

Global boundedness of the gradient for a class of nonlinear elliptic systems

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Abstract

Gradient boundedness up to the boundary for solutions to Dirichlet and Neumann problems for elliptic systems with Uhlenbeck type structure is established. Nonlinearities of possibly non-polynomial type are allowed, and minimal regularity on the data and on the boundary of the domain is assumed. The case of arbitrary bounded convex domains is also included.

1 Introduction

We are concerned with second-order nonlinear elliptic systems of the form

$$(1.1) \quad -\operatorname{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u}) = \mathbf{f}(x) \quad \text{in } \Omega,$$

coupled with either the Dirichlet condition $\mathbf{u} = 0$, or the Neumann condition $\frac{\partial \mathbf{u}}{\partial \nu} = 0$ on $\partial\Omega$. Here, Ω is a domain, namely an open connected set in \mathbb{R}^n , $n \geq 2$, $\mathbf{u} : \Omega \rightarrow \mathbb{R}^N$, $N \geq 1$, is a vector-valued unknown function, $\nabla \mathbf{u} : \Omega \rightarrow \mathbb{R}^{Nn}$ denotes its gradient, $\mathbf{f} : \Omega \rightarrow \mathbb{R}^N$ is a datum, div stands for the \mathbb{R}^N -valued divergence operator, and ν for the outward unit normal to $\partial\Omega$. Only bounded domains Ω will be considered throughout this paper.

We prove the boundedness of the gradient, or, equivalently, the Lipschitz continuity, of the solutions to the relevant boundary-value problems in the whole of Ω . Quite general nonlinearities of the differential operator, non-necessarily of power type, are allowed, and essentially weakest possible integrability conditions on \mathbf{f} , and minimal regularity assumptions on $\partial\Omega$ are imposed. In the case when Ω is convex, no regularity on $\partial\Omega$ is assumed at all. The boundary value problems to be considered involve the Euler equation of variational problems for strictly convex integral functionals depending on the gradient only through its modulus, and hence the solutions to the former agree with the minimizers of the latter. In particular, our results on convex domains provide a version in the vectorial case ($N > 1$) of the so called semi-classical Hilbert-Haar theory of minimization of strictly convex scalar integral functionals of the modulus of gradient on convex domains in classes of Lipschitz functions (see e.g. [Gi, Chapter 1]).

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Nonlinear elliptic systems associated with differential operators as in (1.1), whose coefficient only depends on the modulus of the gradient, are sometimes referred to as systems with Uhlenbeck structure in the literature. Indeed, the regularity theory of their solutions can be traced back to the celebrated paper [Uhl]. Results from [Uhl] imply that, if, for instance, $a(t) = t^{p-2}$ for some $p \geq 2$, a choice which turns (1.1) into the p -Laplacian system, then any local weak solution \mathbf{u} to (1.1) satisfies $\nabla \mathbf{u} \in L_{\text{loc}}^\infty(\Omega, \mathbb{R}^{Nn})$, and, in addition, $\nabla \mathbf{u} \in C_{\text{loc}}^\alpha(\Omega, \mathbb{R}^{Nn})$ for some $\alpha \in (0, 1)$. The regularity of solutions to the p -Laplacian equation in the scalar case ($N = 1$) had instead earlier been derived in [Ur] from theorems on the C^α -regularity of solutions to non-uniformly elliptic quasilinear systems. The contribution [Uhl] was subsequently extended to the situation when $1 < p < 2$ in [AF, CDiB]. Further generalization to elliptic systems with non polynomial growth are the subject of [BSV, DSV, MS, Mar]. Precise inner pointwise gradient estimates, via nonlinear and linear potentials, for local solutions to nonlinear elliptic equations, and to systems with Uhlenbeck structure, are the subject of the recent papers [DM1, DM2, KuM].

Recall that, in striking contrast with the scalar case [Di, Ev, Le, To], solutions to nonlinear elliptic systems with a more general structure than that of (1.1) can be irregular. Examples in this connection are produced in [SY], where the existence of nonlinear elliptic systems, with smooth coefficients depending only on the gradient, but endowed with solutions which are not even bounded, is established. Earlier contributions on irregular solutions to elliptic systems are rooted in the paper [DeG], and include [GM] and [Ne]. Related examples, in the scalar case, of linear and nonlinear higher-order elliptic equations with irregular solutions were independently exhibited in [Ma1].

Let us incidentally mention that, however, solutions to elliptic systems with general structure are well known to enjoy partial regularity properties, in the sense that they are locally regular in some open subset of Ω whose complement has zero Lebesgue measure. This is the subject of a rich literature, starting with the contributions [HKW, GiaMo, Iv] – see the monographs [BF, Gia, Gi] for a comprehensive treatment of this topic. Recent improvements of these results in terms of Hausdorff measures can be found in [Mi1, Mi2, KrM1].

The study of global regularity, that is to say up to the boundary, in boundary value problems for nonlinear elliptic systems has a more recent history. Global gradient boundedness, and gradient Hölder regularity, for the p -Laplacian elliptic system, under homogeneous Dirichlet boundary conditions, were obtained in [CDiB], as a consequence of analogous results for the associated parabolic problem. Right-hand sides which are bounded in x , and domains whose boundary is of class $C^{1,\alpha}$ are considered in that paper. Global gradient Hölder regularity, both for Dirichlet and for Neumann problems, is established in [Ha] for systems with Uhlenbeck structure in domains of class $C^{1,1}$. Further contributions on gradient regularity up to the boundary for systems and variational problems with Uhlenbeck structure, or perturbations of it, are [BC, Fo, FPV]. Partial boundary regularity, i.e. regularity at the boundary outside subsets of zero $(n - 1)$ -dimensional Hausdorff measure, for nonlinear elliptic systems with general structure, is proved in [KrM2]. See also [Ha] for an earlier partial boundary regularity result for special integrands.

The techniques employed in the literature mentioned above, for both inner and boundary regularity, have a local character. In particular, the proofs of global results entail distinguishing between points inside the domain, and boundary points, and reducing the treatment of the latter to the former. Such reduction requires the use of suitable local coordinates in which the boundary of the domain is flat, and the structure of the differential operator is still close enough to the Uhlenbeck type to ensure everywhere inner regularity. Techniques for inner regularity then apply after a reflection argument, which allows to extend the solution beyond the flattened boundary.

By contrast, the approach of the present paper is global in nature. Loosely speaking, an

underlying idea in our proof of the global boundedness of the gradient consists in integrating the system (1.1), after multiplication by $\Delta \mathbf{u}$, over the level sets of $|\nabla \mathbf{u}|$. In particular, no localization via cut-off functions is employed. Besides allowing for weak regularity assumptions on \mathbf{f} and Ω , an approach of this kind enables us to deal not only with Dirichlet, but also with Neumann boundary conditions. The proof of the global boundedness of the gradient in arbitrary bounded convex domains also relies upon the fact that no local change of coordinates near the boundary is required.

Let us finally notice that integration on the level sets of partial derivatives was used in [Ma2, Ma4] to show gradient boundedness for linear scalar problems, and in [CM1] for nonlinear scalar problems. The approximation scheme exploited in those papers does not apply to the vectorial case. Here, we follow an alternative outline, which provides a more self-contained proof also in the scalar case.

2 Main results

Our assumptions on the system (1.1) amount to what follows. The function $a : (0, \infty) \rightarrow (0, \infty)$ is required to be monotone (either non-decreasing or non-increasing), of class $C^1(0, \infty)$, and to fulfil

$$(2.1) \quad -1 < i_a \leq s_a < \infty,$$

where

$$(2.2) \quad i_a = \inf_{t>0} \frac{ta'(t)}{a(t)} \quad \text{and} \quad s_a = \sup_{t>0} \frac{ta'(t)}{a(t)}.$$

In particular, the standard p -Laplace operator for vector-valued functions, corresponding to the choice $a(t) = t^{p-2}$, with $p > 1$, falls within this framework, since $i_a = s_a = p - 2$ in this case.

Thanks to the first inequality in (2.2), the function $b : [0, \infty) \rightarrow [0, \infty)$, defined as

$$(2.3) \quad b(t) = a(t)t \quad \text{if } t > 0, \quad \text{and} \quad b(0) = 0,$$

turns out to be strictly increasing, and hence the function $B : [0, \infty) \rightarrow [0, \infty)$, given by

$$(2.4) \quad B(t) = \int_0^t b(\tau) d\tau \quad \text{for } t \geq 0,$$

is strictly convex. The Orlicz-Sobolev space $W^{1,B}(\Omega, \mathbb{R}^N)$ associated with the function B , or its subspace $W_0^{1,B}(\Omega, \mathbb{R}^N)$ of those functions vanishing in the suitable sense on $\partial\Omega$, are appropriate functional settings where to define weak solutions to the boundary value problems associated with the system (1.1). Precise definitions of function spaces and weak solutions are given in Sections 3 and 6, respectively; existence and uniqueness of such solutions is also discussed in Section 6.

The right-hand side \mathbf{f} in 1.1 is assumed to belong to the Lorentz space $L^{n,1}(\Omega, \mathbb{R}^N)$. This space is borderline, in a sense, for the family of Lebesgue spaces $L^q(\Omega, \mathbb{R}^N)$ with $q > n$, since $L^q(\Omega, \mathbb{R}^N) \subsetneq L^{n,1}(\Omega, \mathbb{R}^N) \subsetneq L^n(\Omega, \mathbb{R}^N)$ for every $q > n$. Let us mention that membership of the right-hand side to the same Lorentz space is already known to yield global gradient boundedness for solutions to scalar boundary value problems [CM1]. It has also been shown, via a different approach relying upon potential theory, to ensure the inner local boundedness of the gradient of local solutions to equations, and to systems with Uhlenbeck structure [DM2], and also its continuity [DM3].

The regularity of $\partial\Omega$ is prescribed in terms of a Lorentz space as well. We impose that $\partial\Omega \in W^2L^{n-1,1}$. This means that Ω is locally the subgraph of a function of $n - 1$ variables whose second-order distributional derivatives belong to the Lorentz space $L^{n-1,1}$. This is the weakest possible integrability assumption on the second-order derivatives of such a function for its first-order derivatives to be continuous, and hence for $\partial\Omega \in C^{1,0}$. Note that, by contrast, the available regularity results at the boundary require $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0, 1]$.

Let us emphasize that both the assumption on \mathbf{f} and that on Ω cannot be essentially relaxed for our conclusions to hold – see Remarks 2.8 and 2.9 at the end of this section.

Our result for the Dirichlet problem

$$(2.5) \quad \begin{cases} -\mathbf{div}(a(|\nabla\mathbf{u}|)\nabla\mathbf{u}) = \mathbf{f}(x) & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega \end{cases}$$

reads as follows (see Section 6 for a precise definition of weak solution to problem (2.5)).

Theorem 2.1 *Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 3$, such that $\partial\Omega \in W^2L^{n-1,1}$. Assume that $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$. Let \mathbf{u} be the (unique) weak solution to the Dirichlet problem (2.5). Then there exists a constant $C = C(i_a, s_a, \Omega)$ such that*

$$(2.6) \quad \|\nabla\mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \leq Cb^{-1}(\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}).$$

In particular, \mathbf{u} is Lipschitz continuous in Ω .

An interesting variant of Theorem 2.1 asserts that the the regularity assumption on $\partial\Omega$ can be replaced by the convexity of Ω . This is stated in the next result.

Theorem 2.2 *The same conclusion as in Theorem 2.1 holds if Ω is any bounded convex domain in \mathbb{R}^n , $n \geq 3$.*

Problem (2.5) is the Euler equation of the minimization problem for the strictly convex functional

$$(2.7) \quad J(u) = \int_{\Omega} (B(|\nabla\mathbf{u}|) - \mathbf{f} \cdot \mathbf{u}) \, dx$$

among trial functions \mathbf{u} in $W_0^{1,B}(\Omega, \mathbb{R}^N)$. Note that J is well defined in this function space under our assumption on \mathbf{f} – the beginning of Section 6. The interpretation of Theorem 2.2 as an existence result for minimizers of the functional J in the space $\text{Lip}_0(\Omega, \mathbb{R}^N)$ of \mathbb{R}^N -valued Lipschitz continuous functions in Ω vanishing on $\partial\Omega$, to which we alluded in Section 1, is the content of the following corollary.

Corollary 2.3 *Let Ω be any bounded convex domain in \mathbb{R}^n , $n \geq 3$. Assume that $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$. Then the functional J admits a (unique) minimizer in the space $\text{Lip}_0(\Omega, \mathbb{R}^N)$.*

Results parallel to Theorems 2.1–2.2 and Corollary 2.3 hold for the solutions to the Neumann problem

$$(2.8) \quad \begin{cases} -\mathbf{div}(a(|\nabla\mathbf{u}|)\nabla\mathbf{u}) = \mathbf{f}(x) & \text{in } \Omega, \\ \frac{\partial\mathbf{u}}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly, here \mathbf{f} has to fulfil the compatibility condition

$$(2.9) \quad \int_{\Omega} \mathbf{f}(x) \, dx = 0.$$

A definition of weak solution to problem (2.8) is given in Section 6.

Theorem 2.4 *Let Ω and \mathbf{f} be as in Theorem 2.1. Assume, in addition, that (2.9) holds. Let \mathbf{u} be the (unique up to additive constant vectors) weak solution to problem (2.8). Then there exists a constant $C = C(i_a, s_a, \Omega)$ such that*

$$(2.10) \quad \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \leq Cb^{-1}(\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}).$$

In particular, \mathbf{u} is Lipschitz continuous in Ω .

A counterpart of Theorem 2.4 for convex domains is contained in the next result.

Theorem 2.5 *The same conclusion as in Theorem 2.4 holds if Ω is any bounded convex domain in \mathbb{R}^n , $n \geq 3$.*

The minimization problem for the functional J in the whole of $W^{1,B}(\Omega, \mathbb{R}^N)$ leads to the Euler equation (2.8). Hence, we have the following corollary of Theorem 2.5.

Corollary 2.6 *Let Ω be any bounded convex domain in \mathbb{R}^n , $n \geq 3$. Assume that $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$ and fulfils (2.9). Then the functional J admits a (unique up to additive constant vectors) minimizer in the class $\text{Lip}(\Omega, \mathbb{R}^N)$.*

Remark 2.7 Versions of the above results can be established via our approach also in the case when $n = 2$, under the slightly stronger assumption that $\mathbf{f} \in L^q(\Omega, \mathbb{R}^N)$ for some $q > n$. This becomes clear from a close inspection of the proofs.

Remark 2.8 The sharpness of assumption $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$ for the boundedness of the gradient of the solution to the Dirichlet problem follows, in the linear case corresponding to the choice $a = 1$, from a result of [Ci2] dealing with the scalar Laplace equation in a ball.

Remark 2.9 The assumption $\partial\Omega \in W^2L^{n-1,1}$ is optimal for the boundedness of the gradient, as long as the regularity of Ω is prescribed in terms of integrability properties of the curvature of its boundary. This can be demonstrated, again even just for scalar problems, by ad hoc examples of Dirichlet and Neumann problems for the p -Laplace equation in domains whose boundaries have conical singularities – see e.g. [CM2]. Examples of the same nature also show that the conclusion of Theorems 2.2 and 2.5 may fail under slight local non-smooth perturbations of convex domains [CM2].

Remark 2.10 A natural question that could be raised is whether, in fact, the stronger conclusion $\mathbf{u} \in C^1(\bar{\Omega}, \mathbb{R}^N)$ holds in the results stated above. The answer is however negative in general. This can be verified, for instance, via the following example. Consider a disk $D \subset \mathbb{R}^2$, fix a point $\bar{x} \in \partial D$, and consider a sequence of points $x_k \in \partial D$ such that $x_k \rightarrow \bar{x}$. Let $\Omega \subset \mathbb{R}^2$ be the convex domain obtained as the union of D and of the sequence of sets bounded by ∂D and by the tangent straight-lines to ∂D at the points x_k . Let f be a nonnegative, smooth, radially symmetric function about the center of D , which has compact support in D . Consider the solution u to the (scalar) equation $-\Delta u = f(x)$ in Ω , subject to the Dirichlet condition $u = 0$ on $\partial\Omega$. If ∇u were continuous on $\partial\Omega$, then it should vanish at the intersection of the tangent straight-lines to ∂D at the points x_k , and hence one would have $|\nabla u(\bar{x})| = \left|\frac{\partial u}{\partial \nu}(\bar{x})\right| = 0$ as well. This contradicts the fact that $\left|\frac{\partial u}{\partial \nu}(\bar{x})\right| \geq \left|\frac{\partial v}{\partial \nu}(\bar{x})\right| > 0$, where v is the solution to the problem $-\Delta v = f(x)$ in D , with $v = 0$ on ∂D .

Variants of this example could be exhibited for Neumann problems. Also, the domain Ω can be modified into another domain whose boundary is of class C^1 . Analogues in dimension $n \geq 3$ can be produced on replacing the disk with a ball, and the tangent straight-lines with the tangent hyperplanes.

3 Function spaces

3.1 Spaces of measurable functions and rearrangements

Let (\mathcal{R}, m) be a positive, finite, non-atomic measure space. The decreasing rearrangement $v^* : [0, \infty) \rightarrow [0, \infty]$ of a real-valued m -measurable function v on \mathcal{R} is the unique right-continuous non-increasing function in $[0, \infty)$ equidistributed with v . Namely, on defining the distribution function $\mu_v : [0, \infty) \rightarrow [0, \infty)$ of v as

$$(3.1) \quad \mu_v(t) = m(\{x \in \mathcal{R} : |v(x)| > t\}) \quad \text{for } t \geq 0,$$

we have that

$$(3.2) \quad v^*(s) = \sup\{t \geq 0 : \mu_v(t) > s\} \quad \text{for } s \in [0, \infty).$$

Clearly, $v^*(s) = 0$ if $s \geq m(\mathcal{R})$.

The function $v^{**} : (0, \infty) \rightarrow [0, \infty)$, defined by

$$v^{**}(s) = \frac{1}{s} \int_0^s v^*(r) dr \quad \text{for } s > 0,$$

is also nondecreasing, and such that $v^*(s) \leq v^{**}(s)$ for $s > 0$.

The Hardy-Littlewood inequality is a basic property of rearrangements, and asserts that

$$(3.3) \quad \int_{\mathcal{R}} |v(x)w(x)| dm(x) \leq \int_0^\infty v^*(s)w^*(s) ds$$

for all measurable functions v and w on \mathcal{R} .

Roughly speaking, a rearrangement-invariant space is a Banach function space whose norm only depends on the rearrangement of functions – see e.g. [BS, Chapter 2] for a more precise definition. Besides the Lebesgue spaces, their generalizations provided by the Lorentz and the Orlicz spaces are classical instances of rearrangement-invariant spaces which will play a role in our discussion.

Given $q \in (1, \infty)$ and $\sigma \in [1, \infty]$, the Lorentz space $L^{q,\sigma}(\mathcal{R})$ is the set of all real-valued measurable functions v on \mathcal{R} for which the quantity

$$(3.4) \quad \|v\|_{L^{q,\sigma}(\mathcal{R})} = \|s^{\frac{1}{q}-\frac{1}{\sigma}} v^*(s)\|_{L^\sigma(0,m(\mathcal{R}))}$$

is finite. One has that $L^{q,\sigma}(\mathcal{R})$ is a Banach space, equipped with the norm, equivalent to $\|\cdot\|_{L^{q,\sigma}(\mathcal{R})}$, obtained on replacing v^* with v^{**} on the right-hand side of (3.4). Furthermore,

$$(3.5) \quad L^{q,q}(\mathcal{R}) = L^q(\mathcal{R}) \quad \text{if } q \in (1, \infty),$$

$$(3.6) \quad L^{q,\sigma_1}(\mathcal{R}) \rightarrow L^{q,\sigma_2}(\mathcal{R}) \quad \text{if } q \in (1, \infty) \text{ and } \sigma_1 < \sigma_2,$$

and

$$(3.7) \quad L^{q_1,\sigma_1}(\mathcal{R}) \rightarrow L^{q_2,\sigma_2}(\mathcal{R}) \quad \text{if } q_1 > q_2 \text{ and } \sigma_1, \sigma_2 \in [1, \infty].$$

Here, and in what follows, the arrow “ \rightarrow ” stands for continuous embedding. Let us notice that the norm of the embedding (3.6) depends on q , σ_1 , σ_2 , and the norm of the embedding (3.7) depends on q_1 , q_2 , σ_1 , σ_2 and $m(\mathcal{R})$.

Denote by q' and σ' the usual Hölder's conjugate exponents of q and σ . A Hölder type inequality in Lorentz spaces tells us that there exists a constant $C = C(q, \sigma)$ such that

$$(3.8) \quad \int_{\mathcal{R}} |v(x)w(x)| dm(x) \leq C \|v\|_{L^{q,\sigma}(\mathcal{R})} \|w\|_{L^{q',\sigma'}(\mathcal{R})}$$

for every $v \in L^{q,\sigma}(\mathcal{R})$ and $w \in L^{q',\sigma'}(\mathcal{R})$.

