A survey of functional and L^p dissipativity theory

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Abstract. Various notions of dissipativity type for partial differential operators and their applications are surveyed. We deal with functional dissipativity and its particular case L^p-dissipativity. Most of the results are due to the authors.

1 Introduction

The present paper contains a survey of recent results concerning dissipativity of partial differential operators. To be more precise, we mean the notion of functional dissipativity introduced in [15] and its particular case, the so called L^p-dissipativity.

Our joint studies in this area started in 2005, when we found necessary and sufficient conditions for the L^p-dissipativity of second order differential operators with complex valued coefficients.

The L^p-dissipativity of a partial differential operator arises in a natural way in the study of partial differential equations with data in L^p. The theory of such problems has a long history. In fact L^p-dissipativity appeared in 1937 in the pioneering work of Cimmino [16] on the Dirichlet problem with boundary data in L^p. Similar ideas were used in [45] and [51]. In [48] the study of degenerate oblique derivative problem hinges on the weighted L^p positivity of the differential operator. Later we give more historical information.

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In order to introduce the topic in a simple way, let us consider the classical Cauchy-Dirichlet problem for the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, & \text{for } t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where φ is a given function in $C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

It is well known that the unique solution of problem (1) in the class of smooth bounded solutions is given by the formula

$$u(x,t) = \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} \varphi(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy, \quad x \in \mathbb{R}^n, t > 0. \tag{2}$$

From (2) it follows immediately

$$|u(x,t)| \leqslant \|\varphi\|_\infty, \quad t > 0, \tag{3}$$

since

$$\frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, dy = 1 \quad (t > 0). \tag{4}$$

Inequality (3) leads to the classical maximum modulus principle

$$\|u(\cdot,t)\|_\infty \leqslant \|\varphi\|_\infty, \quad t > 0,$$

and this in turn implies that the norm $\|u(\cdot,t)\|_\infty$ is a decreasing function of t. In fact, fix $t_0 > 0$ and consider the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v, & \text{for } t > t_0, \\ v(x,t_0) = u(x,t_0), & x \in \mathbb{R}^n. \end{cases} \tag{5}$$

It is clear that the unique solution of (5) is given by $v(x,t) = u(x,t)$ $(t > t_0)$ and we have

$$\|v(\cdot,t)\|_\infty \leqslant \|u(\cdot,t_0)\|_\infty, \quad t > t_0,$$

i.e.,

$$\|u(\cdot,t)\|_\infty \leqslant \|u(\cdot,t_0)\|_\infty, \quad t > t_0.$$

The L^∞ norm is not the only norm for which we have this kind of dissipativity. Let us consider the L^p-norm with $1 < p < \infty$. By Cauchy-Hölder inequality, from (2) we get

$$|u(x,t)| \leqslant \left(\frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} |\varphi(y)|^p e^{-\frac{|x-y|^2}{4t}} dy \right)^{1/p} \left(\frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} dy \right)^{1/p'}$$

$(1/p + 1/p' = 1)$ and then, keeping in mind (4),

$$|u(x,t)|^p \leqslant \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} |\varphi(y)|^p e^{-\frac{|x-y|^2}{4t}} dy.$$

Integrating over $\mathbb{R}^n$ and applying Tonelli's Theorem we find

$$\int_{\mathbb{R}^n} |u(x,t)|^p dx \leqslant \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} |\varphi(y)|^p e^{-\frac{|x-y|^2}{4t}} dy =$$
$$\frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} |\varphi(y)|^p dy \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} dx = \int_{\mathbb{R}^n} |\varphi(y)|^p dy$$

and we have proved that

$$\|u(\cdot,t)\|_p \leqslant \|\varphi\|_p. \tag{6}$$

As before, this inequality implies that the norm $\|u(\cdot,t)\|_p$ is a decreasing function of t.

Let us consider now the more general problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au, & \text{for } t > 0, \\ u(x,t) = 0, & \text{for } x \in \partial\Omega,\ t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases} \tag{7}$$

where Ω is a domain in $\mathbb{R}^n$ and A is a linear elliptic partial differential operator of the second order

$$Au = \sum_{|\alpha| \leqslant 2} a_\alpha(x)\, D^\alpha u\,,$$

the coefficients a_α being complex valued. A natural question arises: under which conditions for the operator A the solution $u(x,t)$ of the problem (7) satisfies the inequality (6) ?

As we know already, (6) implies that $\|u(\cdot,t)\|_p$ is a decreasing function of t and then

$$\frac{d}{dt}\|u(\cdot,t)\|_p \leqslant 0. \tag{8}$$

On the other hand, at least formally, we have for $1<p<\infty$, [1]

$$\frac{d}{dt}\|u(\cdot,t)\|_p^p = \frac{d}{dt}\int_\Omega |u(x,t)|^p dx = p\,\mathbb{R}\mathrm{e}\int_\Omega \langle \partial_t u, u\rangle |u|^{p-2} dx, \tag{9}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{C}$. Since u is a solution of the problem (7), keeping in mind (9), we have that (8) holds if, and only if,

$$\mathbb{R}\mathrm{e}\int_\Omega \langle Au, u\rangle |u|^{p-2} dx \leqslant 0.$$

This leads to the following definition: *let A a linear operator from $D(A)\subset L^p(\Omega)$ to $L^p(\Omega)$; A is said to be L^p-dissipative if*

$$\mathbb{R}\mathrm{e}\int_\Omega \langle Au, u\rangle |u|^{p-2} dx \leqslant 0, \quad \forall\, u\in D(A). \tag{10}$$

It follows from what we have said before that if A is L^p-dissipative and if problem (7) has a solution, then (8) holds. Here we shall not dwell upon details of rigourous justification of the above argument.

We conclude Introduction by a well known result (see e.g. [60, p.215]). Consider the operator in divergence form with real smooth coefficients

$$Au = \partial_i\,(a_{ij}(x)\,\partial_j u) \tag{11}$$

($a_{ji}=a_{ij}\in C^1(\overline{\Omega})$): if $a_{ij}(x)\xi_i\xi_j \geqslant 0$ for any $\xi\in\mathbb{R}^n$, $x\in\Omega$, the operator (11) is L^p-dissipative for any p. If $2\leqslant p<\infty$ this can be deduced easily by integration by parts. Indeed we can write

$$\begin{aligned}\int_\Omega \langle Au, u\rangle |u|^{p-2} dx &= -\int_\Omega a_{ij}\,\partial_j u\,\partial_i\left(\overline{u}\,|u|^{p-2}\right) dx\\ &= -\int_\Omega a_{ij}\partial_j u\left(|u|^{p-2}\partial_i\overline{u} + \overline{u}\,\partial_i(|u|^{p-2})\right) dx\,.\end{aligned}$$

Since

$$\partial_i(|u|^{p-2}) = (p-2)|u|^{p-3}\partial_i|u| = (p-2)|u|^{p-4}\,\mathbb{R}\mathrm{e}\,(\overline{u}\,\partial_i u)$$

[1]Note that $\partial_t|u| = \partial_t\sqrt{u\,\overline{u}} = (u_t\overline{u}+u\overline{u}_t)/(2\sqrt{u\,\overline{u}}) = \mathbb{R}\mathrm{e}(u_t\overline{u}/|u|)$.

we can write

$$\begin{aligned}&\partial_j u \left(|u|^{p-2}\partial_i\overline{u} + \overline{u}\,\partial_i(|u|^{p-2})\right)\\ &= |u|^{p-4}\left[\overline{u}\,u\,\partial_j u\,\partial_i\overline{u} + (p-2)\overline{u}\,\partial_j u\,\mathbb{R}\mathrm{e}\,(u\,\partial_i\overline{u})\right].\end{aligned}$$

Setting

$$|u|^{(p-4)/2}\overline{u}\,\partial_j u = \xi_j + i\eta_j\,,$$

we have

$$a_{ij}\partial_j u \left(|u|^{p-2}\partial_i\overline{u} + \overline{u}\,\partial_i(|u|^{p-2})\right) = a_{ij}\,(\xi_j + i\eta_j)\,(\xi_i - i\eta_i + (p-2)\xi_i).$$

This implies

$$\mathbb{R}\mathrm{e}\left(a_{ij}\partial_j u\left(|u|^{p-2}\partial_i\overline{u} + \overline{u}\,\partial_i(|u|^{p-2})\right)\right) = (p-1)a_{ij}\xi_i\xi_j + a_{ij}\eta_i\eta_j$$

and then

$$\mathbb{R}\mathrm{e}\int_\Omega a_{ij}\partial_j u\,\partial_i\left(\overline{u}\,|u|^{p-2}\right)dx \geqslant 0,$$

i.e., A is L^p-dissipative. Some extra arguments are necessary for the case $1 < p < 2$.

During the last half a century various aspects of the L^p-theory of semigroups generated by linear differential operators were studied in [5, 18, 1, 64, 19, 34, 62, 20, 21, 41, 43, 39, 2, 17, 33, 42, 63, 58, 9, 54, 10] and others. A general account of the subject can be found in the book [59]. Certain of our earlier results have been described in our monograph [12], where they are considered in the more general frame of semi-bounded operators.

Necessary and sufficient conditions for the L^∞-contractivity for general second order strongly elliptic systems with smooth coefficients were given in [36] (see also the monograph [37]). Scalar second order elliptic operators with complex coefficients were handled as a particular case. The operators generating L^∞-contractive semigroups were later characterized in [2] under the assumption that the coefficients are measurable and bounded.

The maximum modulus principle for linear elliptic equations and systems with complex coefficients was considered by Kresin and Maz'ya. They have obtained several results on the best constants in different forms of maximum principles for linear elliptic equations and systems (see the monograph [37] and the recent survey [38]).

The case of higher order operators is quite different. Apparently, only the paper [40] by Langer and Maz'ya dealt with the question of L^p-dissipativity for higher order differential operators. In the case $1 \leqslant p < \infty$, $p \neq 2$, they proved that, in the class of linear partial differential operators of order higher than two, with the domain containing $C_0^\infty(\Omega)$, there are no generators of a contraction semigroup on $L^p(\Omega)$. If u runs over not $C_0^\infty(\Omega)$, but only $(C_0^\infty(\Omega))^+$ (i.e., the class of nonnegative functions of $C_0^\infty(\Omega)$), then the result for operators with real coefficients is quite different and really surprising: if the operator A of order k has real coefficients and the integral

$$\int_\Omega (Au)\, u^{p-1} dx$$

preserves the sign as u runs over $(C_0^\infty(\Omega))^+$, then either $k = 0, 1$ or 2, or $k = 4$ and $3/2 \leqslant p \leqslant 3$.

Let us now give an outline of the paper. Section 2 presents the basic results of Functional Analysis leading to the concept of abstract dissipative operators.

In Section 3 we recall our general definition of L^p-dissipativity of the sesquilinear form related to a scalar second order operator. In Section 4 we give an algebraic condition we found, which provides necessary and sufficient conditions for the L^p-dissipativity of second order operators in divergence form, with no lower order terms.

Section 5 presents a review on p-elliptic operators, which are operators satisfying a strengthened version of our algebraic condition.

Section 6 is concerned with the L^p-dissipativity of operators with lower order terms.

The topic of Section 7 is the linear elasticity system. More general systems are considered in Section 8.

In Section 9 we show how the necessary and sufficient conditions we have obtained lead to determine exactly the angle of dissipativity of certain operators.

Section 10 is devoted to presents some of the results obtained by Kresin and Maz'ya concerning the validity of the classical maximum principle.

In Section 11 we briefly describe some other results we have obtained. They concern the L^p-dissipativity of first order systems, of the "complex oblique derivative" operator and of a certain class of integral operators which includes the fractional powers of Laplacian $(-\Delta)^s$, with $0 < s < 1$.

Section 12 discusses the concept of functional dissipativity, which we have recently introduced.

The final section of this paper, Section 13, briefly shows how our conditions for L^p-dissipativity and its strengthened variant are getting more and more important in many respects.

2 Abstract setting

Let X be a (complex) Banach space. A *semigroup of linear operators on* X is a family of linear and continuous operators $T(t)$ $(0 \leqslant t < \infty)$ from X into itself such that $T(0) = I$, $T(t+s) = T(t)T(s)$ $(s, t \geqslant 0)$.

We say that $T(t)$ is a strongly continuous semigroup (briefly, a C^0-semigroup) if

$$\lim_{t \to 0^+} T(t)x = x, \qquad \forall\, x \in X.$$

The linear operator

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} \tag{12}$$

is the *infinitesimal generator of the semigroup* $T(t)$.

The domain $D(A)$ of the operator A (maybe not continuous) is the set of $x \in X$ such that the limit in (12) does exist.

If $T(t)$ is a C^0-semigroup generated by A and u_0 is a given element in $D(A)$, the function $u(t) = T(t)u_0$ is solution of the evolution problem

$$\begin{cases} \dfrac{du}{dt} = Au, \qquad (t > 0) \\ u(0) = u_0 \,. \end{cases} \tag{13}$$

We remark the it is still possible to solve the Cauchy problem (13) when u_0 is an arbitrary element of X. In order to do that, it is necessary to introduce a concept of generalized solution. For this we refer to [60, Ch.4].

