma}_{\beta',\delta}(\mathcal{K}) \times C^{1,\sigma}_{\beta',\delta}(\mathcal{K})$ is given by (3.14), (3.15), where $G(x,\xi)$ is the Green matrix introduced in Section 3.4 with $\kappa = \beta - \sigma - 5/2$. However, the uniquely determined solution in $C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$ has also the representation (3.14), (3.15) with the same Green matrix $G(x,\xi)$. This proves the theorem in the case $\delta = \delta'$, $\sigma = \sigma'$.

2) By Theorem 3.4, there exists a unique solution $(v,q) \in C^{2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{K})$ of problem (3.1), (3.2). We put $\sigma'' = \min(\sigma, \sigma')$, $\delta'_k = \max(\delta_k - \sigma + \sigma'', \delta'_k - \sigma' + \sigma'', 0)$. Then

$$(u,p) \in C^{2,\sigma''}_{\beta-\sigma+\sigma'',\delta''}(\mathcal{K})^3 \times C^{1,\sigma''}_{\beta-\sigma+\sigma'',\delta''}(\mathcal{K}), \qquad (v,q) \in C^{2,\sigma''}_{\beta'-\sigma'+\sigma'',\delta''}(\mathcal{K})^3 \times C^{1,\sigma''}_{\beta'-\sigma'+\sigma'',\delta''}(\mathcal{K}).$$

From the first part of the proof and from Theorem 3.4 it follows that $(u,p) = (v,q)$. $\square$

3.10 Solvability in $C^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$

Theorem 3.6 *Let $(u,p) \in C^{2,\sigma'}_{\beta',\delta'}(\mathcal{K}) \times C^{1,\sigma'}_{\beta',\delta'}(\mathcal{K})$ be a solution of problem* (3.1), (3.2)*, where*

$$f \in C^{l-2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \cap C^{0,\sigma'}_{\beta',\delta'}(\mathcal{K}), \quad g \in C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K}) \cap C^{1,\sigma'}_{\beta',\delta'}(\mathcal{K}),$$
$$h_j \in C^{l,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j} \cap C^{2,\sigma'}_{\beta',\delta'}(\Gamma_j)^{3-d_j}, \quad \phi_j \in C^{l-1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j} \cap C^{1,\sigma'}_{\beta',\delta'}(\Gamma_j)^{d_j}.$$

We suppose that δ_k, δ'_k are nonnegative numbers such that $\delta_k - \sigma$ and $\delta'_k - \sigma'$ are not integer and

$$l - \mu_k < \delta_k - \sigma < l, \quad 2 - \mu_k < \delta'_k - \sigma' < 2.$$

Furthermore, we assume that there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ in the closed strip between the lines $\operatorname{Re}\lambda = l + \sigma - \beta$ and $\operatorname{Re}\lambda = 2 + \sigma' - \beta'$ and that in the case $\delta_k < l - 1 + \sigma$ the data g, h_j, ϕ_j satisfy the same compatibility conditions as in Theorem 3.4*. Then $(u,p) \in C^{l,\sigma}_{\beta,\delta}(\mathcal{K}) \times C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$.*

P r o o f. 1) Suppose that $l - 2 \le \delta_k < l + \sigma$ for $k = 1, \dots, N$. Then

$$f \in C^{0,\sigma}_{\beta-l+2,\delta-l+2}(\mathcal{D})^3, \quad g \in C^{1,\sigma}_{\beta-l+2,\delta-l+2}(\mathcal{D}), \quad h_j \in C^{2,\sigma}_{\beta-l+2,\delta-l+2}(\Gamma_j)^{3-d_j}, \quad \phi_j \in C^{1,\sigma}_{\beta-l+2,\delta-l+2}(\Gamma_j)^{d_j},$$

and Theorem 3.5 implies $(u,p) \in C^{2,\sigma}_{\beta-l+2,\delta-l+2}(\mathcal{D})^3 \times C^{1,\sigma}_{\beta-l+2,\delta-l+2}(\mathcal{D})$. Using Lemma 3.4, we obtain $(u,p) \in C^{l,\sigma}_{\beta,\delta}(\mathcal{D})^3 \times C^{l-1,\sigma}_{\beta,\delta}(\mathcal{D})$.

2) Let $l - m \le \delta_k < l - m + 1$ for $k = 1, \dots, N$, where m is an integer, $2 \le m \le l$. Then we prove the assertion of the theorem by induction in m. For $m = 2$ we may refer to part 1). Suppose that $m \ge 3$ and the assertion of the theorem is true for $\delta_k \ge l - m + 1$. Since

$$f \in C^{l-2,\sigma}_{\beta,\delta}(\mathcal{K})^3 \subset C^{l-3,\sigma}_{\beta-1,\delta}(\mathcal{K})^3, \quad g \in C^{l-2,\sigma}_{\beta-1,\delta}(\mathcal{K}), \quad h_j \in C^{l-1,\sigma}_{\beta-1,\delta}(\Gamma_j)^{3-d_j}, \quad \phi_j \in C^{l-2,\sigma}_{\beta-1,\delta}(\Gamma_j)^{d_j},$$

the induction hypothesis implies $(u,p) \in C^{l-1,\sigma}_{\beta-1,\delta}(\mathcal{K})^3 \times C^{l-2,\sigma}_{\beta-1,\delta}(\mathcal{K})$. Furthermore, from Lemma 3.4 it follows that $(u,p) \in C^{l,\sigma}_{\beta,\delta+1}(\mathcal{K})^3 \times C^{l-1,\sigma}_{\beta,\delta+1}(\mathcal{K})$. Consequently, $\rho\partial_\rho u \in C^{l-1,\sigma}_{\beta-1,\delta+1}(\mathcal{K})^3 \subset C^{2,\sigma}_{\beta-l+2,2-\varepsilon}(\mathcal{K})^3$, where $0 < \varepsilon < \min(2, l - \delta_k - 2)$. Analogously, $\rho\partial_\rho p \in C^{1,\sigma}_{\beta-l+2,2-\varepsilon}(\mathcal{K})$. From the equalities

$$-\Delta(\rho\partial_\rho u) + \nabla(\rho\partial_\rho p + p) = (\rho\partial_\rho + 2)f \in C^{l-3,\sigma}_{\beta-1,\delta}(\mathcal{K})^3, \quad -\nabla \cdot \rho\partial_\rho u = (\rho\partial_\rho + 1)g \in C^{l-2,\sigma}_{\beta-1,\delta}(\mathcal{K}),$$
$$S_j\rho\partial_\rho u = \rho\partial_\rho h_j \in C^{l-1,\sigma}_{\beta-1,\delta}(\Gamma_j)^{3-d_j}, \quad N_j(\rho\partial_\rho u, \rho\partial_\rho p + p) = (\rho\partial_\rho + 1)\phi_j \in C^{l-2,\sigma}_{\beta-1,\delta}(\Gamma_j)^{d_j}$$

and from the induction hypothesis it follows that $\rho\partial_\rho(u,p) \in C^{l-1,\sigma}_{\beta-1,\delta}(\mathcal{K})^3 \times C^{l-2,\sigma}_{\beta-1,\delta}(\mathcal{K})$. Applying Lemma 3.5, we obtain $(u,p) \in C^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$.

3) Finally, we assume that $l - \delta_j \in (m_j - 1, m_j]$ for $j = 1, \dots, N$ with different $m_j \in \{0, 1, \dots, l\}$. Then let $\psi_1, \dots, \psi_n$ be smooth functions on $\bar{\Omega}$ such that $\psi_j \ge 0$, $\psi_j = 1$ near $M_j \cap S^2$, and $\sum \psi_j = 1$. We extend ψ_j to $\mathcal{K}$ by the equality $\psi_j(x) = \psi_j(x/|x|)$. Then $\partial_x^\alpha \psi_j(x) \le c\,|x|^{-|\alpha|}$. Using the first two parts

of the proof, one can show by induction in l that $\psi_j u \in C^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3$ and $\psi_j p \in C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$ for $j = 1, \dots, N$. The proof of the theorem is complete. $\Box$

In [22], we considered weak solutions $(u,p) \in V^1_\kappa(\mathcal{K})^3 \times V^0_\kappa(\mathcal{K})$ of problem (3.1), (3.2), i.e., solutions of the problem

$$b(u,v) - \int_{\mathcal{K}} p\, \nabla\cdot v\, dx = F(v) \quad \text{for all } v \in V_{-\kappa}, \tag{3.31}$$

$$-\nabla\cdot u = g \ \text{ in } \mathcal{K}, \quad S_j u = h_j \ \text{ on } \Gamma_j,\ j = 1,\dots,N, \tag{3.32}$$

where $V_{-\kappa} = \{v \in V^1_{-\kappa}(\mathcal{K})^3,\ S_j v = 0 \text{ on } \Gamma_j,\ j = 1,\dots,N\}$ and

$$b(u,v) = 2\int_{\mathcal{K}} \sum_{i,j=1}^3 \varepsilon_{i,j}(u)\ \varepsilon_{i,j}(v)\, dx$$

According to [22, Th.4.2], problem (3.31), (3.32) is uniquely solvable in $V^1_\kappa(\mathcal{K})^3 \times V^0_\kappa(\mathcal{K})$ for arbitrary $F \in V^*_{-\kappa}$, $g \in V^0_\kappa(\mathcal{K})$, $h_j \in V^{1/2}_\kappa(\Gamma_j)^{3-d_j}$ provided the line $\operatorname{Re}\lambda = -\kappa - 1/2$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and the boundary data h_j are such that there exists a vector function $w \in V^1_\kappa(\mathcal{K})^3$ satisfying $S_j w = h_j$ on Γ_j for $j = 1,\dots,n$.

