Global boundedness of the gradient for a class of nonlinear elliptic systems

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Abstract

Gradient boundedness up to the boundary for solutions to Dirichlet and Neumann problems for elliptic systems with Uhlenbeck type structure is established. Nonlinearities of possibly non-polynomial type are allowed, and minimal regularity on the data and on the boundary of the domain is assumed. The case of arbitrary bounded convex domains is also included.

1 Introduction

We are concerned with second-order nonlinear elliptic systems of the form

$$-\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u}) = \mathbf{f}(x) \quad \text{in } \Omega\,, \tag{1.1}$$

coupled with either the Dirichlet condition $\mathbf{u} = 0$, or the Neumann condition $\frac{\partial \mathbf{u}}{\partial \nu} = 0$ on $\partial\Omega$. Here, Ω is a domain, namely an open bounded connected set in $\mathbb{R}^n$, $n \geq 2$, $\mathbf{u} : \Omega \to \mathbb{R}^N$, $N \geq 1$, is a vector-valued unknown function, $\nabla \mathbf{u} : \Omega \to \mathbb{R}^{Nn}$ denotes its gradient, $\mathbf{f} : \Omega \to \mathbb{R}^N$ is a datum, $\mathbf{div}$ stands for the $\mathbb{R}^N$-valued divergence operator, and ν for the outward unit normal to $\partial\Omega$.

The boundedness of the gradient, or, equivalently, the Lipschitz continuity, of the solutions to the relevant boundary-value problems is established in the whole of Ω. Quite general nonlinearities of the differential operator, non-necessarily of power type, are allowed, and essentially weakest possible integrability conditions on $\mathbf{f}$, and minimal regularity assumptions on $\partial\Omega$ are imposed. In the case when Ω is convex, no regularity on $\partial\Omega$ is assumed at all. The boundary value problems to be considered are the Euler equation of variational problems for strictly convex integral functionals depending on the gradient only through its modulus, and hence the solutions to the former agree with the minimizers of the latter. In particular, our results on convex domains provide an extension to the vectorial case ($N > 1$) of the so called semi-classical

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Hilbert-Haar theory of minimization of strictly convex scalar integral functionals of the modulus of gradient on convex domains in classes of Lipschitz functions (see e.g. [Gi, Chapter 1]).

Nonlinear elliptic systems involving differential operators as in (1.1), namely whose coefficient only depends on the modulus of the gradient, are sometimes referred to as systems with Uhlenbeck structure in the literature. Indeed, the theory of regularity of their solutions, and of minimizers of associated variational problems, can be traced back to the celebrated paper [Uhl]. Results from [Uhl] imply that, if, for instance, $a(t) = t^{p-2}$ for some $p \geq 2$, a choice which turns (1.1) into the p-Laplacian system, then any local weak solution $\mathbf{u}$ to (1.1) satisfies $\nabla\mathbf{u} \in L^{\infty}_{\rm loc}(\Omega, \mathbb{R}^{Nn})$, and, in addition, $\nabla\mathbf{u} \in C^{\alpha}_{\rm loc}(\Omega, \mathbb{R}^{Nn})$ for some $\alpha \in (0,1)$. The regularity of solutions to the p-Laplacian equation in the scalar case ($N = 1$) had instead earlier been proved in [Ur]. The result of [Uhl] was subsequently extended to the situation when $1 < p < 2$ in [AF, CDiB]. Further generalization to elliptic systems with non polynomial growth are the subject of [BSV, DSV, MS, Mar]. Precise pointwise gradient estimates, via nonlinear and linear potentials, for local solutions to nonlinear elliptic equations, and to systems with Uhlenbeck structure, are the subject of the recent papers [DM1, DM2, KuM].

Let us recall that, in striking contrast with the scalar case [Di, Ev, Le, To], solutions to nonlinear elliptic systems with a more general structure than that of (1.1) can be irregular. Examples in this connection are produced in [SY], where the existence of elliptic systems, with smooth coefficients depending only on the gradient, but endowed with unbounded solutions, is established. Earlier contributions on irregular solutions to elliptic systems are rooted in the paper[DeG], and include [GM] and [Ne]. Related examples, in the scalar case, of linear and nonlinear higher-order elliptic equations with irregular solutions were independently exhibited in [Ma1].

Solutions to elliptic systems with general structure are however well known to enjoy partial regularity properties, in the sense that they are locally regular in some open subset of Ω whose complement has zero Lebesgue measure. This is the subject of a rich literature, starting with the contributions [HKW, GiaMo, Iv] – see the monographs [BF, Gia, Gi] for a comprehensive treatment of this topic. Recent improvements of these results in terms of Hausdorff measures are established in [Mi1, Mi2, KrM1].

The study of global regularity, namely up to the boundary, in boundary value problems for nonlinear elliptic systems has a more recent history. Global gradient boundedness, and Hölder regularity, for the p-Laplacian elliptic system, under homogeneous Dirichlet boundary conditions, were obtained in [CDiB], as a corollary of analogous results for the associated parabolic problem. Right-hand sides which are bounded in x, and domains whose boundary is of class $C^{1,\alpha}$ are considered in that paper. Further contributions on gradient regularity up to the boundary for systems and variational problems with Uhlenbeck structure, or perturbations of it, are [BC, Fo, FPV]. Partial boundary regularity, namely regularity at the boundary outside subsets of zero $(n-1)$-dimensional Hausdorff measure, for nonlinear elliptic systems with general structure, is proved in [JM, DGK, KrM2].

The techniques employed in the literature mentioned above, for both inner and boundary regularity, have a local character. In particular, the proofs of global results entail distinguishing between points inside the domain, and boundary points, and reducing the treatment of the latter to the former. Such reduction requires the use of suitable local coordinates in which the boundary of the domain is flat, and the structure of the differential operator is still close enough to the Uhlenbeck type to ensure everywhere inner regularity. Techniques for inner regularity apply after a reflection argument, which allows to extend the solution beyond the flattened boundary.

A novelty in the approach of the present paper is of being global in nature. An underlying

idea in our proof of the global boundedness of the gradient consists in integrating the system (1.1), just after multiplication by $\Delta \mathbf{u}$, over the level sets of $|\nabla \mathbf{u}|$. In particular, no localization via cut-off functions is employed. Besides yielding optimal results in terms of regularity the datum $\mathbf{f}$ and of the domain Ω, an approach of this kind enables us to deal not only with Dirichlet, but also with Neumann boundary conditions, for which results seem to be missing in the literature. The proof of the global boundedness of the gradient in arbitrary convex domains, without any additional regularity assumption on the boundary, also relies upon the fact that no local change of coordinates near the boundary is required.
Let us finally mention that integration on the level sets of partial derivatives was used in [Ma2, Ma4] to show gradient regularity for linear equations, and in [CM1] for nonlinear equations. The method exploited in those papers rests on regularity results for approximating problems which are not available in the vectorial case. Here, we follow an alternative outline, which provides a more self-contained proof also in the scalar case.

2 Main results

Our assumptions on the system (1.1) amount to what follows. The function $a : (0,\infty) \to (0,\infty)$ is required to be monotone (either non-decreasing or non-increasing), of class $C^1(0,\infty)$, and to fulfil

$$-1 < i_a \leq s_a < \infty, \tag{2.1}$$

where

$$i_a = \inf_{t>0} \frac{ta'(t)}{a(t)} \qquad \text{and} \qquad s_a = \sup_{t>0} \frac{ta'(t)}{a(t)}. \tag{2.2}$$

In particular, the standard p-Laplace operator for vector-valued functions, corresponding to the choice $a(t) = t^{p-2}$, with $p > 1$, falls within this framework, since $i_a = s_a = p-2$ in this case. Thanks to the first inequality in (2.2), the function $b : [0,\infty) \to [0,\infty)$ defined as

$$b(t) = a(t)t \qquad \text{if } t > 0, \text{ and } \quad b(0) = 0, \tag{2.3}$$

turns out to be strictly increasing, and hence the function $B : [0,\infty) \to [0,\infty)$, given by

$$B(t) = \int_0^t b(\tau)\, d\tau \qquad \text{for } t \geq 0, \tag{2.4}$$

is strictly convex. The Orlicz-Sobolev space $W^{1,B}(\Omega, \mathbb{R}^N)$ associated with the function B, or its subspace $W^{1,B}_0(\Omega, \mathbb{R}^N)$ of those functions vanishing in the suitable sense on $\partial\Omega$, are appropriate functional settings where to define weak solutions to the boundary value problems associated with the system (1.1) – see Sections 3 and 6 for precise definitions of function spaces and weak solutions, respectively.

The right-hand side $\mathbf{f}$ is assumed to belong to the Lorentz space $L^{n,1}(\Omega, \mathbb{R}^N)$. This space is borderline, in a sense, for the family of Lebesgue spaces $L^q(\Omega, \mathbb{R}^N)$ with $q > n$, since $L^q(\Omega, \mathbb{R}^N) \subsetneq L^{n,1}(\Omega, \mathbb{R}^N) \subsetneq L^n(\Omega, \mathbb{R}^N)$ for every $q > n$. Let us mention that membership of the right-hand side to the same Lorentz space has also been shown to ensure the local boundedness of the gradient of local solutions to equations, and to systems with Uhlenbeck structure, in [DM2], and also its continuity [DM3].

The regularity of $\partial\Omega$ is prescribed in terms of a Lorentz space as well. We impose that $\partial\Omega \in W^2L^{n-1,1}$. This means that Ω is locally the subgraph of a function of $n-1$ variables whose second-order distributional derivatives belong to the Lorentz space $L^{n-1,1}$. This is the weakest possible integrability assumption on the second-order derivatives of such a function for its first-order derivatives to be continuous, and hence for $\partial\Omega \in C^{1,0}$. Note that, by contrast, the available regularity results at the boundary require $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0,1]$.

Let us emphasize that both the assumption on $\mathbf{f}$ and that on Ω cannot be essentially relaxed for our conclusions to hold – see Remarks 2.8 and 2.9 at the end of this section.

Our result for the Dirichlet problem

$$\begin{cases} -\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u}) = \mathbf{f}(x) & \text{in } \Omega\,, \\ \mathbf{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

reads as follows.

Theorem 2.1 *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in W^2L^{n-1,1}$. Assume that $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$. Let $\mathbf{u}$ be the (unique) weak solution to the Dirichlet problem (2.5). Then there exists a constant C $C = C(i_a, s_a, \Omega)$ such that*

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \leq C b^{-1}\big(\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}\big). \tag{2.6}$$

In particular, $\mathbf{u}$ is Lipschitz continuous in Ω.

An interesting variant of Theorem 2.1 amounts to replacing the regularity assumption on $\partial\Omega$ with the convexity of Ω. This is stated in the next result.

Theorem 2.2 *The same conclusion as in Theorem 2.1 holds if Ω is any convex domain in $\mathbb{R}^n$, $n \geq 3$.*

Problem (2.5) is the Euler equation of the strictly convex functional

$$J(u) = \int_\Omega \big(B(|\nabla \mathbf{u}|) - \mathbf{f}\cdot \mathbf{u}\big)\, dx\,, \tag{2.7}$$

which is well defined for $\mathbf{u} \in W^{1,B}_0(\Omega, \mathbb{R}^N)$ under our assumption on $\mathbf{f}$ (see the discussion at the beginning of Section 6). The interpretation of Theorem 2.2 as an existence result for minimizers of the functional J in the space $\mathrm{Lip}_0(\Omega, \mathbb{R}^N)$ of $\mathbb{R}^N$-valued Lipschitz continuous functions in Ω vanishing on $\partial\Omega$, to which we alluded in Section 1, is the content of the following corollary.