Given $N > 1$, the Lorentz space $L^{q,\sigma}(\mathcal{R}, \mathbb{R}^N)$ of \mathbb{R}^N -valued measurable functions on \mathcal{R} is defined as $L^{q,\sigma}(\mathcal{R}, \mathbb{R}^N) = (L^{q,\sigma}(\mathcal{R}))^N$, and is endowed with the norm defined as $\|\mathbf{v}\|_{L^{q,\sigma}(\mathcal{R}, \mathbb{R}^N)} = \|\mathbf{v}\|_{L^{q,\sigma}(\mathcal{R})}$ for $\mathbf{v} \in L^{q,\sigma}(\mathcal{R}, \mathbb{R}^N)$.

The Orlicz spaces extend the Lebesgue spaces in the sense that the role of powers in the definition of the norms is instead played by Young functions. A Young function $B : [0, \infty) \rightarrow [0, \infty]$ is a convex function such that $B(0) = 0$. If, in addition, $0 < B(t) < \infty$ for $t > 0$ and

$$\lim_{t \rightarrow 0} \frac{B(t)}{t} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{B(t)}{t} = \infty,$$

then B is called an N -function. The Young conjugate of a Young function B is the Young function \tilde{B} defined as

$$\tilde{B}(t) = \sup\{st - B(s) : s \geq 0\} \quad \text{for } t \geq 0.$$

In particular, if B is an N -function, then \tilde{B} is an N -function as well. Moreover, if B is given by (2.4), then

$$(3.9) \quad \tilde{B}(t) = \int_0^t b^{-1}(s) ds \quad \text{for } t \geq 0.$$

Notice that

$$(3.10) \quad s \leq B^{-1}(s)\tilde{B}^{-1}(s) \leq 2s \quad \text{for } s \geq 0.$$

A Young function (and, more generally, an increasing function) B is said to belong to the class Δ_2 if there exists a constant $C > 1$ such that

$$B(2t) \leq CB(t) \quad \text{for } t > 0.$$

The Orlicz space $L^B(\mathcal{R})$ is the Banach function space of those real-valued measurable functions v on \mathcal{R} whose Luxemburg norm

$$\|v\|_{L^B(\mathcal{R})} = \inf \left\{ \lambda > 0 : \int_{\mathcal{R}} B\left(\frac{|v(x)|}{\lambda}\right) dm(x) \leq 1 \right\}$$

is finite. The Hölder type inequality

$$(3.11) \quad \int_{\mathcal{R}} |v(x)w(x)| dm(x) \leq 2 \|v\|_{L^B(\mathcal{R})} \|w\|_{L^{\tilde{B}}(\mathcal{R})}$$

holds for every $v \in L^B(\mathcal{R})$ and $w \in L^{\tilde{B}}(\mathcal{R})$. Let B_1 and B_2 be Young functions. Then

$$(3.12) \quad L^{B_1}(\mathcal{R}) \rightarrow L^{B_2}(\mathcal{R}) \quad \text{if and only if there exist } c, t_0 > 0 \text{ such that } B_2(t) \leq B_1(ct) \text{ for } t > t_0.$$

The Orlicz space $L^B(\mathcal{R}, \mathbb{R}^N)$, with $N > 1$, of \mathbb{R}^N -valued measurable functions on \mathcal{R} is defined as $L^B(\mathcal{R}, \mathbb{R}^N) = (L^B(\mathcal{R}))^N$, and is equipped with the norm given by $\|\mathbf{v}\|_{L^B(\mathcal{R}, \mathbb{R}^N)} = \|\mathbf{v}\|_{L^B(\mathcal{R})}$ for $\mathbf{v} \in L^B(\mathcal{R}, \mathbb{R}^N)$.

3.2 Spaces of Sobolev type

Let Ω be an open set in \mathbb{R}^n , with $n \geq 2$, and let $m \in \mathbb{N}$. Sobolev type spaces of m -th order weakly differentiable functions in Ω , built upon Lorentz and Orlicz spaces, are defined as follows. Given $q \in (1, \infty)$ and $\sigma \in [1, \infty]$, the Lorentz-Sobolev space

$$W^m L^{q,\sigma}(\Omega) = \{u \in L^{q,\sigma}(\Omega) : \text{is } m\text{-times weakly differentiable in } \Omega \\ \text{and } |\nabla^k u| \in L^{q,\sigma}(\Omega) \text{ for } 1 \leq k \leq m\}$$

is a Banach space equipped with the norm $\|u\|_{W^m L^{q,\sigma}(\Omega)} = \sum_{k=0}^m \|\nabla^k u\|_{L^{q,\sigma}(\Omega)}$. Here, $\nabla^k u$ denotes the vector of all weak derivatives of u of order k . By $\nabla^0 u$ we mean u . Moreover, when $k = 1$ we simply write ∇u instead of $\nabla^1 u$.

If $\sigma < \infty$, the space $C^\infty(\Omega) \cap W^m L^{q,\sigma}(\Omega)$ is dense in $W^m L^{q,\sigma}(\Omega)$. This fact follows via an easy variant of a standard argument for classical Sobolev spaces, and makes use the density of $C_0^\infty(\Omega)$ in $L^{q,\sigma}(\Omega)$, and of a version of Young convolution inequality in Lorentz spaces due to O'Neil [Zi, Theorem 2.10.1].

A limiting case of the Sobolev embedding theorem asserts that if Ω has a Lipschitz boundary, then $W^1 L^{n,1}(\Omega) \rightarrow C^0(\Omega)$; moreover, $L^{n,1}(\Omega)$ is optimal, in the sense that it is the largest rearrangement invariant space enjoying this property [CP]. Hence, in particular,

$$(3.13) \quad W^2 L^{n,1}(\Omega) \rightarrow C^{1,0}(\Omega),$$

and $L^{n,1}(\Omega)$ is optimal in the same sense as above.

The Lorentz-Sobolev space $W^m L^{q,\sigma}(\Omega, \mathbb{R}^N)$, $N > 1$, of \mathbb{R}^N -valued functions in Ω is defined as $W^m L^{q,\sigma}(\Omega, \mathbb{R}^N) = (W^m L^{q,\sigma}(\Omega))^N$, and endowed with the norm given by $\|\mathbf{u}\|_{W^m L^{q,\sigma}(\Omega)} = \sum_{k=0}^m \|\nabla^k \mathbf{u}\|_{L^{q,\sigma}(\Omega)}$ for $\mathbf{u} \in W^m L^{q,\sigma}(\Omega, \mathbb{R}^N)$.

The Orlicz-Sobolev space $W^{m,B}(\Omega)$ is the Banach space

$$W^{m,B}(\Omega) = \{u \in L^B(\Omega) : \text{is } m\text{-times weakly differentiable in } \Omega \\ \text{and } |\nabla^k u| \in L^B(\Omega) \text{ for } 1 \leq k \leq m\},$$

and is equipped with the norm $\|u\|_{W^{m,B}(\Omega)} = \sum_{k=0}^m \|\nabla^k u\|_{L^B(\Omega)}$. In what follows, we shall only make use of first-order Orlicz-Sobolev spaces $W^{1,B}(\Omega)$. By $W_0^{1,B}(\Omega)$ and $W_\perp^{1,B}(\Omega)$ we denote the subspaces of $W^{1,B}(\Omega)$ given by

$$W_0^{1,B}(\Omega) = \{u \in W^{1,B}(\Omega) : \text{the continuation of } u \text{ by } 0 \text{ outside } \Omega \text{ is weakly differentiable in } \mathbb{R}^n\},$$

and

$$W_\perp^{1,B}(\Omega) = \left\{ u \in W^{1,B}(\Omega) : \int_\Omega u(x) dx = 0 \right\}.$$

A theorem of [DT] ensures that, if $B \in \Delta_2$, then the space $C_0^\infty(\Omega)$ is dense in $W_0^{1,B}(\Omega)$, and that, if Ω is a bounded open set with a Lipschitz boundary, then $C^\infty(\bar{\Omega})$ is dense in $W^{1,B}(\Omega)$. Let B be a Young function such that

$$(3.14) \quad \int_0 \left(\frac{t}{B(t)} \right)^{\frac{1}{n-1}} dt < \infty.$$

The Sobolev conjugate of B , introduced in [Ci4] (and, in an equivalent form, in [Ci3]), is the Young function B_n defined as

$$(3.15) \quad B_n(t) = B(H_n^{-1}(t)) \quad \text{for } t \geq 0,$$

where

$$(3.16) \quad H_n(s) = \left(\int_0^s \left(\frac{t}{B(t)} \right)^{\frac{1}{n-1}} dt \right)^{1/n'} \quad \text{for } s \geq 0,$$

and H_n^{-1} denotes the (generalized) left-continuous inverse of H_n .

An embedding theorem for Orlicz-Sobolev spaces [Ci3, Ci4] tells us that, if B fulfils (3.14), then there exists a constant $C = C(n, |\Omega|)$ such that

$$(3.17) \quad \|u\|_{L^{B_n}(\Omega)} \leq C \|\nabla u\|_{L^B(\Omega)}$$

for every $u \in W_0^{1,B}(\Omega)$. Moreover, if Ω has a Lipschitz boundary, then inequality (3.17) holds for every $u \in W_{\perp}^{1,B}(\Omega)$. The space $L^{B_n}(\Omega)$ is optimal in (3.17) among all Orlicz spaces. Note that assumption (3.14) is, in fact, immaterial in (3.17), since, owing to (3.12), the Young function B can be replaced, if necessary, with another Young function fulfilling (3.14) in such a way that $W^{1,B}(\Omega)$ remains unchanged, up to equivalent norms.

If B is a Young function, then

$$(3.18) \quad L^{\infty}(\Omega) \rightarrow L^{B_n}(\Omega) \rightarrow L^{n'}(\Omega).$$

The first embedding in (3.18) is trivial. As for the second one, since B is a Young function, there exist constants c_0 and $t_0 > 0$ such that $t \leq B(c_0 t)$ if $t > t_0$. As a consequence, there exist constants c_1 and t_1 such that $t^{n'} \leq B_n(c_1 t)$ for some $t > t_1$. Hence, the second embedding in (3.18) follows via (3.12).

Let us also observe that, if B grows so fast near infinity that

$$(3.19) \quad \int^{\infty} \left(\frac{t}{B(t)} \right)^{\frac{1}{n-1}} dt < \infty,$$

then equality holds in the first embedding in (3.18). Indeed, under (3.19), $H_n^{-1}(t) = \infty$ for large t , and hence $B_n(t) = \infty$ for large t as well. Hence, $L^{B_n}(\Omega) = L^{\infty}(\Omega)$, up to equivalent norms.

The Orlicz-Sobolev space $W^{m,B}(\Omega, \mathbb{R}^N)$ is defined, for $N > 1$, as $W^{m,B}(\Omega, \mathbb{R}^N) = (W^{m,B}(\Omega))^N$, and equipped with the norm $\|\mathbf{u}\|_{W^{m,B}(\Omega)} = \sum_{k=0}^m \|\nabla^k \mathbf{u}\|_{L^B(\Omega)}$. The spaces $W_0^{1,B}(\Omega, \mathbb{R}^N)$ and $W_{\perp}^{1,B}(\Omega, \mathbb{R}^N)$ are defined accordingly.

4 The function a

This section is devoted to the proof of some properties of the function a appearing in (1.1). Hereafter, b and B denote the functions associated with a as in (2.3) and (2.4). Furthermore, we define the function $H : [0, \infty) \rightarrow [0, \infty)$ as

$$(4.1) \quad H(t) = \int_0^t a(\tau) b(\tau) d\tau \quad \text{for } t \geq 0,$$

and the function $F : [0, \infty) \rightarrow [0, \infty)$ as

$$(4.2) \quad F(t) = \int_0^t b(\tau)^2 d\tau \quad \text{for } t \geq 0.$$

Proposition 4.1 *Assume that the function $a : (0, \infty) \rightarrow (0, \infty)$ is of class C^1 and fulfils (2.1). Let b and B be the functions given by (2.3) and (2.4), respectively, and let H and F be defined as above. Then:*

(i)

$$(4.3) \quad a(1) \min\{t^{i_a}, t^{s_a}\} \leq a(t) \leq a(1) \max\{t^{i_a}, t^{s_a}\} \quad \text{for } t > 0.$$

(ii) b is increasing,

$$(4.4) \quad \lim_{t \rightarrow 0} b(t) = 0, \quad \text{and} \quad \lim_{t \rightarrow \infty} b(t) = \infty.$$

(iii) B is a strictly convex N -function,

$$(4.5) \quad B \in \Delta_2 \quad \text{and} \quad \tilde{B} \in \Delta_2.$$

(iv) There exists a constant $C = C(i_a, s_a)$ such that

$$(4.6) \quad \tilde{B}(b(t)) \leq CB(t) \quad \text{for } t \geq 0.$$

(v) For every $C > 0$, there exists a positive constant $C_1 = C_1(s_a, C) > 0$ such that

$$(4.7) \quad Cb^{-1}(s) \leq b^{-1}(C_1s) \quad \text{for } s > 0,$$

and a positive constant $C_2 = C_2(i_a, C) > 0$ such that

$$(4.8) \quad b^{-1}(Cs) \leq C_2b^{-1}(s) \quad \text{for } s > 0.$$

(vi) There exists a positive constant $C = C(s_a)$ such that

$$(4.9) \quad B(t) \leq tb(t) \leq CB(t) \quad \text{for } t \geq 0.$$

(vii) There exists a positive constant $C = C(i_a, s_a)$ such that

$$(4.10) \quad F(t) \leq tb(t)^2 \leq CF(t) \quad \text{for } t \geq 0.$$

(viii) There exist positive constants $C_1 = C_1(i_a)$ and $C_2 = C_2(i_a, s_a)$ such that

$$(4.11) \quad C_1H(t) \leq b(t)^2 \leq C_2H(t) \quad \text{for } t \geq 0.$$

Proof. Assertions (ii)–(vii) are proved in [CM1, Propositions 2.9 and 2.15]. Property (i) can be shown on distinguishing the case when $t \in (0, 1)$ and $t \in [1, \infty)$, and integrating the inequality

$$\frac{i_a}{\tau} \leq \frac{a'(\tau)}{a(\tau)} \leq \frac{s_a}{\tau} \quad \text{for } \tau > 0,$$

on $(t, 1)$ and on $(1, t)$, respectively.

As far as (viii) is concerned, since $b'(t) = a(t) + ta'(t)$ for $t > 0$, one has that

$$(4.12) \quad \frac{b'(t)}{1 + s_a} \leq a(t) \leq \frac{b'(t)}{1 + \min\{i_a, 0\}} \quad \text{for } t > 0,$$

and hence

$$(4.13) \quad \frac{b(t)}{1 + s_a} \leq \int_0^t a(\tau) d\tau \leq \frac{b(t)}{1 + \min\{i_a, 0\}} \quad \text{for } t > 0.$$

Thus, inasmuch as b is increasing,

$$H(t) \leq b(t) \int_0^t a(\tau) d\tau \leq \frac{b(t)^2}{1 + \min\{i_a, 0\}} \quad \text{for } t > 0,$$

whence the first inequality in (4.11) follows. On the other hand, integration by parts and inequalities (4.12) and (4.13) yield

$$\begin{aligned} H(t) &= \int_0^t a(\tau)b(\tau) d\tau = b(t) \int_0^t a(\tau) d\tau - \int_0^t b'(\tau) \int_0^\tau a(r) dr d\tau \\ &\geq \frac{b(t)^2}{1 + s_a} - \int_0^t (1 + s_a)a(\tau) \frac{b(\tau)}{1 + \min\{i_a, 0\}} d\tau = \frac{b(t)^2}{1 + s_a} - \frac{1 + s_a}{1 + \min\{i_a, 0\}} H(t) \quad \text{for } t > 0. \end{aligned}$$

This implies the second inequality in (4.11). \square

Lemma 4.2 *Let a be as in Lemma 4.1. Then*

$$(4.14) \quad (1 + \min\{i_a, 0\})a(|\xi|)|\eta|^2 \leq \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n \frac{\partial(a(|\xi|)\xi_i^\alpha)}{\partial \xi_j^\beta} \eta_i^\alpha \eta_j^\beta \leq (1 + \max\{s_a, 0\})a(|\xi|)|\eta|^2$$

for $\xi, \eta \in \mathbb{R}^{Nn}$. Moreover,

$$(4.15) \quad [a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) \geq (1 + \min\{i_a, 0\})|\xi - \eta|^2 \int_0^1 a(|\eta + s(\xi - \eta)|) ds$$

for $\xi, \eta \in \mathbb{R}^{Nn}$.

Proof. Given i, j, α, β , one has that

$$(4.16) \quad \frac{\partial(a(|\xi|)\xi_i^\alpha)}{\partial \xi_j^\beta} = \frac{a'(|\xi|)}{|\xi|} \xi_i^\alpha \xi_j^\beta + a(|\xi|)\delta_{ij}\delta_{\alpha\beta} \quad \text{for } \xi \in \mathbb{R}^{Nn}.$$

Thus

$$\begin{aligned} (4.17) \quad \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n \frac{\partial(a(|\xi|)\xi_i^\alpha)}{\partial \xi_j^\beta} \eta_i^\alpha \eta_j^\beta &= \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n \frac{a'(|\xi|)}{|\xi|} \xi_i^\alpha \xi_j^\beta \eta_i^\alpha \eta_j^\beta + a(|\xi|) \sum_{\alpha=1}^N \sum_{i=1}^n (\eta_i^\alpha)^2 \\ &= \frac{a'(|\xi|)}{|\xi|} (\xi \cdot \eta)^2 + a(|\xi|)|\eta|^2 \quad \text{for } \xi, \eta \in \mathbb{R}^{Nn}. \end{aligned}$$

Since $|\xi \cdot \eta| \leq |\xi||\eta|$, equation (4.14) follows via (2.1).

Next, set $A_{ij}^{\alpha\beta}(\zeta) = \frac{\partial(a(|\zeta|)\zeta_i^\alpha)}{\partial \zeta_j^\beta}$ for $\zeta \in \mathbb{R}^{Nn}$. Given i and α , we have that

$$[a(|\xi|)\xi_i^\alpha - a(|\eta|)\eta_i^\alpha] = \sum_{\beta=1}^N \sum_{j=1}^n (\xi_j^\beta - \eta_j^\beta) \int_0^1 A_{ij}^{\alpha\beta}(\eta + s(\xi - \eta)) ds \quad \text{for } \xi, \eta \in \mathbb{R}^{Nn}..$$

Therefore,

$$[a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) = \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n (\xi_i^\alpha - \eta_i^\alpha)(\xi_j^\beta - \eta_j^\beta) \int_0^1 A_{ij}^{\alpha\beta}(\eta + s(\xi - \eta)) ds \quad \text{for } \xi, \eta \in \mathbb{R}^{Nn},$$

and hence inequality (4.15) follows via (4.14). \square

Lemma 4.3 *Let a be as in Lemma 4.1. Assume in addition that a is non-decreasing. Then*

$$(4.18) \quad [a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) \geq \frac{1}{3}[a(|\xi|) + a(|\eta|)] |\xi - \eta|^2 \quad \text{for } \xi, \eta \in \mathbb{R}^{Nn}.$$

Proof. Since inequality (4.18) is invariant under replacements of ξ and η by each other, we may assume, without loss of generality, that $|\xi| \geq |\eta|$, and hence $a(|\xi|) \geq a(|\eta|)$. Consider first the case when $a(|\xi|) \leq 2a(|\eta|)$. Then, given any $\xi \neq \eta$,

$$(4.19) \quad \begin{aligned} \frac{(a(|\xi|)\xi - a(|\eta|)\eta) \cdot (\xi - \eta)}{(a(|\xi|) + a(|\eta|))|\xi - \eta|^2} &= \frac{(\frac{a(|\xi|)}{a(|\eta|)}\xi - \eta) \cdot (\xi - \eta)}{(\frac{a(|\xi|)}{a(|\eta|)} + 1)|\xi - \eta|^2} \geq \frac{(\frac{a(|\xi|)}{a(|\eta|)}\xi - \eta) \cdot (\xi - \eta)}{3|\xi - \eta|^2} \\ &= \frac{|\xi - \eta|^2 + (\frac{a(|\xi|)}{a(|\eta|)} - 1)\xi \cdot (\xi - \eta)}{3|\xi - \eta|^2} = \frac{1}{3} + \frac{(\frac{a(|\xi|)}{a(|\eta|)} - 1)(|\xi|^2 - \xi \cdot \eta)}{3|\xi - \eta|^2} \\ &\geq \frac{1}{3} + \frac{(\frac{a(|\xi|)}{a(|\eta|)} - 1)(|\xi|^2 - |\xi||\eta|)}{3|\xi - \eta|^2} \geq \frac{1}{3}. \end{aligned}$$

Assume now that $a(|\xi|) \geq 2a(|\eta|)$. Then, given any $\xi \neq \eta$,

$$(4.20) \quad \begin{aligned} \frac{(a(|\xi|)\xi - a(|\eta|)\eta) \cdot (\xi - \eta)}{(a(|\xi|) + a(|\eta|))|\xi - \eta|^2} &= \frac{(\frac{a(|\xi|)}{a(|\eta|)}\xi - \eta) \cdot (\xi - \eta)}{(\frac{a(|\xi|)}{a(|\eta|)} + 1)|\xi - \eta|^2} \geq \inf_{s \geq 2} \frac{(s\xi - \eta) \cdot (\xi - \eta)}{(s+1)|\xi - \eta|^2} = \frac{(2\xi - \eta) \cdot (\xi - \eta)}{3|\xi - \eta|^2} \\ &= \frac{2|\xi|^2 - 3\xi \cdot \eta + |\eta|^2}{3|\xi - \eta|^2} = \frac{|\xi - \eta|^2 + |\xi|^2 - \xi \cdot \eta}{3|\xi - \eta|^2} \\ &\geq \frac{1}{3} + \frac{|\xi|^2 - |\xi||\eta|}{3|\xi - \eta|^2} \geq \frac{1}{3}. \end{aligned}$$

Inequality (4.18) is fully proved. \square

Lemma 4.4 *Let a be as in Lemma 4.1. Assume, in addition, that a is monotone (either non-decreasing or non-increasing). Then, for every $t, \tau > 0$, there exists a positive constant $\vartheta = \vartheta(a, t, \tau)$ such that*

$$(4.21) \quad \inf\{[a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) : \xi, \eta \in \mathbb{R}^{Nn}, |\xi - \eta| \geq t, |\xi| \leq \tau, |\eta| \leq \tau\} > \vartheta.$$

Proof. Assume first that a is non-decreasing. In particular, $s_a \geq i_a \geq 0$. Then, by Lemma 4.3,

$$(4.22) \quad [a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) \geq \frac{1}{3}[a(|\xi|) + a(|\eta|)] |\xi - \eta|^2 \quad \text{for } \xi, \eta \in \mathbb{R}^{Nn}.$$

Since, by (4.5) and (4.9), $a \in \Delta_2$,

$$(4.23) \quad a(|\xi - \eta|) \leq a(|\xi| + |\eta|) \leq a(2|\xi|) + a(2|\eta|) \leq C[a(|\xi|) + a(|\eta|)] \quad \text{for } \xi, \eta \in \mathbb{R}^{Nn}.$$

By (4.22), (4.23) and the first inequality in (4.3),

$$(4.24) \quad [a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) \geq C \min\{|\xi - \eta|^{i_a+2}, |\xi - \eta|^{s_a+2}\} \quad \text{for } \xi, \eta \in \mathbb{R}^{Nn},$$

for some positive constant $C = C(a)$. Hence (4.21) follows, since $\{\xi, \eta \in \mathbb{R}^{Nn} : |\xi - \eta| \geq t, |\xi| \leq \tau, |\eta| \leq \tau\}$ is a compact set.