One can show that if $T(t)$ is a C_0 semigroup, then there exist two constants $\omega \geqslant 0$, $M \geqslant 1$ such hat

$$\|T(t)\| \leqslant M\, e^{\omega t}, \qquad 0 \leqslant t < \infty. \tag{14}$$

If we can choose $\omega = 0$ and $M = 1$ in the inequality (14), we have

$$\|T(t)\| \leqslant 1, \qquad 0 \leqslant t < \infty$$

and the semigroup is said to be a contraction semigroup or a semigroup of contractions. If the operator A is the generator of a C^0-semigroup of contractions, the solution of the Cauchy problem (13) satisfies the estimates

$$\|u(t)\| \leqslant \|u_0\| \qquad 0 \leqslant t < \infty. \tag{15}$$

If the norm in (15) is the L^∞ norm, we have the classical maximum principle for parabolic equations.

The next famous Hille-Yosida Theorem characterizes the operators which generates C^0 semigroups of contractions

Theorem 1 *A linear operator A generates a C^0 semigroup of contractions $T(t)$ if, and only if,*

(i) A is closed and $D(A)$ is dense in X;

(ii) the resolvent set $\varrho(A)$ contains $\mathbb{R}^+$ and the resolvent operator R_λ satisfies the inequality

$$\|R_\lambda\| \leqslant \frac{1}{\lambda}, \quad \forall\, \lambda > 0.$$

Given $x \in X$, denote by $\mathscr{I}(x)$ the set

$$\mathscr{I}(x) = \{x^* \in X^* \mid \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\},$$

X^* being the (topological) dual space of X. The set $\mathscr{I}(x)$ is called *the dual set of* x. The operator A is said to be dissipative if, for any $x \in D(A)$, there exists $x^* \in \mathscr{I}(x)$ such that

$$\mathbb{R}\mathrm{e}\,\langle x^*, Ax\rangle \leqslant 0. \tag{16}$$

Another characterization of operators generating contractive semigroups is given by the equally famous Lumer-Phillips theorem:

Theorem 2 *If A generates a C^0 semigroup of contractions, then*

(i) $\overline{D(A)} = X$;

(ii) A is dissipative. More precisely, for any $x \in D(A)$, we have

$$\mathbb{R}\mathrm{e}\langle x^*, Ax\rangle \leqslant 0, \forall\, x^* \in \mathscr{I}(x);$$

(iii) $\varrho(A) \supset \mathbb{R}^+$.

Conversely, if

(i') $\overline{D(A)} = X$;

(ii') A is dissipative;

(iii') $\varrho(A) \cap \mathbb{R}^+ \neq \emptyset$,

then A generates a C^0 semigroup of contractions.

Lumer-Phillips theorem shows that in order to have a contractive semigroup the operator A has to be dissipative, i.e., inequality (16) has to be satisfied. If

$X = L^p(\Omega)$ $(1 < p < \infty)$ it is easy to see that the dual set $\mathscr{I}(f)$ contains only the element f^* defined by

$$f^*(x) \begin{cases} = \|f\|_p^{2-p}\overline{f(x)}\,|f(x)|^{p-2} & \text{if } f(x) \neq 0 \\ = 0 & \text{if } f(x) = 0. \end{cases}$$

and then inequality (16) coincide with (10). We remark that, in the case $1 < p < 2$, the integral in (10) has to be understood with the integrand extended by zero on the set where it vanishes.

Maz'ya and Sobolevskiĭ [51] obtained independently of Lumer and Phillips the same result under the assumption that the norm of the Banach space is Gâteaux-differentiable. Their result looks as follows

Theorem 3 *The closed and densely defined operator $A+\lambda I$ has a bounded inverse for all $\lambda \geqslant 0$ and satisfies the inequality*

$$\|[A + \lambda I]^{-1}\| \leqslant [\mathbb{R}\mathrm{e}\,\lambda + \lambda_0]^{-1}$$

$(\lambda_0 > 0)$ *if and only if, for any $v \in D(A)$ and $f \in D(A^*)$,*

$$\mathbb{R}\mathrm{e}\langle \Gamma v, Av\rangle \geqslant \lambda_0 \|v\|,$$

$$\mathbb{R}\mathrm{e}\langle \Gamma^* f, A^* f\rangle \geqslant \lambda_0 \|f\|.$$

Here Γ and Γ^* stand for the Gâteaux gradient of the norm in B and in B^*, respectively. Applications to second order elliptic operators were also given in [51]. It is interesting to note that the paper [51] was sent to the journal in 1960, before the Lumer-Phillips paper of 1961 [44] appeared.

3 Scalar second order operators with complex coefficients

In this section we describe the main results obtained in [9], where we studied the L^p-dissipativity of scalar second order operators with complex coefficients.

To be precise we consider operators of the form

$$Au = \nabla^t(\mathscr{A}\nabla u) + \mathbf{b}\nabla u + \nabla^t(\mathbf{c}u) + au$$

where ∇^t is the divergence operator, defined in a domain $\Omega \subset \mathbb{R}^N$. The coefficients satisfy the following very general assumptions:

— $\mathscr{A}$ is an $n \times n$ matrix whose entries are complex-valued measures a^{hk} belonging to $(C_0(\Omega))^*$. This is the dual space of $C_0(\Omega)$, the space of complex-valued continuous functions with compact support contained in Ω;

— $\mathbf{b} = (b_1, \dots, b_n)$ and $\mathbf{c} = (c_1, \dots, c_n)$ are complex-valued vectors with $b_j, c_j \in (C_0(\Omega))^*$;

— a is a complex-valued scalar distribution in $(C_0^1(\Omega))^*$, where $C_0^1(\Omega) = C^1(\Omega) \cap C_0(\Omega)$.

Consider the related sesquilinear form $\mathscr{L}(u,v)$

$$\mathscr{L}(u,v) = \int_\Omega (\langle \mathscr{A}\nabla u, \nabla v\rangle - \langle \mathbf{b}\nabla u, v\rangle + \langle u, \overline{\mathbf{c}}\nabla v\rangle - a\langle u, v\rangle)$$

on ${C_0}^1(\Omega) \times {C_0}^1(\Omega)$.

The operator A acts from $C_0^1(\Omega)$ to $(C_0^1(\Omega))^*$ through the relation

$$\mathscr{L}(u,v) = -\int_\Omega \langle Au, v\rangle$$

for any $u, v \in C_0^1(\Omega)$. The integration is understood in the sense of distributions.

The following definition was given in [9]. Let $1 < p < \infty$. A form $\mathscr{L}$ is called *L^p-dissipative* if for all $u \in C_0^1(\Omega)$

$$\begin{aligned} \mathbb{R}\mathrm{e}\, \mathscr{L}(u, |u|^{p-2}u) \geqslant 0, \quad \text{if } p \geqslant 2, \\ \mathbb{R}\mathrm{e}\, \mathscr{L}(|u|^{p'-2}u, u) \geqslant 0, \quad \text{if } 1 < p < 2, \end{aligned} \tag{17}$$

where $p' = p/(p-1)$ (we use here that $|u|^{q-2}u \in C_0^1(\Omega)$ for $q \geqslant 2$ and $u \in C_0^1(\Omega)$).

We remark that the form $\mathscr{L}$ is L^p-dissipative if and only if

$$\mathbb{R}\mathrm{e}\, \mathscr{L}(u, |u|^{p-2}u) \geqslant 0 \tag{18}$$

for any $u \in C_0^1(\Omega)$ such that $|u|^{p-2}u \in C_0^1(\Omega)$.

Indeed, if $p \geqslant 2$, $|u|^{p-2}u$ belongs to $C_0^1(\Omega)$ for any $u \in C_0^1(\Omega)$. If $1 < p < 2$, we prove the following simple fact: $u \in C_0^1(\Omega)$ is such that $|u|^{p-2}u$ belongs to $C_0^1(\Omega)$ if and only if we can write $u = \|v\|_{p'}^{2-p'}|v|^{p'-2}\overline{v}$, with $v \in C_0^1(\Omega)$.

In fact, if v is any function in $C_0^1(\Omega)$, then setting $u = \|v\|_{p'}^{2-p'}|v|^{p'-2}\overline{v}$, we have $u \in C_0^1(\Omega)$ and $u^* = v$ belongs to $C_0^1(\Omega)$ too. Conversely, if u is such that $|u|^{p-2}u$ belongs to $C_0^1(\Omega)$, set $v = u^*$. We have $v \in C_0^1(\Omega)$ and $u = \|v\|_{p'}^{2-p'}|v|^{p'-2}\overline{v}$.

Therefore, if $1 < p < 2$, condition (18) for any $u \in C_0^1(\Omega)$ such that $|u|^{p-2}u \in C_0^1(\Omega)$ means

$$\mathbb{R}\mathrm{e}\, \mathscr{L}(|v|^{p'-2}v, v) \geqslant 0$$

for any $v \in C_0^1(\Omega)$. This completes the proof of the equivalence between condition (18) for any $u \in C_0^1(\Omega)$ such that $|u|^{p-2}u \in C_0^1(\Omega)$ and definition (17).

A first property of dissipative operators is given by the lemma

Lemma 1 *If a form $\mathscr{L}$ is L^p-dissipative, then*

$$\langle \operatorname{\mathbb{R}e} \mathscr{A}\xi, \xi\rangle \geqslant 0 \quad \forall \xi \in \mathbb{R}^n. \tag{19}$$

This assertion follows from the following basic lemma wich provides a necessary and sufficient condition for the L^p-dissipativity of the form $\mathscr{L}$.

Lemma 2 ([9]) *A form $\mathscr{L}$ is L^p-dissipative if and only if for all $w \in C_0^1(\Omega)$*

$$\begin{aligned}
&\operatorname{\mathbb{R}e} \int_\Omega \Big[\langle \mathscr{A}\nabla w, \nabla w\rangle - (1-2/p)\langle (\mathscr{A}-\mathscr{A}^*)\nabla(|w|), |w|^{-1}\overline{w}\nabla w\rangle \\
&- (1-2/p)^2 \langle \mathscr{A}\nabla(|w|), \nabla(|w|)\rangle\Big] + \int_\Omega \langle \operatorname{\mathbb{I}m}(\mathbf{b}+\mathbf{c}), \operatorname{\mathbb{I}m}(\overline{w}\nabla w)\rangle \\
&+ \int_\Omega \operatorname{\mathbb{R}e}(\nabla^t(\mathbf{b}/p - \mathbf{c}/p') - a)|w|^2 \geqslant 0.
\end{aligned}$$

Condition (19) is necessary and sufficient for the L^2- dissipativity, but it is not sufficient if $p \neq 2$.

Lemma 2 implies the following sufficient condition.

Corollary 1 ([9]) *Let α and β be two real constants. If*

$$\begin{aligned}
\frac{4}{p\,p'}\langle \operatorname{\mathbb{R}e} \mathscr{A}\xi, \xi\rangle + \langle \operatorname{\mathbb{R}e} \mathscr{A}\eta, \eta\rangle + 2\langle (p^{-1} \operatorname{\mathbb{I}m} \mathscr{A} + p'^{-1} \operatorname{\mathbb{I}m} \mathscr{A}^*)\xi, \eta\rangle \\
+ \langle \operatorname{\mathbb{I}m}(\mathbf{b}+\mathbf{c}), \eta\rangle - 2\langle \operatorname{\mathbb{R}e}(\alpha\mathbf{b}/p - \beta\mathbf{c}/p'), \xi\rangle \\
+ \operatorname{\mathbb{R}e}\left[\nabla^t\left((1-\alpha)\mathbf{b}/p - (1-\beta)\mathbf{c}/p'\right) - a\right] \geqslant 0
\end{aligned} \tag{20}$$

for any $\xi, \eta \in \mathbb{R}^n$, then the form $\mathscr{L}$ is L^p-dissipative.

Putting $\alpha = \beta = 0$ in (20), we find that if

$$\begin{aligned}
\frac{4}{p\,p'}\langle \operatorname{\mathbb{R}e} \mathscr{A}\xi, \xi\rangle + \langle \operatorname{\mathbb{R}e} \mathscr{A}\eta, \eta\rangle + 2\langle (p^{-1} \operatorname{\mathbb{I}m} \mathscr{A} + p'^{-1} \operatorname{\mathbb{I}m} \mathscr{A}^*)\xi, \eta\rangle \\
+\langle \operatorname{\mathbb{I}m}(\mathbf{b}+\mathbf{c}), \eta\rangle + \operatorname{\mathbb{R}e}\left[\nabla^t\left(\mathbf{b}/p - \mathbf{c}/p'\right) - a\right] \geqslant 0
\end{aligned} \tag{21}$$

for any $\xi, \eta \in \mathbb{R}^n$, then the form $\mathscr{L}$ is L^p-dissipative.

Generally speaking, condition (21) (and the more general condition (20)) is not necessary.

Example 1 Let $n = 2$ and

$$\mathscr{A} = \begin{pmatrix} 1 & i\gamma \\ -i\gamma & 1 \end{pmatrix},$$

where γ is a real constant, $\mathbf{b} = \mathbf{c} = a = 0$. In this case, the polynomial (21) is given by

$$(\eta_1 + \gamma\xi_2)^2 + (\eta_2 - \gamma\xi_1)^2 - (\gamma^2 - 4/(pp'))|\xi|^2.$$

For $\gamma^2 > 4/(pp')$ the condition (20) is not satisfied, whereas the L^p-dissipativity holds because the corresponding operator A is the Laplacian.

Note that the matrix $\mathbb{I}\mathrm{m}\,\mathscr{A}$ is not symmetric. Below (after Corollary 3), we give another example showing that the condition (21) is not necessary for the L^p-dissipativity even for symmetric matrices $\mathbb{I}\mathrm{m}\,\mathscr{A}$.

4 The operator $\nabla^t(\mathscr{A}\nabla u)$

The main result proved in [9] concerns a scalar operator in divergence form with no lower order terms:

$$Au = \nabla^t(\mathscr{A}\nabla u). \tag{22}$$

The following assertion gives a necessary and sufficient condition for the L^p-dissipativity of the operator (22), where - as before - the coefficients a^{hk} belong to $(C_0(\Omega))^*$.