Corollary 2.3 *Let Ω be any convex domain in $\mathbb{R}^n$, $n \geq 3$, and let B be defined as in (2.4). Assume that $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$. Then the functional J admits a (unique) minimizer in the space $\mathrm{Lip}_0(\Omega, \mathbb{R}^N)$.*

Results parallel to Theorems 2.1–2.2 and Corollary 2.3 hold for the solutions to the Neumann problem

$$\begin{cases} -\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u}) = \mathbf{f}(x) & \text{in } \Omega\,, \\ \dfrac{\partial \mathbf{u}}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.8}$$

Clearly, here, $\mathbf{f}$ has to fulfil the compatibility condition

$$\int_\Omega \mathbf{f}(x)\, dx = 0. \tag{2.9}$$

Theorem 2.4 *Let Ω and $\mathbf{f}$ be as in Theorem 2.1. Assume, in addition, that* (2.9) *holds. Let $\mathbf{u}$ be the (unique up to additive constant vectors) weak solution to problem* (2.8). *Then there exists a constant $C = C(i_a, s_a, \Omega)$ such that*

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \leq C b^{-1}\big(\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}\big). \tag{2.10}$$

In particular, $\mathbf{u}$ is Lipschitz continuous in Ω.

A counterpart of Theorem 2.4 for convex domains is contained in the next result.

Theorem 2.5 *The same conclusion as in Theorem 2.4 holds if Ω is any convex domain in $\mathbb{R}^n$, $n \geq 3$.*

The minimization problem for the functional J, whose Euler equation is (2.8), is properly set in the subspace $W^{1,B}_\perp(\Omega,\mathbb{R}^N)$ of functions in $W^{1,B}(\Omega,\mathbb{R}^N)$ with vanishing mean-value. An analogue of Corollary 2.3 can thus be formulated in terms of the space $\mathrm{Lip}_\perp(\Omega,\mathbb{R}^N)$ of $\mathbb{R}^N$-valued Lipschitz continuous functions in Ω with vanishing mean-value

Corollary 2.6 *Let Ω be any convex domain in $\mathbb{R}^n$, $n \geq 3$, and let B be defined as in* (2.4). *Assume that $\mathbf{f} \in L^{n,1}(\Omega,\mathbb{R}^N)$. Then the functional J admits a (unique) minimizer in the class $\mathrm{Lip}_\perp(\Omega,\mathbb{R}^N)$.*

Remark 2.7 Versions of the above results can be established via our approach also in the case when $n = 2$, under the slightly stronger assumption that $\mathbf{f} \in L^q(\Omega,\mathbb{R}^N)$ for some $q > n$. This will be clear from a close inspection of the proofs.

Remark 2.8 The sharpness of assumption $\mathbf{f} \in L^{n,1}(\Omega,\mathbb{R}^N)$ for the boundedness of the gradient of the solution to the Dirichlet problem follows, in the linear case corresponding to the choice $a = 1$, from a result of [Ci2] dealing with the scalar Laplace equation in a ball.

Remark 2.9 The assumption $\partial\Omega \in W^2L^{n-1,1}$ is optimal for the boundedness of the gradient, as long as the regularity of Ω is prescribed in terms of integrability properties of its curvature. This can be demonstrated, again even just for scalar problems, by ad hoc examples of Dirichlet and Neumann problems for the p-Laplace equation in domains with conical singularities – see e.g. [CM2]. The examples in question also show that the conclusion of Theorems 2.2 and 2.5 may fails under slight perturbations of convex domains.

3 Function spaces

3.1 Spaces of measurable functions and rearrangements

Let $(\mathcal{R}, m)$ be a positive, finite, non-atomic measure space. The decreasing rearrangement $v^* : [0,\infty) \to [0,\infty]$ of be a real-valued m-measurable function v on $\mathcal{R}$ is the unique right-continuous non-increasing function in $[0,\infty)$ equidistributed with v. Namely, on defining the distribution function $\mu_v : [0,\infty) \to [0,\infty)$ of v as

$$\mu_v(t) = m(\{x \in \mathcal{R} : |v(x)| > t\}) \qquad \text{for } t \geq 0, \tag{3.1}$$

we have that

$$v^*(s) = \sup\{t \geq 0 : \mu_v(t) > s\} \qquad \text{for } s \in [0,\infty). \tag{3.2}$$

Clearly, $v^*(s) = 0$ if $s \geq m(\mathcal{R})$.
The function $v^{**} : (0, \infty) \to [0, \infty)$, defined by

$$v^{**}(s) = \frac{1}{s} \int_0^s v^*(r)\, dr \qquad \text{for } s > 0,$$

is also nondecreasing, and such that $v^*(s) \leq v^{**}(s)$ for $s > 0$.
The Hardy-Littlewood inequality is a basic property of rearrangements, and asserts that

$$\int_{\mathcal{R}} |v(x)w(x)| dm(x) \leq \int_0^\infty v^*(s)w^*(s)ds \tag{3.3}$$

for all measurable functions v and w on $\mathcal{R}$.

Roughly speaking, a rearrangement-invariant space is a Banach function space whose norm only depends on the rearrangement of functions – see e.g. [BS, Chapter 2] for precise definitions. Besides the Lebesgue spaces, their generalizations provided by the Lorentz and the Orlicz spaces are classical instances of rearrangement-invariant spaces which will play a role in our discussion. Given $q \in (1, \infty)$ and $\sigma \in [1, \infty]$, the Lorentz space $L^{q,\sigma}(\mathcal{R})$ is the set of all real-valued measurable functions v on $\mathcal{R}$ for which the quantity

$$\|v\|_{L^{q,\sigma}(\mathcal{R})} = \|s^{\frac{1}{q} - \frac{1}{\sigma}} v^*(s)\|_{L^\sigma(0,m(\mathcal{R}))} \tag{3.4}$$

is finite. One has that $\sigma \in [1, \infty]$, then $L^{q,\sigma}(\mathcal{R})$ is a Banach space, equipped with the norm, equivalent to $\| \cdot \|_{L^{q,\sigma}(\mathcal{R})}$, obtained on replacing v^* with v^{**} on the right-hand side of (3.4). Furthermore,

$$L^{q,q}(\mathcal{R}) = L^q(\mathcal{R}) \qquad \text{for } q \in (1, \infty), \tag{3.5}$$

$$L^{q,\sigma_1}(\mathcal{R}) \to L^{q,\sigma_2}(\mathcal{R}) \qquad \text{if } \sigma_1 < \sigma_2, \tag{3.6}$$

and

$$L^{q_1,\sigma_1}(\mathcal{R}) \to L^{q_2,\sigma_2}(\mathcal{R}) \qquad \text{if } q_1 > q_2 \text{ and } \sigma_1, \sigma_2 \in [1, \infty]. \tag{3.7}$$

Here, and in what follows, the arrow " $\to$ " stands for continuous embedding. Let us notice that the norm of the embedding (3.6) depends on q, σ_1, σ_2 and $m(\mathcal{R})$, and the norm of the embedding (3.7) depends on q_1, q_2, σ_1, σ_2 and $m(\mathcal{R})$. A Hölder type inequality in Lorentz spaces tells us that, if q' and σ' denote the usual Hölder conjugate exponents to q and σ, then there exists a constant $C = C(q, \sigma)$ such that

$$\int_{\mathcal{R}} |v(x)w(x)|\, dm(x) \leq C\|v\|_{L^{q,\sigma}(\mathcal{R})}\|w\|_{L^{q',\sigma'}(\mathcal{R})} \tag{3.8}$$

for every $v \in L^{q,\sigma}(\mathcal{R})$ and $w \in L^{q',\sigma'}(\mathcal{R})$.
Given $N > 1$, the Lorentz space $L^{q,\sigma}(\mathcal{R}, \mathbb{R}^N)$ of $\mathbb{R}^N$-valued functions on $\mathcal{R}$ can be defined as

$$L^{q,\sigma}(\mathcal{R}, \mathbb{R}^N) = (L^{q,\sigma}(\mathcal{R}))^N.$$

Obviously, the norm $\|\mathbf{v}\|_{L^{q,\sigma}(\mathcal{R},\mathbb{R}^N)}$ is equivalent to $\| \, |\mathbf{v}| \, \|_{L^{q,\sigma}(\mathcal{R})}$.

The Orlicz spaces extend the Lebesgue spaces in the sense that the role of powers in the definition of the norms is instead played by Young functions. A Young function $B : [0, \infty) \to [0, \infty]$ is a convex function such that $B(0) = 0$. If, in addition, $0 < B(t) < \infty$ for $t > 0$ and

$$\lim_{t\to 0} \frac{B(t)}{t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \frac{B(t)}{t} = \infty,$$

then B is called an N-function. The Young conjugate of a Young function B is the Young function $\widetilde{B}$ defined as

$$\widetilde{B}(t) = \sup\{st - B(s) : s \geq 0\} \qquad \text{for } t \geq 0.$$

In particular, if B is an N-function, then $\widetilde{B}$ is an N-function as well. Moreover, if B is given by (2.4), then

$$\widetilde{B}(t) = \int_0^t b^{-1}(s)\, ds \qquad \text{for } t \geq 0. \tag{3.9}$$

Notice that

$$s \leq B^{-1}(s)\widetilde{B}^{-1}(s) \leq 2s \qquad \text{for } s \geq 0. \tag{3.10}$$

A Young function (and, more generally, an increasing function) B is said to belong to the class Δ_2 if there exists a constant $C > 1$ such that

$$B(2t) \leq CB(t) \qquad \text{for } t > 0.$$

If $B \in \Delta_2$, then there exist constants $\kappa > 1$ and $C > 0$ such that

$$B(\lambda t) \leq C\lambda^\kappa B(t) \quad \text{for } t \geq 0 \text{ and } \lambda \geq 1. \tag{3.11}$$

The Orlicz space $L^B(\mathcal{R})$ is the Banach function space of those real-valued measurable functions v on $\mathcal{R}$ whose Luxemburg norm

$$\|v\|_{L^B(\mathcal{R})} = \inf\left\{\lambda > 0 : \int_{\mathcal{R}} B\left(\frac{|v(x)|}{\lambda}\right) dm(x) \leq 1\right\}$$

is finite. The Hölder type inequality

$$\int_{\mathcal{R}} |v(x)w(x)|\, dm(x) \leq 2\|v\|_{L^B(\mathcal{R})}\|w\|_{L^{\widetilde{B}}(\mathcal{R})} \tag{3.12}$$

holds for every $v \in L^B(\mathcal{R})$ and $w \in L^{\widetilde{B}}(\mathcal{R})$. Let B_1 and B_2 be Young functions. Then

$$L^{B_1}(\mathcal{R}) \to L^{B_2}(\mathcal{R}) \text{ if and only if there exist } c, t_0 > 0 \text{ such that } B_2(t) \leq B_1(ct) \text{ for } t > t_0. \tag{3.13}$$

The Orlicz space $L^B(\mathcal{R}, \mathbb{R}^N)$, with $N > 1$, of $\mathbb{R}^N$-valued functions on $\mathcal{R}$ is given by

$$L^B(\mathcal{R}, \mathbb{R}^N) = (L^B(\mathcal{R}))^N.$$

The norm $\|\mathbf{v}\|_{L^B(\mathcal{R},\mathbb{R}^N)}$ is equivalent to $\|\,|\mathbf{v}|\,\|_{L^B(\mathcal{R})}$.