Assume next that a is non-increasing. Thus, $0 \geq s_a \geq i_a$. By (4.15),

$$(4.25) \quad [a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) \geq (1 + \min\{i_a, 0\})|\xi - \eta|^2 \int_0^1 a(|\eta + t(\xi - \eta)|) dt \quad \text{for } \xi, \eta \in \mathbb{R}^{Nn}.$$

Owing to (4.3),

$$(4.26) \quad \begin{aligned} a(|\eta + t(\xi - \eta)|) &\geq C \min\{|\eta + t(\xi - \eta)|^{i_a}, |\eta + t(\xi - \eta)|^{s_a}\} \\ &\geq C \min\{(|\eta| + |\xi - \eta|)^{i_a}, (|\eta| + |\xi - \eta|)^{s_a}\} \\ &\geq C' \min\{(|\eta| + |\xi|)^{i_a}, (|\eta| + |\xi|)^{s_a}\}, \end{aligned}$$

for some positive constants $C = C(a)$ and $C' = C'(a)$. Coupling (4.25) with (4.26) yields

$$[a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) \geq C \min\{(|\eta| + |\xi|)^{i_a}, (|\eta| + |\xi|)^{s_a}\} |\xi - \eta|^2 \quad \text{for } \xi, \eta \in \mathbb{R}^{Nn},$$

for some positive constant $C = C(a)$. Hence, (4.21) follows also in this case. \square

In the following lemma, any function a as in the statement of Theorem 2.1 is approximated by a family $\{a_\varepsilon\}$ of functions enjoying the additional property of being bounded from above and from below by positive constants.

Lemma 4.5 *Let a be as in Lemma 4.1. Assume, in addition, that a is monotone (either non-decreasing or non-increasing). Given $\varepsilon \in (0, 1)$, define $a_\varepsilon : [0, \infty) \rightarrow (0, \infty)$ as*

$$(4.27) \quad a_\varepsilon(t) = \frac{a(\sqrt{\varepsilon + t^2}) + \varepsilon}{1 + \varepsilon a(\sqrt{\varepsilon + t^2})} \quad \text{for } t \geq 0.$$

Then a_ε has the same monotonicity property as a ,

$$(4.28) \quad a_\varepsilon \in C^1([0, \infty)),$$

$$(4.29) \quad \varepsilon \leq a_\varepsilon(t) \leq \varepsilon^{-1} \quad \text{for } t \geq 0,$$

$$(4.30) \quad \min\{i_a, 0\} \leq i_{a_\varepsilon} \leq s_{a_\varepsilon} \leq \max\{s_a, 0\},$$

$$(4.31) \quad \lim_{\varepsilon \rightarrow 0} a_\varepsilon(|\xi|)\xi = a(|\xi|)\xi \quad \text{uniformly in } \{\xi \in \mathbb{R}^{Nn} : |\xi| \leq M\} \text{ for every } M > 0.$$

Moreover, if b_ε and B_ε are defined as in (2.3) and (2.4), respectively, with a replaced with a_ε , then

$$(4.32) \quad \lim_{\varepsilon \rightarrow 0} b_\varepsilon = b \quad \text{uniformly in } [0, M] \text{ for every } M > 0,$$

and hence

$$(4.33) \quad \lim_{\varepsilon \rightarrow 0} B_\varepsilon = B \quad \text{uniformly in } [0, M] \text{ for every } M > 0.$$

Proof. Property (4.28) trivially follows from the fact that $a \in C^1(0, \infty)$. Since

$$(4.34) \quad a'_\varepsilon(t) = \frac{(1 - \varepsilon^2) a'(\sqrt{\varepsilon + t^2}) t}{(1 + \varepsilon a(\sqrt{\varepsilon + t^2}))^2 \sqrt{\varepsilon + t^2}} \quad \text{for } t \geq 0,$$

a' and a'_ε have like signs, and hence a and a_ε share the same monotonicity property. Equation (4.30) is an easy consequence of (4.34) and of the very definitions of i_{a_ε} and s_{a_ε} . Equation (4.29) follows from the definition of a_ε , and from the fact that the function $[0, \infty) \ni s \mapsto \frac{s+\varepsilon}{1+\varepsilon s}$ is increasing for every $\varepsilon \in (0, 1)$.

Next, note that

$$\lim_{\varepsilon \rightarrow 0} a_\varepsilon = a \quad \text{uniformly in } [L, M] \text{ for every } M > L > 0.$$

Hence,

$$(4.35) \quad \lim_{\varepsilon \rightarrow 0} b_\varepsilon = b \quad \text{uniformly in } [L, M] \text{ for every } M > L > 0.$$

On the other hand, by (4.3) with a replaced with a_ε and by (4.30),

$$(4.36) \quad 0 \leq b_\varepsilon(t) = t a_\varepsilon(t) \leq a_\varepsilon(1) t^{1+\min\{i_a, 0\}} \leq (\max\{a(\sqrt{2}), a(1)\} + 1) t^{1+\min\{i_a, 0\}} \quad \text{if } 0 < t < 1,$$

whence

$$(4.37) \quad \lim_{t \rightarrow 0} b_\varepsilon(t) = 0 \quad \text{uniformly for } \varepsilon \in (0, 1).$$

Combining (4.35), (4.37) and (4.4) yields (4.32).

The proof of (4.31) is analogous. □

5 Fundamental geometric and differential inequalities

Here, we enucleate some inequalities of geometric and functional nature which are needed in the proofs of our main results.

We begin with a relative isoperimetric inequality, which tells us that if Ω is a bounded Lipschitz domain in \mathbb{R}^n , $n \geq 2$, then there exists a constant C such that

$$(5.1) \quad |E|^{1/n'} \leq C \mathcal{H}^{n-1}(\partial^M E \cap \Omega)$$

for every measurable set $E \subset \Omega$ such that $|E| \leq |\Omega|/2$ [Ma5, Corollary 5.2.1/3]. Here, $|E|$ denotes the Lebesgue measure of E , $\partial^M E$ its essential boundary, and \mathcal{H}^{n-1} stands for $(n-1)$ -dimensional Hausdorff measure.

Inequality (5.1) can be derived via another geometric inequality, which holds in any bounded Lipschitz domain Ω , and asserts that

$$(5.2) \quad \mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega) \leq C \mathcal{H}^{n-1}(\partial^M E \cap \Omega)$$

for some constant $C = C(\Omega)$, and for every measurable set $E \subset \Omega$ such that $|E| \leq |\Omega|/2$ [Ma5, Chapter 6]. Indeed,

$$(5.3) \quad \partial^M E = (\partial^M E \cap \partial\Omega) \cup (\partial^M E \cap \Omega),$$

for every measurable set $E \subset \Omega$, the union being disjoint, and hence

$$(5.4) \quad \mathcal{H}^{n-1}(\partial^M E) = \mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega) + \mathcal{H}^{n-1}(\partial^M E \cap \Omega),$$

inasmuch as \mathcal{H}^{n-1} is a measure when restricted to Borel sets. Thus, inequality (5.1) follows from (5.2), (5.4), and the classical isoperimetric inequality in \mathbb{R}^n , which takes the form

$$|E|^{1/n'} \leq C \mathcal{H}^{n-1}(\partial^M E)$$

for some constant $C = C(n)$, and for every measurable set E in \mathbb{R}^n with finite measure. Notice that, in particular, the constant in (5.1) depends on n and on the constant in (5.2).

A trace inequality for functions from the Sobolev space $W^{1,2}(\Omega)$, whose support has measure not exceeding $|\Omega|/2$, is the content of the following lemma. In the statement, $\text{Tr } v$ denotes the trace on $\partial\Omega$ of a function v , and $\text{supp } v$ its support.

Lemma 5.1 *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n , $n \geq 2$. Assume that either $1 \leq q \leq \frac{2(n-1)}{n-2}$, or $1 \leq q < \infty$, according to whether $n \geq 3$ or $n = 2$. Then there exists a constant C , depending on n , q and on the constant in (5.2), such that*

$$(5.5) \quad \left(\int_{\partial\Omega} |\text{Tr } v|^q d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{q}} \leq C |\text{supp } v|^{\frac{n-1}{qn} - \frac{n-2}{2n}} \left(\int_{\Omega} |\nabla v|^2 dx \right)^{\frac{1}{2}}$$

for every $v \in W^{1,2}(\Omega)$ satisfying $|\text{supp } v| \leq |\Omega|/2$.

Proof of Lemma 5.1 There exists a constant C , depending on n , q and on the constant in (5.2), such that

$$(5.6) \quad \left(\int_{\partial\Omega} |\text{Tr } v|^q d\mathcal{H}^{n-1} \right)^{1/q} \leq C \left(\int_{\Omega} |\nabla v|^{\frac{nq}{q+n-1}} dx \right)^{\frac{q+n-1}{nq}}$$

for every $v \in W^{1,2}(\Omega)$ fulfilling $|\text{supp } v| \leq |\Omega|/2$. Inequality (5.6) can be derived from a subsequent application of [Ma5, Theorem 6.11.4/1] to $|v|^q$ (with $p = 1$), of Hölder's inequality, and of [Ma5, Lemma 6.2]. With inequality (5.6) in place, inequality (5.5) follows via Hölder's inequality. \square

If $\mathbf{u} \in W^{2,1}(\Omega, \mathbb{R}^N)$, then $|\nabla \mathbf{u}| \in W^{1,1}(\Omega)$, by the chain rule for vector-valued functions [MM, Theorem 2.1]. An application of the coarea formula for Sobolev functions in the form of [BZ] then tells us that, for every Borel function $g : \Omega \rightarrow [0, \infty)$,

$$(5.7) \quad \int_{\{|\nabla \mathbf{u}| > t\}} g(x) |\nabla |\nabla \mathbf{u}|| dx = \int_t^\infty \int_{\{|\nabla \mathbf{u}| = \tau\}} g(x) d\mathcal{H}^{n-1}(x) d\tau \quad \text{for } t \geq 0,$$

provided that a suitable precise representative of the function $|\nabla \mathbf{u}|$ is employed. Hence, if the left-hand side is finite for $t > 0$, then it is a (locally) absolutely continuous function of t , and

$$(5.8) \quad -\frac{d}{dt} \int_{\{|\nabla \mathbf{u}| > t\}} g(x) |\nabla |\nabla \mathbf{u}|| dx = \int_{\{|\nabla \mathbf{u}| = t\}} g(x) d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } t > 0.$$

The use of the coarea formula again tells us that $\mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\} \cap \{|\nabla |\nabla \mathbf{u}|| = 0\}) = 0$ for a.e. $t > 0$, and that if g is as above, then

$$(5.9) \quad \int_{\{|\nabla \mathbf{u}| > t\}} g(x) dx = \int_{\{|\nabla \mathbf{u}| > t\} \cap \{|\nabla |\nabla \mathbf{u}|| = 0\}} g(x) dx + \int_t^\infty \int_{\{|\nabla \mathbf{u}| = \tau\}} \frac{g(x)}{|\nabla |\nabla \mathbf{u}||} d\mathcal{H}^{n-1}(x) d\tau$$

for $t \geq 0$. In particular, equation (5.9) entails that, if $g \in L^1(\Omega)$, then

$$(5.10) \quad -\frac{d}{dt} \int_{\{|\nabla \mathbf{u}| > t\}} g(x) dx \geq \int_{\{|\nabla \mathbf{u}| = t\}} \frac{g(x)}{|\nabla |\nabla \mathbf{u}||} d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } t > 0.$$

The following differential inequality involving integrals over the level sets of a Sobolev function relies upon the coarea formula and the relative isoperimetric inequality (5.1), and is established in [Ma3].

Lemma 5.2 *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n , with $n \geq 2$. Let v be a nonnegative function from $W^{1,2}(\Omega)$, and let μ_v and v^* denote the distribution function and the decreasing rearrangement of v defined as in (3.1) and (3.2), respectively. Then there exists a constant C , depending on the constant in (5.2), such that*

$$(5.11) \quad 1 \leq C(-\mu'_v(t))^{1/2} \mu_v(t)^{-1/n'} \left(-\frac{d}{dt} \int_{\{v > t\}} |\nabla v|^2 dx \right)^{1/2} \quad \text{for a.e. } t \geq v^*(|\Omega|/2).$$

The next lemma provides us with a lower estimate for the scalar product between the differential operator appearing on the left-hand side of the equation in (2.5), evaluated at some \mathbb{R}^N -valued smooth function, and its Laplacian, via terms in divergence form and a nonnegative term.

Lemma 5.3 *Assume that $a : (0, \infty) \rightarrow (0, \infty)$ is of class C^1 , and satisfies the first inequality in (2.1). Let Ω be an open set in \mathbb{R}^n , $n \geq 2$, and let $\mathbf{v} \in C^3(\Omega, \mathbb{R}^N)$, with $\mathbf{v} = (v^1, \dots, v^N)$. Then*

$$(5.12) \quad \sum_{\alpha=1}^N \Delta v^\alpha \operatorname{div}(a(|\nabla \mathbf{v}|) \nabla v^\alpha) \geq \sum_{\alpha=1}^N \operatorname{div}(\Delta v^\alpha a(|\nabla \mathbf{v}|) \nabla v^\alpha) - \sum_{\alpha=1}^N \sum_{i,j}^n v_{x_i x_j}^\alpha a(|\nabla \mathbf{v}|) v_{x_i}^\alpha + (1 + \min\{i_a, 0\}) a(|\nabla \mathbf{v}|) \sum_{\alpha=1}^N |\nabla^2 v^\alpha|^2$$

in $\{\nabla \mathbf{v} \neq 0\}$.

Proof. In $\{\nabla \mathbf{v} \neq 0\}$, we have that

$$(5.13) \quad \begin{aligned} \sum_{\alpha=1}^N \Delta v^\alpha \operatorname{div}(a(|\nabla \mathbf{v}|) \nabla v^\alpha) &= \sum_{\alpha=1}^N \operatorname{div}(\Delta v^\alpha a(|\nabla \mathbf{v}|) \nabla v^\alpha) - \sum_{\alpha=1}^N \sum_{i,j=1}^n v_{x_i x_j}^\alpha a(|\nabla \mathbf{v}|) v_{x_i}^\alpha \\ &= \sum_{\alpha=1}^N \operatorname{div}(\Delta v^\alpha a(|\nabla \mathbf{v}|) \nabla v^\alpha) - \sum_{\alpha=1}^N \sum_{i,j}^n (v_{x_i x_j}^\alpha a(|\nabla \mathbf{v}|) v_{x_i}^\alpha)_{x_j} \\ &\quad + \sum_{\alpha=1}^N \sum_{i,j=1}^n (v_{x_i x_j}^\alpha)^2 a(|\nabla \mathbf{v}|) + \sum_{\alpha=1}^N \sum_{i,j=1}^n v_{x_i x_j}^\alpha a(|\nabla \mathbf{v}|)_{x_j} v_{x_i}^\alpha. \end{aligned}$$

Now,

$$\begin{aligned}
(5.14) \quad & \sum_{\alpha=1}^N \sum_{i,j=1}^n (v_{x_i x_j}^\alpha)^2 a(|\nabla \mathbf{v}|) + \sum_{\alpha=1}^N \sum_{i,j=1}^n v_{x_i x_j}^\alpha a(|\nabla \mathbf{v}|)_{x_j} v_{x_i}^\alpha \\
&= \sum_{i,j=1}^n (v_{x_i x_j}^\alpha)^2 a(|\nabla \mathbf{v}|) + \sum_{\alpha,\beta=1}^N \sum_{i,j,k=1}^n v_{x_i x_j}^\alpha a'(|\nabla \mathbf{v}|) \frac{v_{x_k}^\beta}{|\nabla \mathbf{v}|} v_{x_k x_j}^\beta v_{x_i}^\alpha \\
&= a(|\nabla \mathbf{v}|) \left(\sum_{\alpha=1}^N \sum_{i,j=1}^n (v_{x_i x_j}^\alpha)^2 + \sum_{\alpha,\beta=1}^N \sum_{i,j,k=1}^n \frac{a'(|\nabla \mathbf{v}|) |\nabla \mathbf{v}|}{a(|\nabla \mathbf{v}|)} \frac{v_{x_i}^\alpha}{|\nabla \mathbf{v}|} v_{x_i x_j}^\alpha \frac{v_{x_k}^\beta}{|\nabla \mathbf{v}|} v_{x_k x_j}^\beta \right).
\end{aligned}$$

On setting $V^j = (v_{x_1 x_j}^1, \dots, v_{x_n x_j}^1, \dots, v_{x_1 x_j}^N, \dots, v_{x_n x_j}^N)$ and $\omega = \frac{\nabla \mathbf{v}}{|\nabla \mathbf{v}|}$, and making use of the first inequality in (2.1), one obtains that

$$\begin{aligned}
(5.15) \quad & a(|\nabla \mathbf{v}|) \left(\sum_{\alpha=1}^N \sum_{i,j=1}^n (v_{x_i x_j}^\alpha)^2 + \sum_{\alpha,\beta=1}^N \sum_{i,j,k=1}^n \frac{a'(|\nabla \mathbf{v}|) |\nabla \mathbf{v}|}{a(|\nabla \mathbf{v}|)} \frac{v_{x_i}^\alpha}{|\nabla \mathbf{v}|} v_{x_i x_j}^\alpha \frac{v_{x_k}^\beta}{|\nabla \mathbf{v}|} v_{x_k x_j}^\beta \right) \\
&= a(|\nabla \mathbf{v}|) \sum_{j=1}^n \left(|V^j|^2 + \frac{a'(|\nabla \mathbf{v}|) |\nabla \mathbf{v}|}{a(|\nabla \mathbf{v}|)} (V^j \cdot \omega)^2 \right) \\
&\geq a(|\nabla \mathbf{v}|) \sum_{j=1}^n (|V^j|^2 + i_a (V^j \cdot \omega)^2) \geq a(|\nabla \mathbf{v}|) (1 + \min\{i_a, 0\}) \sum_j |V^j|^2.
\end{aligned}$$

Inequality (5.12) follows from (5.13)-(5.15). \square

The last two results of this section provide us with key inequalities involving integrals on level sets and on level surfaces of \mathbb{R}^N -valued smooth functions in a smooth domain Ω , satisfying either Dirichlet, or Neumann homogenous boundary conditions.