Theorem 3.7 *Let $g \in V^0_\kappa(\mathcal{K}) \cap C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$, $h_j \in V^{1/2}_\kappa(\Gamma_j)^{3-d_j} \cap C^{l,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, and let the functional $F \in V^*_{-\kappa}$ have the form*

$$F(v) = \int_{\mathcal{K}} (f + \nabla g)\cdot v\, dx + \sum_{j=1}^n \int_{\Gamma_j} \phi_j \cdot v\, dx,$$

where $f \in C^{l-2,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $\phi_j \in C^{l-1,\sigma}_{\beta,\delta}(\Gamma_j)^3$, $\phi_j = (\phi_j\cdot n)\, n$ if $d_j = 1$, $\phi_j \cdot n = 0$ if $d_j = 2$ (if $d_j = 0$, then the integral over Γ_j does not appear in the representation of F). Suppose that on the lines lines $\operatorname{Re}\lambda = -\kappa - 1/2$ and $\operatorname{Re}\lambda = l + \sigma - \beta$, there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$, the components δ_k of δ satisfy the condition

$$l - \mu_k < \delta_k - \sigma < l, \quad \delta_k \ge 0, \quad \delta_k - \sigma \ \textit{not integer}, \tag{3.33}$$

and the data g, h_j and ϕ_j satisfy the same compatibility conditions as in Theorem 3.4. Then the solution $(u,p) \in V^1_\kappa(\mathcal{K})^3 \times V^0_\kappa(\mathcal{K})$ of problem (3.31), (3.32) admits the decomposition

$$(u,p) = \sum_{\nu=1}^N \sum_{j=1}^{I_\nu} \sum_{s=0}^{\kappa_{\nu,j}-1} c_{\nu,j,s} \sum_{\sigma=0}^s \frac{1}{\sigma!}\, (\log\rho)^\sigma\, \big(\rho^{\lambda_\nu} u^{(\nu,j,s-\sigma)}(\omega), \rho^{\lambda_\nu - 1} p^{(\nu,j,s-\sigma)}(\omega)\big) + (w,q), \tag{3.34}$$

where $(w,q) \in C^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$ is a solution of problem (3.1)–(3.2), λ_ν are the eigenvalues of the pencil $\mathfrak{A}$ between the lines $\operatorname{Re}\lambda = -\kappa - 1/2$ and $\operatorname{Re}\lambda = l + \sigma - \beta$ and $\big(u^{(\nu,j,s)}, p^{(\nu,j,s)}\big)$ are eigenvectors and generalized eigenvectors corresponding to the eigenvalue λ_ν.

P r o o f. By Theorems 3.4 and 3.6, there exists a solution $(w,q) \in C^{l,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{l-1,\sigma}_{\beta,\delta}(\mathcal{K})$ of problem (3.1), (3.2). Let ζ, η be smooth functions on $\bar{\mathcal{K}}$ equal to one near the vertex which vanish outside the unit ball and satisfy the equality $\zeta\eta = \zeta$. Due to the inequality $\delta_k < l + \sigma$, we have

$$\zeta(w,q) \in V^1_{\beta'+\varepsilon}(\mathcal{K})^3 \times V^0_{\beta'+\varepsilon}(\mathcal{K}), \quad (1-\zeta)(w,q) \in V^1_{\beta'-\varepsilon}(\mathcal{K})^3 \times V^0_{\beta'-\varepsilon}(\mathcal{K}),$$

where $\beta' = \beta - l - \sigma - 1/2$ and ε is an arbitrarily small positive number. Since (w,q) satisfies (3.31) for all $v \in C_0^\infty(\bar{\mathcal{K}}\backslash\{0\})^3$, $S_j v = 0$ on Γ_j, we obtain

$$b(\zeta w, v) - \int_{\mathcal{K}} \zeta q\, \nabla\cdot v\, dx = \Phi(v) \stackrel{def}{=} F(\eta v) - b\big((1-\zeta)w, \eta v\big) + \int_{\mathcal{K}} (1-\zeta) q\, \nabla\cdot(\eta v)\, dx.$$

Obviously, $\Phi \in V^*_{-\kappa}$. Furthermore,

$$-\nabla\cdot(\zeta w) = \zeta g - w\cdot\nabla\zeta \in V^0_\kappa(\mathcal{K}) \quad \text{and} \quad S_j(\zeta w) = \zeta h_j \in V^{1/2}_\kappa(\Gamma_j)^{3-d_j}.$$

From [22, Th.4.4] it follows that there exists a sum Σ_1 of the same form as in (3.34) such that $\zeta(w,q)-\Sigma_1 \in V^1_\kappa(\mathcal{K})^3 \times V^0_\kappa(\mathcal{K})$. Analogously, we obtain $(1-\zeta)\,(w,q) - \Sigma_2 \in V^1_\kappa(\mathcal{K})^3 \times V^0_\kappa(\mathcal{K})$, where Σ_2 is also a sum of the same form as in (3.34). Consequently, there is the representation

$$(w,q) = \Sigma_1 + \Sigma_2 + (w',q'),$$

where $(w',q') \in V^1_\kappa(\mathcal{K})^3 \times V^0_\kappa(\mathcal{K})$. Since Σ_1 and Σ_2 are solutions of the homogeneous problem (3.1), (3.2), the vector function (w',q') solves problem (3.31), (3.32). By virtue of the uniqueness of the solution in $V^1_\kappa(\mathcal{K})^3 \times V^0_\kappa(\mathcal{K})$ (see [22, Th.4.2]), we conclude that $(w',q') = (u,p)$. This proves the theorem. $\square$

4 Weak solutions in weighted Hölder spaces

In this section, we prove the existence of solutions $(u,p) \in C^{1,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{0,\sigma}_{\beta,\delta}(\mathcal{K})$ to problem (3.1), (3.2) for data

$$f \in C^{-1,\sigma}_{\beta,\delta}(\mathcal{K})^3, \quad g \in C^{0,\sigma}_{\beta,\delta}(\mathcal{K}), \quad h_j \in C^{1,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}, \quad \phi_j \in C^{1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}.$$

Here, $C^{-1,\sigma}_{\beta,\delta}(\mathcal{K})$ denotes the set of all distributions of the form

$$f = f^{(0)} + \sum_{k=1}^{3} \partial_{x_k} f^{(k)}, \quad \text{where} \quad f^{(0)} \in C^{0,\sigma}_{\beta+1,\delta+1}(\mathcal{K}),\ f^{(k)} \in C^{0,\sigma}_{\beta,\delta}(\mathcal{K}),\ k=1,2,3.$$

4.1 Representation of weak solutions by Green's matrix

Let $G(x,\xi)$ be the Green matrix introduced in Section 3.4. We suppose that β and δ are such that

$$1-\mu_k < \delta_k - \sigma < 1, \quad \delta_k \ge 0, \quad \delta_k \ne \sigma \quad \text{for } k=1,\dots,N, \tag{4.1}$$

$$\Lambda_- < 1+\sigma-\beta < \Lambda_+, \tag{4.2}$$

where Λ_+, Λ_- are as in Section 3.4. We consider the functions (3.14), (3.15) for

$$f_j = F_j + \nabla\cdot\Phi^{(j)}, \quad F_j \in C^{0,\sigma}_{\beta+1,\delta+1}(\mathcal{K}),\ \Phi^{(j)} \in C^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3, \tag{4.3}$$

and $g \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})$. If δ satisfies (4.1), then $C^{0,\sigma}_{\beta+1,\delta+1}(\mathcal{K}) = \mathcal{N}^{0,\sigma}_{\beta+1,\delta+1}(\mathcal{K})$. Furthermore, every $\Phi^{(j)} \in C^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3$ can be written as

$$\Phi^{(j)} = \Psi^{(j)} + f^{(j)}, \quad \text{where } \Psi^{(j)} \in C^{1,\sigma}_{\beta+1,\delta+1}(\mathcal{K})^3,\ f^{(j)} \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3$$

(cf. Lemma 1.2). Consequently, every f_j given by (4.3) has also the representation

$$f_j = f_j^{(0)} + \nabla\cdot f^{(j)}, \quad f_j^{(0)} \in \mathcal{N}^{0,\sigma}_{\beta+1,\delta+1}(\mathcal{K}),\ f^{(j)} \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3. \tag{4.4}$$

Here, $f_j^{(0)} = F_j + \nabla\cdot\Psi^{(j)}$. If the vector function $f^{(j)}$ belongs to the subspace $\mathcal{N}^{1,\sigma}_{\beta+1,\delta+1}(\mathcal{K})^3$ of $\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3$, then integration by parts yields

$$\int_{\mathcal{K}} (\nabla\cdot f^{(j)})\, v\, dx = -\int_{\mathcal{K}} f^{(j)}\cdot\nabla v\, dx + \sum_{\nu=1}^{N} \int_{\Gamma_\nu} f^{(j)} v\cdot n\, dx$$

for arbitrary $v \in C^\infty_{\sigma,\beta,\delta}$. Here, $C^\infty_{\sigma,\beta,\delta}$ denotes the set of all $v \in C^\infty(\bar{\mathcal{K}}\backslash\mathcal{S})$ such that

$$\int_{\mathcal{K}} |x|^{\sigma-\beta} \prod_{k=1}^{N} \Big(\frac{r_k(x)}{|x|}\Big)^{\sigma-\delta_k} \big(r(x)^{-1}\,|v(x)| + |\nabla v(x)|\big)\, dx < \infty.$$

Therefore, it is natural to extend the distributions f_j to $C^\infty_{\sigma,\beta,\delta}$ by

$$f_j(v) = \int_{\mathcal{K}} \big(f_j^{(0)}\, v - f^{(j)}\cdot\nabla v\big)\, dx + \sum_{\nu=1}^{N} \int_{\Gamma_\nu} f^{(j)} v\cdot n\, dx.$$

Then for the functions (3.14), (3.15), we obtain the representation

$$u_i = u_i^+ + u_i^-, \quad p = p^+ + p^-,$$

where

$$\begin{aligned} u_i^\pm(x) &= \sum_{j=1}^3 \int_{\mathcal{K}} \Big(f_j^{(0)}(\xi)\, \chi^\pm(x,\xi)\, G_{i,j}(x,\xi) \, - f^{(j)}(\xi)\cdot \nabla_\xi \big(\chi^\pm(x,\xi)\, G_{i,j}(x,\xi)\big)\Big)\, d\xi \\ &\quad + \int_{\mathcal{K}} g(\xi) \Big(\chi^\pm(x,\xi)\, G_{i,4}(x,\xi) - \sum_{j=1}^3 \partial_{\xi_j}\big(\chi^\pm(x,\xi)\, G_{i,j}(x,\xi)\big)\Big)\, d\xi \\ &\quad + \sum_{j=1}^3 \sum_{\nu=1}^N \int_{\Gamma_\nu} \big(f^{(j)}(\xi)\cdot n + g(\xi)\, n_j\big)\, \chi^\pm(x,\xi)\, G_{i,j}(x,\xi)\, d\xi \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} p^\pm(x) &= -\frac{g(x)}{2} + \sum_{j=1}^3 \int_{\mathcal{K}} \Big(f_j^{(0)}(\xi)\, \chi^\pm(x,\xi)\, G_{4,j}(x,\xi) \, - f^{(j)}(\xi)\cdot \nabla_\xi \big(\chi^\pm(x,\xi)\, G_{4,j}(x,\xi)\big)\Big)\, d\xi \\ &\quad + \int_{\mathcal{K}} g(\xi) \Big(\chi^\pm(x,\xi)\, G_{4,4}(x,\xi) - \sum_{j=1}^3 \partial_{\xi_j}\big(\chi^\pm(x,\xi)\, G_{4,j}(x,\xi)\big)\Big)\, d\xi \\ &\quad + \sum_{j=1}^3 \sum_{\nu=1}^N \int_{\Gamma_\nu} \big(f^{(j)}(\xi)\cdot n + g(\xi)\, n_j\big)\, \chi^\pm(x,\xi)\, G_{4,j}(x,\xi)\, d\xi. \end{aligned} \tag{4.6}$$

Here, χ^+ and χ^- are the same functions as in Section 3.5.