3.2 Spaces of Sobolev type

Let Ω be a domain in $\mathbb{R}^n$, with $n \geq 2$, and let $m \in \mathbb{N}$. Sobolev type spaces of m-th order weakly differentiable functions in Ω, built upon Lorentz and Orlicz spaces, are defined as follows.
Given $q \in (1, \infty)$ and $\sigma \in [1, \infty]$, the Lorentz-Sobolev space $W^m L^{q,\sigma}(\Omega)$ is the Banach

$$W^m L^{q,\sigma}(\Omega) = \{u \in L^{q,\sigma}(\Omega) : \text{is } m\text{-times weakly differentiable in } \Omega \\ \text{and } |\nabla^k u| \in L^{q,\sigma}(\Omega) \text{ for } 1 \leq k \leq m\},$$

and is equipped with the norm $\|u\|_{W^m L^{q,\sigma}(\Omega)} = \sum_{k=0}^m \|\nabla^k u\|_{L^{q,\sigma}(\Omega)}$. Here, $\nabla^k u$ denotes the vector of all weak derivatives of u of order k. By $\nabla^0 u$ we mean u. Moreover, when $k = 1$ we simply write ∇u instead of $\nabla^1 u$.
If $\sigma < \infty$, the space $C^\infty(\Omega)$ is dense in $W^m L^{q,\sigma}(\Omega)$. This fact follows via an easy variant of a standard argument for classical Sobolev spaces, and makes use the density of $C_0^\infty(\Omega)$ in $L^{q,\sigma}(\Omega)$, and of a version of Young convolution inequality in Lorentz spaces due to O'Neil [Zi, Theorem 2.10.1].
A limiting case of the Sobolev embedding theorem asserts that if Ω has a Lipschitz boundary, then $W^1 L^{n,1}(\Omega) \to C^0(\Omega)$; moreover, $L^{n,1}(\Omega)$ is optimal, in the sense that it is the largest rearrangement invariant space enjoying this property [CP]. Hence, in particular,

$$W^2 L^{n,1}(\Omega) \to C^{1,0}(\Omega), \tag{3.14}$$

and $L^{n,1}(\Omega)$ is optimal in the same sense as above.
The Lorentz-Sobolev space $W^m L^{q,\sigma}(\Omega, \mathbb{R}^N)$ of $\mathbb{R}^N$-valued functions in Ω is defined as

$$W^m L^{q,\sigma}(\Omega, \mathbb{R}^N) = \big(W^m L^{q,\sigma}(\Omega)\big)^N.$$

The Orlicz-Sobolev space $W^{m,B}(\Omega)$ is the Banach space

$$W^{m,B}(\Omega) = \{u \in L^B(\Omega) : \text{is } m\text{-times weakly differentiable in } \Omega \\ \text{and } |\nabla^k u| \in L^B(\Omega) \text{ for } 1 \leq k \leq m\},$$

and is equipped with the norm $\|u\|_{W^{m,B}(\Omega)} = \sum_{k=0}^m \|\nabla^k u\|_{L^B(\Omega)}$. In what follows, we shall only make use of first-order Orlicz-Sobolev spaces $W^{1,B}(\Omega)$. By $W_0^{1,B}(\Omega)$ and $W_\perp^{1,B}(\Omega)$ we denote the subspaces of $W^{1,B}(\Omega)$ given by

$$W_0^{1,B}(\Omega) = \{u \in W^{1,B}(\Omega) : \text{the continuation of } u \text{ by 0 outside } \Omega \text{ is weakly differentiable in } \mathbb{R}^n\},$$

and

$$W_\perp^{1,B}(\Omega) = \left\{u \in W^{1,B}(\Omega) : \int_\Omega u(x)\, dx = 0\right\}.$$

A theorem of [DT] ensures that, if $B \in \Delta_2$, then the space $C_0^\infty(\Omega)$ is dense in $W_0^{1,B}(\Omega)$, and that, if Ω is a Lipschitz domain, then $C^\infty(\overline{\Omega})$ is dense in $W^{1,B}(\Omega)$.
Let B be a Young function such that

$$\int_0 \left(\frac{t}{B(t)}\right)^{\frac{1}{n-1}} dt < \infty. \tag{3.15}$$

The Sobolev conjugate of B, introduced in [Ci4] (and, in an equivalent form, in [Ci3]), is the Young function B_n defined as

$$B_n(t) = B\big(H_n^{-1}(t)\big) \qquad \text{for } t \geq 0, \tag{3.16}$$

where

$$(3.17) \qquad H_n(s) = \left(\int_0^s \left(\frac{t}{B(t)} \right)^{\frac{1}{n-1}} dt \right)^{1/n'} \qquad \text{for } s \geq 0,$$

and H_n^{-1} denotes the (generalized) left-continuous inverse of H_n.
An embedding theorem for Orlicz-Sobolev spaces [Ci3, Ci4] tells us that, if B fulfils (3.15), then there exists a constant $C = C(n, |\Omega|)$ such that

$$(3.18) \qquad \|u\|_{L^{B_n}(\Omega)} \leq C \|\nabla u\|_{L^B(\Omega)}$$

for every $u \in W_0^{1,B}(\Omega)$. Moreover, if has a Lipschitz boundary, then inequality (3.18) holds for every $u \in W_\perp^{1,B}(\Omega)$. The space $L^{B_n}(\Omega)$ is optimal in (3.18) among all Orlicz spaces. Note that, in fact, assumption (3.15) is immaterial in (3.18), since, owing to (3.13), the Young function B can be replaced, if necessary, with another Young function fulfilling (3.15) in such a way that $W^{1,B}(\Omega)$ remains unchanged, up to equivalent norms.
If B is a Young function, then

$$(3.19) \qquad L^\infty(\Omega) \to L^{B_n}(\Omega) \to L^{n'}(\Omega).$$

The first embedding in (3.19) is trivial. As for the second one, since B is a Young function, there exist constants c and $t_0 > 0$ such that $t \leq B(ct)$ if $t > t_0$. As a consequence, there exist constants k and t_1 such that $t^{n'} \leq B_n(kt)$ for some $t > t_1$. Hence, the second embedding in (3.19) follows via (3.13).
Let us also observe that, if B grows so fast near infinity that

$$(3.20) \qquad \int^\infty \left(\frac{t}{B(t)} \right)^{\frac{1}{n-1}} dt < \infty,$$

then equality holds in the first embedding in (3.19). Indeed, under (3.20), $H_n^{-1}(t) = \infty$ for large t, and hence $B_n(t) = \infty$ for large t as well. Thus, by (3.13), $L^{B_n}(\Omega) = L^\infty(\Omega)$, up to equivalent norms.
The Orlicz-Sobolev space $W^{m,B}(\Omega, \mathbb{R}^N)$ is defined as

$$W^{m,B}(\Omega, \mathbb{R}^N) = \big(W^{m,B}(\Omega)\big)^N.$$

4 The function a

This section is devoted to the proof of some properties of the function a appearing in (1.1). Hereafter, b and B denote the functions associated with a as in (2.3) and (2.4). Furthermore, we define the functions $\widehat{B}(t) : [0, \infty) \to [0, \infty)$, as

$$(4.1) \qquad \widehat{B}(t) = \frac{B(t)}{t} \quad \text{for } t > 0, \quad \widehat{B}(0) = 0,$$

$H : [0, \infty) \to [0, \infty)$, as

$$(4.2) \qquad H(t) = \int_0^t a(\tau) b(\tau)\, d\tau \qquad \text{for } t \geq 0,$$

and $F : [0, \infty) \to [0, \infty)$, as

$$(4.3) \qquad F(t) = \int_0^t b(\tau)^2\, d\tau \qquad \text{for } t \geq 0.$$

Proposition 4.1 *Assume that the function $a : (0,\infty) \to (0,\infty)$ is of class C^1 and fulfils* (2.1)*. Let b and B be the functions defined by* (2.3) *and* (2.4)*, respectively, and let $\widehat{B}$, H, and F be defined as above. Then:*
(i)

$$a(1)\min\{t^{i_a}, t^{s_a}\} \le a(t) \le a(1)\max\{t^{i_a}, t^{s_a}\} \quad \textit{for } t > 0. \tag{4.4}$$

(ii) b is increasing, and

$$\lim_{t\to 0} b(t) = 0, \quad \textit{and} \quad \lim_{t\to\infty} b(t) = \infty. \tag{4.5}$$

(iii) B is a strictly convex N-function, and

$$B \in \Delta_2 \qquad \textit{and} \qquad \widetilde{B} \in \Delta_2. \tag{4.6}$$

(iv) There exists a constant $C = C(i_a, s_a)$ such that

$$\widetilde{B}(b(t)) \le CB(t) \qquad \textit{for } t \ge 0. \tag{4.7}$$

(v) For every $C > 0$, there exists a positive constant $C_1 = C_1(s_a, C) > 0$ such that

$$Cb^{-1}(s) \le b^{-1}(C_1 s) \qquad \textit{for } s > 0, \tag{4.8}$$

and a positive constant $C_2 = C_2(i_a, C) > 0$ such that

$$b^{-1}(Cs) \le C_2 b^{-1}(s) \qquad \textit{for } s > 0. \tag{4.9}$$

(vi) There exists a positive constant $C = C(s_a)$ such that

$$B(t) \le tb(t) \le CB(t) \qquad \textit{for } t \ge 0. \tag{4.10}$$

(vii) There exists a positive constant $C = C(i_a, s_a)$ such that

$$\widehat{B}^{-1}(s) \le Cb^{-1}(s) \qquad \textit{for } s > 0. \tag{4.11}$$

(viii) There exists a positive constant $C = C(i_a, s_a)$ such that

$$F(t) \le tb(t)^2 \le CF(t) \qquad \textit{for } t \ge 0. \tag{4.12}$$

(ix) There exist positive constants $C_1 = C_1(i_a)$ and $C_2 = C_2(i_a, s_a)$such that

$$C_1 H(t) \le b(t)^2 \le C_2 H(t) \qquad \textit{for } t \ge 0. \tag{4.13}$$

Proof. Assertions (ii)–(viii) are proved in [CM1, Propositions 2.9 and 2.15]. Property (i) can be shown on distinguishing the case when $t \in (0,1)$ and $t \in [1,\infty)$, and integrating the inequality

$$\frac{i_a}{\tau} \le \frac{a'(\tau)}{a(\tau)} \le \frac{s_a}{\tau} \quad \text{for } \tau > 0,$$

on $(t,1)$ and on $(1,t)$, respectively.
As far as (ix) is concerned, since $b'(t) = a(t) + ta'(t)$ for $t > 0$, one has that

$$\frac{b'(t)}{1+s_a} \le a(t) \le \frac{b'(t)}{1+\min\{i_a,0\}} \quad \text{for } t > 0, \tag{4.14}$$

and hence

$$\frac{b(t)}{1+s_a} \le \int_0^t a(s)\,ds \le \frac{b(t)}{1+\min\{i_a,0\}} \quad \text{for } t>0. \tag{4.15}$$

Thus, inasmuch as b is increasing,

$$H(t) \le b(t)\int_0^t a(s)\,ds \le \frac{b(t)^2}{1+\min\{i_a,0\}} \quad \text{for } t>0,$$

whence the first inequality in (4.13) follows. On the other hand, integration by parts and inequalities (4.14) and (4.15) yield

$$\begin{aligned} H(t) &= \int_0^t a(s)b(s)\,ds = b(t)\int_0^t a(s)\,ds - \int_0^t b'(s)\int_0^s a(\tau)\,d\tau\,ds \\ &\ge \frac{b(t)^2}{1+s_a} - \int_0^t (1+s_a)a(s)\frac{b(s)}{1+\min\{i_a,0\}}ds = \frac{b(t)^2}{1+s_a} - \frac{1+s_a}{1+\min\{i_a,0\}}H(t) \quad \text{for } t>0. \end{aligned}$$

This implies the second inequality in (4.13). □

Lemma 4.2 *Let a be as in Lemma 4.1. Then*

$$(1+\min\{i_a,0\})a(|\xi|)|\eta|^2 \le \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\frac{\partial(a(|\xi|)\xi_i^\alpha)}{\partial\xi_j^\beta}\eta_i^\alpha\eta_j^\beta \le (1+\max\{s_a,0\})a(|\xi|)|\eta|^2 \tag{4.16}$$

for $\xi,\eta\in\mathbb{R}^{Nn}$. Moreover,

$$[a(|\xi|)\xi - a(|\eta|)\eta]\cdot(\xi-\eta) \ge (1+\min\{i_a,0\})|\xi-\eta|^2\int_0^1 a(|\eta+s(\xi-\eta)|)ds \tag{4.17}$$

for $\xi,\eta\in\mathbb{R}^{Nn}$.