Theorem 4 ([9]) *Let $\mathbb{I}\mathrm{m}\,\mathscr{A}$ be symmetric, i.e., $\mathbb{I}\mathrm{m}\,\mathscr{A}^t = \mathbb{I}\mathrm{m}\,\mathscr{A}$. The form*

$$\mathscr{L}(u, v) = \int_\Omega \langle \mathscr{A}\nabla u, \nabla v\rangle$$

is L^p-dissipative if and only if

$$|p-2|\,|\langle\, \mathbb{I}\mathrm{m}\,\mathscr{A}\xi, \xi\rangle| \leqslant 2\sqrt{p-1}\,\langle\, \mathbb{R}\mathrm{e}\,\mathscr{A}\xi, \xi\rangle \tag{23}$$

for any $\xi \in \mathbb{R}^n$, where $|\cdot|$ denotes the total variation.

The condition (23) is understood in the sense of comparison of measures. Of course if the coefficients $\{a_{hk}\}$ are complex valued L^∞ functions (or more generally L^1_{loc}), the condition (23) means

$$|p-2|\,|\langle\, \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\xi, \xi\rangle| \leqslant 2\sqrt{p-1}\,\langle\, \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\xi, \xi\rangle$$

for any $\xi \in \mathbb{R}^n$ and for a.e. $x \in \Omega$.

When this result appeared, it was new even for operators with smooth coefficients. For such operators it implies the contractivity of the generated semigroup.

Note that from Theorem 4 we immediately derive the following well known results.

Corollary 2 *Let A be such that $\langle \operatorname{\mathbb{R}e} \mathscr{A}\xi, \xi\rangle \geqslant 0$ for any $\xi \in \mathbb{R}^n$. Then*

1) A is L^2-dissipative,

2) if A is an operator with real coefficients, then A is L^p-dissipative for any p.

The condition (23) is equivalent to the positivity of some polynomial in ξ and η. More exactly, (23) is equivalent to the following condition:

$$\frac{4}{p\,p'}\langle \operatorname{\mathbb{R}e} \mathscr{A}\xi, \xi\rangle + \langle \operatorname{\mathbb{R}e} \mathscr{A}\eta, \eta\rangle - 2(1-2/p)\langle \operatorname{\mathbb{I}m} \mathscr{A}\xi, \eta\rangle \geqslant 0 \tag{24}$$

for any $\xi, \eta \in \mathbb{R}^n$.

More generally, if the matrix $\operatorname{\mathbb{I}m} \mathscr{A}$ is not symmetric, the condition

$$\frac{4}{p\,p'}\langle \operatorname{\mathbb{R}e} \mathscr{A}(x)\xi, \xi\rangle + \langle \operatorname{\mathbb{R}e} \mathscr{A}(x)\eta, \eta\rangle + 2\langle (p^{-1} \operatorname{\mathbb{I}m} \mathscr{A}(x) + p'^{-1} \operatorname{\mathbb{I}m} \mathscr{A}^*(x))\xi, \eta\rangle \geqslant 0 \tag{25}$$

for almost any $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^n$ ($p' = p/(p-1)$) is only sufficient for the L^p-dissipativity.

Let us assume that either A has lower order terms or it has no lower order terms and $\operatorname{\mathbb{I}m} \mathscr{A}$ is not symmetric. Then (23) is still necessary for the L^p-dissipativity of A, but not sufficient, which will be shown in Example 2 (cf. also Theorem 7 below for the case of constant coefficients). In other words, for such general operators the algebraic condition (24) is necessary but not sufficient, whereas the condition (21) is sufficient, but not necessary.

Example 2 Let $n = 2$, and let Ω be a bounded domain. Denote by σ a real function of class ${C_0}^2(\Omega)$ which does not vanish identically. Let $\lambda \in \mathbb{R}$. Consider the operator (22) with

$$\mathscr{A} = \begin{pmatrix} 1 & i\lambda\partial_1(\sigma^2) \\ -i\lambda\partial_1(\sigma^2) & 1 \end{pmatrix},$$

i.e.,

$$Au = \partial_1(\partial_1 u + i\lambda\partial_1(\sigma^2)\,\partial_2 u) + \partial_2(-i\lambda\partial_1(\sigma^2)\,\partial_1 u + \partial_2 u),$$

where $\partial_i = \partial/\partial x_i$ $(i = 1, 2)$. By definition, we have L^2-dissipativity if and only if

$$\mathbb{R}\mathrm{e} \int_\Omega ((\partial_1 u + i\lambda\partial_1(\sigma^2)\, \partial_2 u)\partial_1\overline{u} + (-i\lambda\partial_1(\sigma^2)\, \partial_1 u + \partial_2 u)\partial_2\overline{u})\, dx \geqslant 0$$

for any $u \in C_0{}^1(\Omega)$, i.e., if and only if

$$\int_\Omega |\nabla u|^2 dx - 2\lambda \int_\Omega \partial_1(\sigma^2)\, \mathbb{I}\mathrm{m}(\partial_1\overline{u}\, \partial_2 u)\, dx \geqslant 0$$

for any $u \in C_0{}^1(\Omega)$. Taking $u = \sigma\, \exp(itx_2)$ $(t \in \mathbb{R})$, we obtain, in particular,

$$t^2 \int_\Omega \sigma^2 dx - t\lambda \int_\Omega (\partial_1(\sigma^2))^2 dx + \int_\Omega |\nabla\sigma|^2 dx \geqslant 0. \tag{26}$$

Since

$$\int_\Omega (\partial_1(\sigma^2))^2 dx > 0,$$

we can choose $\lambda \in \mathbb{R}$ so that (26) is impossible for all $t \in \mathbb{R}$. Thus, A is not L^2-dissipative, although (23) is satisfied. Since A can be written as

$$Au = \Delta u - i\lambda(\partial_{21}(\sigma^2)\, \partial_1 u - \partial_{11}(\sigma^2)\, \partial_2 u),$$

this example shows that (23) is not sufficient for the L^2-dissipativity of an operator with lower order terms, even if $\mathbb{I}\mathrm{m}\, \mathscr{A}$ is symmetric.

5 The p-ellipticity

Let us consider the class of operators

$$Au = \nabla(\mathscr{A}\, \nabla u) + b\nabla u + \nabla(cu) + au. \tag{27}$$

with L^∞ coefficients, such that the form (25) is not merely non-negative, but strictly positive, i.e., there exists $\kappa > 0$ such that

$$\frac{4}{p\,p'}\langle \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\xi, \xi\rangle + \langle \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\eta, \eta\rangle + 2\langle (p^{-1}\, \mathbb{I}\mathrm{m}\, \mathscr{A}(x) + p'^{-1}\, \mathbb{I}\mathrm{m}\, \mathscr{A}^*(x))\xi, \eta\rangle \geqslant \kappa(|\xi|^2 + |\eta|^2) \tag{28}$$

for almost any $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^n$. The class of operators (27) whose principal part satisfies (28) and which could be called (*strongly*) *p-elliptic*, was recently considered by several authors.

Carbonaro and Dragičević [6, 7] showed the validity of a so called (dimension free) bilinear embedding. Their main result is the following

Theorem 5 ([6]) *Let* $P_t^A = \exp(-tL_A), t > 0$ *and let* $p > 1$. *Suppose that the matrices* A, B *are* p*-elliptic. Then for all* $f, g \in C_0^\infty(\mathbb{R}^n)$ *we have*

$$\int_0^\infty \int_{\mathbb{R}^n} \left|\nabla P_t^A f(x)\right| \left|\nabla P_t^B g(x)\right| dxdt \leqslant C\|f\|_p \|g\|_{p'} \tag{29}$$

with constant depending on ellipticity parameters, but not dimension.

We note that if A and B are real accretive matrices then (29) holds for the full range of exponents $p \in (1, \infty)$.

In a series of papers [23, 24, 25, 26] Dindoš and Pipher proved several results concerning the L^p solvability of the Dirichlet problem. Their result hinges on a regularity property for the solutions of the Dirichlet problem for the equation

$$\partial_i (a_{ij}(x)\partial_j u) + b_i(x)\partial_i u = 0\,, \tag{30}$$

given by the next result

Lemma 3 ([23], p.269) *Let the matrix* A *be* p*-elliptic for* $p \geqslant 2$ *and let* B *have coefficients* $B_i \in L^\infty_{\text{loc}}(\Omega)$ *satisfying the condition*

$$|B_i(x)| \leqslant K(\delta(x))^{-1}, \quad \forall x \in \Omega\,, \tag{31}$$

where the constant K *is uniform, and* $\delta(x)$ *denotes the distance of* x *to the boundary of* Ω. *Suppose that* $u \in W^{1,2}_{loc}(\Omega)$ *is a weak solution of the equation* (30) *in* Ω, *an open subset of* $\mathbb{R}^n$. *Then, for any ball* $B_r(x)$ *with* $r < \delta(x)/4$,

$$\int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \leqslant C_1 r^{-2} \int_{B_{2r}(x)} |u(y)|^p dy$$

and

$$\left(\int_{B_r(x)} |u(y)|^q dy\right)^{1/q} \leqslant C_2 \left(\int_{B_{2r}(x)} |u(y)|^2 dy\right)^{1/2}$$

for all $q \in \left(2, \frac{np}{n-2}\right]$ *when* $n > 2$, *and where* C_1, C_2 *depend only on* p*-ellipticity constants and* K *of* (31). *When* $n = 2, q$ *can be any number in* $(2, \infty)$. *In particular,* $|u|^{(p-2)/2} u$ *belongs to* $W^{1,2}_{loc}(\Omega)$.

Dindoš and Pipher used this result in an iterative procedure, which is similar to Moser's iteration scheme (used in Moser's proof of the celebrated De Giorgi–Nash–Moser regularity theorem for real divergence form elliptic equations). Differently from the real coefficients case, where the procedure can be applied for any p and leads to the boundedness of the solution (and then to its Hölderianity), here the iteration scheme can be applied only up to a threshold determined by the p-ellipticity of the operator. This is sufficient to obtain an higher integrability of the solution.

Dindoš and Pipher uses this regularity result in the study of the existence for the Dirichlet problem

$$\begin{cases} \partial_i \left(a_{ij}(x)\partial_j u\right) + b_i(x)\partial_i u = 0 & \text{in } \Omega \\ u(x) = f(x) & \text{a.e. on } \partial\Omega \\ \widetilde{N}_{2,a}(u) \in L^p(\partial\Omega) \end{cases} \tag{32}$$

where f is in $L^p(\partial\Omega)$. Here $a > 0$ is a fixed parameter and $\widetilde{N}_{2,a}(u)$ is a nontangential maximal function defined using L^p averages over balls

$$\widetilde{N}_{2,a}(u)(y) = \sup_{x\in\Gamma_a(y)} \left(\fint_{B_{\delta(x)/2}(x)} |u(z)|^2 dz \right)^{1/2}$$

($y \in \partial\Omega$) where the barred integral indicates the average and $\Gamma_a(y)$ is a cone of aperture a. To be precise, they say that the Dirichlet problem (32) is solvable for a given $p \in (1, \infty)$ if there exists a $C = C(p, \Omega) > 0$ such that for all complex valued boundary data $f \in L^p(\partial\Omega) \cap B^{2,2}_{1/2}(\partial\Omega)$ the unique "energy solution" satisfies the estimate

$$\left\| \widetilde{N}_{2,a}(u) \right\|_{L^p(\partial\Omega)} \leqslant C \|f\|_{L^p(\partial\Omega)} .$$

Since the space $\dot{B}^{2,2}_{1/2}(\partial\Omega) \cap L^p(\partial\Omega)$ is dense in $L^p(\partial\Omega)$ for each $p \in (1, \infty)$, there exists a unique continuous extension of the solution operator $f \mapsto u$ to the whole space $L^p(\partial\Omega)$, with u such that $\widetilde{N}_{2,a}(u) \in L^p(\partial\Omega)$ and the accompanying estimate $\left\| \widetilde{N}_{2,a}(u) \right\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{L^p(\partial\Omega)}$ is valid.

Their results have been extended by Dindoš, Li and Pipher to systems and in particular to elasticity in [22].

We mention that - as Carbonaro and Dragičević [6] show - the p-ellipticity comes into play also in the study of the convexity of power functions (Bellman functions) and in the holomorphic functional calculus.

Egert [27] shows that the p-ellipticity condition implies extrapolation to a holomorphic semigroup on Lebesgue spaces in a p-dependent range of exponents.

Finally we remark that, if the partial differential operator has no lower order terms, the concepts of p-ellipticity and strict L^p-dissipativity coincide. By strict

L^p-dissipativity we mean that there exists $\kappa > 0$ such that

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}^{hk}\, \partial_k u, \partial_h(|u|^{p-2}u)\rangle \, dx \geqslant \kappa \int_\Omega |\nabla(|u|^{(p-2)/2}u)|^2 dx$$

for any $u \in C_0^1(\Omega)$ such that $|u|^{p-2}u \in C_0^1(\Omega)$.

It is worthwhile to remark that, if the partial differential operator has no lower order terms, the concepts of p-ellipticity and strict L^p-dissipativity coincide. One can prove that the operator A is strict L^p-dissipative, i.e., p-elliptic, if and only if there exists $\kappa > 0$ such that $A - \kappa\Delta$ is L^p-dissipative.