Lemma 5.4 *Let Ω be a bounded domain with $\partial\Omega \in C^2$ in \mathbb{R}^n , $n \geq 2$, and let a be as in Theorem 2.1. Assume that $\mathbf{v} \in C^\infty(\Omega, \mathbb{R}^N) \cap C^2(\bar{\Omega}, \mathbb{R}^N)$, and $\mathbf{v} = 0$ on $\partial\Omega$. Let \mathcal{B} denote the second fundamental form on $\partial\Omega$, and let $\text{tr}\mathcal{B}$ be its trace. Then*

$$\begin{aligned}
(5.16) \quad & \frac{(1 + \min\{i_a, 0\})^2}{2} b(t) \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla |\nabla \mathbf{v}|| d\mathcal{H}^{n-1}(x) \leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) \\
&+ \frac{\|\nabla \mathbf{v}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}}{b(t)} \int_{\{|\nabla \mathbf{v}|>t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx \\
&+ a(\|\nabla \mathbf{v}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \|\nabla \mathbf{v}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}^2 \int_{\partial\Omega \cap \{|\nabla \mathbf{v}|>t\}} |\text{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)
\end{aligned}$$

for a.e. $t > 0$. Moreover, if $r > n - 1$, then

$$\begin{aligned}
(5.17) \quad & \frac{(1 + \min\{i_a, 0\})^2}{2} b(t) \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla |\nabla \mathbf{v}|| d\mathcal{H}^{n-1}(x) \leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) \\
&+ \int_{\{|\nabla \mathbf{v}|>t\}} \frac{1}{a(|\nabla \mathbf{v}|)} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx + a(t)t^2 \int_{\partial\Omega \cap \{|\nabla \mathbf{v}|>t\}} |\text{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)
\end{aligned}$$

for a.e. $t \geq t_{\mathbf{v}}$, where $t_{\mathbf{v}} = |\nabla \mathbf{v}|^*(\alpha|\Omega|)$, and $\alpha \in (0, \frac{1}{2}]$ is a constant depending on $i_a, s_a, n, r, \|\mathrm{tr}\mathcal{B}\|_{L^r(\partial\Omega)}, |\Omega|$, and on the constant in inequality (5.2).

If Ω is convex, then the integral involving $\mathrm{tr}\mathcal{B}$ can be dropped on the right-hand sides of inequalities (5.16) and (5.17), and the constant α neither depends on r , nor on $\|\mathrm{tr}\mathcal{B}\|_{L^r(\partial\Omega)}$.

Proof. The level set $\{|\nabla \mathbf{v}| > t\}$ is open for $t > 0$. Moreover, for a.e. $t > 0$, the level surface $\partial\{|\nabla \mathbf{v}| > t\}$ is an $(n-1)$ -dimensional manifold of class C^1 outside a set of \mathcal{H}^{n-1} measure zero, and

$$\partial\{|\nabla \mathbf{v}| > t\} = \{|\nabla \mathbf{v}| = t\} \cup (\partial\Omega \cap \partial\{|\nabla \mathbf{v}| > t\}).$$

By inequality (5.12) and the divergence theorem we have that

$$\begin{aligned} (5.18) \quad & \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}| > t\}} \Delta v^\alpha \operatorname{div}(a(|\nabla \mathbf{v}|)\nabla v^\alpha) dx \geq \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}| > t\}} \operatorname{div}(\Delta v^\alpha a(|\nabla \mathbf{v}|)\nabla v^\alpha) dx \\ & - \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}| > t\}} \sum_{i,j=1}^n (v_{x_i x_j}^\alpha a(|\nabla \mathbf{v}|) v_{x_i}^\alpha)_{x_j} dx + (1 + \min\{i_a, 0\}) \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}| > t\}} a(|\nabla \mathbf{v}|) |\nabla^2 v^\alpha|^2 dx \\ & = \sum_{\alpha=1}^N \int_{\partial\{|\nabla \mathbf{v}| > t\}} \Delta v^\alpha a(|\nabla \mathbf{v}|) \frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x) - \sum_{\alpha=1}^N \int_{\partial\{|\nabla \mathbf{v}| > t\}} \sum_{i,j=1}^n v_{x_i x_j}^\alpha a(|\nabla \mathbf{v}|) v_{x_i}^\alpha \nu_j d\mathcal{H}^{n-1}(x) \\ & + (1 + \min\{i_a, 0\}) \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}| > t\}} a(|\nabla \mathbf{v}|) |\nabla^2 v^\alpha|^2 dx \quad \text{for a.e. } t > 0. \end{aligned}$$

Here, ν_j denotes the j -th component of the outer normal vector ν to $\partial\{|\nabla \mathbf{v}| > t\}$. Now, observe that, for a.e. $t > 0$,

$$\nu = -\frac{\nabla|\nabla \mathbf{v}|}{|\nabla|\nabla \mathbf{v}||} \quad \text{on } \{|\nabla \mathbf{v}| = t\}.$$

Moreover,

$$\sum_{\alpha=1}^N \sum_{i=1}^n v_{x_i x_j}^\alpha v_{x_i}^\alpha = |\nabla \mathbf{v}|_{x_j} |\nabla \mathbf{v}|.$$

Thus,

$$\begin{aligned} (5.19) \quad & \sum_{\alpha=1}^N \int_{\partial\{|\nabla \mathbf{v}| > t\}} \Delta v^\alpha a(|\nabla \mathbf{v}|) \frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x) - \sum_{\alpha=1}^N \int_{\partial\{|\nabla \mathbf{v}| > t\}} \sum_{i,j=1}^n v_{x_i x_j}^\alpha a(|\nabla \mathbf{v}|) v_{x_i}^\alpha \nu_j d\mathcal{H}^{n-1}(x) \\ & = a(t) \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}| = t\}} \Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x) + a(t)t \int_{\{|\nabla \mathbf{v}| = t\}} |\nabla|\nabla \mathbf{v}|| d\mathcal{H}^{n-1}(x) \\ & + \sum_{\alpha=1}^N \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}| > t\}} a(|\nabla \mathbf{v}|) \left(\Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^n v_{x_i x_j}^\alpha v_{x_i}^\alpha \nu_j \right) d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } t > 0. \end{aligned}$$

Let us focus on the integrals on the right-hand side of (5.19). Since, for each $\alpha = 1, \dots, N$,

$$(5.20) \quad \operatorname{div}(a(|\nabla \mathbf{v}|)\nabla v^\alpha) = a(|\nabla \mathbf{v}|)\Delta v^\alpha + a'(|\nabla \mathbf{v}|)\nabla v^\alpha \cdot \nabla|\nabla \mathbf{v}|,$$

one has that

$$\begin{aligned}
(5.21) \quad & a(t) \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}|=t\}} \Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1} \\
&= \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}|=t\}} \operatorname{div}(a(|\nabla \mathbf{v}|) \nabla v^\alpha) \frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x) - a'(t) \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}|=t\}} \nabla v^\alpha \cdot \nabla |\nabla \mathbf{v}| \frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x) \\
&= \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}|=t\}} \operatorname{div}(a(|\nabla \mathbf{v}|) \nabla v^\alpha) \frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x) + a'(t) \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla |\nabla \mathbf{v}|| \left(\frac{\partial v^\alpha}{\partial \nu} \right)^2 d\mathcal{H}^{n-1}(x) \\
&\leq t \int_{\{|\nabla \mathbf{v}|=t\}} \mathbf{div}(a(|\nabla \mathbf{v}|) \nabla \mathbf{v}) d\mathcal{H}^{n-1}(x) + a'(t) \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla |\nabla \mathbf{v}|| \left| \frac{\partial \mathbf{v}}{\partial \nu} \right|^2 d\mathcal{H}^{n-1}(x)
\end{aligned}$$

for a.e. $t > 0$. Here, we have exploited the fact that

$$\frac{\partial v^\alpha}{\partial \nu} = - \frac{\nabla v^\alpha \cdot \nabla |\nabla \mathbf{v}|}{|\nabla |\nabla \mathbf{v}||} \quad \text{on } \{|\nabla \mathbf{v}| = t\}, \quad \text{for a.e. } t > 0.$$

Next, we make use of the fact that, for each $\alpha = 1, \dots, N$,

$$\begin{aligned}
(5.22) \quad & \Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^n v_{x_i x_j}^\alpha v_{x_i}^\alpha \nu_j \\
&= \operatorname{div}_T \left(\frac{\partial v^\alpha}{\partial \nu} \nabla_T v^\alpha \right) - \operatorname{tr} \mathcal{B} \left(\frac{\partial v^\alpha}{\partial \nu} \right)^2 - \mathcal{B}(\nabla_T v^\alpha, \nabla_T v^\alpha) - 2 \nabla_T v^\alpha \cdot \nabla_T \frac{\partial v^\alpha}{\partial \nu} \quad \text{on } \partial \Omega,
\end{aligned}$$

where div_T and ∇_T denote the divergence operator and the gradient operator on $\partial \Omega$, respectively [Gr, Equation (3,1,1,2)]. Coupling (5.22) with the condition $\mathbf{v} = 0$ on $\partial \Omega$ tells us that

$$(5.23) \quad \Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^n v_{x_i x_j}^\alpha v_{x_i}^\alpha \nu_j = -\operatorname{tr} \mathcal{B} \left(\frac{\partial v^\alpha}{\partial \nu} \right)^2 \quad \text{on } \partial \Omega.$$

Therefore,

$$\begin{aligned}
(5.24) \quad & \sum_{\alpha=1}^N \int_{\partial \Omega \cap \{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|) \left(\Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^n v_{x_i x_j}^\alpha v_{x_i}^\alpha \nu_j \right) d\mathcal{H}^{n-1}(x) \\
&= - \sum_{\alpha=1}^N \int_{\partial \Omega \cap \{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|) \operatorname{tr} \mathcal{B}(x) \left(\frac{\partial v^\alpha}{\partial \nu} \right)^2 d\mathcal{H}^{n-1}(x) \\
&\geq - \int_{\partial \Omega \cap \{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|) |\nabla \mathbf{v}|^2 \operatorname{tr} \mathcal{B}(x) d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } t > 0.
\end{aligned}$$

By Young's inequality and the inequality $|\Delta \mathbf{v}| \leq |\nabla^2 \mathbf{v}|$, we have that

$$\begin{aligned}
(5.25) \quad & \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}|>t\}} \Delta v^\alpha \operatorname{div}(a(|\nabla \mathbf{v}|) \nabla v^\alpha) dx \leq \frac{1 + \min\{i_a, 0\}}{2} \int_{\{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|) |\nabla^2 \mathbf{v}|^2 dx \\
&\quad + \frac{2}{1 + \min\{i_a, 0\}} \int_{\{|\nabla \mathbf{v}|>t\}} \frac{1}{a(|\nabla \mathbf{v}|)} |\mathbf{div}(a(|\nabla \mathbf{v}|) \nabla \mathbf{v})|^2 dx
\end{aligned}$$

for a.e. $t > 0$. Combining (5.18), (5.19), (5.21), (5.24) and (5.25) yields

$$(5.26) \quad \begin{aligned} & \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla |\nabla \mathbf{v}|| \left(a(t)t + a'(t) \left| \frac{\partial \mathbf{v}}{\partial \nu} \right|^2 \right) d\mathcal{H}^{n-1}(x) + \frac{1 + \min\{i_a, 0\}}{2} \int_{\{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|) |\nabla^2 \mathbf{v}|^2 dx \\ & \leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) + \frac{2}{1 + \min\{i_a, 0\}} \int_{\{|\nabla \mathbf{v}|>t\}} \frac{1}{a(|\nabla \mathbf{v}|)} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx \\ & \quad + \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|) |\nabla \mathbf{v}|^2 |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \end{aligned}$$

for a.e. $t > 0$. Observe that

$$(5.27) \quad a(t)t + a'(t) \left| \frac{\partial \mathbf{v}}{\partial \nu} \right|^2 \geq a(t)t + \min\{0, a'(t)\}t^2 \geq (1 + \min\{i_a, 0\})a(t)t \quad \text{on } \{|\nabla \mathbf{v}| = t\}.$$

From (5.26) and (5.27) we deduce that

$$(5.28) \quad \begin{aligned} & (1 + \min\{i_a, 0\})b(t) \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla |\nabla \mathbf{v}|| d\mathcal{H}^{n-1}(x) + \frac{1 + \min\{i_a, 0\}}{2} \int_{\{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|) |\nabla^2 \mathbf{v}|^2 dx \\ & \leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) \\ & \quad + \frac{2}{1 + \min\{i_a, 0\}} \int_{\{|\nabla \mathbf{v}|>t\}} \frac{1}{a(|\nabla \mathbf{v}|)} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx \\ & \quad + \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|) |\nabla \mathbf{v}|^2 |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } t > 0. \end{aligned}$$

Hence, since $b(t)$ is an increasing function, and hence also $a(t)t^2$ is an increasing function,

$$(5.29) \quad \begin{aligned} & (1 + \min\{i_a, 0\})b(t) \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla |\nabla \mathbf{v}|| d\mathcal{H}^{n-1}(x) \leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) \\ & \quad + \frac{2}{1 + \min\{i_a, 0\}} \frac{\|\nabla \mathbf{v}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}}{b(t)} \int_{\{|\nabla \mathbf{v}|>t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx \\ & \quad + a(\|\nabla \mathbf{v}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \|\nabla \mathbf{v}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}^2 \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}|>t\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \end{aligned}$$

for a.e. $t > 0$. Inequality (5.16) follows.

Let us next focus on (5.17). Inasmuch as $a(t)t^2$ is an increasing function,

$$(5.30) \quad \begin{aligned} & \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|) |\nabla \mathbf{v}|^2 |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \\ & \leq 2 \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}|>t\}} (a(|\nabla \mathbf{v}|)^{1/2} |\nabla \mathbf{v}| - a(t)^{1/2} t)^2 |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \\ & \quad + 2a(t)t^2 \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}|>t\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x). \end{aligned}$$

Denote, for simplicity, the distribution function $\mu_{|\nabla \mathbf{v}|}$ of $|\nabla \mathbf{v}|$ by $\mu : [0, \infty) \rightarrow [0, |\Omega|]$ for a.e. $t > 0$. Set $\delta = \frac{n-1}{nr'} - \frac{n-2}{n}$, and observe that $\delta > 0$ since $r > n - 1$. Thanks to our assumptions on the function a and to the chain rule for vector-valued Sobolev functions [MM, Theorem 2.1], the function $\max\{a(|\nabla \mathbf{v}|)^{1/2}|\nabla \mathbf{v}| - a(t)^{1/2}t, 0\}$ belongs to $W^{1,2}(\Omega)$. Hölder's inequality and an application of Lemma 5.1 with v replaced with $\max\{a(|\nabla \mathbf{v}|)^{1/2}|\nabla \mathbf{v}| - a(t)^{1/2}t, 0\}$ tell us that

(5.31)

$$\begin{aligned}
& \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}| > t\}} (a(|\nabla \mathbf{v}|)^{1/2}|\nabla \mathbf{v}| - a(t)^{1/2}t)^2 |\operatorname{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \\
& \leq \left(\int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}| > t\}} (a(|\nabla \mathbf{v}|)^{1/2}|\nabla \mathbf{v}| - a(t)^{1/2}t)^{2r'} d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{r'}} \left(\int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}| > t\}} |\operatorname{tr} \mathcal{B}(x)|^r d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{r}} \\
& \leq C\mu(t)^\delta \|\operatorname{tr} \mathcal{B}\|_{L^r(\partial\Omega)} \int_{\{|\nabla \mathbf{v}| > t\}} \left| \nabla [a(|\nabla \mathbf{v}|)^{1/2}|\nabla \mathbf{v}|] \right|^2 dx \\
& = C\mu(t)^\delta \|\operatorname{tr} \mathcal{B}\|_{L^r(\partial\Omega)} \int_{\{|\nabla \mathbf{v}| > t\}} \left(\frac{1}{2} a'(|\nabla \mathbf{v}|) a(|\nabla \mathbf{v}|)^{-1/2} |\nabla \mathbf{v}| + a(|\nabla \mathbf{v}|)^{1/2} \right)^2 |\nabla |\nabla \mathbf{v}||^2 dx \\
& \leq C' \mu(t)^\delta \|\operatorname{tr} \mathcal{B}\|_{L^r(\partial\Omega)} \int_{\{|\nabla \mathbf{v}| > t\}} a(|\nabla \mathbf{v}|) |\nabla^2 \mathbf{v}|^2 dx \quad \text{if } t > |\nabla \mathbf{v}|^*(|\Omega|/2) \quad \text{for a.e. } t > 0,
\end{aligned}$$

for some positive constants C , depending on the constant in (5.2) and on r , and C' depending on the same quantities and on s_a . Observe that, in the last inequality in (5.31), we have employed the inequality $|\nabla |\nabla \mathbf{v}|| \leq |\nabla^2 \mathbf{v}|$. Set

$$\beta = \left(\frac{1 + \min\{i_a, 0\}}{4C' \|\operatorname{tr} \mathcal{B}\|_{L^r(\partial\Omega)}} \right)^{\frac{1}{\delta}},$$

where C' is the constant appearing in (5.31),

$$(5.32) \quad \alpha = \min\{\beta/|\Omega|, 1/2\},$$

and $t_{\mathbf{v}} = |\nabla \mathbf{v}|^*(\alpha|\Omega|)$. Thus, α depends on the quantities specified in the statement, and

$$(5.33) \quad \frac{1 + \min\{i_a, 0\}}{2} - 2C' \|\operatorname{tr} \mathcal{B}\|_{L^r(\partial\Omega)} \mu(t)^\delta \geq 0 \quad \text{if } t > t_{\mathbf{v}}.$$

Inequality (5.17) follows from (5.28), (5.30), (5.31) and (5.33), since $\frac{1 + \min\{i_a, 0\}}{2} < 1$.

The assertion concerning the case when Ω is convex follows via the same argument, on observing that the leftmost side of (5.24) can be estimated from below just by 0, inasmuch as $\operatorname{tr} \mathcal{B} \leq 0$ on $\partial\Omega$ in this case. \square

Lemma 5.5 *Let Ω be a bounded domain with $\partial\Omega \in C^2$ in \mathbb{R}^n , $n \geq 2$, and let a be as in Theorem 2.1. Assume that $\mathbf{v} \in C^\infty(\Omega, \mathbb{R}^N) \cap C^2(\bar{\Omega}, \mathbb{R}^N)$, and $\frac{\partial \mathbf{v}}{\partial \nu} = 0$ on $\partial\Omega$. Let \mathcal{B} denote the second fundamental form on $\partial\Omega$, and let $|\mathcal{B}|$ be its operator norm, namely*

$$|\mathcal{B}(x)| = \sup_{0 \neq \zeta \in \mathbb{R}^{n-1}} \frac{|\mathcal{B}(x)(\zeta, \zeta)|}{|\zeta|^2} \quad \text{for } x \in \partial\Omega.$$

Then

$$(5.34) \quad \begin{aligned} \frac{(1 + \min\{i_a, 0\})^2}{2} b(t) \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla |\nabla \mathbf{v}|| d\mathcal{H}^{n-1}(x) &\leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) \\ &+ \frac{\|\nabla \mathbf{v}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}}{b(t)} \int_{\{|\nabla \mathbf{v}|>t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx \\ &+ a(\|\nabla \mathbf{v}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \|\nabla \mathbf{v}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}^2 \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}|>t\}} |\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \end{aligned}$$

for a.e. $t > 0$. Moreover, if $r > n - 1$, then

$$(5.35) \quad \begin{aligned} \frac{(1 + \min\{i_a, 0\})^2}{2} b(t) \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla |\nabla \mathbf{v}|| d\mathcal{H}^{n-1}(x) &\leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) \\ &+ \int_{\{|\nabla \mathbf{v}|>t\}} \frac{1}{a(|\nabla \mathbf{v}|)} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx + a(t)t^2 \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{v}|>t\}} |\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \end{aligned}$$

for a.e. $t \geq t_{\mathbf{v}}$, where $t_{\mathbf{v}} = |\nabla \mathbf{v}|^*(\alpha|\Omega|)$, and $\alpha \in (0, \frac{1}{2}]$ is a constant depending on i_a , s_a , n , r , $\|\mathcal{B}\|_{L^r(\partial\Omega)}$, $|\Omega|$, and on the constant in inequality (5.2).

If Ω is convex, the integral involving \mathcal{B} can be dropped on the right-hand sides of inequalities (5.34) and (5.35), and the constant α neither depends on r , nor on $\|\mathcal{B}\|_{L^r(\partial\Omega)}$.

Sketch of the proof. The proof is completely analogous to that of Lemma 5.4. One has just to observe that, by (5.22) and the condition $\frac{\partial \mathbf{v}}{\partial \nu} = 0$ on $\partial\Omega$, equation (5.23) has to be replaced with

$$(5.36) \quad \Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^n v_{x_i x_j}^\alpha v_{x_i}^\alpha \nu_j = -\mathcal{B}(\nabla_T v^\alpha, \nabla_T v^\alpha) \quad \text{on } \partial\Omega,$$

which, in particular, implies that

$$(5.37) \quad \left| \Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^N v_{x_i x_j}^\alpha v_{x_i}^\alpha \nu_j \right| \leq |\mathcal{B}| |\nabla v^\alpha|^2 \quad \text{on } \partial\Omega.$$

The conclusion concerning convex domains Ω holds owing to the fact that $\mathcal{B} \leq 0$ on $\partial\Omega$ in this case. \square

6 Proof of the main results

Let B be the Young function defined by (2.4), and let B_n be its Sobolev conjugate given by (3.15). Assume that $f \in L^{\tilde{B}_n}(\Omega, \mathbb{R}^N)$. A weak solution to the Dirichlet problem (2.5) is a function $u \in W_0^{1,B}(\Omega, \mathbb{R}^N)$ such that

$$(6.1) \quad \int_{\Omega} a(|\nabla \mathbf{u}|) \nabla \mathbf{u} \cdot \nabla \phi dx = \int_{\Omega} \mathbf{f} \cdot \phi dx$$

for every $\phi \in W_0^{1,B}(\Omega, \mathbb{R}^N)$.