Proof. We have that

$$\frac{\partial(a(|\xi|)\xi_i^\alpha)}{\partial\xi_j^\beta} = \frac{a'(|\xi|)}{|\xi|}\xi_i^\alpha\xi_j^\beta + a(|\xi|)\delta_{ij}\delta_{\alpha\beta} \quad \text{for } \xi\in\mathbb{R}^{Nn}. \tag{4.18}$$

Thus

$$\begin{aligned} \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\frac{\partial(a(|\xi|)\xi_i^\alpha)}{\partial\xi_j^\beta}\eta_i^\alpha\eta_j^\beta &= \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\frac{a'(|\xi|)}{|\xi|}\xi_i^\alpha\xi_j^\beta\eta_i^\alpha\eta_j^\beta + a(|\xi|)\sum_{\alpha=1}^{N}\sum_{i=1}^{n}(\eta_i^\alpha)^2 \\ &= \frac{a'(|\xi|)}{|\xi|}(\xi\cdot\eta)^2 + a(|\xi|)|\eta|^2 \quad \text{for } \xi,\eta\in\mathbb{R}^{Nn}. \end{aligned} \tag{4.19}$$

Since $|\xi\cdot\eta| \le |\xi||\eta|$, equation (4.16) follows via (2.1).
Next, set $A_{ij}^{\alpha\beta}(\zeta) = \frac{\partial(a(|\zeta|)\zeta_i^\alpha)}{\partial\zeta_j^\beta}$. Given i and α, we have that

$$[a(|\xi|)\xi_i^\alpha - a(|\eta|)\eta_i^\alpha] = \sum_{\beta=1}^{N}\sum_{j=1}^{n}(\xi_j^\beta-\eta_j^\beta)\int_0^1 A_{ij}^{\alpha\beta}(\eta+s(\xi-\eta))\,ds.$$

Hence,

$$[a(|\xi|)\xi - a(|\eta|)\eta]\cdot(\xi-\eta) = \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}(\xi_i^\alpha-\eta_i^\alpha)(\xi_j^\beta-\eta_j^\beta)\int_0^1 A_{ij}^{\alpha\beta}(\eta+s(\xi-\eta))\,ds \quad \text{for } \xi,\eta\in\mathbb{R}^{Nn}.$$

Thus, inequality (4.17) is a consequence (4.16). □

Lemma 4.3 *Let a be as in Lemma 4.1. Assume in addition that a is non-decreasing. Then*

$$[a(|\xi|)\xi - a(|\eta|)\eta]\cdot(\xi-\eta) \geq \frac{1}{3}[a(|\xi|)+a(|\eta|)]|\xi-\eta|^2 \quad \textit{for } \xi,\eta\in\mathbb{R}^{Nn}. \tag{4.20}$$

Proof. Since inequality (4.20) is invariant under replacement of ξ and η by each other, may assume, without loss of generality, that $|\xi| \geq |\eta|$, and hence $a(|\xi|) \geq a(|\eta|)$. Consider first the case when $a(|\xi|) \leq 2a(|\eta|)$. Then, given any $\xi \neq \eta$,

(4.21)
$$\begin{aligned}
\frac{(a(|\xi|)\xi - a(|\eta|)\eta)\cdot(\xi-\eta)}{(a(|\xi|)+a(|\eta|))|\xi-\eta|^2} &= \frac{(\frac{a(|\xi|)}{a(|\eta|)}\xi-\eta)\cdot(\xi-\eta)}{(\frac{a(|\xi|)}{a(|\eta|)}+1)|\xi-\eta|^2} \geq \frac{(\frac{a(|\xi|)}{a(|\eta|)}\xi-\eta)\cdot(\xi-\eta)}{3|\xi-\eta|^2}\\
&= \frac{|\xi-\eta|^2+(\frac{a(|\xi|)}{a(|\eta|)}-1)\xi\cdot(\xi-\eta)}{3|\xi-\eta|^2} = \frac{1}{3}+\frac{(\frac{a(|\xi|)}{a(|\eta|)}-1)(|\xi|^2-\xi\cdot\eta)}{3|\xi-\eta|^2}\\
&\geq \frac{1}{3}+\frac{(\frac{a(|\xi|)}{a(|\eta|)}-1)(|\xi|^2-|\xi||\eta|)}{3|\xi-\eta|^2} \geq \frac{1}{3}.
\end{aligned}$$

Assume now that $a(|\xi|) \geq 2a(|\eta|)$. Then, given any $\xi \neq \eta$,

(4.22)
$$\begin{aligned}
\frac{(a(|\xi|)\xi - a(|\eta|)\eta)\cdot(\xi-\eta)}{(a(|\xi|)+a(|\eta|))|\xi-\eta|^2} &= \frac{(\frac{a(|\xi|)}{a(|\eta|)}\xi-\eta)\cdot(\xi-\eta)}{(\frac{a(|\xi|)}{a(|\eta|)}+1)|\xi-\eta|^2} \geq \inf_{s\geq 2}\frac{(s\xi-\eta)\cdot(\xi-\eta)}{(s+1)|\xi-\eta|^2} = \frac{(2\xi-\eta)\cdot(\xi-\eta)}{3|\xi-\eta|^2}\\
&= \frac{2|\xi|^2-3\xi\cdot\eta+|\eta|^2}{3|\xi-\eta|^2} = \frac{|\xi-\eta|^2+|\xi|^2-\xi\cdot\eta}{3|\xi-\eta|^2}\\
&\geq \frac{1}{3}+\frac{|\xi|^2-|\xi||\eta|}{3|\xi-\eta|^2} \geq \frac{1}{3}.
\end{aligned}$$

Inequality (4.20) is fully proved. □

Lemma 4.4 *Let a be as in Lemma 4.1. Assume, in addition, that a is monotone (either non-decreasing or non-increasing). Then, for every $t,\tau>0$, there exists a positive constant $\vartheta = \vartheta(a,t,\tau)$ such that*

$$\inf\{[a_\varepsilon(|\xi|)\xi - a_\varepsilon(|\eta|)\eta]\cdot(\xi-\eta) : \xi,\eta\in\mathbb{R}^{Nn}, |\xi-\eta|\geq t, |\xi|\leq\tau, |\eta|\leq\tau\} > \vartheta\,. \tag{4.23}$$

Proof. Assume first that a is non-decreasing. In particular, $s_a \geq i_a \geq 0$. Then, by Lemma 4.3,

$$[a(|\xi|)\xi - a(|\eta|)\eta]\cdot(\xi-\eta) \geq \tfrac{1}{3}[a(|\xi|)+a(|\eta|)]|\xi-\eta|^2 \quad \text{for } \xi,\eta\in\mathbb{R}^{Nn}. \tag{4.24}$$

Since, by (4.6) and (4.10), $a\in\Delta_2$,

$$a(|\xi-\eta|) \leq a(|\xi|+|\eta|) \leq a(2|\xi|)+a(2|\eta|) \leq C[a(|\xi|)+a(|\eta|)] \quad \text{for } \xi,\eta\in\mathbb{R}^{Nn}. \tag{4.25}$$

By (4.24), (4.25) and the first inequality in (4.4),

$$[a(|\xi|)\xi - a(|\eta|)\eta]\cdot(\xi-\eta) \geq C\min\{|\xi-\eta|^{i_a+2}, |\xi-\eta|^{s_a+2}\} \quad \text{for } \xi,\eta\in\mathbb{R}^{Nn}, \tag{4.26}$$

for some positive constant $C=C(a)$. Hence (4.23) follows, since $\{\xi,\eta\in\mathbb{R}^{Nn}, |\xi-\eta|\geq t, |\xi|\leq\tau, |\eta|\leq\tau\}$ is a compact set.

Assume next that a is non-increasing. Thus, $0 \geq s_a \geq i_a$. By (4.17),

$$[a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) \geq (1 + \min\{i_a, 0\})|\xi - \eta|^2 \int_0^1 a(|\eta + t(\xi - \eta)|)\, dt. \tag{4.27}$$

Owing to (4.4),

$$\begin{aligned} a(|\eta + t(\xi - \eta)|) &\geq C \min\{|\eta + t(\xi - \eta)|^{i_a}, |\eta + t(\xi - \eta)|^{s_a}\} \\ &\geq C \min\{(|\eta| + |\xi - \eta|)^{i_a}, (|\eta| + |\xi - \eta|)^{s_a}\} \\ &\geq C' \min\{(|\eta| + |\xi|)^{i_a}, (|\eta| + |\xi|)^{s_a}\}, \end{aligned} \tag{4.28}$$

for some positive constants $C = C(a)$ and $C' = C'(a)$. Coupling (4.27) with (4.28) yields

$$[a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) \geq C \min\{(|\eta| + |\xi|)^{i_a}, (|\eta| + |\xi|)^{s_a}\}|\xi - \eta|^2 \quad \text{for } \xi, \eta \in \mathbb{R}^{Nn},$$

for some positive constant $C = C(a)$. Hence, (4.23) follows also in this case. □

In the following lemma, any function a as in the statement of Theorem 2.1 is approximated by a family $\{a_\varepsilon\}$ of functions enjoying the additional property of being bounded from above and from below by positive constants.

Lemma 4.5 *Let a be as in Lemma 4.1. Assume, in addition, that a is monotone (either non-decreasing or non-increasing). Given $\varepsilon \in (0, 1)$, define $a_\varepsilon : [0, \infty) \to (0, \infty)$ as*

$$a_\varepsilon(t) = \frac{a(\sqrt{\varepsilon + t^2}) + \varepsilon}{1 + \varepsilon a(\sqrt{\varepsilon + t^2})} \qquad \textit{for } t \geq 0. \tag{4.29}$$

Then a_ε has the same monotonicity property as a,

$$a_\varepsilon \in C^1([0, \infty)), \tag{4.30}$$

$$\varepsilon < a_\varepsilon(t) \leq \varepsilon^{-1} \quad \textit{for } t \geq 0, \tag{4.31}$$

$$\min\{i_a, 0\} \leq i_{a_\varepsilon} \leq s_{a_\varepsilon} \leq \max\{s_a, 0\}, \tag{4.32}$$

$$\lim_{\varepsilon \to 0} a_\varepsilon(|\xi|)\xi = a(|\xi|)\xi \qquad \textit{uniformly in } \{\xi \in \mathbb{R}^{Nn} : |\xi| \leq M\} \textit{ for every } M > 0. \tag{4.33}$$

Moreover, if b_ε and B_ε are defined as in (2.3) and (2.4), respectively, with a replaced with a_ε, then

$$\lim_{\varepsilon \to 0} b_\varepsilon = b \qquad \textit{uniformly in } [0, M] \textit{ for every } M > 0, \tag{4.34}$$

and hence

$$\lim_{\varepsilon \to 0} B_\varepsilon = B \qquad \textit{uniformly in } [0, M] \textit{ for every } M > 0. \tag{4.35}$$

Proof. Property (4.30) trivially follows from the fact that $a \in C^1(0,\infty)$. Since

$$a'_\varepsilon(t) = \frac{(1-\varepsilon^2)a'(\sqrt{\varepsilon+t^2})\,t}{(1+\varepsilon a(\sqrt{\varepsilon+t^2}))^2\sqrt{\varepsilon+t^2}} \quad \text{for } t \geq 0, \tag{4.36}$$

a' and a'_ε have like signs, and hence a and a_ε share the same monotonicity property. Equation (4.32) easily follows from (4.36) and the very definitions of i_{a_ε} and s_{a_ε}. Equation (4.31) is an easy consequence of the definition of a_ε, and of the fact that the function $[0,\infty) \ni s \mapsto \frac{s+\varepsilon}{1+\varepsilon s}$ is increasing for every $\varepsilon \in (0,1)$.
Next, note that

$$\lim_{\varepsilon\to 0} a_\varepsilon = a \qquad \text{uniformly in } [L,M] \text{ for every } M > L > 0.$$

Hence,

$$\lim_{\varepsilon\to 0} b_\varepsilon = b \quad \text{uniformly in } [L,M] \text{ for every } M > L > 0. \tag{4.37}$$

On the other hand, by (4.4) with a replaced with a_ε and by (4.32),

$$0 \leq b_\varepsilon(t) = t a_\varepsilon(t) \leq a_\varepsilon(1) \max\left\{t^{1+\max\{s_a,0\}}, t^{1+\min\{s_i,0\}}\right\} \qquad \text{for } t \geq 0, \tag{4.38}$$

whence,

$$\lim_{t\to 0} b_\varepsilon(t) = 0 \qquad \text{uniformly for } \varepsilon \in (0,1). \tag{4.39}$$

Combining (4.37), (4.39) and (4.5) yields (4.34).
The proof of (4.33) is analogous. $\square$

5 Fundamental geometric and differential inequalities

Here, we enucleate some inequalities of geometric and functional nature of use in the proofs of our main results.
We begin with an isoperimetric inequality, which tells us that if Ω is an open subset of $\mathbb{R}^n$, $n \geq 2$, with a Lipschitz boundary, then there exists a constant C such that

$$|E|^{1/n'} \leq C\mathcal{H}^{n-1}(\Omega \cap \partial^M E) \tag{5.1}$$

for every measurable set $E \subset \Omega$ such that $|E| \leq |\Omega|/2$ [Ma5, Corollary 5.2.1/3]. Here, $|E|$ denotes the Lebesgue measure of E, $\partial^M E$ its essential boundary, and $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure.
Moreover, as a consequence of (in fact, equivalently to) a trace inequality for functions of bounded variation, one has that

$$\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega) \leq C\mathcal{H}^{n-1}(\partial^M E \cap \Omega) \tag{5.2}$$

for some constant $C = C(\Omega)$ and for every measurable set $E \subset \Omega$ such that $|E| \leq |\Omega|/2$ [Ma5, Chapter 9].