6 L^p-dissipativity for operators with lower order terms

Generally speaking, it is impossible to obtain an algebraic characterization for an operator with lower order terms. Indeed, let us consider, for example, the operator

$$Au = \Delta u + a(x)u$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with zero Dirichlet boundary data. Denote by λ_1 the first eigenvalue of the Dirichlet problem for the Laplace equation in Ω. A sufficient condition for the L^2-dissipativity of A has the form $\mathbb{R}\mathrm{e}\, a \leqslant \lambda_1$, and we cannot give an algebraic characterization of λ_1.

Consider, as another example, the operator

$$A = \Delta + \mu \tag{33}$$

where μ is a nonnegative Radon measure on Ω. The operator A is L^p-dissipative if and only if

$$\int_\Omega |w|^2 d\mu \leqslant \frac{4}{pp'} \int_\Omega |\nabla w|^2 dx \tag{34}$$

for any $w \in C_0^\infty(\Omega)$ (cf. Lemma 2). Maz'ya [46, 47, 49] proved that the following condition is sufficient for (34):

$$\frac{\mu(F)}{\mathrm{cap}_\Omega(F)} \leqslant \frac{1}{pp'} \tag{35}$$

for all compact set $F \subset \Omega$ and the following condition is necessary:

$$\frac{\mu(F)}{\mathrm{cap}_\Omega(F)} \leqslant \frac{4}{pp'} \tag{36}$$

for all compact set $F \subset \Omega$. Here, $\mathrm{cap}_\Omega(F)$ is the capacity of F relative to Ω, i.e.,

$$\mathrm{cap}_\Omega(F) = \inf\Big\{ \int_\Omega |\nabla u|^2 dx \ : \ u \in {C_0}^\infty(\Omega), \ u \geqslant 1 \text{ on } F \Big\}.$$

The condition (35) is not necessary and the condition (36) is not sufficient.

It must be pointed out that a theorem by Jaye, Maz'ya and Verbitsky can provide a necessary and sufficient condition of a different nature for the L^p-dissipativity of operator (33). In fact in [32] they proved the following result

Theorem 6 *Let Ω be an open set, and let $\sigma \in (C_0^\infty(\Omega))'$ be a real valued distribution. In addition, let $\mathcal{A}$ be a symmetric matrix function defined on Ω satisfying the conditions*

$$\textit{(1.4)} \quad m|\xi|^2 \leqslant \mathcal{A}(x)\xi \cdot \xi, \quad \textit{and} \quad |\mathcal{A}(x)\xi| \leqslant M|\xi|, \quad \textit{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Then

$$\langle \sigma, h^2 \rangle \leqslant \int_\Omega (\mathcal{A}\nabla h) \cdot \nabla h \, dx$$

holds for all $h \in C_0^\infty(\Omega)$ if and only if there exists a vector field $\vec{\Gamma} \in L^2_{loc}(\Omega)$ so that:

$$\sigma \leqslant \mathrm{div}(\mathcal{A}\vec{\Gamma}) - (\mathcal{A}\vec{\Gamma}) \cdot \vec{\Gamma} \quad \textit{in } (C_0^\infty(\Omega))'.$$

Keeping in mind that the operator (33) is L^p-dissipative if and only if (34) holds for any $w \in C_0^\infty(\Omega)$, by taking $\sigma = \mu$, $\mathcal{A} = (4/(pp'))\mathcal{I}$, we find immediately that (33) is L^p-dissipative if and only if there exists a vector field $\vec{\Gamma} \in L^2_{\mathrm{loc}}(\Omega)$ such that

$$\mu \leqslant \frac{4}{p\,p'} \left(\mathrm{div}\,\vec{\Gamma} - |\vec{\Gamma}|^2 \right)$$

in the sense of distributions.

In the case of an operator with constant coefficients and lower order terms, we have found a necessary and sufficient condition. Consider the operator

$$Au = \nabla^t(\mathscr{A}\nabla u) + \mathbf{b}\nabla u + au \tag{37}$$

with constant complex coefficients. Without loss of generality, we can assume that the matrix $\mathscr{A}$ is symmetric.

The following assertion provides a necessary and sufficient condition for the L^p-dissipativity of the operator A.

Theorem 7 ([9]) *Suppose that Ω is an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. The operator* (37) *is L^p-dissipative if and only if there exists a real constant vector V such that*

$$2\,\mathbb{R}\mathrm{e}\,\mathscr{A}V + \mathbb{I}\mathrm{m}\,\mathbf{b} = 0,$$
$$\mathbb{R}\mathrm{e}\,a + \langle\,\mathbb{R}\mathrm{e}\,\mathscr{A}V, V\rangle \leqslant 0$$

and for any $\xi \in \mathbb{R}^n$

$$|p-2|\,|\langle\,\mathbb{I}\mathrm{m}\,\mathscr{A}\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle\,\mathbb{R}\mathrm{e}\,\mathscr{A}\xi,\xi\rangle. \tag{38}$$

If the matrix $\mathbb{R}\mathrm{e}\,\mathscr{A}$ is not singular, the following assertion holds.

Corollary 3 ([9]; cf. also [36]) *Suppose that Ω is an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. Assume that the matrix $\mathbb{R}\mathrm{e}\,\mathscr{A}$ is not singular. The operator A is L^p-dissipative if and only if* (38) *holds and*

$$4\,\mathbb{R}\mathrm{e}\,a \leqslant -\langle(\mathbb{R}\mathrm{e}\,\mathscr{A})^{-1}\,\mathbb{I}\mathrm{m}\,\mathbf{b}, \mathbb{I}\mathrm{m}\,\mathbf{b}\rangle. \tag{39}$$

Now, we can show that the condition (20) is not necessary for the L^p-dissipativity, even if the matrix $\mathbb{I}\mathrm{m}\,\mathscr{A}$ is symmetric.

Example 3 Let $n = 1$, and let $\Omega = \mathbb{R}^1$. Consider the operator

$$\left(1 + 2\,\frac{\sqrt{p-1}}{p-2}\,i\right)u'' + 2iu' - u,$$

where $p \neq 2$ is fixed. The conditions (38) and (39) are satisfied, and this operator is L^p-dissipative in view of Corollary 3.

On the other hand, the polynomial in (21) has the form

$$\left(2\,\frac{\sqrt{p-1}}{p}\,\xi - \eta\right)^2 + 2\eta + 1,$$

i.e., it is not nonnegative for any $\xi, \eta \in \mathbb{R}$.

Recently Maz'ya and Verbitsky [52] (see also [53]) gave necessary and sufficient conditions for the accretivity of a second order partial differential operator $\mathcal{L}$ containing lower order terms, in the case of Dirichlet data. We observe that the accretivity of $\mathcal{L}$ is equivalent to the L^2-dissipativity of $-\mathcal{L}$.

Their result concern second order operators with distributional coefficients

$$\mathcal{L} = \mathrm{div}(A\nabla\cdot) + \mathbf{b}\cdot\nabla + c \tag{40}$$

where $A \in ((C_0^\infty(\Omega))')^{n\times n}$, $\mathbf{b} \in ((C_0^\infty(\Omega))')^n$ and $c \in (C_0^\infty(\Omega))'$ are complex-valued.

Given $A = \{a_{jk}\} \in ((C_0^\infty(\Omega))')^{n\times n}$, we denote by A^s and A^c its symmetric part and skew-symmetric part respectively. The accretivity property for $-\mathcal{L}$ can be characterized in terms of the following real-valued expressions:

$$P = \mathbb{R}\mathrm{e}\, A^s, \quad \mathbf{d} = \frac{1}{2}\left[\mathbb{I}\mathrm{m}\,\mathbf{b} - \mathrm{Div}\,(\mathbb{I}\mathrm{m}\, A^c)\right], \quad \sigma = \mathbb{R}\mathrm{e}\, c - \frac{1}{2}\,\mathrm{div}(\mathbb{R}\mathrm{e}\,\mathbf{b})\,. \tag{41}$$

We note that $P = \{p_{ik}\} \in ((C_0^\infty(\Omega))')^{n\times n}, \mathbf{d} = \{d_j\} \in ((C_0^\infty(\Omega))')^n$, and $\sigma \in (C_0^\infty(\Omega))'$.

Moreover, in order that $-\mathcal{L}$ be accretive, the matrix P must be nonnegative definite, i.e., $P\xi\cdot\xi \geqslant 0$ in $(C_0^\infty(\Omega))'$ for all $\xi \in \mathbb{R}^n$. In particular, each p_{jj} $(j = 1, \dots, n)$ is a nonnegative Radon measure.

The characterization of accretive operators $-\mathcal{L}$ is given in the following criterion obtained in [52, Proposition 2.1]

Theorem 8 *Let $\mathcal{L}$ be the operator* (40). *Suppose that $P, \mathbf{d}$, and σ are defined by* (41). *The operator $-\mathcal{L}$ is accretive if and only if P is a nonnegative definite matrix, and the following two conditions hold:*

$$[h]^2_{\mathcal{H}} = \langle P\nabla h, \nabla h\rangle - \langle \sigma h, h\rangle \geqslant 0$$

for all real-valued $h \in C_0^\infty(\Omega)$, and the commutator inequality

$$|\langle \mathbf{d}, u\nabla v - v\nabla u\rangle| \leqslant [u]_{\mathcal{H}}[v]_{\mathcal{H}} \tag{42}$$

holds for all real-valued $u, v \in C_0^\infty(\Omega)$.

Under some mild restrictions on $\mathcal{H}$, the "norms" $[u]_{\mathcal{H}}$ and $[v]_{\mathcal{H}}$ on the right-hand side of (42) can be replaced, up to a constant multiple, with the corresponding Dirichlet norms

$$|\langle \mathbf{d}, u\nabla v - v\nabla u\rangle| \leqslant C\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \tag{43}$$

where $C > 0$ is a constant which does not depend on real-valued $u, v \in C_0^\infty(\Omega)$. This leads to explicit criteria of accretivity (see [53, Section 4] for the details). Indeed Maz'ya and Verbitsky have found necessary and sufficient conditions for the validity of commutator inequality (43). For example, when $\Omega = \mathbb{R}^n$ and $\mathbf{d}$ has L^1_{loc} components, they prove the following result

Theorem 9 *Let* $\mathbf{d} \in [L^1_{loc}(\mathbb{R}^n)]^n, n \geqslant 2$*. The inequality*

$$\left| \int_{\mathbb{R}^n} \langle \mathbf{d}, u\nabla v - v\nabla u \rangle dx \right| \leqslant C \|\nabla u\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)} \tag{44}$$

holds for any real-valued $u, v \in C_0^\infty(\mathbb{R}^n)$ *if and only if*

$$\mathbf{d} = \mathbf{c} + \operatorname{Div} F \tag{45}$$

where $F \in \mathrm{BMO}\,(\mathbb{R}^n)^{n\times n}$ *is a skew-symmetric matrix field, and* $\mathbf{c}$ *satisfies the condition*

$$\int_{\mathbb{R}^n} |\mathbf{c}|^2 |u|^2 dx \leqslant C \|\nabla u\|^2_{L^2(\mathbb{R}^n)} \tag{46}$$

where the constant C *does not depend on* $u \in C_0^\infty(\mathbb{R}^n)$*. Moreover, if* (44) *holds, then* (45) *is valid with* $\mathbf{c} = \nabla\Delta^{-1}(\operatorname{div}\mathbf{d})$ *satisfying* (46)*, and* $F = \Delta^{-1}(\operatorname{Curl}\mathbf{d}) \in \mathrm{BMO}\,(\mathbb{R}^n)^{n\times n}$.

In the case $n = 2$*, necessarily* $\mathbf{c} = 0$*, and* $\mathbf{d} = (-\partial_2 f, \partial_1 f)$ *with* $f \in \mathrm{BMO}\,(\mathbb{R}^2)$ *in the above statements.*

7 Elasticity

Consider the classical operator of two-dimensional elasticity

$$Eu = \Delta u + (1-2\nu)^{-1}\nabla\nabla^t u, \tag{47}$$

where ν is the Poisson ratio. As is known, E is strongly elliptic if and only if either $\nu > 1$ or $\nu < 1/2$. To obtain a necessary and sufficient condition for the L^p-dissipativity of this elasticity system, we formulate some facts about systems of partial differential equations of the form

$$A = \partial_h(\mathscr{A}^{hk}(x)\partial_k), \tag{48}$$

where $\mathscr{A}^{hk}(x) = \{a^{hk}_{ij}(x)\}$ are $m \times m$ matrices whose entries are complex locally integrable functions defined in an arbitrary domain Ω of $\mathbb{R}^n$ $(1 \leqslant i, j \leqslant m,\ 1 \leqslant h, k \leqslant n)$.

Lemma 4 ([10]) *An operator A of the form* (48) *is L^p-dissipative in $\Omega \subset \mathbb{R}^n$ if and only if*

$$\int_\Omega \Big(\operatorname{Re} \langle \mathscr{A}^{hk} \partial_k w, \partial_h w\rangle - (1-2/p)^2 |w|^{-4} \operatorname{Re} \langle \mathscr{A}^{hk} w, w\rangle \operatorname{Re}\langle w, \partial_k w\rangle \operatorname{Re} \langle w, \partial_h w\rangle - (1-2/p)|w|^{-2} \operatorname{Re} \big(\langle \mathscr{A}^{hk} w, \partial_h w\rangle \operatorname{Re} \langle w, \partial_k w\rangle - \langle \mathscr{A}^{hk} \partial_k w, w\rangle \operatorname{Re} \langle w, \partial_h w\rangle \big) \Big) dx \geqslant 0$$

for any $w \in ({C_0}^1(\Omega))^m$.