4.2 Weighted L_∞ estimates for u^+, p^+

Lemma 4.1 *Under the assumption of Section* 4.1*, there is the estimate*

$$\|u^+\|_{\mathcal{N}^0_{\beta-1-\sigma,\delta-1-\sigma}(\mathcal{K})^3} + \|p^+\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})} \le c \Big(\sum_{j=1}^3 \|f_j^{(0)}\|_{\mathcal{N}^{0,\sigma}_{\beta+1,\delta+1}(\mathcal{K})} + \sum_{j=1}^3 \|f^{(j)}\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})}\Big) \tag{4.7}$$

with a constant independent of $f_j^{(0)}, f^{(j)}$*, and* g.

P r o o f. The estimate for u^+ holds easily by means of the estimate

$$\big|\partial_\xi^\alpha \big(\chi^+(x,\xi)\, G_{i,j}(x,\xi)\big)\big| \le c\, |x-\xi|^{-1-|\alpha|-\delta_{i,4}-\delta_{j,4}} \tag{4.8}$$

and of the inequalities

$$|x-\xi| \le r_k(x)/2, \quad |x|/2 \le |\xi| \le 3|x|/2, \quad r_k(x)/2 \le r_k(\xi) \le 3r_k(x)/2 \qquad \text{on } \operatorname{supp} \chi^+ \tag{4.9}$$

(cf. proof of Lemma 3.6). In the same way, the integrals

$$\int_{\mathcal{K}} f_j^{(0)}(\xi)\, \chi^+(x,\xi)\, G_{4,j}(x,\xi)\, d\xi$$

in the representation of p^+ can be estimated. We consider the expressions

$$\begin{aligned} A_j &= -\int_{\mathcal{K}} f^{(j)}(\xi)\cdot \nabla_\xi \big(\chi^+(x,\xi)\, G_{4,j}(x,\xi)\big)\, d\xi + \sum_{\nu=1}^N \int_{\Gamma_\nu} f^{(j)}(\xi)\cdot \chi^+(x,\xi)\, G_{4,j}(x,\xi)\, n\, d\xi, \\ B_j &= -\int_{\mathcal{K}} g(\xi)\, \partial_{\xi_j}\big(\chi^+(x,\xi)\, G_{4,j}(x,\xi)\big)\, d\xi + \sum_{\nu=1}^N \int_{\Gamma_\nu} g(\xi)\, \chi^+(x,\xi)\, G_{4,j}(x,\xi)\, n_j\, d\xi, \\ C &= \int_{\mathcal{K}} g(\xi)\, \chi^+(x,\xi)\, G_{4,4}(x,\xi)\, d\xi \end{aligned}$$

in the representation of p^+. Since

$$\begin{aligned} A_j &= \int_{\mathcal{K}} \big(\nabla_\xi \cdot f^{(j)}(\xi)\big)\, \chi^+(x,\xi)\, G_{4,j}(x,\xi)\, d\xi = \int_{\mathcal{K}} \big(\nabla_\xi \cdot (f^{(j)}(\xi) - f^{(j)}(x))\big)\, \chi^+(x,\xi)\, G_{4,j}(x,\xi)\, d\xi \\ &= -\int_{\mathcal{K}} \big(f^{(j)}(\xi) - f^{(j)}(x)\big) \cdot \nabla_\xi \big(\chi^+(x,\xi)\, G_{4,j}(x,\xi)\big)\, d\xi \\ &\quad + \sum_{\nu=1}^N \int_{\Gamma_\nu} \big(f^{(j)}(\xi) - f^{(j)}(x)\big) \cdot \chi^+(x,\xi)\, G_{4,j}(x,\xi)\, n\, d\xi, \end{aligned}$$

we obtain by means of (4.8) and (4.9),

$$\begin{aligned} |A_j| &\le c\, |x|^{-\beta} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{-\delta_k} \|f^{(j)}\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} \\ &\quad \times \Big(\int_{\mathcal{K}} |x-\xi|^\sigma \, \big|\nabla_\xi \big(\chi^+(x,\xi)\, G_{4,j}(x,\xi)\big)\big|\, d\xi + \sum_{\nu=1}^N \int_{\Gamma_\nu} |x-\xi|^\sigma \, \big|\chi^+(x,\xi)\, G_{4,j}(x,\xi)\big|\, d\xi \Big) \\ &\le c\, |x|^{-\beta} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{-\delta_k} r(x)^\sigma \|f^{(j)}\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} \le c\, |x|^{-\beta+\sigma} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{-\delta_k+\sigma} \|f^{(j)}\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3}. \end{aligned}$$

Analogously, this estimates holds for B_j. For the estimation of C, we use the representation $G_{4,4}(x,\xi) = -\nabla_\xi \cdot \mathcal{P}(x,\xi) + \mathcal{Q}(x,\xi)$, where $\mathcal{P}(x,\xi)\cdot n = 0$ for $\xi \in \Gamma_\nu$, $|\partial_\xi^\alpha \mathcal{P}(x,\xi)| \le c\, |x-\xi|^{-2-|\alpha|}$, $|\mathcal{Q}(x,\xi)| \le c\, r(\xi)^{-3}$ for $|x-\xi| < r(x)/2$ (see Theorem 3.1). This implies

$$C = \int_{\mathcal{K}} \Big(\big(g(\xi) - g(x)\big)\, \chi^+(x,\xi)\, \nabla_\xi \cdot \mathcal{P}(x,\xi) - g(x)\, \nabla_\xi \chi^+(x,\xi) \cdot \mathcal{P}(x,\xi) + g(\xi)\, \chi^+(x,\xi) \mathcal{Q}(x,\xi) \Big)\, d\xi.$$

Due to (4.8) and (4.9), we have

$$\begin{aligned} & |x|^\beta \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k} \Big| \int_{\mathcal{K}} \Big(\big(g(\xi) - g(x)\big)\, \chi^+(x,\xi)\, \nabla_\xi \cdot \mathcal{P}(x,\xi)\, d\xi \Big| \\ & \le c\, \|g\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})} \int\limits_{|x-\xi|<r(x)/2} |x-\xi|^{-3+\sigma}\, d\xi \le c\, \|g\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})}\, r(x)^\sigma \end{aligned}$$

and

$$|x|^{\beta-\sigma} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k-\sigma} \Big| \int_{\mathcal{K}} \Big(g(x)\, \nabla_\xi \chi^+(x,\xi) \cdot \mathcal{P}(x,\xi) - g(\xi)\, \chi^+(x,\xi) \mathcal{Q}(x,\xi) \Big)\, d\xi \Big| \le c\, \|g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})}.$$

This proves the desired inequality for p^+. □

4.3 Hölder estimates for u^-, p^-

First, we show that the functions u^- and p^- defined by (4.5), (4.6) satisfy the estimates

$$\begin{aligned} & |x|^{\beta-1-\sigma+|\alpha|} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-1-\sigma+|\alpha|)} \big|\partial_x^\alpha u^-(x)\big| + |x|^{\beta-\sigma+|\alpha|} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-\sigma+|\alpha|)} \big|\partial_x^\alpha p^-(x)\big| \\ & \le c \Big(\sum_{j=1}^3 \|f_j^{(0)}\|_{\mathcal{N}^{0,\sigma}_{\beta+1,\delta+1}(\mathcal{K})} + \sum_{j=1}^3 \|f^{(j)}\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})} \Big) \end{aligned} \tag{4.10}$$

for arbitrary $x \in \mathcal{K}$ and for arbitrary multi-index α if β and δ satisfy conditions (4.1) and (4.2). The integrals over $\mathcal{K}$ in the representations for u^- and p^- can be estimated by means of Lemmas 3.7, 3.8 and 3.10. In order to estimate the integrals over the sides Γ_ν, we prove the following lemma.

Lemma 4.2 *Let $g \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})$ and let*

$$v(x) = \int_{\Gamma_\nu} K(x,\xi)\, g(\xi)\, d\xi,$$

where $K(x,\xi)$ vanishes for $|x-\xi| < r(x)/4$ and satisfies the inequalities

$$\begin{aligned}
\big|K(x,\xi)\big| &\le c\, |x|^{\Lambda_- - s+\varepsilon}\, |\xi|^{-\Lambda_- -1-\varepsilon} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k-s-\varepsilon)} && \text{for } |\xi| < |x|/2,\\
\big|K(x,\xi)\big| &\le c\, |x|^{\Lambda_+ - s-\varepsilon}\, |\xi|^{-\Lambda_+ -1+\varepsilon} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k-s-\varepsilon)} && \text{for } |\xi| > 2|x|,\\
\big|K(x,\xi)\big| &\le c\, |x-\xi|^{-1-s} \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-s-\varepsilon)} && \text{for } |x|/2 < |\xi| < 2|x|
\end{aligned}$$

with an arbitrarily small positive ε. Suppose that β an δ satisfy conditions (4.1) *and* (4.2)*. Then there is the estimate*

$$|x|^{\beta-1-\sigma+s} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-1-\sigma+s)} \big|v(x)\big| \le c\, \|g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})} \tag{4.11}$$

P r o o f. By the assumptions on $K(x,\xi)$, we have

$$\begin{aligned}
&|x|^{\beta-1-\sigma+s} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-1-\sigma+s)} \Big| \int\limits_{\substack{\Gamma_\nu\\ |\xi|<|x|/2}} K(x,\xi)\, g(\xi)\, d\xi\Big| \\
&\le c\, |x|^{\beta-1-\sigma+\Lambda_-+\varepsilon} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-1-\sigma+s)+\min(0,\mu_k-s-\varepsilon)} \|g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})} \\
&\quad\times \int\limits_{|\xi|<|x|/2} |\xi|^{\sigma-\beta-\Lambda_- -1-\varepsilon} \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\sigma-\delta_k} d\xi.
\end{aligned}$$