Assume now, in addition that Ω has a Lipschitz boundary. A weak solution to the Neumann problem (2.8) is a function $\mathbf{u} \in W^{1,B}(\Omega, \mathbb{R}^N)$ such that

$$(6.2) \quad \int_{\Omega} a(|\nabla \mathbf{u}|) \nabla \mathbf{u} \cdot \nabla \phi \, dx = \int_{\Omega} \mathbf{f} \cdot \phi \, dx$$

for every $\phi \in W^{1,B}(\Omega, \mathbb{R}^N)$.

Note that the left-hand sides of (6.1) and (6.2) are well defined by inequalities (3.11) and (4.6). The right-hand sides are also well defined, owing to the Sobolev inequality (3.17) and inequality (3.11) with B replaced with B_n . In particular, the right-hand sides of (6.1) and (6.2) are well defined if $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$, since $L^{n,1}(\Omega, \mathbb{R}^N) \rightarrow L^{\widetilde{B}_n}(\Omega, \mathbb{R}^N)$, as shown in [CM1, Remark 2.12].

The following existence and uniqueness result holds for weak solutions to problems (2.5) and (2.8).

Theorem 6.1 *Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$, and let $N \geq 1$. Assume that $a : (0, \infty) \rightarrow (0, \infty)$ is of class C^1 , and fulfills (2.1). Let $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$. Then there exists a unique solution $\mathbf{u} \in W_0^{1,B}(\Omega, \mathbb{R}^N)$ to problem (2.5).*

Assume, in addition, that Ω has a Lipschitz boundary, and \mathbf{f} fulfills (2.9). Then there exists a solution $\mathbf{u} \in W^{1,B}(\Omega, \mathbb{R}^N)$ to problem (2.8), which is unique up to additive constant vectors in \mathbb{R}^N . In particular, there exists a unique solution in $W_{\perp}^{1,B}(\Omega, \mathbb{R}^N)$.

A proof of Theorem 6.1 in the case when $N = 1$ can be found in [CM1]; the proof for $N > 1$ is completely analogous.

The next Proposition provides us with a basic energy estimate for the weak solutions to problems (2.5) and (2.8).

Proposition 6.2 *Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$, and let $N \geq 1$. Assume that a is as in Theorem 6.1. Let $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^{Nn})$.*

(i) Let $\mathbf{u} \in W_0^{1,B}(\Omega, \mathbb{R}^N)$ be the weak solution to problem (2.5). Then

$$(6.3) \quad \int_{\Omega} B(|\nabla \mathbf{u}|) \, dx \leq C \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)} b^{-1}(\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}),$$

where $C = C'|\Omega|$, and C' is a constant depending on n , N and i_a .

(ii) Assume, in addition, that Ω has a Lipschitz boundary, and \mathbf{f} fulfills (2.9). Let $\mathbf{u} \in W^{1,B}(\Omega, \mathbb{R}^N)$ be a weak solution to problem (2.8). Then inequality (6.3) holds for some constant C depending on n , N , i_a and on the constant in (5.1)

Proof. (i) Making use of \mathbf{u} as test function ϕ in the definition of weak solution (6.1) tells us that

$$\int_{\Omega} a(|\nabla \mathbf{u}|) |\nabla \mathbf{u}|^2 \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, dx.$$

By the first inequality in (4.9), Hölder's inequality in Lorentz spaces (3.8), and (3.6), there exist constants C and C' , depending on n , such that

$$(6.4) \quad \int_{\Omega} B(|\nabla \mathbf{u}|) \, dx \leq C \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)} \|\mathbf{u}\|_{L^{n',\infty}(\Omega, \mathbb{R}^N)} \leq C' \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)} \|\mathbf{u}\|_{L^{n'}(\Omega, \mathbb{R}^N)}.$$

By the Poincaré inequality in $W_0^{1,1}(\Omega, \mathbb{R}^N)$, there exists a constant $C = C(n, N)$ such that

$$(6.5) \quad \|\mathbf{u}\|_{L^{n'}(\Omega, \mathbb{R}^N)} \leq C \int_{\Omega} |\nabla \mathbf{u}| \, dx.$$

On the other hand, Jensen's inequality entails that

$$(6.6) \quad B\left(\frac{1}{|\Omega|} \int_{\Omega} |\nabla \mathbf{u}| \, dx\right) \leq \frac{1}{|\Omega|} \int_{\Omega} B(|\nabla \mathbf{u}|) \, dx.$$

Combining inequalities (6.4)–(6.6), and making use of the second inequality in (3.10) yields

$$(6.7) \quad \frac{1}{|\Omega|} \int_{\Omega} B(|\nabla \mathbf{u}|) \, dx \leq \tilde{B}(2C \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}).$$

Since b^{-1} is an increasing function, equation (3.9) ensures that $\tilde{B}(t) \leq tb^{-1}(t)$ for $t \geq 0$. Thus, (6.3) follows from (6.7), via (4.8).

(ii) The proof follows along the same lines as above. One has just to make use of the fact that inequality (6.5) holds, for every $\mathbf{u} \in W_{\perp}^{1,B}(\Omega, \mathbb{R}^N)$, with a constant C depending on n, N and on the constant in (5.1) [Ma5, Theorem 5.2.3], and that any solution u to (2.8) differs from the solution in $W_{\perp}^{1,B}(\Omega, \mathbb{R}^N)$ by a constant vector in \mathbb{R}^N . \square

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. We split the proof in steps.

Step 1. We assume in addition, for the time being, that

$$(6.8) \quad \partial\Omega \in C^{\infty},$$

and there exist positive constants c and C such that

$$(6.9) \quad c \leq a(t) \leq C \quad \text{for } t \geq 0.$$

Since $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$, in particular $\mathbf{f} \in L^2(\Omega, \mathbb{R}^N)$, owing to (3.7). A result by Elcrat and Meyers implies that the weak solution \mathbf{u} to problem (2.5) belongs to $W^{2,2}(\Omega)$ [BF, Theorem 8.2]. Notice that the hypotheses of that result are fulfilled under our additional assumptions (6.8)–(6.9), owing to equation (4.14). Thus, $\mathbf{u} \in W_0^{1,2}(\Omega, \mathbb{R}^N) \cap W^{2,2}(\Omega, \mathbb{R}^N)$. By standard approximation, there exists a sequence $\{\mathbf{u}_k\} \subset C^{\infty}(\Omega, \mathbb{R}^N) \cap C^2(\bar{\Omega}, \mathbb{R}^N)$ such that $\mathbf{u}_k = 0$ on $\partial\Omega$,

$$(6.10) \quad \mathbf{u}_k \rightarrow \mathbf{u} \quad \text{in } W_0^{1,2}(\Omega, \mathbb{R}^N), \quad \mathbf{u}_k \rightarrow \mathbf{u} \quad \text{in } W^{2,2}(\Omega, \mathbb{R}^N), \quad \nabla \mathbf{u}_k \rightarrow \nabla \mathbf{u} \quad \text{a.e. in } \Omega,$$

as $k \rightarrow \infty$. Furthermore, $|\nabla \mathbf{u}_k| \in W^{1,2}(\Omega)$ and $|\nabla |\nabla \mathbf{u}_k|| \leq |\nabla^2 \mathbf{u}_k|$ a.e. in Ω , by the chain rule for vector-valued Sobolev functions [MM, Theorem 2.1]. Thus, owing to the compactness of the trace embedding $\text{Tr} : W^{1,2}(\Omega) \rightarrow L^1(\partial\Omega)$, we may also assume that

$$(6.11) \quad \text{Tr} |\nabla \mathbf{u}_k| \rightarrow \text{Tr} |\nabla \mathbf{u}| \quad \mathcal{H}^{n-1} \text{ a.e. on } \partial\Omega,$$

as $k \rightarrow \infty$. We claim that

$$(6.12) \quad -\text{div}(a(|\nabla \mathbf{u}_k|) \nabla \mathbf{u}_k) \rightarrow \mathbf{f} \quad \text{in } L^2(\Omega, \mathbb{R}^N),$$

as $k \rightarrow \infty$. Let us verify this claim. First, since $\nabla \mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^{Nn})$, an application of the chain rule for vector-valued Sobolev functions again tells us that, for each $\alpha = 1, \dots, N$,

$$(6.13) \quad \text{div}(a(|\nabla \mathbf{u}|) \nabla u^{\alpha}) = \frac{a'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \sum_{\beta=1}^N \sum_{i,j=1}^n u_{x_i}^{\alpha} u_{x_j}^{\beta} u_{x_j x_i}^{\beta} \chi_{\{\nabla \mathbf{u} \neq 0\}} + a(|\nabla \mathbf{u}|) \Delta u^{\alpha} \quad \text{a.e. in } \Omega,$$

and that the same equation holds with \mathbf{u} replaced with \mathbf{u}_k . Here, and in what follows, we adhere to the convention that $0 \cdot \infty = 0$, so that $\frac{\chi_{\{\nabla \mathbf{u} \neq 0\}}}{|\nabla \mathbf{u}|} = 0$ in $\{\nabla \mathbf{u} = 0\}$. Now, for each $k \in \mathbb{N}$ and $\alpha = 1, \dots, N$,

$$\begin{aligned}
(6.14) \quad & \left(\int_{\Omega} |\operatorname{div}(a(|\nabla \mathbf{u}_k|)\nabla u_k^\alpha) + f^\alpha(x)|^2 dx \right)^{\frac{1}{2}} = \left(\int_{\Omega} |\operatorname{div}(a(|\nabla \mathbf{u}_k|)\nabla u_k^\alpha) - \operatorname{div}(a(|\nabla \mathbf{u}|)\nabla u^\alpha)|^2 dx \right)^{\frac{1}{2}} \\
& = \left(\int_{\Omega} \left| \frac{a'(|\nabla \mathbf{u}_k|)}{|\nabla \mathbf{u}_k|} \sum_{\beta=1}^N \sum_{i,j=1}^n u_{k x_i}^\alpha u_{k x_j}^\beta u_{k x_j x_i}^\beta \chi_{\{\nabla \mathbf{u}_k \neq 0\}} + a(|\nabla \mathbf{u}_k|)\Delta u_k^\alpha \right. \right. \\
& \quad \left. \left. - \frac{a'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \sum_{\beta=1}^N \sum_{i,j=1}^n u_{x_i}^\alpha u_{x_j}^\beta u_{x_j x_i}^\beta \chi_{\{\nabla \mathbf{u} \neq 0\}} - a(|\nabla \mathbf{u}|)\Delta u^\alpha \right|^2 dx \right)^{\frac{1}{2}} \\
& \leq \left(\int_{\Omega} |a(|\nabla \mathbf{u}_k|)(\Delta u_k^\alpha - \Delta u^\alpha)|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} |(a(|\nabla \mathbf{u}_k|) - a(|\nabla \mathbf{u}|))\Delta u^\alpha|^2 dx \right)^{\frac{1}{2}} \\
& \quad + \left(\int_{\Omega} \left| \frac{a'(|\nabla \mathbf{u}_k|)}{|\nabla \mathbf{u}_k|} \sum_{\beta=1}^N \sum_{i,j=1}^n u_{k x_i}^\alpha u_{k x_j}^\beta \chi_{\{\nabla \mathbf{u}_k \neq 0\}} (u_{k x_j x_i}^\beta - u_{x_j x_i}^\beta) \right|^2 dx \right)^{\frac{1}{2}} \\
& \quad + \left(\int_{\Omega} \left| \sum_{\beta=1}^N \left(\frac{a'(|\nabla \mathbf{u}_k|)}{|\nabla \mathbf{u}_k|} \sum_{i,j=1}^n u_{k x_i}^\alpha u_{k x_j}^\beta \chi_{\{\nabla \mathbf{u}_k \neq 0\}} - \frac{a'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \sum_{i,j=1}^n u_{x_i}^\alpha u_{x_j}^\beta \chi_{\{\nabla \mathbf{u} \neq 0\}} \right) u_{x_j x_i}^\beta \right|^2 dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Since the functions $a(t)$ and $a'(t)t$ are bounded, the first and the third addend on the rightmost side of (6.14) converge to 0 as $k \rightarrow \infty$, inasmuch as $u_{k x_j x_i}^\beta \rightarrow u_{x_j x_i}^\beta$ in $L^2(\Omega)$ as $k \rightarrow \infty$, for $\beta = 1, \dots, N$ and $i, j = 1, \dots, n$. The boundedness of the functions $a(t)$ and $a'(t)t$, and the convergence of $\nabla \mathbf{u}_k$ to $\nabla \mathbf{u}$ a.e. in Ω implies that the second and the fourth addend also converge to 0 by the dominated convergence theorem for integrals. Hence, (6.12) follows.

Step 2. Let $\{\mathbf{u}_k\}$ be the sequence considered in Step 1. For each $k \in \mathbb{N}$, the function \mathbf{u}_k satisfies the same assumptions as the function \mathbf{v} in Lemma 5.4. Hence, inequality (5.17) holds with \mathbf{v} replaced with \mathbf{u}_k . This tells us that

$$\begin{aligned}
(6.15) \quad & Cb(t) \int_{\{|\nabla \mathbf{u}_k|=t\}} |\nabla|\nabla \mathbf{u}_k|| d\mathcal{H}^{n-1}(x) \leq t \int_{\{|\nabla \mathbf{u}_k|=t\}} |\operatorname{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)| d\mathcal{H}^{n-1}(x) \\
& \quad + \int_{\{|\nabla \mathbf{u}_k|>t\}} \frac{1}{a(|\nabla \mathbf{u}_k|)} |\operatorname{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)|^2 \\
& \quad + a(t)t^2 \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{u}_k|>t\}} |\operatorname{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } t > t_{\mathbf{u}_k},
\end{aligned}$$

where $C = \frac{(1+\min\{i_a, 0\})^2}{2}$, and $t_{\mathbf{u}_k}$ is defined analogously to $t_{\mathbf{v}}$, with \mathbf{v} replaced with \mathbf{u}_k . We claim that inequality (6.15) continues to hold with \mathbf{u}_k replaced with \mathbf{u} , namely that

$$\begin{aligned}
(6.16) \quad & Cb(t) \int_{\{|\nabla \mathbf{u}|=t\}} |\nabla|\nabla \mathbf{u}|| d\mathcal{H}^{n-1}(x) \leq t \int_{\{|\nabla \mathbf{u}|=t\}} |\mathbf{f}(x)| d\mathcal{H}^{n-1}(x) + \int_{\{|\nabla \mathbf{u}|>t\}} \frac{1}{a(|\nabla \mathbf{u}|)} |\mathbf{f}(x)|^2 dx \\
& \quad + a(t)t^2 \int_{\partial\Omega \cap \partial^M \{|\nabla \mathbf{u}|>t\}} |\operatorname{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } t > t_{\mathbf{u}}.
\end{aligned}$$

To verify this claim, observe that $t_{\mathbf{u}_k} \rightarrow t_{\mathbf{u}}$ as $k \rightarrow \infty$, fix any $t > t_{\mathbf{u}}$ and $h > 0$, and, for sufficiently large k , integrate inequality (6.15) over the interval $(t, t+h)$, and make use of the coarea formula (5.7) to obtain

$$(6.17) \quad \begin{aligned} C \int_{\{t < |\nabla \mathbf{u}_k| < t+h\}} b(|\nabla \mathbf{u}_k|) |\nabla |\nabla \mathbf{u}_k||^2 dx &\leq \int_{\{t < |\nabla \mathbf{u}_k| < t+h\}} |\nabla \mathbf{u}_k| |\nabla |\nabla \mathbf{u}_k|| |\mathbf{div}(a(|\nabla \mathbf{u}_k|) \nabla \mathbf{u}_k)| dx \\ &+ \int_t^{t+h} \int_{\{|\nabla \mathbf{u}_k| > \tau\}} \frac{1}{a(|\nabla \mathbf{u}_k|)} |\mathbf{div}(a(|\nabla \mathbf{u}_k|) \nabla \mathbf{u}_k)|^2 dx d\tau \\ &+ \int_t^{t+h} a(\tau) \tau^2 \int_{\partial \Omega \cap \partial \{|\nabla \mathbf{u}_k| > \tau\}} |\mathrm{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x) d\tau. \end{aligned}$$

We have that

$$(6.18) \quad \begin{aligned} \int_{\{t < |\nabla \mathbf{u}_k| < t+h\}} b(|\nabla \mathbf{u}_k|) |\nabla |\nabla \mathbf{u}_k||^2 dx &= \int_{\Omega} \chi_{\{t < |\nabla \mathbf{u}_k| < t+h\}}(x) b(|\nabla \mathbf{u}_k|) \sum_{i=1}^n \left(\sum_{\beta=1}^N \sum_{j=1}^n \frac{u_{kx_j}^\beta}{|\nabla \mathbf{u}_k|} u_{kx_j x_i}^\beta \right)^2 dx \\ &= \int_{\Omega} \chi_{\{t < |\nabla \mathbf{u}_k| < t+h\}}(x) b(|\nabla \mathbf{u}_k|) \sum_{i=1}^n \left(\sum_{\beta=1}^N \sum_{j=1}^n \frac{u_{kx_j}^\beta}{|\nabla \mathbf{u}_k|} (u_{kx_j x_i}^\beta - u_{x_j x_i}^\beta) + \sum_{\beta=1}^N \sum_{j=1}^n \frac{u_{kx_j}^\beta}{|\nabla \mathbf{u}_k|} u_{x_j x_i}^\beta \right)^2 dx \\ &= \int_{\Omega} \chi_{\{t < |\nabla \mathbf{u}_k| < t+h\}}(x) b(|\nabla \mathbf{u}_k|) \sum_{i=1}^n \left(\sum_{\beta=1}^N \sum_{j=1}^n \frac{u_{kx_j}^\beta}{|\nabla \mathbf{u}_k|} (u_{kx_j x_i}^\beta - u_{x_j x_i}^\beta) \right)^2 dx \\ &+ 2 \int_{\Omega} \chi_{\{t < |\nabla \mathbf{u}_k| < t+h\}}(x) b(|\nabla \mathbf{u}_k|) \sum_{i=1}^n \left(\sum_{\beta=1}^N \sum_{j=1}^n \frac{u_{kx_j}^\beta}{|\nabla \mathbf{u}_k|} (u_{kx_j x_i}^\beta - u_{x_j x_i}^\beta) \right) \left(\sum_{\beta=1}^N \sum_{j=1}^n \frac{u_{kx_j}^\beta}{|\nabla \mathbf{u}_k|} u_{x_j x_i}^\beta \right) dx \\ &+ \int_{\Omega} \chi_{\{t < |\nabla \mathbf{u}_k| < t+h\}}(x) b(|\nabla \mathbf{u}_k|) \sum_{i=1}^n \left(\sum_{\beta=1}^N \sum_{j=1}^n \frac{u_{kx_j}^\beta}{|\nabla \mathbf{u}_k|} u_{x_j x_i}^\beta \right)^2 dx. \end{aligned}$$