In the case $n = 2$, Lemma 4 yields a necessary algebraic condition.

Theorem 10 ([10]) *Let Ω be a domain of $\mathbb{R}^2$. If an operator A of the form* (48) *is L^p-dissipative, then*

$$\operatorname{Re} \langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\lambda, \lambda\rangle - (1-2/p)^2 \operatorname{Re} \langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega, \omega\rangle (\operatorname{Re}\langle \lambda, \omega\rangle)^2 - (1-2/p) \operatorname{Re}(\langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega, \lambda\rangle - \langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\lambda, \omega\rangle) \operatorname{Re} \langle \lambda, \omega\rangle \geqslant 0$$

for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^2$, λ, $\omega \in \mathbb{C}^m$, $|\omega| = 1$.

Based on Lemma 4 and Theorem 10, it is possible to obtain the following criterion for the L^p-dissipativity of the two-dimensional elasticity system.

Theorem 11 ([10]) *The operator* (47) *is L^p-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}. \tag{49}$$

By Theorems 10 and 11, it is easy to compare E and Δ from the point of view of L^p-dissipativity.

Corollary 4 ([10]) *There exists $k > 0$ such that $E - k\Delta$ is L^p-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 < \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}.$$

There exists $k < 2$ such that $k\Delta - E$ is L^p-dissipative if and only if

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 < \frac{2\nu(2\nu-1)}{(1-4\nu)^2}.$$

As remarkd at p.17 for scalar operators, this is equivalent to say that E is strict L^p-dissipative, i.e., E is p-elliptic. The last result was recently extended to variable Lamé parameters by Dindoš, Li and Pipher [22]. It must be pointed out that these authors introduce an auxiliary function $r(x)$ (see [22, pp.390–391]) which generates some first order terms in the partial differential operator. In the definition of p-ellipticity these terms do not play any role, while they have some role in the dissipativity. Therefore our and their results do not seem to be completely equivalent.

In [11] we showed that condition (49) is necessary for the L^p-dissipativity of operator (47) in any dimension, even when the Poisson ratio is not constant. At the present it is not known if condition (49) is also sufficient for the L^p-dissipativity of elasticity operator for $n > 2$, in particular for $n = 3$ (see [50, Problem 43]). Nevertheless, in the same paper, we gave a more strict explicit condition which is sufficient for the L^p-dissipativity of (47). Indeed we proved that if

$$(1-2/p)^2 \leqslant \begin{cases} \dfrac{1-2\nu}{2(1-\nu)} & \text{if } \nu < 1/2 \\[2ex] \dfrac{2(1-\nu)}{1-2\nu} & \text{if } \nu > 1, \end{cases}$$

then the operator (47) is L^p-dissipative.

In [11] we gave also necessary and sufficient conditions for a weighted L^p-negativity of the Dirichlet-Lamé operator, i.e. for the validity of the inequality

$$\int_\Omega (\Delta u + (1-2\nu)^{-1}\nabla \operatorname{div} u)\, |u|^{p-2} u \,\frac{dx}{|x|^\alpha} \leqslant 0 \tag{50}$$

under the condition that the vector u is rotationally invariant, i.e. u depends only on $\varrho = |x|$ and u_ϱ is the only nonzero spherical component of u. Namely we showed that (50) holds for any such u belonging to $(C_0^\infty(\mathbb{R}^N \setminus \{0\}))^N$ if and only if

$$-(p-1)(n+p'-2) \leqslant \alpha \leqslant n+p-2.$$

8 A Class of Systems of Partial Differential Operators

In this section, we consider systems of partial differential operators of the form

$$Au = \partial_h(\mathscr{A}^h(x)\partial_h u), \tag{51}$$

where $\mathscr{A}^h(x) = \{a^h_{ij}(x)\}$ $(i,j = 1,\dots,m)$ are matrices with complex locally integrable entries defined in a domain $\Omega \subset \mathbb{R}^n$ $(h = 1,\dots,n)$. Note that the elasticity system is not a system of this kind.

To characterize the L^p-dissipativity of such operators, one can reduce the consideration to the one-dimensional case. Auxiliary facts are given in the following two subsections.

Langer and Maz'ya considered the L^p-dissipativity of weakly coupled systems in [39].

8.1 Dissipativity of systems of ordinary differential equations

In this subsection, we consider the operator

$$Au = (\mathscr{A}(x)u')', \tag{52}$$

where $\mathscr{A}(x) = \{a_{ij}(x)\}$ $(i,j = 1,\dots,m)$ is a matrix with complex locally integrable entries defined in a bounded or unbounded interval (a,b). The corresponding sesquilinear form $\mathscr{L}(u,w)$ takes the form

$$\mathscr{L}(u,w) = \int_a^b \langle \mathscr{A}u', w'\rangle\, dx.$$

Theorem 12 ([10]) *The operator A is L^p-dissipative if and only if*

$$\begin{aligned}&\mathbb{R}\mathrm{e}\,\langle \mathscr{A}(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathbb{R}\mathrm{e}\,\langle \mathscr{A}(x)\omega,\omega\rangle(\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle)^2\\ &-(1-2/p)\,\mathbb{R}\mathrm{e}(\langle \mathscr{A}(x)\omega,\lambda\rangle - \langle \mathscr{A}(x)\lambda,\omega\rangle)\,\mathbb{R}\mathrm{e}\,\langle\lambda,\omega\rangle \geqslant 0\end{aligned}$$

for almost every $x \in (a,b)$ and for any $\lambda,\omega \in \mathbb{C}^m$, $|\omega| = 1$.

This theorem implies the following assertion.

Corollary 5 ([10]) *If the operator A is L^p-dissipative, then*

$$\mathbb{R}\mathrm{e}\,\langle \mathscr{A}(x)\lambda,\lambda\rangle \geqslant 0$$

for almost every $x \in (a,b)$ and for any $\lambda \in \mathbb{C}^m$.

As a consequence of Theorem 12 is the possibility to compare the operators A and $I(d^2/dx^2)$.

Corollary 6 ([10]) *There exists* $k > 0$ *such that* $A - kI(d^2/dx^2)$ *is* L^p*-dissipative if and only if*

$$\operatorname*{ess\,inf}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m\\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) > 0.$$

There exists $k > 0$ *such that* $kI(d^2/dx^2) - A$ *is* L^p*-dissipative if and only if*

$$\operatorname*{ess\,sup}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m\\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) < \infty.$$

There exists $k \in \mathbb{R}$ *such that* $A - kI(d^2/dx^2)$ *is* L^p*-dissipative if and only if*

$$\operatorname*{ess\,inf}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m\\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) > -\infty.$$

8.2 Criteria in terms of eigenvalues of $\mathscr{A}(x)$

If the coefficients a_{ij} of the operator (52) are real, it is possible to give a necessary and sufficient condition for the L^p-dissipativity of A in terms of eigenvalues of the matrix $\mathscr{A}$.

Theorem 13 ([10]) *Let* $\mathscr{A}$ *be a real matrix* $\{a_{hk}\}$ *with* $h,k = 1,\dots,m$*. Suppose that* $\mathscr{A} = \mathscr{A}^t$ *and* $\mathscr{A} \geqslant 0$ *(in the sense that* $\langle \mathscr{A}(x)\xi,\xi\rangle \geqslant 0$ *for almost every* $x \in (a,b)$ *and for any* $\xi \in \mathbb{R}^m$*). The operator* A *is* L^p*-dissipative if and only if*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 (\mu_1(x)+\mu_m(x))^2 \leqslant \mu_1(x)\mu_m(x)$$

almost everywhere, where $\mu_1(x)$ *and* $\mu_m(x)$ *are the smallest and largest eigenvalues of the matrix* $\mathscr{A}(x)$ *respectively. In the particular case* $m = 2$*, this condition is equivalent to*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 (\mathrm{tr}\mathscr{A}(x))^2 \leqslant \det \mathscr{A}(x)$$

almost everywhere.

Corollary 7 ([10]) *Let* $\mathscr{A}$ *be a real symmetric matrix. Let* $\mu_1(x)$ *and* $\mu_m(x)$ *be the smallest and largest eigenvalues of* $\mathscr{A}(x)$ *respectively. There exists* $k > 0$ *such that* $A - kI(d^2/dx^2)$ *is* L^p*-dissipative if and only if*

$$\operatorname*{ess\,inf}_{x\in(a,b)} \Big[(1+\sqrt{p\,p'}/2)\,\mu_1(x) + (1-\sqrt{p\,p'}/2)\,\mu_m(x)\Big] > 0. \tag{53}$$

In the particular case $m=2$, the condition (53) *is equivalent to*

$$\operatorname*{ess\,inf}_{x\in(a,b)}\left[\mathrm{tr}\mathscr{A}(x)-\frac{\sqrt{p\,p'}}{2}\sqrt{(\mathrm{tr}\mathscr{A}(x))^2-4\det\mathscr{A}(x)}\right]>0.$$

Under an extra condition on the matrix $\mathscr{A}$, the following assertion holds.

Corollary 8 ([10]) *Let $\mathscr{A}$ be a real symmetric matrix. Suppose that $\mathscr{A}\geqslant 0$ almost everywhere. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and largest eigenvalues of $\mathscr{A}(x)$ respectively. If there exists $k>0$ such that $A-kI(d^2/dx^2)$ is L^p-dissipative, then*

$$\operatorname*{ess\,inf}_{x\in(a,b)}\left[\mu_1(x)\mu_m(x)-\left(\frac{1}{2}-\frac{1}{p}\right)^2(\mu_1(x)+\mu_m(x))^2\right]>0. \tag{54}$$

If, in addition, there exists C such that

$$\langle\mathscr{A}(x)\xi,\xi\rangle\leqslant C|\xi|^2 \tag{55}$$

for almost every $x\in(a,b)$ and for any $\xi\in\mathbb{R}^m$, the converse assertion is also true. In the particular case $m=2$, the condition (54) *is equivalent to*

$$\operatorname*{ess\,inf}_{x\in(a,b)}\left[\det\mathscr{A}(x)-\left(\frac{1}{2}-\frac{1}{p}\right)^2(\mathrm{tr}\mathscr{A}(x))^2\right]>0.$$

We remark that $A-kI(d^2/dx^2)$ is L^p-dissipative means that A is p-elliptic. Generally speaking, the assumption (55) cannot be removed even if $\mathscr{A}\geqslant 0$.

Example 4 Consider $(a,b)=(1,\infty)$, $m=2$, $\mathscr{A}(x)=\{a_{ij}(x)\}$, where

$$a_{11}(x)=(1-2/\sqrt{pp'})x+x^{-1},\quad a_{12}(x)=a_{21}(x)=0,$$
$$a_{22}(x)=(1+2/\sqrt{pp'})x+x^{-1}.$$

Then

$$\mu_1(x)\mu_2(x)-\left(\frac{1}{2}-\frac{1}{p}\right)^2(\mu_1(x)+\mu_2(x))^2=(8+4x^{-2})/(p\,p')$$

and (54) holds. But (53) is not satisfied because

$$(1+\sqrt{p\,p'}/2)\,\mu_1(x)+(1-\sqrt{p\,p'}/2)\,\mu_2(x)=2x^{-1}.$$

Corollary 9 ([10]) *Let $\mathscr{A}$ be a real symmetric matrix. Let $\mu_1(x)$ and $\mu_m(x)$ be the smallest and largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k > 0$ such that $kI(d^2/dx^2) - A$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,sup}_{x\in(a,b)} \Big[(1 - \sqrt{p\,p'}/2)\,\mu_1(x) + (1 + \sqrt{p\,p'}/2)\,\mu_m(x)\Big] < \infty. \tag{56}$$

In the particular case $m = 2$, the condition (56) *is equivalent to*

$$\operatorname*{ess\,sup}_{x\in(a,b)} \left[\mathrm{tr}\mathscr{A}(x) + \frac{\sqrt{p\,p'}}{2}\sqrt{(\mathrm{tr}\mathscr{A}(x))^2 - 4\det\mathscr{A}(x)}\right] < \infty.$$

If $\mathscr{A}$ is positive, the following assertion holds.

Corollary 10 ([10]) *Let $\mathscr{A}$ be a real symmetric matrix. Suppose that $\mathscr{A} \geqslant 0$ almost everywhere. Let $\mu_1(x)$ and $\mu_m(x)$ be the smallest and largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k > 0$ such that $kI(d^2/dx^2) - A$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,sup}_{x\in(a,b)} \mu_m(x) < \infty.$$

8.3 L^p-dissipativity of the operator (51)

We represent necessary and sufficient conditions for the L^p-dissipativity of the system (51), obtained in [10].

Denote by y_h the $(n-1)$-dimensional vector $(x_1, \dots, x_{h-1}, x_{h+1}, \dots, x_n)$ and set $\omega(y_h) = \{x_h \in \mathbb{R} \mid x \in \Omega\}$.

Lemma 5 ([10]) *The operator* (51) *is L^p-dissipative if and only if the ordinary differential operators*

$$A(y_h)[u(x_h)] = d(\mathscr{A}^h(x)du/dx_h)/dx_h$$

are L^p-dissipative in $\omega(y_h)$ for almost every $y_h \in \mathbb{R}^{n-1}$ ($h = 1, \dots, n$). This condition is void if $\omega(y_h) = \emptyset$.