Since $\sigma-\beta-\Lambda_- -1 > -2$, $\sigma-\delta_k > -1$ and $\max(0,\delta_k-1-\sigma+s)+\min(0,\mu_k-s-\varepsilon) \ge 0$, the right-hand side of the last inequality does not exceed the right hand side of (4.11). Analogously,

$$|x|^{\beta-1-\sigma+s} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-1-\sigma+s)} \Big| \int\limits_{\substack{\Gamma_\nu\\ |\xi|>2|x|}} K(x,\xi)\, g(\xi)\, d\xi\Big| \le c\, \|g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})} .$$

We consider the expression

$$A = |x|^{\beta-1-\sigma+s} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-1-\sigma+s)} \Big| \int\limits_{\substack{\Gamma_\nu\\ |x|/2<|\xi|<2|x|}} K(x,\xi)\, g(\xi)\, d\xi\Big|$$

and assume that M_1 is the nearest edge to x. If $M_1 \cap \bar{\Gamma}_\nu = \{0\}$, then there exists a positive constant c such that $|x-\xi| > c|x|$ for $\xi \in \Gamma_\nu$, and we obtain

$$A \le c\, |x|^{\beta-2-\sigma} \Big(\frac{r_1(x)}{|x|}\Big)^{\max(0,\delta_1-1-\sigma+s)+\min(0,\mu_1-s-\varepsilon)} \|g\| \int\limits_{\substack{\Gamma_\nu\\ |x|/2<|\xi|<2|x|}} |\xi|^{\sigma-\beta} \prod_{k=1}^N \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\sigma-\delta_k} d\xi.$$

Here by $\|g\|$, we mean the $\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}$ norm of g. Since the integral on the right-hand side of the last inequality does not exceed $c\,|x|^{\sigma-\beta+2}$, we obtain $A \le c\, \|g\|$. If $M_1 \subset \bar{\Gamma}_\nu$ and $g(\xi) = 0$ for $r_1(\xi) < 2\, r(\xi)$, then $|x-\xi| > c|x|$ for $\xi \in \Gamma_\nu \cap \operatorname{supp} g$. Therefore, as in the case $M_1 \cap \bar{\Gamma}_\nu = \{0\}$, we obtain $A \le c\,\|g\|$. Suppose finally that $M_1 \subset \bar{\Gamma}_\nu$ and $g(\xi) = 0$ for $r_1(\xi) > 3\,r(\xi)$. Then

$$|g(\xi)| \le c\,|\xi|^{\sigma-\beta} \Big(\frac{r_1(\xi)}{|\xi|}\Big)^{\sigma-\delta_1} \|g\|$$

and

$$A \le c\,|x|^{\delta_1-1-\sigma+s} \Big(\frac{r_1(x)}{|x|}\Big)^{\max(0,\delta_1-1-\sigma+s)} r_1(x)^{\min(0,\mu_1-s-\varepsilon)}\,\|g\| \int |x-\xi|^{-1-s-\min(0,\mu_1-s-\varepsilon)}\, r_1(\xi)^{\sigma-\delta_1}\, d\xi,$$

where the domain of integration is the set of all $\xi \in \Gamma_\nu$ satisfying the inequalities $|x|/2 < |\xi| < 2|x|$ and $|x-\xi| > r(x)/4$. For the estimation of the integral in the last inequality, we use the following two inequalities analogous to (3.24) and (3.25):

$$\int\limits_{\substack{\Gamma_\nu \\ r_1(x)/4<|x-\xi|<R}} |x-\xi|^{-\alpha}\, r_1(\xi)^{-\delta}\, d\xi \le c\,R^{2-\alpha-\delta} \quad \text{if } \alpha+\delta<2,\ \delta<1, \tag{4.12}$$

$$\int\limits_{\substack{\Gamma_\nu \\ |x-\xi|>R}} |x-\xi|^{-\alpha}\, r_1(\xi)^{-\delta}\, d\xi \le c\,R^{2-\alpha-\delta} \quad \text{if } \alpha+\delta>2,\ \delta<1, \tag{4.13}$$

where c is independent of x and R, $R > r_1(x)/4$. If $\delta_1 - 1 - \sigma + s < 0$, then

$$\mu_1 > s, \quad -1-s-\min(0,\mu_1-s-\varepsilon)+\sigma-\delta_1 = -1-s+\sigma-\delta_1 > -2,$$

and (4.13) with $R = 3|x|$ implies

$$A \le c\,|x|^{\delta_1-1-\sigma+s}\,\|g\| \int |x-\xi|^{-1-s}\, r_1(\xi)^{\sigma-\delta_1}\, d\xi \le c\,\|g\|$$

If $\delta_1-1-\sigma+s > 0$, then $-1-s-\min(0,\mu_1-s-\varepsilon)+\sigma-\delta_1 < -2$, and we obtain the same estimate using (4.12) with $R = r_1(x)/4$. Thus, the estimate $A \le c\,\|g\|$ is true for functions g vanishing for $r_1(\xi) < 2r(\xi)$ or for $r_1(\xi) > 3r(\xi)$. It remains to note that every $g \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})$ can be written as a sum $g = g_1 + g_2$, where $g_1(\xi) = 0$ for $r_1(\xi) < 2r(\xi)$, $g_2(\xi) = 0$ for $r_1(\xi) > 3r(\xi)$, and

$$\|g_1\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})} + \|g_2\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})} \le c\,\|g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})}.$$

This completes the proof. □

Lemma 4.3 *Let $f_j^{(0)} \in \mathcal{N}^0_{\beta+1,\delta+1}(\mathcal{K})$, $f^{(j)} \in \mathcal{N}^0_{\beta,\delta}(\mathcal{K})^3$, $j = 1,2,3$, and $g \in \mathcal{N}^0_{\beta,\delta}(\mathcal{K})$. Suppose that β and δ satisfy conditions* (4.1) *and* (4.2)*, respectively. Then estimate* (4.10) *is valid for the functions* (4.5), (4.6).

P r o o f. If β and δ_k satisfy conditions (4.1) and (4.2), then $\beta' = \beta+1$ and $\delta_k' = \delta_k+1$ satisfy conditions (3.12) and (3.13), respectively. By Theorem 3.1, the functions $K(x,\xi) = \partial_x^\alpha\big(\chi^-(x,\xi)\,G_{i,j}(x,\xi)\big)$, $j = 1,2,3$, satisfy the assumptions of Lemmas 3.7, 3.8 and 3.10 with $s = |\alpha| + \delta_{i,4}$ and $t = 0$. Consequently,

$$|x|^{\beta'-2-\sigma+|\alpha|+\delta_{i,4}} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k'-2-\sigma+|\alpha|+\delta_{i,4})} \Big|\partial_x^\alpha \int_{\mathcal{K}} f_j^{(0)}(\xi)\,\chi^-(x,\xi)\,G_{i,j}(x,\xi)\,d\xi\Big|$$
$$\le c\,\|f_j^{(0)}\|_{\mathcal{N}^0_{\beta'-\sigma,\delta'-\sigma}(\mathcal{K})} = c\,\|f_j^{(0)}\|_{\mathcal{N}^{0,\sigma}_{\beta+1-\sigma,\delta+1-\sigma}(\mathcal{K})}\,.$$

Using the fact that $K(x,\xi) = \partial_x^\alpha \partial_{\xi_\nu}\big(\chi^-(x,\xi)\,G_{i,j}(x,\xi)\big)$, $j = 1,2,3$, satisfies the conditions of Lemmas 3.7, 3.8 and 3.10 with $s = |\alpha| + \delta_{i,4}$ and $t = 1$, we obtain

$$|x|^{\beta'-2-\sigma+|\alpha|+\delta_{i,4}} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k'-2-\sigma+|\alpha|+\delta_{i,4})} \Big|\partial_x^\alpha \int_{\mathcal{K}} f^{(j)}(\xi)\cdot\nabla_\xi\big(\chi^-(x,\xi)\,G_{i,j}(x,\xi)\big)\,d\xi\Big|$$
$$\le c\,\|f^{(j)}\|_{\mathcal{N}^0_{\beta'-1-\sigma,\delta'-1-\sigma}(\mathcal{K})^3} = c\,\|f^{(j)}\|_{\mathcal{N}^{0,\sigma}_{\beta-\sigma,\delta-\sigma}(\mathcal{K})^3}\,.$$

Since the functions $\partial_x^\alpha\big(\chi^-(x,\xi)\, G_{i,4}(x,\xi)\big)$ and $\partial_x^\alpha \partial_{\xi_j}\big(\chi^-(x,\xi)\, G_{i,j}(x,\xi)\big)$ also satisfy the conditions of Lemmas 3.7, 3.8 and 3.10 with $s=|\alpha|+\delta_{i,4}$ and $t=1$, the same inequality with g instead of $f^{(j)}$ holds for the integral

$$\int_{\mathcal{K}} g(\xi)\, \Big(\chi^-(x,\xi)\, G_{i,4}(x,\xi) - \sum_{j=1}^3 \partial_{\xi_j}\big(\chi^-(x,\xi)\, G_{i,j}(x,\xi)\big)\Big)\, d\xi.$$

Finally, using Lemma 4.2 with $K(x,\xi)=\partial_x^\alpha\big(\chi^-(x,\xi)\, G_{i,j}(x,\xi)\big)$, $s=|\alpha|+\delta_{i,4}$, we obtain

$$\begin{aligned} & |x|^{\beta-1-\sigma+|\alpha|+\delta_{i,4}} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\max(0,\delta_k-1-\sigma+|\alpha|+\delta_{i,4})} \Big|\int_{\Gamma_\nu} \big(f^{(j)}(\xi)\cdot n + g(\xi)\, n_j\big)\, \chi^-(x,\xi)\, G_{i,j}(x,\xi)\, d\xi\Big| \\ & \le c\, \Big(\|f^{(j)}\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})}\Big). \end{aligned}$$

The proof of the theorem is complete. □

The proof of the following lemma proceeds analogously to Lemma 3.11.