Note that the first equality holds by the chain rule for vector-valued functions. Since b is an increasing function,

$$\chi_{\{t < |\nabla \mathbf{u}_k| < t+h\}}(x) b(|\nabla \mathbf{u}_k|) \leq b(t+h) \quad \text{for } x \in \Omega,$$

for every $k \in \mathbb{N}$. Moreover, $|u_{kx_j}^\beta|/|\nabla \mathbf{u}_k| \leq 1$ for every $k \in \mathbb{N}$, $\beta = 1, \dots, N$, $j = 1, \dots, n$. Thus, the first integral on the rightmost side of (6.18) converges to 0 as $k \rightarrow \infty$, since $u_{kx_j x_i}^\beta \rightarrow u_{x_j x_i}^\beta$ in $L^2(\Omega)$, for $\beta = 1, \dots, N$ and $i, j = 1, \dots, n$. The same observation, combined with Hölder's inequality, ensures that also the second integral converges to 0 as $k \rightarrow \infty$. Since $\nabla \mathbf{u}_k \rightarrow \nabla \mathbf{u}$ a.e. in Ω , the last integral in (6.18) tends to

$$(6.19) \quad \int_{\{t < |\nabla \mathbf{u}| < t+h\}} b(|\nabla \mathbf{u}|) \sum_{i=1}^n \left(\sum_{\beta=1}^N \sum_{j=1}^n \frac{u_{x_j}^\beta}{|\nabla \mathbf{u}|} u_{x_j x_i}^\beta \right)^2 dx,$$

by to the dominated convergence theorem for integrals, and the expression (6.19) agrees with $\int_{\{t < |\nabla \mathbf{u}| < t+h\}} b(|\nabla \mathbf{u}|) |\nabla |\nabla \mathbf{u}||^2 dx$. Thus, we have shown that

$$(6.20) \quad \int_{\{t < |\nabla \mathbf{u}_k| < t+h\}} b(|\nabla \mathbf{u}_k|) |\nabla |\nabla \mathbf{u}_k||^2 dx \rightarrow \int_{\{t < |\nabla \mathbf{u}| < t+h\}} b(|\nabla \mathbf{u}|) |\nabla |\nabla \mathbf{u}||^2 dx$$

as $k \rightarrow \infty$. A similar argument implies, via (6.12), that

$$(6.21) \quad \int_{\{t < |\nabla \mathbf{u}_k| < t+h\}} |\nabla \mathbf{u}_k| |\nabla |\nabla \mathbf{u}_k|| |\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)| dx \rightarrow \int_{\{t < |\nabla \mathbf{u}| < t+h\}} |\nabla \mathbf{u}| |\nabla |\nabla \mathbf{u}|| |\mathbf{f}(x)| dx$$

as $k \rightarrow \infty$. Moreover, equation (6.12) and the boundedness of $\frac{1}{a}$ entail that the sequence

$$\int_{\{|\nabla \mathbf{u}_k| > \tau\}} \frac{1}{a(|\nabla \mathbf{u}_k|)} |\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)|^2 dx$$

is uniformly bounded for $\tau > 0$, and that, for every $\tau > 0$,

$$(6.22) \quad \int_{\{|\nabla \mathbf{u}_k| > \tau\}} \frac{1}{a(|\nabla \mathbf{u}_k|)} |\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)|^2 dx \rightarrow \int_{\{|\nabla \mathbf{u}| > \tau\}} \frac{1}{a(|\nabla \mathbf{u}|)} |\mathbf{f}(x)|^2 dx$$

as $k \rightarrow \infty$. Consequently,

$$(6.23) \quad \int_t^{t+h} \int_{\{|\nabla \mathbf{u}_k| > \tau\}} \frac{1}{a(|\nabla \mathbf{u}_k|)} |\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)|^2 dx d\tau \rightarrow \int_t^{t+h} \int_{\{|\nabla \mathbf{u}| > \tau\}} \frac{1}{a(|\nabla \mathbf{u}|)} |\mathbf{f}(x)|^2 dx d\tau$$

as $k \rightarrow \infty$. Let us finally focus on the last integral on the right-hand side of (6.15). For a.e. $\tau > 0$,

$$(6.24) \quad \partial\Omega \cap \partial\{|\nabla \mathbf{u}_k| > \tau\} = \{\text{Tr} |\nabla \mathbf{u}_k| > \tau\} \quad \text{up to subsets of } \partial\Omega \text{ of } \mathcal{H}^{n-1} \text{ measure zero}$$

for $k \in \mathbb{N}$, and

$$(6.25) \quad \partial\Omega \cap \partial^M\{|\nabla \mathbf{u}| > \tau\} = \{\text{Tr} |\nabla \mathbf{u}| > \tau\} \quad \text{up to subsets of } \partial\Omega \text{ of } \mathcal{H}^{n-1} \text{ measure zero.}$$

Equations (6.24) follow, for instance, from a close inspection of the proof of [Ma5, Lemma 6.5.1/2]. By (6.11), for a.e. $\tau > 0$,

$$(6.26) \quad \chi_{\{\text{Tr} |\nabla \mathbf{u}_k| > \tau\}}(x) |\text{tr} \mathcal{B}(x)| \rightarrow \chi_{\{\text{Tr} |\nabla \mathbf{u}| > \tau\}} |\text{tr} \mathcal{B}(x)| \quad \mathcal{H}^{n-1} \text{ a.e. on } \partial\Omega.$$

Hence, by the dominated convergence theorem for integrals,

$$(6.27) \quad \int_{\partial\Omega} \chi_{\{\text{Tr} |\nabla \mathbf{u}_k| > \tau\}}(x) |\text{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \rightarrow \int_{\partial\Omega} \chi_{\{\text{Tr} |\nabla \mathbf{u}| > \tau\}}(x) |\text{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x),$$

and the first integral in (6.27) is uniformly bounded for $\tau > 0$. Thus,

$$(6.28) \quad \begin{aligned} \int_t^{t+h} a(\tau) \tau^2 \int_{\partial\Omega} \chi_{\{\text{Tr} |\nabla \mathbf{u}_k| > \tau\}}(x) |\text{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x) d\tau \\ \rightarrow \int_t^{t+h} a(\tau) \tau^2 \int_{\partial\Omega} \chi_{\{\text{Tr} |\nabla \mathbf{u}| > \tau\}}(x) |\text{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x) d\tau \end{aligned}$$

as $k \rightarrow \infty$, whence, by (6.24) and (6.25),

$$(6.29) \quad \int_t^{t+h} a(\tau) \tau^2 \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{u}_k| > \tau\}} |\text{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x) d\tau \rightarrow \int_t^{t+h} a(\tau) \tau^2 \int_{\partial\Omega \cap \partial^M\{|\nabla \mathbf{u}| > \tau\}} |\text{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x) d\tau$$

as $k \rightarrow \infty$. Combining (6.17), (6.20), (6.21), (6.23) and (6.29) tells that

$$(6.30) \quad \begin{aligned} C \int_{\{t < |\nabla \mathbf{u}| < t+h\}} b(|\nabla \mathbf{u}|) |\nabla |\nabla \mathbf{u}||^2 dx &\leq \int_{\{t < |\nabla \mathbf{u}| < t+h\}} |\nabla \mathbf{u}| |\nabla |\nabla \mathbf{u}|| |\mathbf{f}(x)| dx \\ &+ \int_t^{t+h} \int_{\{|\nabla \mathbf{u}| > \tau\}} \frac{1}{a(|\nabla \mathbf{u}|)} |\mathbf{f}(x)|^2 dx d\tau \\ &+ \int_t^{t+h} a(\tau) \tau^2 \int_{\partial \Omega \cap \partial^M \{|\nabla \mathbf{u}| > \tau\}} |\text{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x) d\tau. \end{aligned}$$

Dividing through by h in (6.30), making use of the coarea formula again, and passing to the limit as $h \rightarrow 0^+$ yields (6.16).

Step 3. Here, we show that, given $r > n - 1$,

$$(6.31) \quad \|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \leq C b^{-1}(\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)})$$

for some constant C depending on $i_a, s_a, n, N, r, \|\text{tr} \mathcal{B}\|_{L^r(\partial \Omega)}, |\Omega|$, and on the constant in (5.2). More precisely, hereafter dependence on i_a and $|\Omega|$ will mean just through a lower bound, and dependence on $s_a, \|\text{tr} \mathcal{B}\|_{L^r(\partial \Omega)}$, and on the constant in (5.2) is just through an upper bound. By the Hardy-Littlewood inequality (3.3),

$$(6.32) \quad \int_{\partial \Omega \cap \partial^M \{|\nabla \mathbf{u}| > t\}} |\text{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \leq \int_0^{\mathcal{H}^{n-1}(\partial \Omega \cap \partial^M \{|\nabla \mathbf{u}| > t\})} (\text{tr} \mathcal{B})^*(r) dr \quad \text{for a.e. } t > 0,$$

where $(\text{tr} \mathcal{B})^*$ denotes the decreasing rearrangement of $\text{tr} \mathcal{B}$ with respect to the measure \mathcal{H}^{n-1} on $\partial \Omega$. Since $|\nabla \mathbf{u}|$ is a suitably represented Sobolev function, for a.e. $t > 0$,

$$\Omega \cap \partial^M \{|\nabla \mathbf{u}| > t\} = \{|\nabla \mathbf{u}| = t\}, \quad \text{up to sets of } \mathcal{H}^{n-1} \text{ measure zero}$$

(see e.g. [BZ].) Thus,

$$(6.33) \quad \mathcal{H}^{n-1}(\partial \Omega \cap \partial^M \{|\nabla \mathbf{u}| > t\}) \leq C \mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\}) \quad \text{for a.e. } t \geq |\nabla \mathbf{u}|^*(|\Omega|/2),$$

where C is the constant in (5.2). Denote the distribution function $\mu_{|\nabla \mathbf{u}|}$ of $|\nabla \mathbf{u}|$, defined as in (3.1), simply by μ . By (5.1),

$$(6.34) \quad \mu(t)^{1/n'} \leq C \mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\}) \quad \text{for a.e. } t \geq |\nabla \mathbf{u}|^*(|\Omega|/2),$$

where C is a constant depending on n and on the constant in (5.2). From (6.33) and (6.34), we obtain that

$$(6.35) \quad \begin{aligned} \int_0^{\mathcal{H}^{n-1}(\partial \Omega \cap \partial^M \{|\nabla \mathbf{u}| > t\})} (\text{tr} \mathcal{B})^*(r) dr &\leq \int_0^{C \mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\})} (\text{tr} \mathcal{B})^*(r) dr \\ &= C \mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\}) (\text{tr} \mathcal{B})^{**}(C \mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\})) \\ &\leq C \mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\}) (\text{tr} \mathcal{B})^{**}(C' \mu(t)^{1/n'}) \end{aligned}$$

for a.e. $t \geq |\nabla \mathbf{u}|^*(|\Omega|/2)$. Observe that the last inequality holds since $(\text{tr} \mathcal{B})^{**}$ is a non-increasing function. Coupling (6.16) with (6.35) tells us that

$$(6.36) \quad \begin{aligned} C b(t) \int_{\{|\nabla \mathbf{u}| = t\}} |\nabla |\nabla \mathbf{u}|| d\mathcal{H}^{n-1}(x) &\leq t \int_{\{|\nabla \mathbf{u}| = t\}} |\mathbf{f}(x)| d\mathcal{H}^{n-1}(x) + \int_{\{|\nabla \mathbf{u}| > t\}} \frac{1}{a(|\nabla \mathbf{u}|)} |\mathbf{f}(x)|^2 dx \\ &+ a(t) t^2 \mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\}) (\text{tr} \mathcal{B})^{**}(C' \mu(t)^{1/n'}) \quad \text{for a.e. } t > t_{\mathbf{u}}, \end{aligned}$$

for some constants $C = C(\Omega, \min\{i_a, 0\})$ and $C' = C'(\Omega)$.

Now, we distinguish into the cases when a is non-decreasing or non-increasing.

First, assume that a is non-decreasing. Then we infer from (6.36) that

$$(6.37) \quad \begin{aligned} Cb(t) \int_{\{|\nabla \mathbf{u}|=t\}} |\nabla |\nabla \mathbf{u}|| d\mathcal{H}^{n-1}(x) &\leq t \int_{\{|\nabla \mathbf{u}|=t\}} |\mathbf{f}(x)| d\mathcal{H}^{n-1}(x) \\ &+ \frac{1}{a(t)} \int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx \\ &+ a(t)t^2 \mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\}) (\text{tr} \mathcal{B})^{**} (C' \mu(t)^{1/n'}) \text{ for a.e. } t > t_{\mathbf{u}}. \end{aligned}$$

By Hölder's inequality, (5.10) and (5.8),

$$(6.38) \quad \begin{aligned} \int_{\{|\nabla \mathbf{u}|=t\}} |\mathbf{f}(x)| d\mathcal{H}^{n-1}(x) &\leq \left(\int_{\{|\nabla \mathbf{u}|=t\}} \frac{|\mathbf{f}(x)|^2}{|\nabla |\nabla \mathbf{u}||} d\mathcal{H}^{n-1}(x) \right)^{1/2} \left(\int_{\{|\nabla \mathbf{u}|=t\}} |\nabla |\nabla \mathbf{u}|| d\mathcal{H}^{n-1}(x) \right)^{1/2} \\ &\leq \left(-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx \right)^{1/2} \left(-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}|>t\}} |\nabla |\nabla \mathbf{u}||^2 dx \right)^{1/2} \end{aligned}$$

for a.e. $t > 0$. An analogous chain as in (6.38), with $|\mathbf{f}(x)|$ replaced with 1 yields

$$(6.39) \quad \mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\}) \leq (-\mu'(t))^{1/2} \left(-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}|>t\}} |\nabla |\nabla \mathbf{u}||^2 dx \right)^{1/2} \text{ for a.e. } t > 0.$$

By the Hardy-Littlewood inequality (3.3),

$$(6.40) \quad \int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx \leq \int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr \text{ for } t > 0.$$

Inequalities (6.37) – (6.40), and inequality (5.11) applied with $v = |\nabla \mathbf{u}|$ entail that

$$(6.41) \quad \begin{aligned} Cb(t) \left(-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}|>t\}} |\nabla |\nabla \mathbf{u}||^2 dx \right) \\ \leq t \left(-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx \right)^{1/2} \left(-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}|>t\}} |\nabla |\nabla \mathbf{u}||^2 dx \right)^{1/2} \\ + \frac{1}{a(t)} (-\mu'(t))^{1/2} \mu(t)^{-1/n'} \int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr \left(-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}|>t\}} |\nabla |\nabla \mathbf{u}||^2 dx \right)^{1/2} \\ + a(t)t^2 (-\mu'(t))^{1/2} (\text{tr} \mathcal{B})^{**} (C' \mu(t)^{1/n'}) \left(-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}|>t\}} |\nabla |\nabla \mathbf{u}||^2 dx \right)^{1/2} \end{aligned}$$

for a.e. $t > t_{\mathbf{u}}$, for some constants $C = C(\Omega, \min\{i_a, 0\})$ and $C' = C'(\Omega)$. By (5.11) with $v = |\nabla \mathbf{u}|$, we have that $-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}|>t\}} |\nabla |\nabla \mathbf{u}||^2 dx > 0$ for a.e. $t > t_{\mathbf{u}}$. Hence, we may divide through by $-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}|>t\}} |\nabla |\nabla \mathbf{u}||^2 dx$ in (6.41), and exploit (5.11) with $v = |\nabla \mathbf{u}|$ again to obtain

$$(6.42) \quad \begin{aligned} Cb(t) &\leq t (-\mu'(t))^{1/2} \mu(t)^{-1/n'} \left(-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx \right)^{1/2} \\ &+ \frac{1}{a(t)} (-\mu'(t)) \mu(t)^{-2/n'} \int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr \\ &+ a(t)t^2 (-\mu'(t)) \mu(t)^{-1/n'} (\text{tr} \mathcal{B})^{**} (C' \mu(t)^{1/n'}) \text{ for a.e. } t > t_{\mathbf{u}}, \end{aligned}$$

for some constants $C = C(\Omega, \min\{i_a, 0\})$ and $C' = C'(\Omega)$. Since $|\nabla \mathbf{u}|$ is a Sobolev function, the function $|\nabla \mathbf{u}|^*$ is (locally absolutely) continuous [CEG, Lemma 6.6], and $|\nabla \mathbf{u}|^*(\mu(t)) = t$ for $t > 0$. Define the function $\phi_{\mathbf{f}} : (0, |\Omega|) \rightarrow [0, \infty)$ as

$$(6.43) \quad \phi_{\mathbf{f}}(s) = \left(\frac{d}{ds} \int_{\{|\nabla \mathbf{u}| > |\nabla \mathbf{u}|^*(s)\}} |\mathbf{f}(x)|^2 dx \right)^{1/2} \quad \text{for a.e. } s \in (0, |\Omega|).$$

As a consequence,

$$(6.44) \quad \left(-\frac{d}{dt} \int_{\{|\nabla \mathbf{u}| > t\}} |\mathbf{f}(x)|^2 dx \right)^{1/2} = (-\mu'(t))^{1/2} \phi_{\mathbf{f}}(\mu(t)) \quad \text{for a.e. } t > 0,$$

and, by [CM1, Proposition 3.4],

$$(6.45) \quad \int_0^s \phi_{\mathbf{f}}^*(r)^2 dr \leq \int_0^s |\mathbf{f}|^*(r)^2 dr \quad \text{for } s \in (0, |\Omega|).$$

We thus deduce from inequality (6.42) that

$$(6.46) \quad Ca(t)b(t) \leq b(t)(-\mu'(t))\mu(t)^{-1/n'} \phi_{\mathbf{f}}(\mu(t)) \\ + (-\mu'(t))\mu(t)^{-2/n'} \int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr + b(t)^2(-\mu'(t))\mu(t)^{-1/n'} (\text{tr} \mathcal{B})^{**} (C' \mu(t))^{1/n'}$$

for a.e. $t > t_{\mathbf{u}}$. Let $t_{\mathbf{u}} \leq t_0 < T < \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}$, and let H be the function defined by (4.1). On estimating $b(t)$ by $b(T)$ for $t \in (t_0, T)$ on the right-hand side of (6.46), and integrating the resulting inequality over (t_0, T) yields

$$(6.47) \quad CH(T) \leq CH(t_0) + b(T) \int_{t_0}^T (-\mu'(t))\mu(t)^{-1/n'} \phi_{\mathbf{f}}(\mu(t)) dt \\ + \int_{t_0}^T (-\mu'(t))\mu(t)^{-2/n'} \int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr dt + b(T)^2 \int_{t_0}^T (-\mu'(t))\mu(t)^{-1/n'} (\text{tr} \mathcal{B})^{**} (C' \mu(t))^{1/n'} dt \\ \leq CH(t_0) + b(T) \int_{\mu(T)}^{\mu(t_0)} s^{-1/n'} \phi_{\mathbf{f}}(s) ds \\ + \int_{\mu(T)}^{\mu(t_0)} s^{-2/n'} \int_0^s |\mathbf{f}|^*(r)^2 dr ds + b(T)^2 \int_{\mu(T)^{1/n'}}^{\mu(t_0)^{1/n'}} (\text{tr} \mathcal{B})^{**} (C' s) s^{\frac{1}{n-1}} \frac{ds}{s}.$$

Hence, owing to (4.11),

$$(6.48) \quad b(T)^2 \leq Cb(t_0)^2 + Cb(T) \int_0^{\mu(t_0)} s^{-1/n'} \phi_{\mathbf{f}}(s) ds \\ + C \int_0^{\mu(t_0)} s^{-2/n'} \int_0^s |\mathbf{f}|^*(r)^2 dr ds + Cb(T)^2 \int_0^{\mu(t_0)^{1/n'}} (\text{tr} \mathcal{B})^{**} (C' s) s^{\frac{1}{n-1}} \frac{ds}{s}$$

for some constants $C = C(\Omega, i_a, s_a)$ and $C' = C'(\Omega)$. Note that the last integral is actually finite, since $\text{tr} \mathcal{B} \in L^r(\partial\Omega)$, and $L^r(\partial\Omega) \rightarrow L^{n-1,1}(\partial\Omega)$ for $r > n-1$, by (3.7). Define the function $G : [0, \infty) \rightarrow [0, \infty)$ as

$$(6.49) \quad G(s) = C \int_0^{s^{1/n'}} (\text{tr} \mathcal{B})^{**} (C' r) r^{\frac{1}{n-1}} \frac{dr}{r} \quad \text{for } s \geq 0,$$

where C and C' are as in (6.48). Set $s_0 = \min\{\alpha|\Omega|, G^{-1}(\frac{1}{2C})\}$, where α is given by (5.32), with \mathbf{v} replaced with \mathbf{u} , and choose

$$t_0 = |\nabla \mathbf{u}|^*(s_0).$$

One has that $t_0 \geq t_{\mathbf{u}}$, inasmuch as $s_0 \leq \alpha|\Omega|$. Moreover, since $\mu(t_0) \leq G^{-1}(\frac{1}{2C})$,

$$C \int_0^{\mu(t_0)^{1/n'}} (\text{tr} \mathcal{B})^{**} (C'r) r^{\frac{1}{n-1}} \frac{dr}{r} \leq \frac{1}{2}.$$

From (6.48) we thus infer that

$$(6.50) \quad b(T)^2 \leq Cb(t_0)^2 + Cb(T) \int_0^{|\Omega|} s^{-1/n'} \phi_{\mathbf{f}}(s) ds + C \int_0^{|\Omega|} s^{-2/n'} \int_0^s |\mathbf{f}|^*(r)^2 dr ds$$

for some constant $C = C(\Omega, i_a, s_a)$. By (6.45) and [CM1, Lemma 3.5], there exists a constant $C = C(n)$ such that

$$(6.51) \quad \int_0^{|\Omega|} s^{-1/n'} \phi_{\mathbf{f}}(s) ds \leq C \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}.$$

Moreover, by [CM1, Lemma 3.6], there exists a constant $C = C(n)$ such that

$$(6.52) \quad \int_0^{|\Omega|} s^{-2/n'} \int_0^s |\mathbf{f}|^*(r)^2 dr ds \leq C \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}^2.$$

Owing to (6.50)–(6.52), there exists a constant $C = C(\Omega, i_a, s_a)$ such that

$$(6.53) \quad b(T)^2 \leq Cb(t_0)^2 + Cb(T) \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)} + C \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}^2.$$

Thus,

$$(6.54) \quad b(T) \leq Cb(t_0) + C \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}$$

for some constant $C = C(\Omega, i_a, s_a)$. Next, let $\beta, \psi : [0, \infty) \rightarrow [0, \infty)$ be the functions defined by $\beta(t) = b(t)t$ for $t \geq 0$ and $\psi(s) = sb^{-1}(s)$ for $s \geq 0$. Proposition 6.2 and inequality (4.9) ensure that

$$(6.55) \quad C\psi(\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}) \geq \int_{\Omega} \beta(|\nabla \mathbf{u}|) dx \geq \int_{\{|\nabla \mathbf{u}| \geq t_0\}} \beta(|\nabla \mathbf{u}|) dx \geq \beta(t_0) \lim_{t \rightarrow t_0^-} \mu(t) \geq \beta(t_0) s_0,$$

for some constant $C = C(\Omega, i_a, s_a)$, whence, by (4.7),

$$(6.56) \quad \beta(t_0) \leq \psi(C \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}),$$

for some constant $C = C(\Omega, r, i_a, s_a)$. Since $b(\beta^{-1}(\psi(s))) = s$ for $s \geq 0$, inequality (6.56) implies that

$$(6.57) \quad b(t_0) \leq C \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}.$$

Hence, by (6.54),

$$(6.58) \quad b(T) \leq C \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}$$

for some constant $C = C(\Omega, r, i_a, s_a)$. Taking the limit as $T \rightarrow \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}$ in (6.58), and making use of (4.8), yields inequality (6.31). An inspection of the proof shows that the constant in (6.31) actually depends on the specified quantities.