Lemma 5.1 *Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 2$, with a Lipschitz boundary. Assume that either $1 \leq q \leq \frac{2(n-1)}{n-2}$, or $1 \leq q < \infty$, according to whether $n \geq 3$ or $n = 2$. Then there exists a constant C, depending on the constant appearing in inequality* (5.2) *and on q, such that*

$$\left(\int_{\partial\Omega} |\mathrm{Tr}\, v|^q \, d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{q}} \leq C|\mathrm{supp}\, v|^{\frac{n-1}{qn} - \frac{n-2}{2n}} \left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}} \tag{5.3}$$

for every $v \in W^{1,2}(\Omega)$ satisfying $|\mathrm{supp}\, v| \leq |\Omega|/2$. Here, $\mathrm{Tr}\, v$ denotes the trace of v on $\partial\Omega$.

Proof of Lemma 5.1 By a Poincaré type trace inequality for Sobolev functions (see e.g. [Ma5, ????]), there exists a constant C, depending on the constant in (5.2) and on q, such that

$$\left(\int_{\partial\Omega} |\mathrm{Tr}\, v|^q d\mathcal{H}^{n-1}\right)^{1/q} \leq C\left(\int_\Omega |\nabla v|^{\frac{nq}{q+n-1}} dx\right)^{\frac{q+n-1}{nq}} \tag{5.4}$$

for every $v \in W^{1,2}(\Omega, \mathbb{R}^N)$ fulfilling $|\mathrm{supp}\, v| \leq |\Omega|/2$. Inequality (5.3) follows from (5.4), via Hölder's inequality. $\square$

If $\mathbf{u} \in W^{2,1}(\Omega, \mathbb{R}^N)$, then $|\nabla \mathbf{u}| \in W^{1,1}(\Omega)$, by the chain rule for vector-valued functions [MM, Theorem 2.1]. An application of the coarea formula then tells us that, for every Borel function $g : \Omega \to [0, \infty)$,

$$\int_{\{|\nabla \mathbf{u}|>t\}} g(x)|\nabla|\nabla \mathbf{u}||dx = \int_t^\infty \int_{\{|\nabla \mathbf{u}|=\tau\}} g(x)\, d\mathcal{H}^{n-1}(x)\, d\tau\,. \tag{5.5}$$

Hence, if the left-hand side is finite for $t > 0$, then it is (locally) absolutely continuous as a function of t, and

$$-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} g(x)|\nabla|\nabla \mathbf{u}||dx = \int_{\{|\nabla \mathbf{u}|=t\}} g(x)\, d\mathcal{H}^{n-1}(x) \qquad \text{for a.e. } t > 0. \tag{5.6}$$

The use of the coarea formula again tells us that $\mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\} \cap \{|\nabla|\nabla \mathbf{u}|| = 0\}) = 0$ for a.e. $t > 0$, and that if g is as above, then

$$\int_{\{|\nabla \mathbf{u}|>t\}} g(x)\, dx = \int_{\{|\nabla \mathbf{u}|>t\}\cap\{|\nabla|\nabla \mathbf{u}||=0\}} g(x)\, dx + \int_t^\infty \int_{\{|\nabla \mathbf{u}|=\tau\}} \frac{g(x)}{|\nabla|\nabla \mathbf{u}||}\, d\mathcal{H}^{n-1}(x)\, d\tau\,. \tag{5.7}$$

In particular, equation (5.7) entails that, if $g \in L^1(\Omega)$, then

$$-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} g(x)dx \geq \int_{\{|\nabla \mathbf{u}|=t\}} \frac{g(x)}{|\nabla|\nabla \mathbf{u}||}\, d\mathcal{H}^{n-1}(x) \qquad \text{for a.e. } t > 0. \tag{5.8}$$

The following differential inequality involving level sets of Sobolev function relies upon the coarea formula and the relative isoperimetric inequality (5.1), and is established in [Ma3].

Lemma 5.2 *Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 2$, with a Lipschitz boundary. Let v be a nonnegative function from $W^{1,2}(\Omega)$, and let μ_v and v^* denote the distribution function and the decreasing rearrangement of v defined as in* (3.1) *and* (3.2)*, respectively. Then there exists a constant C, depending on the constant in* (5.1)*, such that*

$$1 \leq C(-\mu_v'(t))^{1/2}\mu_v(t)^{-1/n'}\left(-\frac{d}{dt}\int_{\{v>t\}} |\nabla v|^2 dx\right)^{1/2} \textit{for a.e. } t \geq v^*(|\Omega|/2). \tag{5.9}$$

The next lemma provides us with a lower estimate for the scalar product between the left-hand side of the equation in (2.5) and $\Delta\mathbf{u}$, via terms in divergence form and a signed term.

Lemma 5.3 *Assume that* $a:(0,\infty)\to(0,\infty)$ *is of class* C^1*, and satisfies the first inequality in* (2.1). *Let* Ω *be an open set in* $\mathbb{R}^n$, $n\geq 2$, *and let* $\mathbf{v}\in C^3(\Omega,\mathbb{R}^N)$, *with* $\mathbf{v}=(v^1,\dots,v^N)$. *Then*

(5.10)
$$\begin{aligned}\sum_{\alpha=1}^N \Delta v^\alpha \operatorname{div}(a(|\nabla\mathbf{v}|)\nabla v^\alpha) \geq & \sum_{\alpha=1}^N \operatorname{div}(\Delta v^\alpha\, a(|\nabla\mathbf{v}|)\nabla v^\alpha)\\ & -\sum_{\alpha=1}^N\sum_{i,j}^n \big(v^\alpha_{x_ix_j}a(|\nabla\mathbf{v}|)v^\alpha_{x_i}\big)_{x_j} + (1+\min\{i_a,0\})a(|\nabla\mathbf{v}|)\sum_{\alpha=1}^N|\nabla^2 v^\alpha|^2\end{aligned}$$

in $\{\nabla\mathbf{v}\neq 0\}$.

Proof. In $\{\nabla\mathbf{v}\neq 0\}$, we have that

$$\begin{aligned}\sum_{\alpha=1}^N \Delta v^\alpha \operatorname{div}(a(|\nabla\mathbf{v}|)\nabla v^\alpha) &= \sum_{\alpha=1}^N \operatorname{div}(\Delta v^\alpha\, a(|\nabla\mathbf{v}|)\nabla v^\alpha) - \sum_{\alpha=1}^N\sum_{i,j=1}^n v^\alpha_{x_ix_jx_j}a(|\nabla\mathbf{v}|)v^\alpha_{x_i}\\ &= \sum_{\alpha=1}^N \operatorname{div}(\Delta v^\alpha\, a(|\nabla\mathbf{v}|)\nabla v^\alpha) - \sum_{\alpha=1}^N\sum_{i,j}^n \big(v^\alpha_{x_ix_j}a(|\nabla\mathbf{v}|)v^\alpha_{x_i}\big)_{x_j}\\ &\quad +\sum_{\alpha=1}^N\sum_{i,j=1}^n \big(v^\alpha_{x_ix_j}\big)^2 a(|\nabla\mathbf{v}|) + \sum_{\alpha=1}^N\sum_{i,j=1}^n v^\alpha_{x_ix_j}a(|\nabla\mathbf{v}|)_{x_j}v^\alpha_{x_i}\,.\end{aligned}\tag{5.11}$$

Now,

$$\begin{aligned}&\sum_{\alpha=1}^N\sum_{i,j=1}^n \big(v^\alpha_{x_ix_j}\big)^2 a(|\nabla\mathbf{v}|) + \sum_{\alpha=1}^N\sum_{i,j=1}^n v^\alpha_{x_ix_j}a(|\nabla\mathbf{v}|)_{x_j}v^\alpha_{x_i}\\ &\qquad = \sum_{i,j=1}^n \big(v^\alpha_{x_ix_j}\big)^2 a(|\nabla\mathbf{v}|) + \sum_{\alpha,\beta=1}^N\sum_{i,j,k=1}^n v^\alpha_{x_ix_j}a'(|\nabla\mathbf{v}|)\frac{v^\beta_{x_k}}{|\nabla\mathbf{v}|}v^\beta_{x_kx_j}v^\alpha_{x_i}\\ &\qquad = a(|\nabla\mathbf{v}|)\Bigg(\sum_{\alpha=1}^N\sum_{i,j=1}^n (v^\alpha_{x_ix_j})^2 + \sum_{\alpha,\beta=1}^N\sum_{i,j,k=1}^n \frac{a'(|\nabla\mathbf{v}|)|\nabla\mathbf{v}|}{a(|\nabla\mathbf{v}|)}\frac{v^\alpha_{x_i}}{|\nabla\mathbf{v}|}v^\alpha_{x_ix_j}\frac{v^\beta_{x_k}}{|\nabla\mathbf{v}|}v^\beta_{x_kx_j}\Bigg).\end{aligned}\tag{5.12}$$

On setting $U^j=(v^1_{x_1x_j},\dots,v^1_{x_nx_j},\dots,v^N_{x_1x_j},\cdots,v^N_{x_nx_j})$ and $\omega=\frac{\nabla\mathbf{v}}{|\nabla\mathbf{v}|}$, and making use of the first inequality in (2.1), one obtains that

$$\begin{aligned}&a(|\nabla\mathbf{v}|)\Bigg(\sum_{\alpha=1}^N\sum_{i,j=1}^n (v^\alpha_{x_ix_j})^2 + \sum_{\alpha,\beta=1}^N\sum_{i,j,k=1}^n \frac{a'(|\nabla\mathbf{v}|)|\nabla\mathbf{v}|}{a(|\nabla\mathbf{v}|)}\frac{v^\alpha_{x_i}}{|\nabla\mathbf{v}|}v^\alpha_{x_ix_j}\frac{v^\beta_{x_k}}{|\nabla\mathbf{v}|}v^\beta_{x_kx_j}\Bigg)\\ &\qquad = a(|\nabla\mathbf{v}|)\sum_{j=1}^n\left(|U^j|^2+\frac{a'(|\nabla\mathbf{v}|)|\nabla\mathbf{v}|}{a(|\nabla\mathbf{v}|)}(U^j\cdot\omega)^2\right)\\ &\qquad \geq a(|\nabla\mathbf{v}|)\sum_{j=1}^n\big(|U^j|^2+i_a(U^j\cdot\omega)^2\big)\geq a(|\nabla\mathbf{v}|)(1+\min\{i_a,0\})\sum_j|U^j|^2.\end{aligned}\tag{5.13}$$

Inequality (5.10) follows from (5.11)-(5.13). □

The last two results of this section provide us with key inequalities involving integrals on level sets and level surfaces of vector-valued functions in Ω satisfying either Dirichlet of Neumann homogenous boundary conditions.