Lemma 4.4 *Let $x\in\mathcal{K}_\nu=\{\xi\in\mathcal{K}:\ r_\nu(\xi)<3r(\xi)/2\}$, $f\in\mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})$, and*

$$v(x)=\int_{\Gamma_j} K(x,\xi)\, f(\xi)\, d\xi.$$

We assume that β,δ satisfy conditions (4.1), (4.2), $K(x,\xi)$ vanishes for $|x-\xi|>r(x)/4$ and that there exists a constant c independent of x and ξ such that

$$\begin{aligned} |\partial_\rho^s K(x,\xi)| &\le c\, |x-\xi|^{-2-s+k_\nu} \quad \text{for } |x|/4<|\xi|<2|x|,\ s=0,1, \\ |\partial_\rho K(x,\xi)| &\le c\, |x|^{\Lambda_--2+k_\nu+\varepsilon}\, |\xi|^{-\Lambda_--1-\varepsilon} \quad \text{for } |\xi|<3|x|/4, \\ |\partial_\rho K(x,\xi)| &\le c\, |x|^{\Lambda_+-2+k_\nu-\varepsilon}\, |\xi|^{-\Lambda_+-1-t+\varepsilon} \quad \text{for } |\xi|>3|x|/2, \end{aligned}$$

where $k_\nu=1+[\delta_\nu-\sigma]$ and $\rho=|x|$. Then there ist the estimate

$$|x|^{\beta-\delta_\nu}\, \frac{|v(x)-v(y)|}{|x-y|^{\sigma-\delta_\nu+k_\nu}} \le c\, \|f\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})}$$

for $y=\tau x$, $4/5<\tau<5/4$, $|x-y|>r(x)/4$, where c is independent of f,x,τ.

Theorem 4.1 *Let $f_j^{(0)}\in\mathcal{N}^0_{\beta+1,\delta+1}(\mathcal{K})$, $f^{(j)}\in\mathcal{N}^0_{\beta,\delta}(\mathcal{K})^3$, $j=1,2,3$, $g\in\mathcal{N}^0_{\beta,\delta}(\mathcal{K})$, and let u^-,p^- be defined by (4.5), (4.6). If β and δ satisfy conditions (4.1) and (4.2), then $u^-\in C^{1,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $p^-\in C^{0,\sigma}_{\beta,\delta}(\mathcal{K})$, and*

$$\|u^-\|_{C^{1,\sigma}_{\beta,\delta}(\mathcal{K})^3} + \|p^-\|_{C^{0,\sigma}_{\beta,\delta}(\mathcal{K})} \le c\, \Big(\sum_{j=1}^3 \|f_j^{(0)}\|_{\mathcal{N}^0_{\beta+1,\delta+1}(\mathcal{K})} + \sum_{j=1}^3 \|f^{(j)}\|_{\mathcal{N}^0_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{\mathcal{N}^0_{\beta,\delta}(\mathcal{K})}\Big)$$

with a constant c independent of $f_j^{(0)}$, $f^{(j)}$, and g.

P r o o f. By Lemma 4.3, u^- and p^- satisfy (4.10). From this it follows analogously to the proof of Theorem 3.3 that

$$|x|^\beta \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k} \frac{|\partial_x^\alpha u^-(x)-\partial_y^\alpha u^-(y)|}{|x-y|^\sigma} \le c\, \Big(\sum_{j=1}^3 \big(\|f_j^{(0)}\|_{\mathcal{N}^0_{\beta+1,\delta+1}(\mathcal{K})} + \|f^{(j)}\|_{\mathcal{N}^0_{\beta,\delta}(\mathcal{K})^3}\big) + \|g\|_{\mathcal{N}^0_{\beta,\delta}(\mathcal{K})}\Big)$$

for $|\alpha|=1$, $x,y\in\mathcal{K}$, $|x-y|<r(x)/2$, and

$$|x|^{\beta-\delta_\nu} \frac{|\partial_x^\alpha u^-(x)-\partial_y^\alpha u^-(y)|}{|x-y|^{k_\nu+\sigma-\delta_\nu}} \le c\, \Big(\sum_{j=1}^3 \big(\|f_j^{(0)}\|_{\mathcal{N}^0_{\beta+1,\delta+1}(\mathcal{K})} + \|f^{(j)}\|_{\mathcal{N}^0_{\beta,\delta}(\mathcal{K})^3}\big) + \|g\|_{\mathcal{N}^0_{\beta,\delta}(\mathcal{K})}\Big)$$

for $|\alpha| = 1-k_\nu$, $x, y \in \mathcal{K}_\nu$, $|x-y| < r(x)/2$, $\nu = 1, \dots, N$. Here, $k_\nu = [\delta_\nu - \sigma]+1$. We have $u_i^- = v_i^- + w_i^-$, where

$$\begin{aligned} v_i^-(x) &= \sum_{j=1}^3 \int_{\mathcal{K}} \Big(f_j^{(0)}(\xi)\, \chi^-(x,\xi)\, G_{i,j}(x,\xi) \;-\; f^{(j)}(\xi) \cdot \nabla_\xi \big(\chi^-(x,\xi)\, G_{i,j}(x,\xi)\big)\Big)\, d\xi \\ &\quad + \int_{\mathcal{K}} g(\xi) \Big(\chi^-(x,\xi)\, G_{i,4}(x,\xi) - \sum_{j=1}^3 \partial_{\xi_j} \big(\chi^-(x,\xi)\, G_{i,j}(x,\xi)\big)\Big)\, d\xi \\ w_i^-(x) &= \sum_{j=1}^3 \sum_{\nu=1}^N \int_{\Gamma_\nu} \big(f^{(j)}(\xi) \cdot n + g(\xi)\, n_j\big)\, \chi^-(x,\xi)\, G_{i,j}(x,\xi)\, d\xi. \end{aligned}$$

If β and δ satisfy (4.1) and (4.2), then $\beta' = \beta+1$ and $\delta'_k = \delta_k + 1$ satisfy (3.12) and (3.13). Consequently using Lemma 3.11, we obtain

$$|x|^{\beta'-\delta'_\nu} \frac{|\partial_x^\alpha v_i^-(x) - \partial_y^\alpha v_i^-(y)|}{|x-y|^{k'_\nu+\sigma-\delta'_\nu}} \le c \Big(\sum_{j=1}^3 \big(\|f_j^{(0)}\|_{\mathcal{N}^0_{\beta'-\sigma,\delta'-\sigma}(\mathcal{K})} + \|f^{(j)}\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})^3}\big) + \|g\|_{\mathcal{N}^0_{\beta-\sigma,\delta-\sigma}(\mathcal{K})}\Big)$$

for $x \in \mathcal{K}_\nu$, $y = \tau x$, $1/2 < \tau < 3/2$, $|\alpha| = 2 - k'_\nu$, where $k'_\nu = k_\nu + 1$. According to Lemma 4.4, the same estimate is valid for w_i^-. Thus, $u^- \in C^{1,\sigma}_{\beta,\delta}(\mathcal{K})^3$. Analogously, the inclusion $p^- \in C^{0,\sigma}_{\beta,\delta}(\mathcal{K})$ holds. □

Remark 4.1 From Lemma 4.3 it follows that even $u^- \in C^{2,\sigma}_{\beta+1,\delta+1}(\mathcal{K})^3$ and $p^- \in C^{1,\sigma}_{\beta+1,\delta+1}(\mathcal{K})$.

4.4 Existence of weak solutions

Analogously to Lemma 3.3, the following regularity result holds. For the proof, one has to apply a regularity assertion for weak solutions of elliptic boundary value problems in domains with smooth boundaries (see [24, Th.6.4.8]).

Lemma 4.5 *Let $(u,p) \in W^{1,s}_{loc}(\bar{\mathcal{K}}\backslash\mathcal{S})^3 \times W^{0,s}_{loc}(\bar{\mathcal{K}}\backslash\mathcal{S})$ be a weak solution of problem* (3.1), (3.2) *such that*

$$\sup |x|^{\beta-1-\sigma} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k-1-\sigma} |u(x)| + \sup |x|^{\beta-\sigma} \prod_{k=1}^N \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k-\sigma} |p(x)| < \infty.$$

If $f \in \mathcal{N}^{-1,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $g \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})$, $h_j \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}$, $j = 1, \dots, N$, then $u \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})^3$ and $p \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})$.

Theorem 4.2 *Let $f \in C^{-1,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $g \in C^{0,\sigma}_{\beta,\delta}(\mathcal{K})$, $h_j \in C^{1,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in C^{0,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}$, $j = 1, \dots, n$, where δ satisfies condition* (4.1) *and β is such that the line $\operatorname{Re}\lambda = 1 + \sigma - \beta$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$. Suppose that the boundary data h_j satisfy the compatibility condition* (3.6) *and that in the case $\delta_k < \sigma$ the functions $g, h_{k\pm}, \phi_{k\pm}$ satisfy the compatibility conditions given in Lemma* 2.3. *Then problem* (3.1), (3.2) *has a unique solution $(u,p) \in C^{1,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{0,\sigma}_{\beta,\delta}(\mathcal{K})$.*

P r o o f. Lemma 3.2 allows us to restrict ourselves to the case $g \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})$, $h_j = 0$, $\phi_j = 0$. Let κ be an arbitrary real number such that the closed strip between the lines $\operatorname{Re}\lambda = 1 + \sigma - \beta$ and $\operatorname{Re}\lambda = -\kappa - 1/2$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$, and let $G(x,\xi)$ be the Green matrix introduced in Section 3.4. Then condition (4.2) is satisfied. We consider the vector-functions (4.5), (4.6) and define $u = u^+ + u^-$, $p = p^+ + p^-$. Since $(u^-, p^-) \in C^{2,\sigma}_{\beta+1,\delta+1}(\mathcal{K})^3 \times C^{1,\sigma}_{\beta+1,\delta+1}(\mathcal{K})$, we have $\Delta u^- - \nabla p^- \in \mathcal{N}^{0,\sigma}_{\beta+1,\delta+1}(\mathcal{K})^3$ and therefore,

$$-\Delta u^+ + \nabla p^+ = f + \Delta u^- - \nabla p^- \in \mathcal{N}^{-1,\sigma}_{\beta,\delta}(\mathcal{K})^3.$$

Furthermore analogously to the proof of Theorem 3.4,

$$-\nabla \cdot u^+ \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K}), \quad S_j u^+|_{\Gamma_j} = 0, \quad N_j(u^+, p^+)|_{\Gamma_j} \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}.$$

Using Lemmas 4.1 and 4.5, we conclude that $(u^+, p^+) \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{K})$. This together with Theorem 4.1 implies $(u, p) \in C^{1,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{0,\sigma}_{\beta,\delta}(\mathcal{K})$. The uniqueness of the solution (u, p) holds analogously to the proof of Theorem 3.4. $\square$

Analogously to Theorem 3.7, the following result holds.