Theorem 14 ([10]) *The operator* (51) *is L^p-dissipative if and only if*

$$\begin{aligned}&\mathrm{Re}\langle\mathscr{A}^h(x_0)\lambda, \lambda\rangle - (1 - 2/p)^2\,\mathrm{Re}\langle\mathscr{A}^h(x_0)\omega, \omega\rangle(\mathrm{Re}\langle\lambda, \omega\rangle)^2 \\ &\quad - (1 - 2/p)\,\mathrm{Re}(\langle\mathscr{A}^h(x_0)\omega, \lambda\rangle - \langle\mathscr{A}^h(x_0)\lambda, \omega\rangle)\,\mathrm{Re}\,\langle\lambda, \omega\rangle \geqslant 0\end{aligned} \tag{57}$$

for almost every $x_0 \in \Omega$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \dots, n$.

In the scalar case ($m = 1$), the operator (51) can be considered as an operator from Section 4.

In fact, if $Au = \sum_{h=1}^{n} \partial_h(a^h \partial_h u)$, a^h is a scalar function, then A can be written in the form (22) with $\mathscr{A} = \{c_{hk}\}$, $c_{hh} = a^h$, $c_{hk} = 0$ if $h \neq k$. The conditions obtained in Section 4 can be directly compared with (57). We know that the operator A is L^p-dissipative if and only if (24) holds. In this particular case, it is clear that (24) is equivalent to the following n conditions:

$$\frac{4}{p\,p'}\,(\mathbb{R}\mathrm{e}\, a^h)\,\xi^2 + (\mathbb{R}\mathrm{e}\, a^h)\,\eta^2 - 2(1-2/p)(\mathbb{I}\mathrm{m}\, a^h)\,\xi\eta \geqslant 0 \tag{58}$$

almost everywhere and for any $\xi, \eta \in \mathbb{R}$, $h = 1, \ldots, n$. On the other hand, in this case, (57) reads as

$$\begin{aligned}(\mathbb{R}\mathrm{e}\, a^h)|\lambda|^2 - (1-2/p)^2(\mathbb{R}\mathrm{e}\, a^h)(\mathbb{R}\mathrm{e}(\lambda\overline{\omega})^2 \\ - 2(1-2/p)(\mathbb{I}\mathrm{m}\, a^h)\,\mathbb{R}\mathrm{e}(\lambda\overline{\omega})\,\mathbb{I}\mathrm{m}(\lambda\overline{\omega}) \geqslant 0\end{aligned} \tag{59}$$

almost everywhere and for any $\lambda, \omega \in \mathbb{C}$, $|\omega| = 1$, $h = 1, \ldots, n$. Setting $\xi + i\eta = \lambda\overline{\omega}$ and observing that $|\lambda|^2 = |\lambda\overline{\omega}|^2 = (\mathbb{R}\mathrm{e}(\lambda\overline{\omega}))^2 + (\mathbb{I}\mathrm{m}(\lambda\overline{\omega}))^2$, we see that the conditions (58) (hence (24)) are equivalent to (59).

If A has real coefficients, we can characterize the L^p-dissipativity in terms of the eigenvalues of the matrices $\mathscr{A}^h(x)$.

Theorem 15 ([10]) *Let A be an operator of the form* (51)*, where $\mathscr{A}^h$ are real matrices $\{a^h_{ij}\}$ with $i, j = 1, \ldots, m$. Suppose that $\mathscr{A}^h = (\mathscr{A}^h)^t$ and $\mathscr{A}^h \geqslant 0$ ($h = 1, \ldots, n$). The operator A is L^p-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1^h(x) + \mu_m^h(x))^2 \leqslant \mu_1^h(x)\,\mu_m^h(x)$$

for almost every $x \in \Omega$, $h = 1, \ldots, n$, where $\mu_1^h(x)$ and $\mu_m^h(x)$ are the smallest and largest eigenvalues of the matrix $\mathscr{A}^h(x)$ respectively. In the particular case $m = 2$, this condition is equivalent to

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mathrm{tr}\mathscr{A}^h(x))^2 \leqslant \det \mathscr{A}^h(x)$$

for almost every $x \in \Omega$, $h = 1, \ldots, n$.

9 The angle of dissipativity

By means of the necessary and sufficient conditions we have obtained, we can determine exactly the angle of dissipativity. Determining the angle of dissipativity of an operator A, means to find necessary and sufficient conditions for the L^p-dissipativity of the differential operator zA, where $z \in \mathbb{C}$.

Consider first the scalar operator

$$A = \nabla^t(\mathscr{A}(x)\nabla),$$

where $\mathscr{A}(x) = \{a_{ij}(x)\}$ $(i, j = 1, \ldots, n)$ is a matrix with complex locally integrable entries defined in a domain $\Omega \subset \mathbb{R}^n$. If $\mathscr{A}$ is a real matrix, it is well known (cf., for example, [29, 30, 56]) that the dissipativity angle is independent of the operator and is given by

$$|\arg z| \leqslant \arctan\left(\frac{2\sqrt{p-1}}{|p-2|}\right). \tag{60}$$

If the entries of the matrix $\mathscr{A}$ are complex, the situation is different because the dissipativity angle depends on the operator, as the next theorem shows.

Theorem 16 ([10]) *Let a matrix $\mathscr{A}$ be symmetric. Suppose that the operator A is L^p-dissipative. Let*

$$\Lambda_1 = \operatorname*{ess\,inf}_{(x,\xi)\in\Xi} \frac{\langle\, \mathbb{I}\mathrm{m}\, \mathscr{A}(x)\xi, \xi\rangle}{\langle\, \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\xi, \xi\rangle}, \quad \Lambda_2 = \operatorname*{ess\,sup}_{(x,\xi)\in\Xi} \frac{\langle\, \mathbb{I}\mathrm{m}\, \mathscr{A}(x)\xi, \xi\rangle}{\langle\, \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\xi, \xi\rangle},$$

where

$$\Xi = \{(x, \xi) \in \Omega \times \mathbb{R}^n \mid \langle\, \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\xi, \xi\rangle > 0\}.$$

The operator zA is L^p-dissipative if and only if

$$\vartheta_- \leqslant \arg z \leqslant \vartheta_+\,,$$

where [2]

$$\vartheta_- = \begin{cases} \operatorname{arccot}\left(\frac{2\sqrt{p-1}}{|p-2|} - \frac{p^2}{|p-2|}\frac{1}{2\sqrt{p-1}+|p-2|\Lambda_1}\right) - \pi & \textit{if } p \neq 2, \\ \operatorname{arccot}(\Lambda_1) - \pi & \textit{if } p = 2, \end{cases}$$

$$\vartheta_+ = \begin{cases} \operatorname{arccot}\left(-\frac{2\sqrt{p-1}}{|p-2|} + \frac{p^2}{|p-2|}\frac{1}{2\sqrt{p-1}-|p-2|\Lambda_2}\right) & \textit{if } p \neq 2 \\ \operatorname{arccot}(\Lambda_2) & \textit{if } p = 2. \end{cases}$$

Note that for a real matrix $\mathscr{A}$ we have $\Lambda_1 = \Lambda_2 = 0$ and, consequently,

$$\frac{2\sqrt{p-1}}{|p-2|} - \frac{p^2}{2\sqrt{p-1}|p-2|} = -\frac{|p-2|}{2\sqrt{p-1}}\,.$$

Theorem 16 asserts that zA is dissipative if and only if

$$\operatorname{arccot}\left(-\frac{|p-2|}{2\sqrt{p-1}}\right) - \pi \leqslant \arg z \leqslant \operatorname{arccot}\left(\frac{|p-2|}{2\sqrt{p-1}}\right),$$

i.e., if and only if (60) holds.

We can precisely determine the angle of dissipativity also for the matrix ordinary differential operator (52) with complex coefficients.

Theorem 17 ([10]) *Let the operator* (52) *be L^p-dissipative. The operator zA is L^p-dissipative if and only if*

$$\vartheta_- \leqslant \arg z \leqslant \vartheta_+$$

where

$$\vartheta_- = \operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\lambda,\omega)\in\Xi}(Q(x,\lambda,\omega)/P(x,\lambda,\omega))\right) - \pi,$$

$$\vartheta_+ = \operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\lambda,\omega)\in\Xi}(Q(x,\lambda,\omega)/P(x,\lambda,\omega))\right),$$

[2]Here, $0 < \operatorname{arccot} y < \pi$, $\operatorname{arccot}(+\infty) = 0$, $\operatorname{arccot}(-\infty) = \pi$, and

$$\operatorname*{ess\,inf}_{(x,\xi)\in\Xi}\frac{\langle\,\mathbb{I}\mathrm{m}\,\mathscr{A}(x)\xi,\xi\rangle}{\langle\,\mathbb{R}\mathrm{e}\,\mathscr{A}(x)\xi,\xi\rangle} = +\infty, \quad \operatorname*{ess\,sup}_{(x,\xi)\in\Xi}\frac{\langle\,\mathbb{I}\mathrm{m}\,\mathscr{A}(x)\xi,\xi\rangle}{\langle\,\mathbb{R}\mathrm{e}\,\mathscr{A}(x)\xi,\xi\rangle} = -\infty$$

if Ξ has zero measure.

$$
\begin{aligned}
P(x,\lambda,\omega) = \mathbb{R}\mathrm{e}\,\langle \mathscr{A}(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathbb{R}\mathrm{e}\,\langle \mathscr{A}(x)\omega,\omega\rangle(\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle)^2 \\
- (1-2/p)\,\mathbb{R}\mathrm{e}(\langle \mathscr{A}(x)\omega,\lambda\rangle - \langle \mathscr{A}(x)\lambda,\omega\rangle)\,\mathbb{R}\mathrm{e}\,\langle\lambda,\omega\rangle,
\end{aligned}
$$

$$
\begin{aligned}
Q(x,\lambda,\omega) = \mathbb{I}\mathrm{m}\,\langle \mathscr{A}(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathbb{I}\mathrm{m}\,\langle \mathscr{A}(x)\omega,\omega\rangle(\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle)^2 \\
- (1-2/p)\,\mathbb{I}\mathrm{m}(\langle \mathscr{A}(x)\omega,\lambda\rangle - \langle \mathscr{A}(x)\lambda,\omega\rangle)\,\mathbb{R}\mathrm{e}\,\langle\lambda,\omega\rangle
\end{aligned}
$$

and Ξ *is the set*

$$
\Xi = \{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m \mid |\omega|=1,\ P^2(x,\lambda,\omega)+Q^2(x,\lambda,\omega)>0\}.
$$

Finally Theorem 14 allows us to determine the angle of dissipativity of the operator (51).

Theorem 18 ([10]) *Let the operator* (51) *be* L^p*-dissipative. The operator* zA *is* L^p*-dissipative if and only if*

$$
\vartheta_- \leqslant \arg z \leqslant \vartheta_+,
$$

where

$$
\begin{aligned}
\vartheta_- &= \max_{h=1,\dots,n} \operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\lambda,\omega)\in\Xi_h}(Q_h(x,\lambda,\omega)/P_h(x,\lambda,\omega))\right) - \pi, \\
\vartheta_+ &= \min_{h=1,\dots,n} \operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\lambda,\omega)\in\Xi_h}(Q_h(x,\lambda,\omega)/P_h(x,\lambda,\omega))\right)
\end{aligned}
$$

and

$$
\begin{aligned}
P_h(x,\lambda,\omega) = \mathbb{R}\mathrm{e}\,\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathbb{R}\mathrm{e}\,\langle \mathscr{A}^h(x)\omega,\omega\rangle(\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle)^2 \\
- (1-2/p)\,\mathbb{R}\mathrm{e}(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\,\mathbb{R}\mathrm{e}\,\langle\lambda,\omega\rangle,
\end{aligned}
$$

$$
\begin{aligned}
Q_h(x,\lambda,\omega) = \mathbb{I}\mathrm{m}\,\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathbb{I}\mathrm{m}\,\langle \mathscr{A}^h(x)\omega,\omega\rangle(\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle)^2 \\
- (1-2/p)\,\mathbb{I}\mathrm{m}(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\,\mathbb{R}\mathrm{e}\,\langle\lambda,\omega\rangle,
\end{aligned}
$$

$$
\Xi_h = \{(x,\lambda,\omega)\in\Omega\times\mathbb{C}^m\times\mathbb{C}^m \mid |\omega|=1,\ P_h^2(x,\lambda,\omega)+Q_h^2(x,\lambda,\omega)>0\}.
$$

10 Maximum principles for linear elliptic equations and systems

As said in the Introduction, Kresin and Maz'ya have obtained results on different forms of maximum principles for linear elliptic equations and systems. Here we recall some of their results.

Let us consider the operator

$$\mathfrak{A}_0 (D_x) = \sum_{j,k=1}^{n} \mathcal{A}_{jk} \partial_{jk} \tag{61}$$

where $D_x = (\partial_1, \ldots, \partial_n)$ and $\mathcal{A}_{jk} = \mathcal{A}_{kj}$ are constant real $(m \times m)$-matrices. Assume that the operator $\mathfrak{A}_0$ is strongly elliptic, i.e., that for all $\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{R}^m$ and $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$, with $\zeta, \sigma \neq 0$, we have the inequality

$$\left\langle \sum_{j,k=1}^{n} \mathcal{A}_{jk} \sigma_j \sigma_k \zeta, \zeta \right\rangle > 0$$

Let Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$. Let $\left[\mathrm{C_b}(\overline{\Omega})\right]^m$ denote the space of bounded m-component vector-valued functions which are continuous in $\overline{\Omega}$. The norm on $\left[\mathrm{C_b}(\overline{\Omega})\right]^m$ is $\|u\| = \sup\{|u(x)| : x \in \overline{\Omega}\}$. The notation $[\mathrm{C_b}(\partial\Omega)]^m$ has a similar meaning. By $[\mathrm{C}^2(\Omega)]^m$ we denote the space of m-component vector-valued functions with continuous derivatives up to the second order in Ω.