Assume next that a is non-increasing. From (6.36) we deduce that

$$(6.59) \quad Cb(t) \int_{\{|\nabla \mathbf{u}|=t\}} |\nabla |\nabla \mathbf{u}|| d\mathcal{H}^{n-1}(x) \leq t \int_{\{|\nabla \mathbf{u}|=t\}} |\mathbf{f}(x)| d\mathcal{H}^{n-1}(x) + \frac{1}{a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega)})} \int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx \\ + a(t)t^2 \mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\}) (\text{tr} \mathcal{B})^{**} (C' \mu(t)^{1/n'}) \text{ for a.e. } t > t_{\mathbf{u}}.$$

Observe that, although $\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}$ is not yet known to be finite at this stage, the quantity $\frac{1}{a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})}$ is finite, since assumption (6.9) is still in force. On starting from (6.59), instead of (6.37), and arguing as in the proof of (6.46), one can now show that

$$(6.60) \quad Ca(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})b(t) \leq ta(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})(-\mu'(t))\mu(t)^{-1/n'} \phi_{\mathbf{f}}(\mu(t)) \\ + (-\mu'(t))\mu(t)^{-2/n'} \int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr \\ + a(t)t^2 a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})(-\mu'(t))\mu(t)^{-1/n'} (\text{tr} \mathcal{B})^{**} (C' \mu(t)^{1/n'})$$

for a.e. $t > t_{\mathbf{u}}$, for some constants $C = C(\Omega, \min\{i_a, 0\})$ and $C' = C'(\Omega)$. Let us fix t_0 and T as above. For every $t \in (t_0, T)$, the expression $ta(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})$ on the right-hand side of (6.60) can be estimated from above by $Ta(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})$. Also, owing to (4.9), the quantity $a(t)t^2$ can be bounded by $CB(T)$ for some constant $C = C(s_a)$. Integrating the resulting inequality over (t_0, T) tells us that

$$(6.61) \quad B(T)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \leq CB(t_0)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) + CTa(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \int_0^{\mu(t_0)} s^{-1/n'} \phi_{\mathbf{f}}(s) ds \\ + C \int_0^{\mu(t_0)} s^{-2/n'} \int_0^s |\mathbf{f}|^*(r)^2 dr ds + Ca(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})B(T) \int_0^{\mu(t_0)^{1/n'}} (\text{tr} \mathcal{B})^{**} (C' s)^{\frac{1}{s^{n-1}}} \frac{ds}{s}$$

for some constants $C = C(\Omega, i_a, s_a)$ and $C' = C'(\Omega)$. Exploiting inequality (6.61) instead of (6.48), and arguing as in the proof of (6.53) lead to

$$(6.62) \quad B(T)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \\ \leq CB(t_0)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) + CTa(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)} + C\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}^2$$

for some constant $C = C(\Omega, i_a, s_a)$. Dividing through by T , and recalling (4.9) entail that

$$(6.63) \quad b(T)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \\ \leq C \frac{B(t_0)}{T} a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) + Ca(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)} + \frac{C}{T} \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}^2$$

for some constant $C = C(\Omega, i_a, s_a)$. The limit as T goes to $\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}$ of the right-hand side of (6.63) is obviously finite, and hence the limit of the left-hand side is finite as well. Thus,

$\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} < \infty$, since $\lim_{T \rightarrow \infty} b(T) = \infty$. Taking $T = \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}$ in (6.63), and multiplying through by $\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}$ yields

$$(6.64) \quad b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})^2 \leq CB(t_0)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) + Cb(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)} + C\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}^2.$$

Observe that, by (4.9) and (6.57),

$$(6.65) \quad B(t_0)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \leq Ct_0b(t_0)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \leq Cb(t_0)b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \leq C'b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)},$$

for some constants $C = C(s_a)$ and $C' = C'(\Omega, r, i_a, s_a)$. Coupling (6.64) with (6.65) tells us that

$$b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})^2 \leq Cb(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)} + C\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}^2$$

for some constant $C = C(\Omega, r, i_a, s_a)$. Hence,

$$(6.66) \quad b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \leq C\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}$$

for some constant $C = C(\Omega, r, i_a, s_a)$, and (6.31) follows also in this case, with a constant C depending on the specified quantities.

Step 4. The present step exploits some variants of the arguments of Steps 1-3 in order to show that inequality (6.31) holds with a constant C which only depends on $i_a, s_a, n, |\Omega|$, the constant in (5.2), and on $\text{tr} \mathcal{B}$ just through (an upper bound for) the norm $\|\text{tr} \mathcal{B}\|_{L^{n-1,1}(\partial\Omega)}$, instead of a stronger norm $\|\text{tr} \mathcal{B}\|_{L^r(\partial\Omega)}$ with $r > n - 1$. The piece of information that was missing until this stage, and makes this further step possible, is that the solution \mathbf{u} is now already known to satisfy

$$(6.67) \quad \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} < \infty,$$

and hence we can exploit inequality (5.16) in the place of (5.17). By (6.67), $\mathbf{u} \in W_0^{1,\infty}(\Omega, \mathbb{R}^N) \cap W^{2,2}(\Omega, \mathbb{R}^N)$. Hence, there exists a sequence $\{\mathbf{u}_k\} \subset C^\infty(\Omega, \mathbb{R}^N) \cap C^2(\bar{\Omega}, \mathbb{R}^N)$ fulfilling (6.10)–(6.12), such that $\mathbf{u}_k = 0$ on $\partial\Omega$, and, in addition,

$$(6.68) \quad \|\nabla \mathbf{u}_k\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \rightarrow \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \quad \text{as } k \rightarrow \infty.$$

Inequality (5.16) holds with \mathbf{v} replaced with \mathbf{u}_k . An analogous argument as in Step 2 shows that the same inequality continues to hold for \mathbf{u} , that is

$$(6.69) \quad Cb(t) \int_{\{|\nabla \mathbf{u}|=t\}} |\nabla |\nabla \mathbf{u}|| d\mathcal{H}^{n-1}(x) \leq t \int_{\{|\nabla \mathbf{u}|=t\}} |\mathbf{f}(x)| d\mathcal{H}^{n-1}(x) + \frac{\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}}{b(t)} \int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx \\ + a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}^2 \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{u}|>t\}} |\text{tr} \mathcal{B}(x)| d\mathcal{H}^{n-1}(x)$$

for a.e. $t > 0$, where $\frac{(1+\min\{i_a, 0\})^2}{2}$. We now start from (6.69), make use of arguments similar to – and even simpler than – those which lead to either (6.46) or (6.60) from (6.16) (in particular,

now we do not need to distinguish into the cases when a is non-decreasing or non-increasing), and show that

(6.70)

$$\begin{aligned} Cb(t)^2 &\leq b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} (-\mu'(t)) \mu(t)^{-1/n'} \phi_{\mathbf{f}}(\mu(t)) \\ &\quad + \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} (-\mu'(t)) \mu(t)^{-2/n'} \int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr \\ &\quad + b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})^2 \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} (-\mu'(t)) \mu(t)^{-1/n'} (\operatorname{tr} \mathcal{B})^{**} (C' \mu(t)^{1/n'}) \\ &\quad \text{for a.e. } t \in [|\nabla \mathbf{u}|^*(|\Omega|/2), \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}], \end{aligned}$$

for some positive constants $C = C(\Omega, \min\{i_a, 0\})$ and $C' = C'(\Omega)$. Moreover, the dependence on Ω is only through the constant in (5.2).

Let F be the function defined by (4.2). Given $t_1 \in [|\nabla \mathbf{u}|^*(|\Omega|/2), \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}]$, an integration in (6.70) yields, via (4.10),

$$\begin{aligned} (6.71) \quad F(|\nabla \mathbf{u}|^*(s)) &\leq CF(t_1) + Cb(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \int_0^{\mu(t_1)} r^{-1/n'} \phi_{\mathbf{f}}(r) dr \\ &\quad + C \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \int_0^{\mu(t_1)} r^{-2/n'} \int_0^r |\mathbf{f}|^*(\rho)^2 d\rho dr \\ &\quad + CF(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \int_0^{\mu(t_1)^{1/n'}} (\operatorname{tr} \mathcal{B})^{**} (C' r) r^{\frac{1}{n}-1} \frac{dr}{r} \quad \text{for } s \in [0, \mu(t_1)], \end{aligned}$$

for some constants $C = C(\Omega, i_a, s_a)$ and $C' = C'(\Omega)$. Let G be the function defined as in (6.49), save that now C and C' are the constants appearing in (6.71). Set $s_1 = \min\{\frac{|\Omega|}{2}, G^{-1}(\frac{1}{2C})\}$, and choose

$$t_1 = |\nabla \mathbf{u}|^*(s_1).$$

Since $\mu(t_1) \leq G^{-1}(\frac{1}{2C})$,

$$C \int_0^{\mu(t_1)^{1/n'}} (\operatorname{tr} \mathcal{B})^{**} (C' r) r^{\frac{1}{n}-1} \frac{dr}{r} \leq \frac{1}{2}.$$

From this choice of t_1 , and the choice $s = 0$ in (6.71) we infer that

(6.72)

$$\begin{aligned} F(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) &\leq CF(t_1) + Cb(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \int_0^{\mu(t_1)} r^{-1/n'} \phi_{\mathbf{f}}(r) dr \\ &\quad + C \|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \int_0^{\mu(t_1)} r^{-2/n'} \int_0^r |\mathbf{f}|^*(\rho)^2 d\rho dr \end{aligned}$$

for some constant $C = C(\Omega, i_a, s_a)$. From (6.72), via (6.51), (6.52) and (4.10), we deduce that there exists a constant $C = C(\Omega, i_a, s_a)$ such that

$$(6.73) \quad b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})})^2 \leq Cb(t_1)^2 + Cb(\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)} + C \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}^2.$$

An inspection of the proof shows that, in fact, the dependence of C on Ω is only through $|\Omega|$ and on the constant in (5.2). Starting from (6.73) instead of (6.53), and arguing as in Step 3, yield (6.31) with a constant C depending on $i_a, s_a, n, N, |\Omega|, \|\operatorname{tr} \mathcal{B}\|_{L^{n-1,1}(\partial\Omega)}$ and on the constant in (5.2).

Step 5. Here we remove the additional assumption (6.8).

Since the space $C^\infty(U) \cap W^2L^{n-1,1}(U)$ is dense in $W^2L^{n-1,1}(U)$ for every open set $U \subset \mathbb{R}^{n-1}$, there exists a sequence $\{\Omega_m\}_{m \in \mathbb{N}}$ of bounded domains $\Omega_m \supset \Omega$ such that $\partial\Omega_m \in C^\infty$, $|\Omega_m \setminus \Omega| \rightarrow 0$, $\Omega_m \rightarrow \Omega$ with respect to the Hausdorff distance, and $\|\text{tr}\mathcal{B}_m\|_{L^{n-1,1}(\partial\Omega_m)} \leq C$ for some constant $C = C(\Omega)$, where $\text{tr}\mathcal{B}_m$ denotes the trace of the second fundamental form on $\partial\Omega_m$. The sequence $\{\Omega_m\}_{m \in \mathbb{N}}$ can be chosen in such a way that the constant in (5.2), and hence the constants in (5.5) and (5.11), with Ω replaced with Ω_m , are bounded, up to a multiplicative constant independent of m , by the corresponding constants for Ω . This fact depends, in particular, on the embedding $W^2L^{n-1,1}(U) \rightarrow W^{1,\infty}(U)$ for $U \subset \mathbb{R}^{n-1}$, which entails the convergence of the Lipschitz constants of the functions whose graphs locally agree with $\partial\Omega_m$ to the Lipschitz constant of the function whose graph coincides with $\partial\Omega$. Let \mathbf{f} be continued by 0 in $\Omega_m \setminus \Omega$, and let \mathbf{u}_m be the solution to (2.5) with Ω replaced with Ω_m . Owing to the estimates for \mathbf{u}_m in $W_{\text{loc}}^{2,2}(\Omega_m)$ [BF, Theorem 8.1], for every open set Ω' such that $\overline{\Omega'} \subset \Omega$, there exists a constant C such that

$$(6.74) \quad \|\mathbf{u}_m\|_{W^{2,2}(\Omega')} \leq C$$

for $m \in \mathbb{N}$. Furthermore, by estimate (6.31) (in the form established in Step 4) with Ω replaced with Ω_m and \mathbf{u} replaced with \mathbf{u}_m , there exists a constant C such that

$$(6.75) \quad \|\nabla\mathbf{u}_m\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \leq C$$

for $m \in \mathbb{N}$. Note that the constant C in (6.74) and (6.75) is independent of m . Let $s \in [1, \frac{2n}{n-2})$. If $\partial\Omega'$ is smooth, then the embedding $W^{2,2}(\Omega', \mathbb{R}^N) \rightarrow W^{1,s}(\Omega', \mathbb{R}^N)$ is compact. Thus, by (6.74) and (6.75), there exists a function $\mathbf{u} \in W^{1,\infty}(\Omega, \mathbb{R}^N)$, and a subsequence of $\{\mathbf{u}_m\}$, still denoted by $\{\mathbf{u}_m\}$, such that

$$\mathbf{u}_m \rightarrow \mathbf{u} \quad \text{in } W_{\text{loc}}^{1,s}(\Omega, \mathbb{R}^N)$$

and

$$(6.76) \quad \nabla\mathbf{u}_m \rightarrow \nabla\mathbf{u} \quad \text{a.e. in } \Omega.$$

Since $\mathbf{u}_m = 0$ on $\partial\Omega_m$, and $\Omega_m \rightarrow \Omega$ in the Hausdorff distance, one can deduce from (6.75) that $\mathbf{u} = 0$ on $\partial\Omega$. Thus, in particular, $\mathbf{u} \in W_0^{1,B}(\Omega, \mathbb{R}^N)$. The function \mathbf{u} is the weak solution to the Dirichlet problem (2.5). Indeed, inasmuch as $\Omega \subset \Omega_m$ for each $m \in \mathbb{N}$,

$$(6.77) \quad \int_{\Omega_m} a(|\nabla\mathbf{u}_m|) \nabla\mathbf{u}_m \cdot \nabla\phi \, dx = \int_{\Omega_m} \mathbf{f} \cdot \phi \, dx$$

for every $\phi \in C_0^\infty(\Omega, \mathbb{R}^N)$. Passing to the limit as $m \rightarrow \infty$ in (6.77) yields

$$(6.78) \quad \int_{\Omega} a(|\nabla\mathbf{u}|) \nabla\mathbf{u} \cdot \nabla\phi \, dx = \int_{\Omega} \mathbf{f} \cdot \phi \, dx$$

for every $\phi \in C_0^\infty(\Omega, \mathbb{R}^N)$, owing to (6.76) and (6.75), via the dominated convergence theorem for integrals. Since $B \in \Delta_2$, the space $C_0^\infty(\Omega, \mathbb{R}^N)$ is dense in $W_0^{1,B}(\Omega, \mathbb{R}^N)$. Thus, (6.78) holds for every $\phi \in W_0^{1,B}(\Omega, \mathbb{R}^N)$ as well. Note that, by (6.76), the solution u satisfies

$$(6.79) \quad \|\nabla\mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \leq Cb^{-1}(\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}),$$

since such an estimate is fulfilled, with \mathbf{u} replaced with \mathbf{u}_m , by (6.31). Here, the constant C depends on i_a , s_a , n , N , $|\Omega|$, $\|\text{tr}\mathcal{B}\|_{L^{n-1,1}(\partial\Omega)}$ and on the constant in (5.2).

Step 6 We conclude by removing assumption (6.9).

Let $\{a_\varepsilon\}_{\varepsilon \in (0,1)}$ be the family of functions defined in Lemma 4.5, and let b_ε and B_ε be as in its statement. Let \mathbf{u} be the weak solution in $W_0^{1,B}(\Omega, \mathbb{R}^N)$ to problem (2.5), and let \mathbf{u}_ε denote the solution in $W_0^{1,B_\varepsilon}(\Omega, \mathbb{R}^N)$ to the problem

$$(6.80) \quad \begin{cases} -\operatorname{div}(a_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon) = \mathbf{f}(x) & \text{in } \Omega, \\ \mathbf{u}_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

We claim that

$$(6.81) \quad \nabla \mathbf{u}_\varepsilon \rightarrow \nabla \mathbf{u} \quad \text{in measure}$$

as $\varepsilon \rightarrow 0^+$, and hence there exists a sequence $\varepsilon_k \rightarrow 0$ such that

$$(6.82) \quad \nabla \mathbf{u}_{\varepsilon_k} \rightarrow \nabla \mathbf{u} \quad \text{a.e. in } \Omega$$

as $k \rightarrow \infty$. By the previous steps, there exists a constant $C = C(i_a, s_a, \Omega)$ (in particular, independent of ε , owing to (4.30)), such that

$$(6.83) \quad b_\varepsilon(C\|\nabla \mathbf{u}_\varepsilon\|_{L^\infty(\Omega, \mathbb{R}^{Nn})}) \leq \|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}.$$

Thus, by the definition of b_ε and by (4.32), it is easily seen that

$$(6.84) \quad \|\nabla \mathbf{u}_\varepsilon\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \leq C$$

for some constant C independent of ε , whence

$$(6.85) \quad \int_{\Omega} B(|\nabla \mathbf{u}_\varepsilon|) dx \leq C$$

for some constant C independent of ε .

We preliminarily observe that, although the function \mathbf{u} need not belong to $W_0^{1,B_\varepsilon}(\Omega, \mathbb{R}^N)$ in the case when a is non-increasing, it can still be used as a test function in the weak formulation of problem (6.80). Indeed, by (6.84), $a_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon \in L^\infty(\Omega, \mathbb{R}^N)$. Therefore, since $\mathbf{u} \in W_0^{1,B}(\Omega, \mathbb{R}^N)$, and the latter space is embedded into $W_0^{1,1}(\Omega, \mathbb{R}^N)$, the function \mathbf{u} can be approximated by a sequence $\{\mathbf{u}_k\} \in C_0^\infty(\Omega)$ of functions such that $\mathbf{u}_k \rightarrow \mathbf{u}$ in $L^{n'}(\Omega, \mathbb{R}^N)$, and hence in $L^{n',\infty}(\Omega, \mathbb{R}^N)$, and $\nabla \mathbf{u}_k \rightarrow \nabla \mathbf{u}$ in $L^1(\Omega, \mathbb{R}^N)$. This allows one to employ \mathbf{u}_k as a test function in the weak formulation of problem (6.80), and then pass to the limit as $k \rightarrow \infty$.