Theorem 4.3 *Let $g \in V^0_\kappa(\mathcal{K}) \cap C^{0,\sigma}_{\beta,\delta}(\mathcal{K})$, $h_j \in V^{1/2}_\kappa(\Gamma_j)^{3-d_j} \cap C^{1,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, and let the functional $F \in V^*_{-\kappa}$ have the form*

$$F(v) = f(v) + \sum_{j=1}^n \int_{\Gamma_j} \phi_j \cdot v \, dx,$$

where $f \in C^{-1,\sigma}_{\beta,\delta}(\mathcal{K})^3$, $\phi_j \in C^{0,\sigma}_{\beta,\delta}(\Gamma_j)^3$, $\phi_j = (\phi_j \cdot n)\, n$ if $d_j = 1$, $\phi_j \cdot n = 0$ if $d_j = 2$ (if $d_j = 0$, then the integral over Γ_j does not appear in the representation of F). Suppose that on the lines lines $\mathrm{Re}\,\lambda = -\kappa - 1/2$ and $\mathrm{Re}\,\lambda = 1 + \sigma - \beta$, there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$, δ satisfy condition (4.1), and the data g, h_j and ϕ_j satisfy the same compatibility conditions as in Theorem 3.4. Then the solution $(u, p) \in V^1_\kappa(\mathcal{K})^3 \times V^0_\kappa(\mathcal{K})$ of problem (3.31), (3.32) admits the decomposition (3.34), where $(w, q) \in C^{1,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{0,\sigma}_{\beta,\delta}(\mathcal{K})$ and λ_ν are the eigenvalues of the pencil $\mathfrak{A}$ between the lines $\mathrm{Re}\,\lambda = -\kappa - 1/2$ and $\mathrm{Re}\,\lambda = 1+\sigma-\beta$. In particular, if the closed strip between the lines $\mathrm{Re}\,\lambda = -\kappa-1/2$ and $\mathrm{Re}\,\lambda = 1+\sigma-\beta$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$, then $(u, p) \in C^{1,\sigma}_{\beta,\delta}(\mathcal{K})^3 \times C^{0,\sigma}_{\beta,\delta}(\mathcal{K})$.

E x a m p l e. We consider the weak solution $(u, p) \in V^1_0(\mathcal{K})^3 \times L_2(\mathcal{K})$ of the Dirichlet problem

$$b(u, v) - \int_{\mathcal{K}} p \, \nabla \cdot v \, dx = F(v) \quad \text{for all } v \in V^1_0(\mathcal{K})^3, \ v|_{\Gamma_j} = 0, \tag{4.14}$$

$$-\nabla \cdot u = g \ \text{ in } \mathcal{K}, \quad u = 0 \ \text{ on } \Gamma_j, \ j = 1, \dots, N. \tag{4.15}$$

We assume that $g \in C^{0,\sigma}(\mathcal{K})$, $g = 0$ on M_j for $j = 1, \dots, n$, and $F \in C^{-1,\sigma}(\mathcal{K})^3$, i.e. F is a distribution of the form

$$F = F^{(0)} + \sum_{j=1}^3 \partial_{x_j} F^{(j)}, \quad \text{where } F^{(j)} \in C^{0,\sigma}(\mathcal{K})^3, \ j = 0, 1, 2, 3. \tag{4.16}$$

One can write any distribution (4.16) also in the form

$$F = \Phi^{(0)} + \sum_{j=1}^3 \partial_{x_j} \Phi^{(j)}, \quad \text{where } \Phi^{(j)} \in C^{0,\sigma}_{0,0}(\mathcal{K})^3, \ j = 0, 1, 2, 3,$$

For example, we can put $\Phi^{(0)} = F^{(0)} - (\chi(x) + \frac{1}{3}\, x \cdot \nabla\chi)\, F^{(0)}(0)$, and $\Phi^{(j)} = F^{(j)} - F^{(j)}(0) + \frac{1}{3}\, \chi(x)\, x_j F^{(0)}(0)$, where χ is a differentiable function on $\bar{\mathcal{K}}$, $\chi(x) = 1$ for $|x| \le 1$, $\chi(x) = 0$ for $|x| \ge 2$. Thus, we have $F \in C^{-1,\sigma}_{0,0}(\mathcal{K})^3 = \mathcal{N}^{-1,\sigma}_{0,0}(\mathcal{K})^3$. Furthermore, every $g \in C^{0,\sigma}(\mathcal{K})$ satisfying $g = 0$ on M_j for $j = 1, \dots, n$ belongs to the space $\mathcal{N}^{0,\sigma}_{0,0}(\mathcal{K})$.

According to [13, Th.5.5.5,Th.5.5.6], the strip $-1 \le \mathrm{Re}\,\lambda \le 0$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$. If the cone is convex, then the eigenvalue with smallest positive real part is $\Lambda_1 = 1$. This eigenvalue is simple and has the eigenvector $(0, 0, 0, 1)$. Moreover for $\theta_k < \pi$, we have $\mu_k = \pi/\theta_k > 1$. Consequently we obtain the following regularity result.

> *Let $F \in V^*_0 \cap C^{-1,\sigma}(\mathcal{K})^3$, $g \in L_2(\mathcal{K}) \cap C^{0,\sigma}(\mathcal{K})$, $g = 0$ on M_j for $j = 1, \dots, N$. If $\mathcal{K}$ is convex and σ is such that $1 + \sigma < \pi/\theta_k$ and the strip $1 < \mathrm{Re}\,\lambda < 1 + \sigma$ contains only the eigenvalue $\Lambda_1 = 1$ of the pencil $\mathfrak{A}(\lambda)$, then $\zeta(u, p) \in C^{1,\sigma}(\mathcal{K})^3 \times C^{0,\sigma}(\mathcal{K})$ for every smooth function ζ with compact support.*

Under the above assumptions, the decomposition (3.34) for (u, p) in Theorem 4.3 has the simple form $(u, p) = c\,(0, 1) + (w, q)$, where $(w, q) \in C^{1,\sigma}_{0,0}(\mathcal{K})^3 \times C^{0,\sigma}_{0,0}(\mathcal{K})$.

5 The problem in a bounded domain

5.1 Formulation of the problem

Let $\mathcal{G}$ be a bounded domain of polyhedral type in $\mathbb{R}^3$. This means that

(i) the boundary $\partial\mathcal{G}$ consists of smooth (of class C^∞) open two-dimensional manifolds Γ_j (the faces of $\mathcal{G}$), $j = 1,\dots,N$, smooth curves M_k (the edges), $k = 1,\dots,N'$, and corners $x^{(1)},\dots,x^{(d)}$,

(ii) for every $\xi \in M_k$ there exist a neighborhood $\mathcal{U}_\xi$ and a diffeomorphism (a C^∞ mapping) κ_ξ which maps $\mathcal{G}\cap\mathcal{U}_\xi$ onto $\mathcal{D}_\xi \cap B_1$, where $\mathcal{D}_\xi$ is a dihedron of the form $K_\xi \times \mathbb{R}$ with a plane wedge K_ξ and B_1 is the unit ball,

(iii) for every corner $x^{(j)}$ there exist a neighborhood $\mathcal{U}_j$ and a diffeomorphism κ_j mapping $\mathcal{G}\cap\mathcal{U}_j$ onto $\mathcal{K}_j \cap B_1$, where $\mathcal{K}_j$ is a cone with vertex at the origin.

We consider the problem

$$-\Delta u + \nabla p = f, \quad -\nabla\cdot u = g \ \text{ in } \mathcal{G}, \tag{5.1}$$
$$S_j u = h_j, \quad N_j(u,p) = \phi_j \ \text{ on } \Gamma_j,\ j = 1,\dots,N, \tag{5.2}$$

where S_j, N_j are the same operators as in Section 3.

5.2 Model problems and corresponding operator pencils

We introduce the operator pencils generated by problem (5.1), (5.2) for the singular boundary points.

1) Let $\xi \in M_k$ be an edge point, and let $\Gamma_{k_+}, \Gamma_{k_-}$ be the faces of $\mathcal{G}$ adjacent to ξ. Then by $\mathcal{D}_\xi$, we denote the dihedron which is bounded by the half-planes $\Gamma^\circ_{k_\pm}$ tangential to $\Gamma_{k_\pm}$ at ξ. The angle between $\Gamma^\circ_{k_+}$ and $\Gamma^\circ_{k_-}$ is denoted by $\theta(\xi)$. We consider the model problem

$$-\Delta u + \nabla p = f, \quad -\nabla\cdot u = g \ \text{ in } \mathcal{D}_\xi,$$
$$S_{k_\pm} u = h^\pm, \quad N_{k_\pm}(u,p) = \phi^\pm \ \text{ on } \Gamma^\circ_{k_\pm}$$

and denote the operator pencil corresponding to this problem (see Section 2.2) by $A_\xi(\lambda)$. Furthermore, we denote by $\lambda_1(\xi)$ the eigenvalue with smallest positive real part and by $\lambda_2(\xi)$ the eigenvalue of $A_\xi(\lambda)$ with smallest real part greater than 1. In the case when $d_{k_+}+d_{k_-}$ is even and $\theta(\xi) < 2\pi/(|d_{k_+}-d_{k_-}|+2)$, we define $\mu(\xi) = \operatorname{Re}\lambda_2(\xi)$. Otherwise, we put $\mu(\xi) = \operatorname{Re}\lambda_1(\xi)$. Furthermore, let

$$\mu_k = \inf_{\xi\in M_k} \mu(\xi).$$

2) Let $x^{(j)}$ be a corner of $\mathcal{G}$, and let $I^{(j)}$ be the set of all indices k such that $x^{(j)} \in \bar{\Gamma}_k$. By our assumptions, there exist a neighborhood $\mathcal{U}$ of $x^{(j)}$ and a diffeomorphism κ mapping $\mathcal{G}\cap\mathcal{U}$ onto $\mathcal{K}\cap B_1$ and $\Gamma_k\cap\mathcal{U}$ onto $\Gamma^\circ_k \cap B_1$ for $k \in I^{(j)}$, where $\mathcal{K}$ is a polyhedral cone with vertex 0 and Γ°_k are the faces of this cone. Without loss of generality, we may assume that the Jacobian matrix $\kappa'(x)$ coincides with the identity matrix I at $x^{(j)}$. We consider the model problem

$$-\Delta u + \nabla p = f, \quad -\nabla\cdot u = g \ \text{ in } \mathcal{K},$$
$$S_k u = h_k, \quad N_k(u,p) = \phi_k \ \text{ on } \Gamma^\circ_k \text{ for } k \in I^{(j)},$$

The operator pencil generated by this model problem (see Section 3.1) is denoted by $\mathfrak{A}_j(\lambda)$.