Let

$$\mathcal{K}(\Omega) = \sup \frac{\|u\|_{\left[\mathrm{C_b}(\overline{\Omega})\right]^m}}{\|u\|_{[\mathrm{C_b}(\partial\Omega)]^m}},$$

where the supremum is taken over all vector-valued functions in the class $\left[\mathrm{C_b}(\overline{\Omega})\right]^m \cap [\mathrm{C}^2(\Omega)]^m$ satisfying the system $\mathfrak{A}_0 (D_x) u = 0$.

Clearly, $\mathcal{K}(\Omega)$ is the best constant in the inequality

$$|u(x)| \leqslant \mathcal{K}(\Omega) \sup\{|u(y)| : y \in \partial\Omega\}$$

where $x \in \Omega$ and u is a solution of the system $\mathfrak{A}_0 (D_x) u = 0$ in the class $\left[\mathrm{C_b}(\overline{\Omega})\right]^m \cap [\mathrm{C}^2(\Omega)]^m$

If $\mathcal{K}(\Omega) = 1$, then the classical maximum modulus principle holds for the system $\mathfrak{A}_0 (D_x) u = \mathbf{0}$.

Kresin and Maz'ya proved the following criterion for the validity of this classical modulus principle.

Theorem 19 *Let Ω be a domain in $\mathbb{R}^n$ with compact closure and C^1-boundary. The equality $\mathcal{K}(\Omega) = 1$ holds if and only if the operator $\mathfrak{A}_0(D_x)$ is defined by*

$$\mathfrak{A}_0(D_x) = \mathcal{A} \sum_{j,k=1}^{n} a_{jk} \partial_{jk} \tag{62}$$

where $\mathcal{A}$ and $\{a_{jk}\}$ are positive-definite constant matrices of orders m and n, respectively.

Suppose now that the operator (61) has complex coefficients, i.e., suppose that $\mathcal{A}_{jk} = \mathcal{A}_{kj}$ are constant complex $(m \times m)$-matrices. Assume that the operator is strongly elliptic. This means that

$$\mathbb{R}\mathrm{e} \left\langle \sum_{j,k=1}^{n} \mathcal{A}_{jk} \sigma_j \sigma_k \zeta, \zeta \right\rangle > 0$$

for all $\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m$ and $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$, with $\zeta, \sigma \neq 0$

A necessary and sufficient condition for validity of the classical modulus principle for operator (61) with complex coefficients in a bounded domain runs as follows.

Theorem 20 *Let Ω be a domain in $\mathbb{R}^n$ with compact closure and C^1-boundary. The equality $\mathcal{K}(\Omega) = 1$ holds if and only if the operator $\mathfrak{A}_0(D_x)$ has the form* (62)*, where now $\mathcal{A}$ is a constant complex-valued $(m \times m)$-matrix such that $\mathbb{R}\mathrm{e}(\mathcal{A}\zeta, \boldsymbol{\zeta}) > 0$ for all $\zeta \in \mathbb{C}^m, \zeta \neq \mathbf{0}$, and $\{a_{jk}\}$ is a real positive-definite $(n \times n)$ matrix.*

These results have been extended to more general systems and we refer to the survey [38] for all the details.

11 Other results

In this section we briefly mention other results we have obtained.

In [13] we found necessary and sufficient conditions for the L^p-dissipativity of systems of the first order. Namely we have considered the matrix operator

$$Eu = \mathscr{B}^h(x) \partial_h u + \mathscr{D}(x) u\,, \tag{63}$$

where $\mathscr{B}^h(x) = \{b^h_{ij}(x)\}$ and $\mathscr{D}(x) = \{d_{ij}(x)\}$ are matrices with complex locally integrable entries defined in the domain Ω of $\mathbb{R}^n$ and $u = (u_1, \ldots, u_m)$ $(1 \leqslant i, j \leqslant$

$m,\ 1 \leqslant h \leqslant n$). It states that, if $p \neq 2$, E is L^p-dissipative if, and only if,

$$\mathscr{B}^h(x) = b_h(x) I \text{ a.e.}, \tag{64}$$

$b_h(x)$ being real locally integrable functions, and the inequality

$$\mathbb{R}\mathrm{e}\langle (p^{-1}\partial_h \mathscr{B}^h(x) - \mathscr{D}(x))\zeta, \zeta\rangle \geqslant 0$$

holds for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$. If $p = 2$ condition (64) is replaced by the more general requirement that the matrices $\mathscr{B}^h(x)$ are self-adjoint a.e..

We have applied this result also to second order operators, obtaining a sufficient condition for their L^p-dissipativity. We have also determined the angle of dissipativity of operator (63).

In [14] we have considered the "complex oblique derivative" operator

$$\lambda \cdot \nabla u = \frac{\partial u}{\partial x_n} + \sum_{j=1}^{n-1} a_j \frac{\partial u}{\partial x_j} \tag{65}$$

where $\lambda = (1, a_1, \ldots, a_{n-1})$ and a_j are complex valued functions. We gave necessary and, separately, sufficient conditions under which such boundary operator is L^p-dissipative on $\mathbb{R}^{n-1}$. If the coefficients a_j are real valued, we have obtained a necessary and sufficient condition: the operator (65) is L^p-dissipative if and only if there exists a real vector $\Gamma \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$-\partial_j(\mathbb{R}\mathrm{e}\, a_j)\, \delta(x_n) \leqslant \frac{2}{p'}(\mathrm{div}\, \Gamma - |\Gamma|^2)$$

in the sense of distributions.

In the same paper we have considered also the class of integral operators which can be written as

$$\int^*_{\mathbb{R}^n} [u(x) - u(y)]\, K(dx, dy) \tag{66}$$

where the integral has to be understood as a principal value in the sense of Cauchy and the kernel $K(dx, dy)$ is a Borel positive measure defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying certain conditions. The class of operators we considered includes the fractional powers of Laplacian $(-\Delta)^s$, with $0 < s < 1$. For the latter we previously had proved the following theorem

Theorem 21 ([12], p.230–231) *Let $0 < \alpha < 1$. We have, for any $u \in C_0^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \langle (-\Delta)^\alpha u, u\rangle |u|^{p-2} dx \geqslant \frac{2\, c_\alpha}{p\, p'} \, \| |u|^{p/2} \|^2_{\mathcal{L}^{\alpha,2}(\mathbb{R}^n)} \,,$$

where

$$c_\alpha = -\pi^{-n/2} 4^\alpha \Gamma(\alpha + n/2)/\Gamma(-\alpha) > 0$$

and

$$\|v\|_{\mathcal{L}^{\alpha,2}(\mathbb{R}^n)} = \left(\iint_{\mathbb{R}^n \times \mathbb{R}^n} |v(y) - v(x)|^2 \frac{dx dy}{|y-x|^{n+2\alpha}} \right)^{1/2} .$$

In [14] we have established the L^p-positivity of operator (66), extending in this way Theorem 21.

12 The functional dissipativity

In [15] we have introduced the new concept of functional dissipativity. Roughly speaking the idea is to replace $|u|^{p-2}$ by a more general $\varphi(|u|)$, φ being a positive function.

Let us consider the operator (22) with L^∞ complex valued coefficients. We say that it is functional dissipative or L^Φ-dissipative if

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}\, \nabla u, \nabla(\varphi(|u|)\, u)\rangle \, dx \geqslant 0$$

for any $u \in \mathring{H}^1(\Omega)$ such that $\varphi(|u|)\, u \in \mathring{H}^1(\Omega)$. Here φ is a positive function defined on $\mathbb{R}^+ = (0, +\infty)$ which satisfies the following conditions:

(i) $\varphi \in C^1((0, +\infty))$;

(ii) $(s\, \varphi(s))' > 0$ for any $s > 0$;

(iii) the range of the strictly increasing function $s\, \varphi(s)$ is $(0, +\infty)$;

(iv) there exist two positive constants C_1, C_2 and a real number $r > -1$ such that

$$C_1 s^r \leqslant (s\varphi(s))' \leqslant C_2\, s^r, \qquad s \in (0, s_0)$$

for a certain $s_0 > 0$. If $r = 0$ we require more restrictive conditions: there exists the finite limit $\lim_{s\to 0^+} \varphi(s) = \varphi_+(0) > 0$ and $\lim_{s\to 0^+} s\, \varphi'(s) = 0$.

(v) There exists $s_1 > s_0$ such that

$$\varphi'(s) \geqslant 0 \text{ or } \varphi'(s) \leqslant 0 \qquad \forall\, s \geqslant s_1.$$

The reason for requiring that function $s\,\varphi(s)$ is increasing is that in such a way the function

$$\Phi(s) = \int_0^s \sigma\,\varphi(\sigma)\,d\sigma \tag{67}$$

is a Young function (i.e., a convex positive function such that $\Phi(0) = 0$ and $\Phi(+\infty) = +\infty$) . We note that, if $t\,\psi(t)$ is the inverse function of $s\,\varphi(s)$, then

$$\Psi(s) = \int_0^s \sigma\,\psi(\sigma)\,d\sigma$$

is the conjugate Young function of Φ.

The condition (iv) prescribes the behaviour of the function φ in a neighborhood of the origin, while (v) concerns the behaviour for large s.

The function $\varphi(s) = s^{p-2}$ $(p > 1)$ provides an example of such a function.

A motivation for the study of the concept of functional dissipativity comes from the decrease of the Luxemburg norm of solutions of the Cauchy–Dirichlet problem

$$\begin{cases} u' = Au \\ u(0) = u_0\,. \end{cases} \tag{68}$$

Indeed let us consider the Orlicz space of functions u for which there exists $\alpha > 0$ such that

$$\int_\Omega \Phi(\alpha\,|u|)\,dx < +\infty\,.$$

For the general theory of Orlicz spaces we refer to Krasnosel'skiĭ, Rutickiĭ [35] and Rao, Ren [61]. As in (9), if $u(x, y)$ is a solution of the Cauchy-Dirichlet problem (68), we have the decrease of the integrals

$$\int_\Omega \Phi(|u(x,t)|)\,dx$$

if

$$\mathbb{R}\mathrm{e} \int_\Omega \langle Eu, u\rangle |u|^{-1}\Phi'(|u|)\,dx \leqslant 0.$$

This implies the decrease of the Luxemburg norm in the related Orlicz space

$$\|u(\cdot,t)\| = \inf\left\{\langle > 0 \mid \int_\Omega \Phi(|u(x,t)|/\lambda)\, dx \leqslant 1\right\}.$$

In paper [15] we proved the following technical lemma, which played a key role.

Lemma 6 *The operator A is L^Φ-dissipative if and only if*

$$\mathbb{R}\mathrm{e}\int_\Omega \Big[\langle \mathscr{A}\, \nabla v, \nabla v\rangle + \Lambda(|v|)\,\langle(\mathscr{A} - \mathscr{A}^*)\nabla|v|, |v|^{-1}\overline{v}\nabla v)\rangle + \\ -\Lambda^2(|v|)\,\langle \mathscr{A}\,\nabla|v|, \nabla|v|\rangle\Big]\, dx \geqslant 0, \qquad \forall v \in \mathring{H}^1(\Omega),$$

where the function Λ is is the function defined by the relation

$$\Lambda\left(s\sqrt{\varphi(s)}\right) = -\frac{s\,\varphi'(s)}{s\,\varphi'(s) + 2\,\varphi(s)}\,.$$

We remark that if $\varphi(s) = s^{p-2}$, the function Λ is constant and

$$\Lambda(t) = -(1 - 2/p), \quad 1 - \Lambda^2(t) = 4/(p\,p').$$

As Corollaries of Lemma 6 we have obtained necessary and, separately, sufficient conditions for the functional dissipativity of the operator E.

Corollary 11 *If the operator A is L^Φ-dissipative, we have*

$$\langle \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\xi, \xi\rangle \geqslant 0 \tag{69}$$

for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$.

Corollary 12 *If*

$$[1 - \Lambda^2(t)]\langle \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\,\xi, \xi\rangle + \langle \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\,\eta, \eta\rangle + \\ [1 + \Lambda(t)]\langle \mathbb{I}\mathrm{m}\, \mathscr{A}(x)\,\xi, \eta\rangle + [1 - \Lambda(t)]\langle \mathbb{I}\mathrm{m}\, \mathscr{A}^*(x)\,\xi, \eta\rangle \geqslant 0 \tag{70}$$

for almost every $x \in \Omega$ and for any $t > 0, \xi, \eta \in \mathbb{R}^n$, the operator A is L^Φ-dissipative.

Corollary 13 *If the operator A has real coefficients and satisfies condition* (69) *for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$, than it is L^Φ-dissipative with respect to any Φ.*

The main result obtained in [15] is the following necessary and sufficient condition

Theorem 22 *Let the matrix $\mathbb{I}\mathrm{m}\,\mathscr{A}$ be symmetric, i.e., $\mathbb{I}\mathrm{m}\,\mathscr{A}^t = \mathbb{I}\mathrm{m}\,\mathscr{A}$. Then the operator A is L^Φ-dissipative if, and only if,*

$$|s\,\varphi'(s)|\,|\langle \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\,\xi,\xi\rangle| \leqslant 2\,\sqrt{\varphi(s)\,[s\,\varphi(s)]'}\,\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\,\xi,\xi\rangle \tag{71}$$

for almost every $x \in \Omega$ and for any $s > 0, \xi \in \mathbb{R}^n$.