The test function $\phi = \mathbf{u} - \mathbf{u}_\varepsilon$ can thus be used both in the weak formulation of problem (2.5), and in that of problem (6.80). Subtracting the resulting equations yields

$$(6.86) \quad \begin{aligned} \int_{\Omega} [a_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon - a(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon] \cdot (\nabla \mathbf{u} - \nabla \mathbf{u}_\varepsilon) dx \\ = \int_{\Omega} [a(|\nabla \mathbf{u}|)\nabla \mathbf{u} - a(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon] \cdot (\nabla \mathbf{u} - \nabla \mathbf{u}_\varepsilon) dx. \end{aligned}$$

Fix any $\sigma \in (0, 1)$. By the definition of Young conjugate,

$$(6.87) \quad \begin{aligned} \left| \int_{\Omega} [a_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon - a(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon] \cdot (\nabla \mathbf{u} - \nabla \mathbf{u}_\varepsilon) dx \right| &\leq \int_{\Omega} |a_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon - a(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon| |\nabla \mathbf{u} - \nabla \mathbf{u}_\varepsilon| dx \\ &\leq \int_{\Omega} \tilde{B}\left(\frac{1}{\sigma}|a_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon - a(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon\right) dx + \int_{\Omega} B(\sigma|\nabla \mathbf{u} - \nabla \mathbf{u}_\varepsilon|) dx. \end{aligned}$$

Since B is a Young function of class Δ_2 and $\sigma \in (0, 1)$, there exists a constant $C = C(B)$ such that $B(\sigma(t+s)) \leq C\sigma(B(t) + B(s))$ for $t, s \geq 0$. Hence, owing to (6.85), there exist positive constants C and C' , independent of ε , such that

$$(6.88) \quad \int_{\Omega} B(\sigma|\nabla\mathbf{u} - \nabla\mathbf{u}_{\varepsilon}|) dx \leq C\sigma \left(\int_{\Omega} B(|\nabla\mathbf{u}_{\varepsilon}|) dx + \int_{\Omega} B(|\nabla\mathbf{u}|) dx \right) \leq C'\sigma.$$

Next, fix any $\delta > 0$. Let C be the constant appearing in (6.84), and let $t > C$. We have that

$$(6.89) \quad \begin{aligned} & \int_{\Omega} \tilde{B}(\sigma|a_{\varepsilon}(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon} - a(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon}|) dx \\ &= \int_{\{|\nabla\mathbf{u}_{\varepsilon}| \leq t\}} \tilde{B}(\sigma|a_{\varepsilon}(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon} - a(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon}|) dx \\ &+ \int_{\{|\nabla\mathbf{u}_{\varepsilon}| > t\}} \tilde{B}(\sigma|a_{\varepsilon}(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon} - a(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon}|) dx. \end{aligned}$$

By the choice of t , the last integral vanishes for every $\varepsilon \in (0, 1)$. On the other hand, by (4.31),

$$(6.90) \quad \int_{\{|\nabla\mathbf{u}_{\varepsilon}| \leq t\}} \tilde{B}(\sigma|a_{\varepsilon}(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon} - a(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon}|) dx < \delta,$$

if ε is sufficiently small. Thanks to the arbitrariness of σ and δ , we infer from (6.87)–(6.90) that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} [a_{\varepsilon}(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon} - a(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon}] \cdot (\nabla\mathbf{u} - \nabla\mathbf{u}_{\varepsilon}) dx = 0,$$

and hence, by (6.86),

$$(6.91) \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} [a(|\nabla\mathbf{u}|)\nabla\mathbf{u} - a(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon}] \cdot (\nabla\mathbf{u} - \nabla\mathbf{u}_{\varepsilon}) dx = 0.$$

We now follow an argument from [BBGGPV]. Fix any $\delta > 0$. Given $t, \tau > 0$, we have that

$$(6.92) \quad \begin{aligned} & |\{|\nabla\mathbf{u} - \nabla\mathbf{u}_{\varepsilon}| > t\}| \\ & \leq |\{|\nabla\mathbf{u}_{\varepsilon}| > \tau\}| + |\{|\nabla\mathbf{u}| > \tau\}| + |\{|\nabla\mathbf{u} - \nabla\mathbf{u}_{\varepsilon}| > t, |\nabla\mathbf{u}_{\varepsilon}| \leq \tau, |\nabla\mathbf{u}| \leq \tau\}|. \end{aligned}$$

If τ is sufficiently large, then

$$(6.93) \quad |\{|\nabla\mathbf{u}| > \tau\}| < \delta,$$

and, by (6.84),

$$(6.94) \quad |\{|\nabla\mathbf{u}_{\varepsilon}| > \tau\}| = 0 \quad \text{for } \varepsilon \in (0, 1).$$

Next, define

$$\vartheta(t, \tau) = \inf\{[a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) : |\xi - \eta| \geq t, |\xi| \leq \tau, |\eta| \leq \tau\},$$

and observe that $\vartheta(t, \tau) > 0$, by Lemma 4.4. Thus, since

$$\begin{aligned} & \int_{\Omega} [a(|\nabla\mathbf{u}|)\nabla\mathbf{u} - a(|\nabla\mathbf{u}_{\varepsilon}|)\nabla\mathbf{u}_{\varepsilon}] \cdot (\nabla\mathbf{u} - \nabla\mathbf{u}_{\varepsilon}) dx \\ & \geq \vartheta(t, \tau) |\{|\nabla\mathbf{u} - \nabla\mathbf{u}_{\varepsilon}| > t, |\nabla\mathbf{u}_{\varepsilon}| \leq \tau, |\nabla\mathbf{u}| \leq \tau\}|, \end{aligned}$$

by (6.91)

$$|\{|\nabla \mathbf{u} - \nabla \mathbf{u}_\varepsilon| > t, |\nabla \mathbf{u}_\varepsilon| \leq \tau, |\nabla \mathbf{u}| \leq \tau\}| < \delta$$

if ε is sufficiently small. Consequently, by (6.92), (6.94), and (6.93),

$$|\{|\nabla \mathbf{u} - \nabla \mathbf{u}_\varepsilon| > t\}| < 2\delta$$

if ε is sufficiently small. This proves (6.81). Inequality (2.6) follows from (6.82) and (6.83).

□

Remark 6.3 We take this opportunity to correct an inaccuracy in an approximation argument, parallel to that of Step 5 above, in the proof of Theorem 1.1 of our earlier paper [CM1], which only applies under somewhat stronger regularity assumptions on $\partial\Omega$. It suffices to replace that argument by the one exploited in the present paper.

Proof of Theorem 2.2. The proof is the same as that of Theorem 2.1, save that the simplified versions of inequalities (5.16) and (5.17) without the term depending on $\text{tr}\mathcal{B}$, described in the last part of Lemma 5.4, have to be exploited in Steps 2 and 4, respectively. Also, a sequence of bounded (smooth) convex approximating domains Ω_m has to be employed in Step 5. □

Proof of Theorem 2.4. The outline of the proof is analogous to that of Theorem 2.1 for the Dirichlet problem. Hereafter, we point out some minor required variants.

Step 1. Under assumptions (6.8)–(6.9), the solution \mathbf{u} to the Neumann problem (2.8) belongs to $W^{2,2}(\Omega, \mathbb{R}^N)$. A proof of this fact parallels that of [BF, Theorem 8.2] in the case of homogeneous Dirichlet boundary conditions. One can make use of this piece of information to conclude that $a(|\nabla \mathbf{u}|)\nabla \mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^N)$, and hence derive from an integration by parts in (6.2) that the boundary condition $\frac{\partial \mathbf{u}}{\partial \nu} = 0$ is fulfilled in the sense of traces.

As a consequence, a sequence $\{\mathbf{u}_k\} \subset C^\infty(\Omega, \mathbb{R}^N) \cap C^2(\overline{\Omega}, \mathbb{R}^N)$ can be constructed such that

$$(6.95) \quad \mathbf{u}_k \rightarrow \mathbf{u} \quad \text{in } W^{2,2}(\Omega, \mathbb{R}^N), \quad \nabla \mathbf{u}_k \rightarrow \nabla \mathbf{u} \quad \text{a.e. in } \Omega, \quad \text{and} \quad \frac{\partial \mathbf{u}_k}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Such construction can be accomplished as follows. First, one can (locally) reduce the problem in some neighborhood of each point $x_0 \in \partial\Omega$ to the case when $\partial\Omega$ is flat via a change of variables. In order to preserve the boundary condition $\frac{\partial \mathbf{u}}{\partial \nu} = 0$, the new system of (curvilinear) orthogonal coordinates (y_1, \dots, y_n) can be chosen in such a way that the level surfaces $\{y_n = c_n\}$, with $c_n \in \mathbb{R}$, agree with the level surfaces of the distance function to $\partial\Omega$, and the curves $\{y_1 = c_1, \dots, y_{n-1} = c_{n-1}\}$, with $c_1 \in \mathbb{R}, \dots, c_{n-1} \in \mathbb{R}$, are orthogonal to these level surfaces. Second, the function \mathbf{u} can be extended to a function $\tilde{\mathbf{u}}$ beyond the flattened boundary of Ω by reflection, so that $\tilde{\mathbf{u}}$ is symmetric with respect to the boundary. The function $\tilde{\mathbf{u}}$ is now defined in a complete neighborhood U of x_0 . The fact that $\frac{\partial \mathbf{u}}{\partial \nu} = 0$ on the flattened boundary ensures that such an extension is twice weakly differentiable, and hence belongs to $W^{2,2}(U)$. Standard mollification of $\tilde{\mathbf{u}}$ by a symmetric kernel provides an approximation of $\tilde{\mathbf{u}}$ in $W^{2,2}(U)$ by a sequence of smooth functions $\tilde{\mathbf{u}}_k$ which satisfy

$$\tilde{\mathbf{u}}_k \rightarrow \mathbf{u} \quad \text{in } W^{2,2}(U, \mathbb{R}^N), \quad \nabla \tilde{\mathbf{u}}_k \rightarrow \nabla \mathbf{u} \quad \text{a.e. in } U,$$

and are symmetric about the boundary of Ω . The latter property ensures that $\frac{\partial \tilde{\mathbf{u}}_k}{\partial \nu} = 0$ on the boundary of Ω . The function \mathbf{u}_k is then just defined as the restriction of $\tilde{\mathbf{u}}_k$ to Ω .

Via the same argument as in the case of the Dirichlet problem, one can prove that the sequence \mathbf{u}_k just obtained also fulfills (6.11) and (6.12).

Step 2. Here one shows that inequality (5.35) is fulfilled when \mathbf{v} equals the solution \mathbf{u} to (2.8). This follows on applying (5.35) of Lemma 5.5 with $\mathbf{v} = \mathbf{u}_k$ (defined in Step 1), and passing to the limit as $k \rightarrow \infty$ via the same argument as in the Dirichlet case.

Step 3. This step is exactly the same as in the Dirichlet case, save that $|\mathcal{B}|$ replaces $\text{tr}\mathcal{B}$ everywhere.

Step 4. Here one applies inequality (5.34) with $\mathbf{v} = \mathbf{u}_k$, and obtains the same inequality for \mathbf{u} on passing to the limit as $k \rightarrow \infty$ as in the case of the Dirichlet problem.

Step 5. We construct the sequence of bounded domains Ω_m as in the case of Dirichlet problems, and obtain a corresponding sequence $\{\mathbf{u}_m\}$ of solutions to the Neumann problems in Ω_m satisfying (6.74), (6.75), and (6.77) for every function $\phi \in \text{Lip}(\mathbb{R}^n, \mathbb{R}^N)$. Thanks to (6.75), passing to the limit as $m \rightarrow \infty$ yields (6.78) for every $\phi \in \text{Lip}(\mathbb{R}^n, \mathbb{R}^N)$. Since Ω has a Lipschitz boundary, the space of the restrictions to Ω of the functions from $\text{Lip}(\mathbb{R}^n, \mathbb{R}^N)$ is dense in $W^{1,B}(\Omega, \mathbb{R}^N)$. Hence, (6.78) also holds for every $\phi \in W^{1,B}(\Omega, \mathbb{R}^N)$.

Step 6. This step is the same as in the case of the Dirichlet problem. □

Proof of Theorem 2.5. The proof consists in a slight modification of that of Theorem 2.4. Specifically, the versions of inequalities (5.34) and (5.35) where the term depending on $|\mathcal{B}|$ is dropped, described in the last part of Lemma 5.5, play a role in Steps 2 and 4, respectively. Moreover, the approximating domains Ω_m in Step 5 have to be chosen convex. □

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References

- [AF] E.Acerbi & N.Fusco, Regularity for minimizers of nonquadratic functionals: the case $1 < p < 2$, *J. Math. Anal. Appl.* **140** (1989), 115–135.
- [BC] H.Beirão da Veiga & F.Crispo, On the global $W^{2,q}$ regularity for nonlinear N -systems of the p -Laplacian type in n space variables, *Nonlinear Anal.* **75** (2012), 4346–4354.
- [BBGGPV] P.Bénilan, L.Boccardo, T.Gallouët, R.Gariepy, M.Pierre & J.L.Vazquez, An L^1 -theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Ann. Sc. Norm. Sup. Pisa* **22** (1995), 241–273.
- [BF] A.Bensoussan & J.Frehse, ‘‘Regularity results for nonlinear elliptic systems and applications’’, Springer-Verlag, Berlin, 2002.

- [BS] C.Bennett & R.Sharpley, “Interpolation of operators”, Academic Press, Boston, 1988.
- [BSV] D.Breit, B.Stroffolini & A.Verde, A general regularity theorem for functionals with φ -growth, *J. Math. Anal. Appl.*, **383** (2011), 226–233.
- [BZ] J.E.Brothers and W.P.Ziemer, Minimal rearrangements of Sobolev functions, *J. Reine Angew. Math* **384** (1988), 153–179.
- [CDiB] Y.Z.Chen & E.Di Benedetto, Boundary estimates for solutions of nonlinear degenerate parabolic systems, *J. Reine Angew. Math.* **395** (1989), 102–131
- [Ci2] A.Cianchi, Maximizing the L^∞ norm of the gradient of solutions to the Poisson equation, *J. Geom. Anal.* **2** (1992), 499–515.
- [Ci3] A.Cianchi, A sharp embedding theorem for Orlicz-Sobolev spaces, *Indiana Univ. Math. J.* **45** (1996), 39–65.
- [Ci4] A.Cianchi, Boundedness of solutions to variational problems under general growth conditions, *Comm. Part. Diff. Equat.* **22** (1997), 1629–1646.
- [CEG] A.Cianchi, D.E.Edmunds & P.Gurka, On weighted Poincaré inequalities, *Math. Nachr.* **180** (1996), 15–41.
- [CM1] A.Cianchi & V.Maz’ya, Global Lipschitz regularity for a class of quasilinear elliptic equations, *Comm. Part. Diff. Equat.* **36** (2011), 100–133.
- [CM2] A.Cianchi & V.Maz’ya, Gradient regularity via rearrangements for p -Laplacian type elliptic problems, *J. Europ. Math. Soc.*, to appear.
- [CP] A.Cianchi & L.Pick, Sobolev embeddings into BMO, VMO and L^∞ , *Ark. Math.* **36** (1998), 317–340.
- [Di] E.Di Benedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.* **7** (1983), 827–850.
- [DT] D.T.Donaldson & N.S.Trudinger, Orlicz-Sobolev spaces and embedding theorems, *J. Funct. Anal.* **8** (1971), 52–75.
- [DeG] E.De Giorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, *Boll. Un. Mat. Ital.* **1** (1968), 135–137.
- [DSV] L.Diening, B.Stroffolini & A.Verde, Everywhere regularity of functionals with ϕ -growth, *Manus. Math.* **129** (2009), 449–481.
- [DM1] F.Duzaar & G.Mingione, Gradient estimates via non-linear potentials, *Amer. J. Math.* **133** (2011), 1093–1149.
- [DM2] F.Duzaar & G.Mingione, Local Lipschitz regularity for degenerate elliptic systems, *Ann. Inst. Henri Poincaré* **27** (2010), 1361–1396.
- [DM3] F.Duzaar & G.Mingione, Gradient continuity estimates, *Calc. Var. Part. Diff. Equat.* **39** (2010), 379–418.
- [Ev] L.C.Evans, A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic P.D.E., *J. Diff. Eq.* **45** (1982), 356–373.

- [Fo] M.Foss, Global regularity for almost minimizers of nonconvex variational problems, *Ann. Mat. Pura Appl.* **187** (2008), 263–321.
- [FPV] M.Foss, A.Passarelli di Napoli & A.Verde, Global Lipschitz regularity for almost minimizers of asymptotically convex variational problems, *Ann. Mat. Pura Appl.* **189** (2010), 127–162.
- [FM] M.Fuchs & G.Mingione, Full $C^{1,\alpha}$ -regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth, *Manus. Math.* **102** (2000), 227–250.
- [Gia] M.Giaquinta, “Multiple integrals in the calculus of variations and nonlinear elliptic systems”, Annals of Mathematical Studies, Princeton University Press, Princeton, NJ, 1983.
- [GiaMo] M.Giaquinta & G.Modica, Almost-everywhere regularity for solutions of nonlinear elliptic systems, *Manuscr. Math.* **28** (1979), 109–158.
- [Gi] E.Giusti, “Direct methods in the calculus of variations”, World Scientific, River Edge, NJ, 2003.
- [GM] E.Giusti & M.Miranda, Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni, *Boll. Un. Mat. Ital.* **1** (1968), 219–226.
- [Gr] P.Grisvard, “Elliptic problems in nonsmooth domains”, Pitman, Boston, MA, 1985.
- [Ha] C.Hamburger, Regularity of differential forms minimizing degenerate elliptic functionals, *J. Reine Angew. Math.* **431** (1992), 7–64.
- [HKW] S.Hildebrandt, H.Kaul & K.-O.Widman, An existence theorem for harmonic mappings of Riemannian manifolds, *Acta Math.* **138** (1977), 1–16.
- [Iv] P.-A.Ivert, Regularitätsuntersuchungen von Lösungen elliptischer Systeme von quasilinearen Differentialgleichungen zweiter Ordnung, *Manuscr. Math.* **30** (1979), 53–88.
- [KrM1] J.Kristensen & G.Mingione, The singular set of minima of integral functionals, *Arch. Ration. Mech. Anal.* **180** (2006), 331–398.
- [KrM2] J.Kristensen & G.Mingione, Boundary regularity in variational problems, *Arch. Ration. Mech. Anal.* **198** (2010), 369–455.
- [KuM] T.Kuusi & G.Mingione, Linear potentials in nonlinear potential theory, *Arch. Ration. Mech. Anal.*, to appear.
- [LU1] O.A.Ladyzenskaya & N.N.Ural’ceva, Quasilinear elliptic equations and variational problems with many independent variables, *Usp. Mat. Nauk.* **16** (1961), 19–92 (Russian); English translation: *Russian Math. Surveys* **16** (1961), 17–91.
- [LU2] O.A.Ladyzenskaya & N.N.Ural’ceva, “Linear and quasilinear elliptic equations”, Academic Press, New York, 1968.
- [Le] J.L.Lewis, Regularity of the derivatives of solutions to certain degenerate elliptic equations, *Indiana Univ. Math. J.* **32** (1983), 849–858.
- [Li1] G.M.Lieberman, The Dirichlet problem for quasilinear elliptic equations with continuously differentiable data, *Comm. Part. Diff. Eq.* **11** (1986), 167–229.

- [Mar] P.Marcellini, Everywhere regularity for a class of elliptic systems without growth conditions, *Annali Scuola Norm. Sup. Pisa* **23** (1996), 1–25.
- [MM] M.Marcus & V.J.Mizel, *Absolute continuity of tracks and mappings of Sobolev spaces*, Arch. Ration. Mech. Anal. **45** (1972), 294–320.
- [Ma1] V.G.Maz'ya, Examples of nonregular solutions of quasilinear elliptic equations with analytic coefficients, *Funkc. Anal. Priložen.* **2** (1968), 53–57 (Russian); English translation: *Funct. Anal. Appl.* **2** (1968), 230–234.
- [Ma2] V.G.Maz'ya, The boundedness of the first derivatives of the solution of the Dirichlet problem in a region with smooth nonregular boundary, *Vestnik Leningrad. Univ.* **24** (1969), 72–79 (Russian); English translation: *Vestnik Leningrad. Univ. Math.* **2** (1975), 59–67.
- [Ma3] V.G.Maz'ya, On weak solutions of the Dirichlet and Neumann problems, *Trudy Moskov. Mat. Obsc.* **20** (1969), 137–172 (Russian); English translation: *Trans. Moscow Math. Soc.* **20** (1969), 135–172.
- [Ma4] V.G.Maz'ya, On the boundedness of the first derivatives for solutions to the Neumann-Laplace problem in a convex domain, *J. Math. Sci. (N.Y.)* **159** (2009), 104–112.
- [Ma5] V.G.Maz'ya, “Sobolev spaces with applications to elliptic partial differential equations”, Springer, Heidelberg, 2011.
- [MS] G.Mingione & F.Siepe, Full $C^{1,\alpha}$ -regularity for minimizers of integral functionals with LlogL-growth, *Z. Anal. Anw.* **18** (1999), 1083–1100
- [Mi1] G.Mingione, The singular set of solutions to non-differentiable elliptic systems, *Arch. Ration. Mech. Anal.* **166** (2003), 287–301.
- [Mi2] G.Mingione, Bounds for the singular set of solutions to non linear elliptic systems, *Calc. Var. Part. Diff. Equat.* **18** (2003), 373–400.
- [Ne] J.Necas, Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity, in Theor. Nonlin. Oper., Constr. Aspects. Proc. 4th Int. Summer School. Akademie-Verlag, Berlin, 1975, 197–206.
- [SY] V.Sverák & X.Yan, Non-Lipschitz minimizers of smooth uniformly convex variational integrals, *Proc. Natl. Acad. Sci. USA* **99** (2002), 15269–15276.
- [To] P.Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Diff. Equat.* **51** (1983), 126–150.
- [Uhl] K.Uhlenbeck, Regularity for a class of non-linear elliptic systems, *Acta Math.* **138** (1977), 219–240.
- [Ur] N.N.Ural'ceva, Degenerate quasilinear elliptic systems, *Zap. Nauch. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **7** (1968), 184–222 (Russian).
- [Zi] W.P.Ziemer, “Weakly differentiable functions”, Springer, Berlin, 1989.