Lemma 5.4 *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, with $\partial\Omega \in C^2$, and let a be as in Theorem 2.1. Let $r > n-1$. Assume that $\mathbf{v} \in C^\infty(\Omega, \mathbb{R}^N) \cap C^2(\overline{\Omega}, \mathbb{R}^N)$, and $\mathbf{v} = 0$ on $\partial\Omega$. Let $\mathcal{B}$ denote the second fundamental form on $\partial\Omega$, and let $\mathrm{tr}\mathcal{B}$ be its trace. Then*

(5.14)
$$\begin{aligned}\frac{(1+\min\{i_a,0\})^2}{2} b(t) \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla |\nabla \mathbf{v}||\, d\mathcal{H}^{n-1}(x) &\leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) \\ &+ \frac{\|\nabla \mathbf{v}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}}{b(t)} \int_{\{|\nabla \mathbf{v}|>t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx \\ &+ a\big(\|\nabla \mathbf{v}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\big) \|\nabla \mathbf{v}\|^2_{L^\infty(\Omega,\mathbb{R}^{Nn})} \int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\end{aligned}$$

for a.e. $t > 0$. Moreover,

(5.15)
$$\begin{aligned}\frac{(1+\min\{i_a,0\})^2}{2} b(t) \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla |\nabla \mathbf{v}||\, d\mathcal{H}^{n-1}(x) &\leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) \\ + \int_{\{|\nabla \mathbf{v}|>t\}} \frac{1}{a(|\nabla \mathbf{v}|)} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx &+ a(t)t^2 \int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\end{aligned}$$

for a.e. $t \geq t_{\mathbf{v}}$, where $t_{\mathbf{v}} = |\nabla \mathbf{v}|^*(\alpha|\Omega|)$, and $\alpha \in (0, \frac{1}{2}]$ is a constant depending on i_a, s_a, n, r, $\|\mathrm{tr}\mathcal{B}\|_{L^r(\partial\Omega)}$, Ω, and on the constant in inequality (5.2).
If Ω is convex, then the integral involving $\mathrm{tr}\mathcal{B}$ can be dropped on the right-hand sides of inequalities (5.14) *and* (5.15)*, and the constant α neither depends on r, nor on $\|\mathrm{tr}\mathcal{B}\|_{L^r(\partial\Omega)}$.*

Proof. The level set $\{|\nabla \mathbf{v}| > t\}$ is open for $t > 0$. Moreover, for a.e. $t > 0$, the level surface $\partial\{|\nabla \mathbf{v}| > t\}$ is an $(n-1)$-dimensional manifold of class C^1 outside a set of $\mathcal{H}^{n-1}$ measure zero, and
$$\partial\{|\nabla \mathbf{v}| > t\} = \{|\nabla \mathbf{v}| = t\} \cup \big(\partial\Omega \cap \partial\{|\nabla \mathbf{v}| > t\}\big).$$
By inequality (5.10) and the divergence theorem we have that

(5.16)
$$\begin{aligned}&\sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}|>t\}} \Delta v^\alpha \,\mathrm{div}(a(|\nabla \mathbf{v}|)\nabla v^\alpha) dx \geq \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}|>t\}} \mathrm{div}(\Delta v^\alpha \, a(|\nabla \mathbf{v}|)\nabla v^\alpha) dx \\ &\quad - \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}|>t\}} \sum_{i,j=1}^n \big(v^\alpha_{x_i x_j} a(|\nabla \mathbf{v}|) v^\alpha_{x_i}\big)_{x_j} dx + (1+\min\{i_a,0\}) \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|)|\nabla^2 v^\alpha|^2 dx \\ &\quad = \sum_{\alpha=1}^N \int_{\partial\{|\nabla \mathbf{v}|>t\}} \Delta v^\alpha \, a(|\nabla \mathbf{v}|) \frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x) - \sum_{\alpha=1}^N \int_{\partial\{|\nabla \mathbf{v}|>t\}} \sum_{i,j=1}^n v^\alpha_{x_i x_j} a(|\nabla \mathbf{v}|) v^\alpha_{x_i} \nu_j d\mathcal{H}^{n-1}(x) \\ &\quad\quad + (1+\min\{i_a,0\}) \sum_{\alpha=1}^N \int_{\{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|)|\nabla^2 v^\alpha|^2 dx \qquad \text{for a.e. } t > 0.\end{aligned}$$

Here, ν_j denotes the $j-th$ component of the outer normal vector ν to $\partial\{|\nabla \mathbf{v}|>t\}$. Now, observe that, for a.e. $t>0$,

$$\nu = -\frac{\nabla|\nabla \mathbf{v}|}{|\nabla|\nabla \mathbf{v}||} \quad \text{on} \quad \{|\nabla \mathbf{v}|=t\}.$$

Moreover,

$$\sum_{\alpha=1}^{N}\sum_{i=1}^{n} v^\alpha_{x_i x_j} v^\alpha_{x_i} = |\nabla \mathbf{v}|_{x_j}|\nabla \mathbf{v}|.$$

Thus,

(5.17)

$$\begin{aligned}
&\sum_{\alpha=1}^{N}\int_{\partial\{|\nabla \mathbf{v}|>t\}} \Delta v^\alpha\, a(|\nabla \mathbf{v}|)\frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x) - \sum_{\alpha=1}^{N}\int_{\partial\{|\nabla \mathbf{v}|>t\}} \sum_{i,j=1}^{n} v^\alpha_{x_i x_j} a(|\nabla \mathbf{v}|) v^\alpha_{x_i}\nu_j d\mathcal{H}^{n-1}(x)\\
&\quad = a(t)\sum_{\alpha=1}^{N}\int_{\{|\nabla \mathbf{v}|=t\}} \Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x) + a(t)t\int_{\{|\nabla \mathbf{v}|=t\}} |\nabla|\nabla \mathbf{v}||\, d\mathcal{H}^{n-1}(x)\\
&\qquad + \sum_{\alpha=1}^{N}\int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|)\Big(\Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^{n} v^\alpha_{x_i x_j} v^\alpha_{x_i}\nu_j\Big) d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } t>0.
\end{aligned}$$

Let us focus on the integrals on the right-hand side of (5.17). Since, for each $\alpha = 1,\dots,N$,

$$\operatorname{div}(a(|\nabla \mathbf{v}|)\nabla v^\alpha) = a(|\nabla \mathbf{v}|)\Delta v^\alpha + a'(|\nabla \mathbf{v}|)\nabla v^\alpha\cdot\nabla|\nabla \mathbf{v}|\,, \tag{5.18}$$

one has that

(5.19)

$$\begin{aligned}
&a(t)\sum_{\alpha=1}^{N}\int_{\{|\nabla \mathbf{v}|=t\}} \Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}\\
&= \sum_{\alpha=1}^{N}\int_{\{|\nabla \mathbf{v}|=t\}} \operatorname{div}(a(|\nabla \mathbf{v}|)\nabla v^\alpha)\,\frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x) - a'(t)\sum_{\alpha=1}^{N}\int_{\{|\nabla \mathbf{v}|=t\}} \nabla v^\alpha\cdot\nabla|\nabla \mathbf{v}|\frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x)\\
&= \sum_{\alpha=1}^{N}\int_{\{|\nabla \mathbf{v}|=t\}} \operatorname{div}(a(|\nabla \mathbf{v}|)\nabla v^\alpha)\,\frac{\partial v^\alpha}{\partial \nu} d\mathcal{H}^{n-1}(x) + a'(t)\sum_{\alpha=1}^{N}\int_{\{|\nabla \mathbf{v}|=t\}} |\nabla|\nabla \mathbf{v}||\Big(\frac{\partial v^\alpha}{\partial \nu}\Big)^2 d\mathcal{H}^{n-1}(x)\\
&\le t\int_{\{|\nabla \mathbf{v}|=t\}} \mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})\, d\mathcal{H}^{n-1}(x) + a'(t)\int_{\{|\nabla \mathbf{v}|=t\}} |\nabla|\nabla \mathbf{v}||\Big|\frac{\partial \mathbf{v}}{\partial \nu}\Big|^2 d\mathcal{H}^{n-1}(x)
\end{aligned}$$

for a.e. $t>0$. Here, we have exploited the fact that

$$\frac{\partial v^\alpha}{\partial \nu} = -\frac{\nabla v^\alpha\cdot\nabla|\nabla \mathbf{v}|}{|\nabla|\nabla \mathbf{v}||} \qquad \text{on} \quad \{|\nabla \mathbf{v}|=t\}\,, \quad \text{for a.e. } t>0.$$

Next recall that, for each $\alpha = 1,\dots,N$,

$$\begin{aligned}
(5.20)\quad &\Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^{N} v^\alpha_{x_i x_j} v^\alpha_{x_i}\nu_j\\
&= \operatorname{div}_T\Big(\frac{\partial v^\alpha}{\partial \nu}\nabla_T v^\alpha\Big) - \operatorname{tr}\mathcal{B}\Big(\frac{\partial v^\alpha}{\partial \nu}\Big)^2 - \mathcal{B}(\nabla_T v^\alpha, \nabla_T v^\alpha) - 2\nabla_T v^\alpha\cdot\nabla_T \frac{\partial v^\alpha}{\partial \nu} \qquad \text{on } \partial\Omega,
\end{aligned}$$

where div_T and ∇_T denote the divergence operator and the gradient operator on $\partial\Omega$, respectively [Gr, Equation (3,1,1,2)]. Hence, since $\mathbf{v} = 0$ on $\partial\Omega$,

$$\Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^{N} v^\alpha_{x_i x_j} v^\alpha_{x_i} \nu_j = -\mathrm{tr}\mathcal{B}\left(\frac{\partial v^\alpha}{\partial \nu}\right)^2 \qquad \text{on } \partial\Omega\,. \tag{5.21}$$

Therefore,

$$\begin{aligned}\sum_{\alpha=1}^{N} \int_{\partial\Omega\cap\partial\{|\nabla\mathbf{v}|>t\}} & a(|\nabla\mathbf{v}|)\Big(\Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^{n} v^\alpha_{x_i x_j} v^\alpha_{x_i} \nu_j\Big) d\mathcal{H}^{n-1}(x) \\ &= -\sum_{\alpha=1}^{N} \int_{\partial\Omega\cap\partial\{|\nabla\mathbf{v}|>t\}} a(|\nabla\mathbf{v}|)\mathrm{tr}\mathcal{B}(x)\left(\frac{\partial v^\alpha}{\partial \nu}\right)^2 d\mathcal{H}^{n-1}(x) \\ &\geq -\int_{\partial\Omega\cap\partial\{|\nabla\mathbf{v}|>t\}} a(|\nabla\mathbf{v}|)|\nabla\mathbf{v}|^2 |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \qquad \text{for a.e. } t>0.\end{aligned} \tag{5.22}$$