Suppose that the condition $\mathbb{I}\mathrm{m}\,\mathscr{A} = \mathbb{I}\mathrm{m}\,\mathscr{A}^t$ is not satisfied. Arguing as in the proof of Theorem 22, one can prove that condition (71) is still necessary for the L^Φ-dissipativity of the operator E. However in general it is not sufficient, whatever the function φ may be. This is shown by the next example.

Example 5 Let $n = 2$, Ω be a bounded domain, λ be a real parameter and

$$\mathscr{A} = \begin{pmatrix} 1 & i\lambda x_1 \\ -i\lambda x_1 & 1 \end{pmatrix}$$

Since $\langle \mathbb{R}\mathrm{e}\,\mathscr{A}\,\xi,\xi\rangle = |\xi|^2$ and $\langle \mathbb{I}\mathrm{m}\,\mathscr{A}\,\xi,\xi\rangle = 0$ for any $\xi \in \mathbb{R}^n$, condition (71) is satisfied.

If the corresponding operator $Eu = \Delta u + i\,\lambda\,\partial_2 u$ is L^Φ-dissipative, then

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \Delta u + i\,\lambda\,\partial_2 u, u\rangle\,\varphi(|u|)\,dx \leqslant 0, \qquad \forall\, u \in C_0^\infty(\Omega). \tag{72}$$

Take $u(x) = \varrho(x)\,e^{i\,t\,x_2}$, where $\varrho \in C_0^\infty(\Omega)$ is real valued and $t \in \mathbb{R}$. Since $\langle Eu, u\rangle = \varrho[\Delta\varrho + 2\,i\,t\,\partial_2\varrho - t^2\varrho + i\,\lambda\,(\partial_2\varrho + it\varrho)]$, condition (72) implies

$$\int_\Omega \varrho\,\Delta\varrho\,\varphi(|\varrho|)\,dx - \lambda\,t \int_\Omega \varrho^2\varphi(|\varrho|)\,dx - t^2 \int_\Omega \varrho^2\varphi(|\varrho|)\,dx \leqslant 0 \tag{73}$$

for any $t, \lambda \in \mathbb{R}$. The function φ being positive, we can choose ϱ in such a way

$$\int_\Omega \varrho^2\varphi(|\varrho|)\,dx > 0.$$

Taking

$$\lambda^2 > 4 \int_\Omega \varrho\, \Delta \varrho\, \varphi(|\varrho|)\, dx \left(\int_\Omega \varrho^2 \varphi(|\varrho|)\, dx \right)^{-1},$$

inequality (73) is impossible for all $t \in \mathbb{R}$. Thus E is not L^Φ-dissipative, although (71) is satisfied.

We have also

Corollary 14 *Let the matrix* $\mathbb{I}\mathrm{m}\, \mathscr{A}$ *be symmetric, i.e.,* $\mathbb{I}\mathrm{m}\, \mathscr{A}^t = \mathbb{I}\mathrm{m}\, \mathscr{A}$. *If*

$$\lambda_0 = \sup_{s>0} \frac{|s\, \varphi'(s)|}{2\sqrt{\varphi(s)\, [s\, \varphi(s)]'}} < +\infty, \tag{74}$$

then the operator E *is* L^Φ*-dissipative if, and only if,*

$$\lambda_0\, |\langle \mathbb{I}\mathrm{m}\, \mathscr{A}(x)\, \xi, \xi\rangle| \leqslant \langle \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\, \xi, \xi\rangle \tag{75}$$

for almost every $x \in \Omega$ *and for any* $\xi \in \mathbb{R}^n$. *If* $\lambda_0 = +\infty$ *the operator* E *is* L^Φ*-dissipative if and only if* $\mathbb{I}\mathrm{m}\, \mathscr{A} = 0$ *and condition* (69) *is satisfied.*

If we use the function Φ (see (67)), condition (71) can be written as

$$|s\, \Phi''(s) - \Phi'(s)|\, |\langle \mathbb{I}\mathrm{m}\, \mathscr{A}(x)\, \xi, \xi\rangle| \leqslant 2\sqrt{s\, \Phi'(s)\, \Phi''(s)}\, \langle \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\, \xi, \xi\rangle$$

for almost every $x \in \Omega$ and for any $s > 0, \xi \in \mathbb{R}^n$. In the same way, formula (74) becomes

$$\lambda_0 = \sup_{s>0} \frac{|s\, \Phi''(s) - \Phi'(s)|}{2\sqrt{s\, \Phi'(s)\, \Phi''(s)}} < +\infty.$$

We consider now some examples in which we indicate both the functions Φ and φ. It is easy to verify that in each example the function φ satisfies conditions (i)-(v) (see p.35).

Example 6 If $\Phi(s) = s^p$, i.e., $\varphi(s) = p\, s^{p-2}$, which corresponds to L^p norm, the function in (74) is constant and $\lambda_0 = |p-2|/(2\sqrt{p-1})$. In this way we reobtain Theorem 4.

Example 7 Let us consider $\Phi(s) = s^p \log(s+e)$ $(p > 1)$, which is the Young function corresponding to the Zygmund space $L^p\, \log L$. This is equivalent to say

$\varphi(s) = ps^{p-2}\log(s+e) + s^{p-1}(s+e)^{-1}$. By a direct computation we find

$$\frac{|s\,\Phi''(s) - \Phi'(s)|}{2\sqrt{s\,\Phi'(s)\,\Phi''(s)}} = \frac{\left|p(p-2)\log(s+e) + \frac{(2p-1)s}{s+e} - \frac{s^2}{(s+e)^2}\right|}{2\sqrt{\left(p\log(s+e) + \frac{s}{s+e}\right)\left(p(p-1)\log(s+e) + \frac{2ps}{s+e} - \frac{s^2}{(s+e)^2}\right)}}\,. \tag{76}$$

Since

$$\lim_{s\to 0^+} \frac{|s\,\Phi''(s) - \Phi'(s)|}{2\sqrt{s\,\Phi'(s)\,\Phi''(s)}} = \lim_{s\to +\infty} \frac{|s\,\Phi''(s) - \Phi'(s)|}{2\sqrt{s\,\Phi'(s)\,\Phi''(s)}} = \frac{|p-2|}{2\sqrt{p-1}}$$

the function is bounded. Then we have the L^Φ-dissipativity of the operator A if, and only if, (75) holds, where λ_0 is the $\sup$ of the function (76) in $\mathbb{R}^+$.

Example 8 Let us consider the function $\Phi(s) = \exp(s^p) - 1$, i.e., $\varphi(s) = p\,s^{p-2}\exp(s^p)$. In this case

$$\frac{|s\,\Phi''(s) - \Phi'(s)|}{2\sqrt{s\,\Phi'(s)\,\Phi''(s)}} = \frac{|p\,s^p + p - 2|}{2\sqrt{(p\,s^p + p - 1)}}$$

and $\lambda_0 = +\infty$. In view of Corollary (14), the operator A is L^Φ-dissipative, i.e.,

$$\mathbb{R}\mathrm{e}\int_\Omega \langle \mathscr{A}\,\nabla u, \nabla[u\,|u|^{p-2}\exp(|u|^p)]\rangle dx \geqslant 0$$

for any $u \in \mathring{H}^1(\Omega)$ such that $|u|^{p-2}\exp(|u|^p)\,u \in \mathring{H}^1(\Omega)$, if and only if the operator A has real coefficients and condition (69) is satisfied.

Example 9 Let $\Phi(s) = s - \arctan s$, i.e., $\varphi(s) = s/(s^2+1)$. In this case

$$\frac{|s\,\Phi''(s) - \Phi'(s)|}{2\sqrt{s\,\Phi'(s)\,\Phi''(s)}} = \frac{|s^2 - 1|}{2\sqrt{2(s^2+1)}}$$

and $\lambda_0 = +\infty$. As in the previous example, we have that

$$\mathbb{R}\mathrm{e}\int_\Omega \langle \mathscr{A}\,\nabla u, \nabla\left(\frac{|u|\,u}{|u|^2+1}\right)\rangle dx \geqslant 0$$

for any $u \in \mathring{H}^1(\Omega)$ such that $|u|\,u/(|u|^2+1) \in \mathring{H}^1(\Omega)$, if and only if the operator A has real coefficients and condition (69) is satisfied.

Example 10 Let $\Phi(s) = s^4/(s^2+1)$, i.e., $\varphi(s) = 2\, s^2(2+s^2)/(s^2+1)^2$. In this case

$$\frac{|s\,\Phi''(s) - \Phi'(s)|}{2\sqrt{s\,\Phi'(s)\,\Phi''(s)}} = \frac{2}{\sqrt{(s^2+1)(s^2+2)(s^4+3s^2+6)}}\,.$$

This function is decreasing and λ_0 is equal to its value at 0, i.e., $\lambda_0 = 1/\sqrt{3}$. The operator A is L^Φ-dissipative, i.e.,

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla \left(\frac{|u|^2(2+|u|^2)u}{(|u|^2+1)^2} \right) \rangle dx \geqslant 0$$

for any $u \in \mathring{H}^1(\Omega)$ such that $|u|^2(2+|u|^2)u/(|u|^2+1)^2 \in \mathring{H}^1(\Omega)$, if and only if

$$|\langle \mathbb{I}\mathrm{m}\, \mathscr{A}(x)\,\xi, \xi\rangle| \leqslant \sqrt{3}\, \langle \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\,\xi, \xi\rangle$$

for almost any $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$.

Example 11 Let $\Phi(s) = s^2(s^2+2)/(s^2+1) - 2\log(s^2+1)$, i.e., $\varphi(s) = 2\, s^4/(s^2+1)^2$. In this case

$$\frac{|s\,\Phi''(s) - \Phi'(s)|}{2\sqrt{s\,\Phi'(s)\,\Phi''(s)}} = \frac{2}{\sqrt{(s^2+1)(s^2+5)}}\,.$$

This function is decreasing and λ_0 is equal to its value at 0, i.e., $\lambda_0 = 2/\sqrt{5}$. The operator A is L^Φ-dissipative, i.e.,

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla \left(\frac{|u|^4 u}{(|u|^2+1)^2} \right) \rangle dx \geqslant 0$$

for any $u \in \mathring{H}^1(\Omega)$ such that $|u|^4 u/(|u|^2+1)^2 \in \mathring{H}^1(\Omega)$, if and only if

$$2\,|\langle \mathbb{I}\mathrm{m}\, \mathscr{A}(x)\,\xi, \xi\rangle| \leqslant \sqrt{5}\, \langle \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\,\xi, \xi\rangle$$

for almost any $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$.

By analogy to the L^p case, if we have an operator with lower order term (27) and if the principal part is such that the left-hand side of (70) is not merely non negative but strictly positive, i.e.

$$[1-\Lambda^2(t)]\langle \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\,\xi, \xi\rangle + \langle \mathbb{R}\mathrm{e}\, \mathscr{A}(x)\,\eta, \eta\rangle +$$
$$[1+\Lambda(t)]\langle \mathbb{I}\mathrm{m}\, \mathscr{A}(x)\,\xi, \eta\rangle + [1-\Lambda(t)]\langle \mathbb{I}\mathrm{m}\, \mathscr{A}^*(x)\,\xi, \eta\rangle \geqslant \kappa(|\xi|^2 + |\eta|^2)$$

for a certain $\kappa > 0$ and for almost every $x \in \Omega$ and for any $t > 0, \xi, \eta \in \mathbb{R}^n$, we say that the operator A is (strongly) Φ-elliptic.

We note that, if A is a (strongly) Φ-elliptic operator, then there exists a constant κ such that for any nonnegative $\chi \in L^\infty(\Omega)$ and any complex valued $u \in H^1(\Omega)$ such that $\varphi(|u|)\, u \in H^1(\Omega)$ we have

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}\, \nabla u, \nabla(\varphi(|u|)\, u)\rangle\, \chi(x) dx \geqslant \kappa \int_\Omega |\nabla(\sqrt{\varphi(|u|)}\, u)|^2 \chi(x)\, dx$$

(see [15, Corollary 4]).

13 Concluding remarks

Our condition (25) and its strengthened variant are getting more and more important in many respects. We said already something about p-ellipticity, but there are also other applications.

We mention that Hömberg, Krumbiegel and Rehberg [31] used some of the techniques introduced in [9] to show the L^p-dissipativity of a certain operator connected to the problem of the existence of an optimal control for the heat equation with dynamic boundary condition.

Beyn and Otten [3, 4] considered the semilinear system

$$A\Delta v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0, \qquad x \in R^N,$$

where A is a $m \times m$ matrix, S is a $N \times N$ skew-symmetric matrix and f is a sufficiently smooth vector function. Among the assumptions they made, they require the existence of a constant $\gamma_A > 0$ such that

$$|z|^2\, \mathbb{R}\mathrm{e}\langle w, Aw\rangle + (p-2)\, \mathbb{R}\mathrm{e}\langle w, z\rangle\, \mathbb{R}\mathrm{e}\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2$$

for any $z, w \in \mathbb{C}^m$. This condition originates from our (57).

The results of [9] allowed Nittka [55] to consider the case of partial differential operators with complex coefficients.

Ostermann and Schratz [57] have obtained the stability of a numerical procedure for solving a certain evolution problem. The necessary and sufficient condition (23) show that their result does not require the contractivity of the corresponding semigroup.

Chill, Meinlschmidt and Rehberg [8] used some ideas from [9] in the study of the numerical range of second order elliptic operators with mixed boundary conditions in L^p.

ter Elst, Haller-Dintelmann, Rehberg and Tolksdorf [28] considered second order divergence form operators with complex coefficients, complemented with Dirichlet, Neumann or mixed boundary conditions. They proved several results related to the generation of strongly continuous semigroup on L^p.

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