5.3 Smoothness of solutions

We denote by $r_k(x)$ the distance of x to the edge M_k, by $\rho_j(x)$ the distance to the corner $x^{(j)}$, by $r(x)$ the distance to $\mathcal{S}$ (the set of all edge points and corners), and by $\rho(x)$ the distance to the set $X = \{x^{(1)},\dots,x^{(d)}\}$. Let $\mathcal{G}_{j,k} = \{x \in \mathcal{G} : r_j(x) < 3r(x)/2,\ \rho_k(x) < 3\rho(x)/2\}$ and $k_j = [\delta_j - \sigma] + 1$. Furthermore, let $\beta = (\beta_1,\dots,\beta_d) \in \mathbb{R}^d$, $\delta = (\delta_1,\dots,\delta_{N'}) \in \mathbb{R}^{N'}$, $\delta_k \geq 0$ for $k = 1,\dots,N'$, and $0 < \sigma < 1$.

Then $C^{l,\sigma}_{\beta,\delta}(\mathcal{G})$, is defined as the set of all l times continuously differentiable functions on $\bar{\mathcal{G}}\backslash\mathcal{S}$ with finite norm

$$\begin{aligned}
\|u\|_{C^{l,\sigma}_{\beta,\delta}(\mathcal{G})} &= \sum_{|\alpha|\le l} \sup_{x\in\mathcal{G}} \prod_{k=1}^{d} \rho_k(x)^{\beta_k-l-\sigma+|\alpha|} \prod_{j=1}^{N'} \Big(\frac{r_j(x)}{\rho(x)}\Big)^{\max(0,\delta_j-l-\sigma+|\alpha|)} \big|\partial_x^\alpha u(x)\big| \\
&\quad + \sum_{k=1}^{d}\sum_{j=1}^{N'}\sum_{|\alpha|=l-k_j} \sup_{\substack{x,y\in\mathcal{G}_{j,k}\\ |x-y|<\rho_k(x)/2}} \rho_k(x)^{\beta_k-\delta_j} \frac{\big|\partial_x^\alpha u(x)-\partial_y^\alpha u(y)\big|}{|x-y|^{k_j+\sigma-\delta_j}} \\
&\quad + \sum_{|\alpha|=l} \sup_{|x-y|<r(x)/2} \prod_{k=1}^{d}\rho_k(x)^{\beta_k} \prod_{j=1}^{N'}\Big(\frac{r_j(x)}{\rho(x)}\Big)^{\delta_j} \frac{\big|\partial_x^\alpha u(x)-\partial_y^\alpha u(y)\big|}{|x-y|^{\sigma}}\,.
\end{aligned}$$

The trace space on Γ_j for $C^{l,\sigma}_{\beta,\delta}(\mathcal{G})$ is denoted by $C^{l,\sigma}_{\beta,\delta}(\Gamma_j)$.

We consider the solution $(u,p)\in W^1(\mathcal{G})\times L_2(\mathcal{G})$ of problem (5.1), (5.2). This means (see [23, Sec.5.3]) that u and p satisfy

$$b(u,v)-\int_{\mathcal{G}} p\,\nabla\cdot v\,dx = F(v) \quad \text{for all } v\in W^1(\mathcal{G}),\ S_jv=0 \ \text{ on } \Gamma_j, \tag{5.3}$$

$$-\nabla\cdot u = g \ \text{ in } \mathcal{G}, \quad S_ju=h_j \ \text{ on } \Gamma_j,\ j=1,\dots,N, \tag{5.4}$$

where

$$b(u,v)=2\int_{\mathcal{G}}\sum_{i,j=1}^{3}\varepsilon_{i,j}(u)\,\varepsilon_{i,j}(v)\,dx \quad\text{and}\quad F(v)=\int_{\mathcal{G}}(f+\nabla g)\cdot v\,dx+\sum_{j=1}^{N}\int_{\Gamma_j}\phi_j\cdot v\,dx$$

The vectors ϕ_j are orthogonal to Γ_j if $d_j=1$ and tangent to Γ_j if $d_j=2$. In the case $d_j=0$, the integral over Γ_j does not appear in (5.3). Using Theorems 3.7 (with $\kappa=0$) and 3.6, one can easily prove the following theorem.

Theorem 5.1 *Suppose that* $f\in C^{l-2,\sigma}_{\beta,\delta}(\mathcal{G})^3$, $g\in C^{l-1,\sigma}_{\beta,\delta}(\mathcal{G})$, $h_j\in C^{l,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j\in C^{l,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, *where* $l\ge 2$, $0<\sigma<1$, *the strip* $-1/2<\operatorname{Re}\lambda<l+\sigma-\beta_j$ *contains no eigenvalues of the pencil* $\mathfrak{A}_j(\lambda)$, $j=1,\dots,d$, *and that* $l-\mu_k<\delta_k-\sigma<l$, $\delta_k-\sigma$ *is not integer for* $k=1,\dots,N'$. *Suppose furthermore that* g,h_j,ϕ_j *satisfy the same compatibility conditions on the edges as in Theorem* 3.4. *Then the solution* $(u,p)\in W^1(\mathcal{G})^3\times L_2(\mathcal{G})$ *of problem* (5.3), (5.4) *belongs to* $C^{l,\sigma}_{\beta,\delta}(\mathcal{G})^3\times C^{l-1,\sigma}_{\beta,\delta}(\mathcal{G})$.

Note that under the conditions of the theorem, the right-hand side of (5.3) defines a continuous functional on $W^1(\mathcal{G})^3$.

5.4 Examples

We consider the boundary value problem (5.1), (5.2) in a polyhedron $\mathcal{G}$ with sides Γ_j, edges M_j and corners $x^{(j)}$. Under certain compatibility conditions on the data $f\in C^{l-2,\sigma}_{\beta,\delta}(\mathcal{G})^3$, $g\in C^{l-1,\sigma}_{\beta,\delta}(\mathcal{G})$, $h_j\in C^{l,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$ and $\phi_j\in C^{l-1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}$, there exists a weak solution $(u,p)\in W^1(\mathcal{G})^3\times L_2(\mathcal{G})$ (see [23, Th.5.1]). We establish regularity assertions for this solution. For the sake of simplicity, we restrict ourselves to homogeneous boundary conditions.

1) *The Dirichlet problem.* Let $(u,p)\in W^1(\mathcal{G})^3\times L_2(\mathcal{G})$ be a weak solution of the problem

$$-\Delta u+\nabla p=f,\quad -\nabla\cdot u=g \ \text{ in } \mathcal{G}, \qquad u=0 \ \text{ on } \Gamma_j,\ j=1,\dots,N.$$

It is known for this problem (see [13, Th.5.5.6]) that the strip $-1\le\operatorname{Re}\lambda\le 0$ is free of eigenvalues of the pencil $\mathfrak{A}_j(\lambda)$. We denote by Λ_j the eigenvalue with smallest positive real part. By [13, Th.5.5.6], there is the estimate

$$\operatorname{Re}\Lambda_j>\frac{a_j}{a_j+4}\,,$$

where $a_j(a_j+1)$ is the first eigenvalue of the Beltrami operator $-\delta$ with Dirichlet boundary condition on the intersection Ω_j of the cone $\mathcal{K}_j$ with the unit sphere, $a_j > 0$. If the polyhedron $\mathcal{G}$ is convex, then $\Lambda_j = 1$ is the smallest positive eigenvalue of the pencil $\mathfrak{A}_j(\lambda)$. The numbers μ_k depend on the angles θ_k at the edges M_k. If $\theta_k > \pi$, then μ_k is the smallest positive solution of the equation $\sin(\mu\theta_k) + \mu \sin\theta_k = 0$. It can be easily verified that $\mu_k > 1/2$. If $\theta_k < \pi$, then $\mu_k = \pi/\theta_k > 1$ (see [13, Sec.5.1]). Consequently, Theorem 5.1 implies the following results.

- If $f \in C^{l-2,\sigma}_{l,l}(\mathcal{G})^3$, $g \in C^{l-1,\sigma}_{l,l}(\mathcal{G})$, and $0 < \sigma < \min(\operatorname{Re}\Lambda_j, \mu_k)$, then $(u,p) \in C^{l,\sigma}_{l,l}(\mathcal{G})^3 \times C^{l-1,\sigma}_{l,l}(\mathcal{G})$. In particular, we have $u \in C^{0,\sigma}(\mathcal{G})^3$. If the polyhedron $\mathcal{G}$ is convex, then this result is true for arbitrary $\sigma \in (0,1)$.
- If $f \in C^{l-2,\sigma}$, $g \in C^{l-1,\sigma}$ in a neighborhood of an edge point $\xi \in M_k$, $g|_{M_k=0}$ and $l+\sigma < \pi/\theta_k$, then $u \in C^{l,\sigma}$ and $p \in C^{l-1,\sigma}$ in a neighborhood of ξ.

For the last result, we refer also to the regularity results in weighted L_p Sobolev spaces given in [23]. Furthermore, there is the following result (see the example at the end of Section 4).

- If $\mathcal{G}$ is convex, $f \in C^{-1,\sigma}(\mathcal{G})^3$, $g \in C^{0,\sigma}(\mathcal{G})$, $g|_{M_k=0}$ for all k, and σ is sufficiently small (such that $1+\sigma < \pi/\theta_k$ and there are no eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$ in the strip $1 < \operatorname{Re}\lambda \le 1+\sigma$), then $(u,p) \in C^{1,\sigma}(\mathcal{G}) \times C^{0,\sigma}(\mathcal{G})$.

2) *The Neumann problem.* Let $(u,p) \in W^1(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a weak solution of the problem

$$-\Delta u + \nabla p = f, \quad -\nabla\cdot u = g \ \text{ in } \mathcal{G}, \qquad -pn + 2\varepsilon_n(u) = 0 \ \text{ on } \Gamma_j,\ j = 1,\dots,N.$$

By [13, Th.6.3.2], the strip $-1 \le \operatorname{Re}\lambda \le 0$ contains only the eigenvalues $\lambda = 0$ and $\lambda = 1$ of the operator pencils $\mathfrak{A}_j(\lambda)$ if the polyhedron $\mathcal{G}$ is Lipschitz. The eigenvectors corresponding to $\lambda = 0$ have the form $(c, 0)$, where $c \in \mathbb{C}^3$ is an arbitrary constant vector, while generalized eigenvectors do not exist. The numbers μ_k are the same as for the Dirichlet problem. Let Λ_j be the eigenvalue of $\mathfrak{A}_j(\lambda)$ with smallest positive real part. Applying Theorems 3.7, 3.6 and [22, Th.4.4], we obtain the following results.