By Young inequality and the fact that $|\Delta\mathbf{v}| \leq |\nabla^2\mathbf{v}|$,

$$\begin{aligned}&\sum_{\alpha=1}^{N} \int_{\{|\nabla\mathbf{v}|>t\}} \Delta v^\alpha \,\mathrm{div}(a(|\nabla\mathbf{v}|)\nabla v^\alpha) dx \\ &\leq \frac{1+\min\{i_a,0\}}{2} \int_{\{|\nabla\mathbf{v}|>t\}} a(|\nabla\mathbf{v}|)|\nabla^2\mathbf{v}|^2 dx + \frac{2}{1+\min\{i_a,0\}} \int_{\{|\nabla\mathbf{v}|>t\}} \frac{1}{a(|\nabla\mathbf{v}|)} |\mathbf{div}(a(|\nabla\mathbf{v}|)\nabla\mathbf{v})|^2 dx\end{aligned} \tag{5.23}$$

for a.e. $t>0$. Combining (5.16), (5.17), (5.19), (5.22) and (5.23) yields

$$\begin{aligned}&\int_{\{|\nabla\mathbf{v}|=t\}} |\nabla|\nabla\mathbf{v}||\left(a(t)t + a'(t)\left|\frac{\partial\mathbf{v}}{\partial\nu}\right|^2\right) d\mathcal{H}^{n-1}(x) + \frac{1+\min\{i_a,0\}}{2} \int_{\{|\nabla\mathbf{v}|>t\}} a(|\nabla\mathbf{v}|)|\nabla^2\mathbf{v}|^2 dx \\ &\leq t \int_{\{|\nabla\mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla\mathbf{v}|)\nabla\mathbf{v})| d\mathcal{H}^{n-1}(x) + \frac{2}{1+\min\{i_a,0\}} \int_{\{|\nabla\mathbf{v}|>t\}} \frac{1}{a(|\nabla\mathbf{v}|)} |\mathbf{div}(a(|\nabla\mathbf{v}|)\nabla\mathbf{v}|^2 dx \\ &\qquad + \int_{\partial\Omega\cap\partial\{|\nabla\mathbf{v}|>t\}} a(|\nabla\mathbf{v}|)|\nabla\mathbf{v}|^2 |\mathrm{tr}\mathcal{B}(x)|\, d\mathcal{H}^{n-1}(x)\end{aligned} \tag{5.24}$$

for a.e. $t>0$. Observe that

$$a(t)t + a'(t)\left|\frac{\partial\mathbf{v}}{\partial\nu}\right|^2 \geq a(t)t + \min\{0, a'(t)\}t^2 \geq (1+\min\{i_a,0\})a(t)t \quad \text{on } \{|\nabla\mathbf{v}| = t\}. \tag{5.25}$$

From (5.24) and (5.25) we deduce that

$$\begin{aligned}(1+\min\{i_a,0\})b(t) &\int_{\{|\nabla\mathbf{v}|=t\}} |\nabla|\nabla\mathbf{v}||\, d\mathcal{H}^{n-1}(x) + \tfrac{1+\min\{i_a,0\}}{2} \int_{\{|\nabla\mathbf{v}|>t\}} a(|\nabla\mathbf{v}|)|\nabla^2\mathbf{v}|^2 dx \\ &\leq t \int_{\{|\nabla\mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla\mathbf{v}|)\nabla\mathbf{v})| d\mathcal{H}^{n-1}(x) \\ &\quad + \tfrac{2}{1+\min\{i_a,0\}} \int_{\{|\nabla\mathbf{v}|>t\}} \frac{1}{a(|\nabla\mathbf{v}|)} |\mathbf{div}(a(|\nabla\mathbf{v}|)\nabla\mathbf{v})|^2 dx \\ &\quad + \int_{\partial\Omega\cap\partial\{|\nabla\mathbf{v}|>t\}} a(|\nabla\mathbf{v}|)|\nabla\mathbf{v}|^2\, |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \qquad \text{for a.e. } t>0.\end{aligned} \tag{5.26}$$

Hence, since $b(t)$ is an increasing function, and hence also $a(t)t^2 = b(t)t$ is an increasing function,

(5.27)

$$
\begin{aligned}
(1+\min\{i_a,0\})b(t) &\int_{\{|\nabla \mathbf{v}|=t\}} |\nabla|\nabla \mathbf{v}||\, d\mathcal{H}^{n-1}(x) \\
&\leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathrm{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) \\
&\quad + \tfrac{2}{1+\min\{i_a,0\}} \frac{\|\nabla \mathbf{v}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}}{b(t)} \int_{\{|\nabla \mathbf{v}|>t\}} |\mathrm{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx \\
&\quad + a\big(\|\nabla \mathbf{v}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\big)\|\nabla \mathbf{v}\|^2_{L^\infty(\Omega,\mathbb{R}^{Nn})} \int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)
\end{aligned}
$$

for a.e. $t > 0$. Inequality (5.14) follows.

Let us next focus on (5.15). Inasmuch as $a(t)t^2$ is an increasing function,

(5.28)
$$\int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|)|\nabla \mathbf{v}|^2 |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)$$

(5.29)
$$
\begin{aligned}
&\leq 2 \int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} \big(a(|\nabla \mathbf{v}|)^{1/2}|\nabla \mathbf{v}| - a(t)^{1/2}t\big)^2 |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \\
&\quad + 2a(t)t^2 \int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\,.
\end{aligned}
$$

Denote, for simplicity, the distribution function $\mu_{|\nabla \mathbf{v}|}$ of $|\nabla \mathbf{v}|$ by $\mu : [0,\infty) \to [0,|\Omega|]$. Set $\delta = \frac{n-1}{nr'} - \frac{n-2}{n}$, and observe that $\delta > 0$ since $r > n-1$. Thanks to our assumptions on the function a and to the chain rule for vector-valued Sobolev functions [MM, Theorem 2.1], the function $\max\{a(|\nabla \mathbf{v}|)^{1/2}|\nabla \mathbf{v}| - a(t)^{1/2}t, 0\}$ belongs to $W^{1,2}(\Omega)$. Hölder's inequality and an application of Lemma 5.1 with v replaced with $\max\{a(|\nabla \mathbf{v}|)^{1/2}|\nabla \mathbf{v}| - a(t)^{1/2}t, 0\}$ tell us that

(5.30)
$$
\begin{aligned}
&\int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} \big(a(|\nabla \mathbf{v}|)^{1/2}|\nabla \mathbf{v}| - a(t)^{1/2}t\big)^2 |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \\
&\leq \bigg(\int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} \big(a(|\nabla \mathbf{v}|)^{1/2}|\nabla \mathbf{v}| - a(t)^{1/2}t\big)^{2r'} d\mathcal{H}^{n-1}(x)\bigg)^{\frac{1}{r'}} \bigg(\int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} |\mathrm{tr}\mathcal{B}(x)|^r d\mathcal{H}^{n-1}(x)\bigg)^{\frac{1}{r}} \\
&\leq C\mu(t)^\delta \|\mathrm{tr}\mathcal{B}\|_{L^r(\partial\Omega)} \int_{\{|\nabla \mathbf{v}|>t\}} \Big|\nabla\big[a(|\nabla \mathbf{v}|)^{1/2}|\nabla \mathbf{v}|\big]\Big|^2 dx \\
&= C\mu(t)^\delta \|\mathrm{tr}\mathcal{B}\|_{L^r(\partial\Omega)} \int_{\{|\nabla \mathbf{v}|>t\}} \Big(\tfrac{1}{2}a'(|\nabla \mathbf{v}|)a(|\nabla \mathbf{v}|)^{-1/2}|\nabla \mathbf{v}| + a(|\nabla \mathbf{v}|)^{1/2}\Big)^2 |\nabla|\nabla \mathbf{v}||^2 dx \\
&\leq C'\mu(t)^\delta \|\mathrm{tr}\mathcal{B}\|_{L^r(\partial\Omega)} \int_{\{|\nabla \mathbf{v}|>t\}} a(|\nabla \mathbf{v}|)|\nabla^2 \mathbf{v}|^2 dx \quad \text{if } t > |\nabla \mathbf{v}|^*(|\Omega|/2),
\end{aligned}
$$

for some positive constants C, depending on the constant in (5.2) and on r, and C' depending on the same quantities and on s_a. Observe that, in the last inequality in (5.30), we have employed the inequality $|\nabla|\nabla \mathbf{v}|| \leq |\nabla^2 \mathbf{v}|$ a.e. in Ω. Set

$$\beta = \Big(\frac{1+\min\{i_a,0\}}{4C'\|\mathrm{tr}\mathcal{B}\|_{L^r(\partial\Omega)}}\Big)^{\frac{1}{\delta}},$$

where C' is the constant appearing in (5.30),

$$\alpha = \min\{\beta/|\Omega|, 1/2\}, \tag{5.31}$$

and $t_{\mathbf{v}} = |\nabla \mathbf{v}|^*(\alpha|\Omega|)$. Thus, α depends on the quantities specified in the statement, and

$$\tfrac{1+\min\{i_a,0\}}{2} - 2C'\|\mathrm{tr}\mathcal{B}\|_{L^r(\partial\Omega)}\mu(t)^\delta \geq 0 \qquad \text{if } t > t_{\mathbf{v}}. \tag{5.32}$$

Inequality (5.15) follows from (5.26), (5.28), (5.30) and (5.32), since $\frac{1+\min\{i_a,0\}}{2} < 1$.

The assertion concerning the case when Ω is convex follows via the same argument, on observing that the left-most side of (5.22) can be estimated from below just by 0, inasmuch as $\mathrm{tr}\mathcal{B} \leq 0$ on $\partial\Omega$ in this case. $\square$

Lemma 5.5 *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, with $\partial\Omega \in C^2$, and let a be as in Theorem 2.1. Let $r > n-1$. Assume that $\mathbf{v} \in C^\infty(\Omega, \mathbb{R}^N) \cap C^2(\overline{\Omega}, \mathbb{R}^N)$, and $\frac{\partial \mathbf{v}}{\partial \nu} = 0$ on $\partial\Omega$. Let $\mathcal{B}$ denote the second fundamental form on $\partial\Omega$, and let $|\mathcal{B}|$ be its operator norm, namely*

$$|\mathcal{B}(x)| = \sup_{0 \neq \zeta \in \mathbb{R}^{n-1}} \frac{|\mathcal{B}(x)(\zeta,\zeta)|}{|\zeta|^2} \quad \text{for } x \in \partial\Omega.$$

Then

(5.33)
$$\begin{aligned}\frac{(1+\min\{i_a,0\})^2}{2} b(t) \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla|\nabla \mathbf{v}||\, d\mathcal{H}^{n-1}(x) &\leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) \\ &+ \frac{\|\nabla \mathbf{v}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}}{b(t)} \int_{\{|\nabla \mathbf{v}|>t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx \\ &+ a\big(\|\nabla \mathbf{v}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\big)\|\nabla \mathbf{v}\|^2_{L^\infty(\Omega,\mathbb{R}^{Nn})} \int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} |\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\end{aligned}$$

for a.e. $t > 0$. Moreover,

(5.34)
$$\begin{aligned}\frac{(1+\min\{i_a,0\})^2}{2} b(t) \int_{\{|\nabla \mathbf{v}|=t\}} |\nabla|\nabla \mathbf{v}||\, d\mathcal{H}^{n-1}(x) &\leq t \int_{\{|\nabla \mathbf{v}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})| d\mathcal{H}^{n-1}(x) \\ &+ \int_{\{|\nabla \mathbf{v}|>t\}} \frac{1}{a(|\nabla \mathbf{v}|)} |\mathbf{div}(a(|\nabla \mathbf{v}|)\nabla \mathbf{v})|^2 dx + a(t)t^2 \int_{\partial\Omega\cap\partial\{|\nabla \mathbf{v}|>t\}} |\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\end{aligned}$$

for a.e. $t \geq t_{\mathbf{v}}$, where $t_{\mathbf{v}} = |\nabla \mathbf{v}|^*(\alpha|\Omega|)$, and $\alpha \in (0, \frac{1}{2}]$ is a constant depending on i_a, s_a, n, r, $\|\,|\mathcal{B}|\,\|_{L^r(\partial\Omega)}$, $|\Omega$, and on the constant appearing in inequality (5.2).
If Ω is convex, the integral involving $|\mathcal{B}|$ can be dropped on the right-hand sides of inequalities (5.33) and (5.34), and the constant α neither depends on r, nor on $\|\,|\mathcal{B}|\,\|_{L^r(\partial\Omega)}$.