- If $f \in C^{l-2,\sigma}_{l,l}(\mathcal{G})^3$ and $g \in C^{l-1,\sigma}_{l,l}(\mathcal{G})$, where $0 < \sigma < \min(\operatorname{Re}\Lambda_j, \mu_k)$, then $p \in C^{l-1,\sigma}_{l,l}(\mathcal{G})$ and $\psi_j\big(u - u(x^{(j)})\big) \in C^{l,\sigma}_{l,l}(\mathcal{G})^3$ for every smooth cut-off function ψ_j equal to one near $x^{(j)}$ and to zero near the other corners. In particular, we have $u \in C^{0,\sigma}(\mathcal{G})^3$.
- If $f \in C^{l-2,\sigma}$, $g \in C^{l-1,\sigma}$ in a neighborhood of an edge point $\xi \in M_k$ and $l+\sigma < \pi/\theta_k$, then $u \in C^{l,\sigma}$ and $p \in C^{l-1,\sigma}$ in a neighborhood of ξ.

3) *The mixed problem with boundary conditions* (i)–(iii). Let $(u,p) \in W^1(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a weak solution of problem (5.1), (5.2), where $d_j \le 2$ for all j (i.e., the Neumann condition does not appear in the boundary conditions). For the sake of simplicity, we restrict ourselves to homogeneous boundary conditions. We assume that the Dirichlet condition is given on at least one of the adjoining sides of every edge. Then $\mu_k > 1/4$ for all k. For $\theta_k < \pi/2$, we have $\mu_k \ge \pi/(2\theta_k)$. If the Dirichlet condition is given on both adjoining sides of the edge M_k, then there are the sharper estimates for μ_k given in Example 1. Furthermore by [13, Th.6.1.5], the strip $-1 \le \operatorname{Re}\lambda \le 0$ is free of eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$. Thus, the following results hold.

- If $f \in C^{l-2,\sigma}_{l,l}(\mathcal{G})^3$ and $g \in C^{l-1,\sigma}_{l,l}(\mathcal{G})$, where $0 < \sigma < \min(\operatorname{Re}\Lambda_j, \mu_k)$, then $(u,p) \in C^{l,\sigma}_{l,l}(\mathcal{G})^3 \times C^{l-1,\sigma}_{l,l}(\mathcal{G})$. In particular, we have $u \in C^{0,\sigma}(\mathcal{G})^3$.
- Let $f \in C^{l-2,\sigma}$, $g \in C^{l-1,\sigma}$ in a neighborhood of an edge point $\xi \in M_k$, and $g|_{M_k} = 0$ (if the boundary condition (i) is given on one of the adjoining sides of M_k, then the last assumption can be omitted.) We suppose furthermore that $l+\sigma < \pi/\theta_k$ if the Dirichlet condition is given on both adjoining sides of M_k. Otherwise, let $l+\sigma < \pi/(2\theta_k)$. Then $u \in C^{l,\sigma}$ and $p \in C^{l-1,\sigma}$ in a neighborhood of ξ.

Finally, we describe two situations, where a global $C^{1,\sigma}$ regularity result holds for the solution of the mixed boundary value problem.

- Let (u,p) be a weak solution of equation (5.1), (5.2), where $d_j \le 1$ for all j (i.e., only the boundary conditions (i) and (ii) appear in (5.2)). We assume that $\mathcal{G}$ is convex and that $\theta_k < \pi/2$ if the boundary condition (ii) is given on at least one of the adjoining sides of the edge M_k. Furthermore, let $f \in C^{-1,\sigma}(\mathcal{G})^3$, $g \in C^{0,\sigma}(\mathcal{G})$, and $g|_{M_k} = 0$ if the Dirichlet condition (i) is given on both adjoining sides of M_k. Then $(u,p) \in C^{1,\sigma}(\mathcal{G})^3 \times C^{0,\sigma}(\mathcal{G})$.

- Suppose the polyhedron $\mathcal{G}$ is convex, the Dirichlet condition is given on all sides except Γ_N, where the boundary condition (iii) is prescribed, and the angles between Γ_N and the adjoining sides are less than $\pi/2$. If $f \in C^{-1,\sigma}(\mathcal{G})^3$, $g \in C^{0,\sigma}(\mathcal{G})$, $g|_{M_k} = 0$ for all k, then the weak solution $(u,p) \in W^1(\mathcal{G})^3 \times L_2(\mathcal{G})$ of problem (5.1), (5.2) belongs to the space $C^{1,\sigma}(\mathcal{G})^3 \times C^{0,\sigma}(\mathcal{G})$.

For the last two results, we refer to [13, Th.6.2.6,Th.6.2.7]. In the first case, we have $\mathrm{Re}\,\Lambda_j > 1$ and $\mu_k > 1$, while in the second case the eigenvalue with smallest positive real part of the pencil $\mathfrak{A}_j(\lambda)$ is $\Lambda_j = 1$. However, this eigenvalue is simple, and the corresponding eigenvector is $(0,0,0,1)$. Thus as in the case of the Dirichlet problem, the desired result can be easily deduced from Theorem 4.3.

References

[1] Agmon, S., Douglis, A., Nirenberg, L., *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II*, Comm. Pure Appl. Math. **17** (1964) 35-92.

[2] Brown, R. M., Shen, Z., *Estimates for the Stokes operator in Lipschitz domains*, Indiana Univ. Math. J. **44** (1995) 4, 1183-1206.

[3] Dauge, M., *Elliptic boundary value problems on corner domains*, Springer-Verlag Berlin, Heidelberg, New York, London, Paris, Tokyo 1988.

[4] Dauge, M., *Stationary Stokes and Navier-Stokes systems on two- and three-dimensional with corners. Part 1: Linearized equations,* SIAM J. Math. Anal. **20** (1989) 74-97.

[5] Deuring, P., von Wahl, W., *Strong solutions of the Navier-Stokes system in Lipschitz bounded domains,* Math. Nachr. **171** (1995) 111-148.

[6] Dindos, M., Mitrea, M., *The stationary Navier-Stokes system in nonsmooth manifolds: The Poisson problem in Lipschitz and C^1 domains*, Arch. Ration. Mech. Anal. **174** (2004) 1, 1-47.

[7] Fabes, E. B., Kenig, C. E., Verchota, G. C., *The Dirichlet problem for the Stokes system on Lipschitz domains,* Duke Math. J. **57** (1988) 3, 769-793.

[8] Gilbarg, D., Trudinger, N. S., *Elliptic partial differential equations of second order*, Springer-Verlag Berlin, Heidelberg, New York 1977.

[9] Grachev, N. V., Maz'ya, V. G., *Estimates for the Neumann problem in a polyhedral cone*, Preprint LiTH-MAT-R-91-28, University of Linköping 1991.

[10] Grisvard, P., *Elliptic problems in nonsmooth domains*, Pitman Boston, London, Melbourne 1985.

[11] Jerison, D. S., Kenig, C. E., *Boundary value problems on Lipschitz domains*, Studies in Partial Differential Equations, MAA Stud. Math. **23** (1982) 1-68.

[12] Kenig, C. E., *Elliptic boundary value problems on Lipschitz domains*, Beijing Lectures in Harmonic Analysis, Ann. Math. Stud. **112** (1986) 131-183.

[13] Kozlov, V. A., Maz'ya, V. G., Rossmann, J., *Spectral problems associated with corner singularities of solutions to elliptic equations*, Mathematical Surveys and Monographs **85**, Amer. Math. Soc., Providence, Rhode Island 2001.

[14] Kozlov, V. A., Maz′ya, V. G., Schwab, C., *On singularities of solutions to the Dirichlet problem of hydrodynamics near the vertex of a cone,* J. Reine Angew. Math. **456** (1994) 65-97.

[15] Maz'ya, V. G., Plamenevskij, B. A., *Weighted spaces with nonhomogeneous norms and boundary value problems in domains with conical points,* Ellipt. Differentialgleichungen (Meeting, Rostock, 1977) Univ. Rostock, 1978, 161-189, Engl. transl. in: Amer. Math. Soc. Transl. **123** (1984) 89-107.

[16] Maz'ya, V. G., Plamenevskiĭ, B. A., *Estimates of Green's function and Schauder estimates of solutions of elliptic boundary value problems in a dihedral angle,* Sibirsk. Mat. Zh. **19** (1978) 5, 1065-1082.

[17] Maz′ya, V. G., Plamenevskiĭ, B. A., *Schauder estimates of solutions of elliptic boundary value problems in domains with edges on the boundary,* Amer. Math. Soc. Transl., Vol. 123 (1984) 141-169.

[18] Maz'ya, V. G., Plamenevskiĭ, B. A., *The first boundary value problem for classical equations of mathematical physics in domains with piecewise smooth boundaries*, Part 1: Zeitschr. Anal. Anw. **2** (4) 1983, 335-359, Part 2: Zeitschr. Anal. Anw. **2** (6) 1983, 523-551.

[19] Maz'ya, V. G., Plamenevskiĭ, B. A., Stupelis, L., *The three-dimensional problem of steady-state motion of a fluid with a free surface*, Amer. Math. Soc. Transl. **123** (1984) 171-268.

[20] Maz'ya, V. G., Rossmann, J., *Weighted L_p of solutions to boundary value problems for second order elliptic systems in polyhedral domains*, Z. angew. Math. Mech. **83** (7), 435-467.

[21] Maz'ya, V. G., Rossmann, J., *Schauder estimates for solutions to boundary value problems for second order elliptic systems in polyhedral domains*, Applicable Analysis **83** (2004) 3, 271-308.

[22] Maz'ya, V. G., Rossmann, J., *Pointwise estimates for Green's kernel of a mixed boundary value problem to the Stokes system in a polyhedral cone*, to appear in Math. Nachr.

[23] Maz'ya, V. G., Rossmann, J., *L_p estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains*, to appear.

[24] Morrey, C. B., *Multiple integrals in the calculus of variations*, springer-Verlag Berlin, Heidelberg, New York 1966.

[25] Nazarov, S. A., Plamenevskiĭ, B. A., *Elliptic problems in domains with piecewise smooth boundaries,* De Gruyter Expositions in Mathematics **13**, Berlin - New York 1994.

[26] Solonnikov, V. A., *Estimates of solutions to the Neumann problem for elliptic second order equations in domains with edges on the boundary*, Preprint P-4-83, Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) (in Russian)

Addresses: V. G. Maz'ya, Ohio State University, Department of Mathematics,
University of Liverpool, Department of Mathematical Sciences, and
University of Linköping, Department of Mathematics, S-58183 Linköping, Sweden,
e-mail: vlmaz@mai.liu.se
J. Rossmann, University of Rostock, Institute of Mathematics, D-18051 Rostock, Germany,
e-mail: juergen.rossmann@mathematik.uni-rostock.de