Sketch of the proof. The proof is completely analogous to that of Lemma 5.4. One has just to observe that, by (5.20) an the boundary condition $\frac{\partial \mathbf{v}}{\partial \nu} = 0$ on $\partial\Omega$, equation (5.21) has to be replaced with

$$\Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^n v^\alpha_{x_i x_j} v^\alpha_{x_i} \nu_j = -\mathcal{B}(\nabla_T\, v^\alpha, \nabla_T\, v^\alpha) \qquad \text{on } \partial\Omega\,, \tag{5.35}$$

which, in particular, implies that

$$\left|\Delta v^\alpha \frac{\partial v^\alpha}{\partial \nu} - \sum_{i,j=1}^{N} v^\alpha_{x_i x_j} v^\alpha_{x_i} \nu_j\right| \leq |\mathcal{B}||\nabla v^\alpha|^2 \qquad \text{on } \partial\Omega\,. \tag{5.36}$$

The conclusion concerning convex domains Ω holds owing to the fact that $\mathcal{B} \leq 0$ on $\partial\Omega$ in this case. □

6 Proof of the main results

Let B be the Young function defined by (2.4), and let B_n be its Sobolev conjugate given by (3.16). Assume that $f \in L^{\widetilde{B_n}}(\Omega, \mathbb{R}^N)$. A weak solution to the Dirichlet problem (2.5) is a function $u \in W^{1,B}_0(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega a(|\nabla \mathbf{u}|)\nabla \mathbf{u} \cdot \nabla \phi \, dx = \int_\Omega \mathbf{f} \cdot \phi \, dx \tag{6.1}$$

for every $\phi \in W^{1,B}_0(\Omega, \mathbb{R}^N)$.
Assume now, in addition that Ω has a Lipschitz boundary. A weak solution to the Neumann problem (2.8) is a function $\mathbf{u} \in W^{1,B}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega a(|\nabla \mathbf{u}|)\nabla \mathbf{u} \cdot \nabla \phi \, dx = \int_\Omega \mathbf{f} \cdot \phi \, dx \tag{6.2}$$

for every $\phi \in W^{1,B}(\Omega, \mathbb{R}^N)$.

Note that the left-hand sides of (6.1) and (6.2) are well defined by the second inequality in (3.12), and by (4.7). The right-hand sides are also well defined, owing to the Sobolev inequality (3.18) and to the Hölder type inequality (3.12) with B replaced with B_n. In particular, the right-hand sides of (6.1) and (6.2) are well defined if $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$, since $L^{n,1}(\Omega, \mathbb{R}^N) \to L^{\widetilde{B_n}}(\Omega, \mathbb{R}^N)$, as shown in [CM1, Remark 2.12].

The following existence and uniqueness result holds for weak solutions to problems (2.5) and (2.8).

Theorem 6.1 *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $N \geq 1$. Assume that $a : (0, \infty) \to (0, \infty)$ is of class C^1, and fulfils* (2.1)*. Let $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$. Then there exist a unique solution $\mathbf{u} \in W^{1,B}_0(\Omega, \mathbb{R}^N)$ to problem* (2.5)*.*
Assume, in addition, that Ω is a Lipschitz domain, and $\mathbf{f}$ fulfills (2.9)*. Then there exists a solution $\mathbf{u} \in W^{1,B}(\Omega, \mathbb{R}^N)$ to problem* (2.8)*, which is unique up to additive constant vectors in $\mathbb{R}^N$. In particular, there exists a unique solution in $W^{1,B}_\perp(\Omega, \mathbb{R}^N)$.*

A proof of Theorem 6.1 in the case when $N = 1$ can be found in [CM1]; the proof for $N > 1$ is completely analogous.

The next Proposition provides us with a basic energy estimate for the weak solutions to problems (2.5) and (2.8).

Proposition 6.2 *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, and let $N \geq 1$. Assume that a is as in Theorem 6.1. Let $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^{Nn})$.*

(i) Let $\mathbf{u} \in W^{1,B}_0(\Omega, \mathbb{R}^N)$ *be the weak solution to problem* (2.5). *Then there exists a constant* C *depending on* n, i_a, *and* $|\Omega|$, *such that*

$$\int_\Omega B(|\nabla \mathbf{u}|)dx \le C|\Omega|\, \|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)} b^{-1}\big(\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}\big). \tag{6.3}$$

(ii) Assume, in addition, that Ω *is a Lipschitz domain, and* $\mathbf{f}$ *fulfills* (2.9). *Let* $\mathbf{u} \in W^{1,B}(\Omega, \mathbb{R}^N)$ *be a weak solution to problem* (2.8). *Then inequality holds for some constant* C *depending on* n, i_a *and on the constant in* (5.1).

Proof. (i) Making use of $\mathbf{u}$ as test function ϕ in the definition of weak solution (6.1) tells us that

$$\int_\Omega a(|\nabla \mathbf{u}|)|\nabla \mathbf{u}|^2\, dx = \int_\Omega \mathbf{f}\cdot \mathbf{u}\, dx.$$

By the first inequality in (4.10), Hölder's inequality in Lorentz spaces (3.8), and (3.6), there exist constants $C = C(n)$ and $C' = C'(n, |\Omega|)$ such that

$$\int_\Omega B(|\nabla \mathbf{u}|)\, dx \le C\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}\|\mathbf{u}\|_{L^{n',\infty}(\Omega,\mathbb{R}^N)} \le C'\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}\|\mathbf{u}\|_{L^{n'}(\Omega,\mathbb{R}^N)}. \tag{6.4}$$

By the Poincaré inequality in $W^{1,1}_0(\Omega, \mathbb{R}^N)$, there exists a constant $C = C(n)$ such that

$$\|\mathbf{u}\|_{L^{n'}(\Omega,\mathbb{R}^N)} \le C\int_\Omega |\nabla \mathbf{u}|\, dx. \tag{6.5}$$

On the other hand, Jensen's inequality entails that

$$B\bigg(\frac{1}{|\Omega|}\int_\Omega |\nabla \mathbf{u}|\, dx\bigg) \le \frac{1}{|\Omega|}\int_\Omega B(|\nabla \mathbf{u}|)\, dx. \tag{6.6}$$

Combining inequalities (6.4)–(6.6), and making use of the second inequality in (3.10) yields

$$\frac{1}{|\Omega|}\int_\Omega B(|\nabla \mathbf{u}|)\, dx \le \widetilde{B}\big(2C\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}\big). \tag{6.7}$$

Since b^{-1} is an increasing function, equation (3.9) ensures that $\widetilde{B}(t) \le tb^{-1}(t)$ for $t \ge 0$. Thus, (6.3) follows from (6.7), via (4.9).
(ii) The proof follows along the same lines as above. One has just to make use of the fact that inequality (6.5) holds, for every $\mathbf{u} \in W^{1,B}_\perp(\Omega, \mathbb{R}^N)$, with a constant C depending on n and on the constant in (5.1) [Ma5, Theorem 5.2.3], and that any solution u to (2.8) differs from the solution in $W^{1,B}_\perp(\Omega, \mathbb{R}^N)$ by a constant vector in $\mathbb{R}^N$. □

We are now in a position to prove Theorem 2.1.
Proof of Theorem2.1. We split the proof in steps.
Step 1. For the time being, we assume in addition that

$$\partial\Omega \in C^\infty, \tag{6.8}$$

and there exist positive constants c and C such that

$$c \le a(t) \le C \quad \text{for } t \ge 0\,. \tag{6.9}$$

Since $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$, in particular, owing to (3.7), $\mathbf{f} \in L^2(\Omega, \mathbb{R}^N)$. By a result of Elcrat and Meyers, the weak solution $\mathbf{u}$ to problem (2.5) belongs to $W^{2,2}(\Omega)$ [BF, Theorem 8.2]. Notice that the hypotheses of that result are fulfilled under our additional assumptions (6.8)–(6.9), owing to equation (4.16). Thus, $\mathbf{u} \in W^{1,2}_0(\Omega, \mathbb{R}^N) \cap W^{2,2}(\Omega, \mathbb{R}^N)$. By standard approximation, there exists a sequence $\{\mathbf{u}_k\} \subset C^\infty(\Omega, \mathbb{R}^N) \cap C^2(\overline{\Omega}, \mathbb{R}^N)$ such that $\mathbf{u}_k = 0$ on $\partial\Omega$,

$$\text{(6.10)} \qquad \mathbf{u}_k \to \mathbf{u} \quad \text{in } W^{1,2}_0(\Omega, \mathbb{R}^N), \quad \mathbf{u}_k \to \mathbf{u} \quad \text{in } W^{2,2}(\Omega, \mathbb{R}^N), \quad \nabla\mathbf{u}_k \to \nabla\mathbf{u} \quad \text{a.e. in } \Omega.$$

Furthermore, $|\nabla \mathbf{u}_k| \in W^{1,2}(\Omega)$ and $|\nabla|\nabla \mathbf{u}_k|| \leq |\nabla^2 \mathbf{u}_k|$ a.e. in Ω, by the chain rule for vector-valued Sobolev functions [MM, Theorem 2.1]. Thus, owing to the compactness of the trace embedding $\mathrm{Tr} : W^{1,2}(\Omega) \to L^1(\partial\Omega)$, we may also assume that

$$\text{(6.11)} \qquad \mathrm{Tr}\,|\nabla \mathbf{u}_k| \to \mathrm{Tr}\,|\nabla \mathbf{u}| \qquad \mathcal{H}^{n-1} \text{ a.e. on } \partial\Omega.$$

We claim that

$$\text{(6.12)} \qquad -\mathbf{div}(a(|\nabla \mathbf{u_k}|)\nabla \mathbf{u_k}) \to \mathbf{f} \quad \text{in } L^2(\Omega, \mathbb{R}^N).$$

Let us verify this claim. First, since $\nabla \mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^{Nn})$, an application of the chain rule for vector-valued Sobolev functions again tells us that, for each $\alpha = 1, \dots N$,

$$\text{(6.13)} \quad \mathrm{div}(a(|\nabla \mathbf{u}|)\nabla u^\alpha) = \frac{a'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \sum_{\beta=1}^{N} \sum_{i,j=1}^{n} u^\alpha_{x_i} u^\beta_{x_j} u^\beta_{x_j x_i} \chi_{\{\nabla \mathbf{u} \neq 0\}} + a(|\nabla \mathbf{u}|)\Delta u^\alpha \quad \text{a.e. in } \Omega,$$

and that the same equation holds with $\mathbf{u}$ replaced with $\mathbf{u}_k$. Here, and in what follows, we adhere the convention that $0 \cdot \infty = 0$, so that $\frac{\chi_{\{\nabla \mathbf{u} \neq 0\}}}{|\nabla \mathbf{u}|} = 0$ in $\{\nabla \mathbf{u} = 0\}$. Now, for each $k \in \mathbb{N}$ and $\alpha = 1, \dots N$,

(6.14)

$$\begin{aligned}
&\Big(\int_\Omega \big|\mathrm{div}(a(|\nabla \mathbf{u}_k|)\nabla u^\alpha_k) + f^\alpha(x)\big|^2 dx\Big)^{\frac12} = \Big(\int_\Omega \big|\mathrm{div}(a(|\nabla \mathbf{u}_k|)\nabla u^\alpha_k) - \mathrm{div}(a(|\nabla \mathbf{u}|)\nabla u^\alpha)\big|^2 dx\Big)^{\frac12} \\
&\quad = \Big(\int_\Omega \Big| \frac{a'(|\nabla \mathbf{u_k}|)}{|\nabla \mathbf{u_k}|} \sum_{\beta=1}^{N} \sum_{i,j=1}^{n} u^\alpha_{k\,x_i} u^\beta_{k\,x_j} u^\beta_{k\,x_j x_i} \chi_{\{\nabla \mathbf{u}_k \neq 0\}} + a(|\nabla \mathbf{u}_k|)\Delta u^\alpha_k \\
&\qquad - \frac{a'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \sum_{\beta=1}^{N} \sum_{i,j=1}^{n} u^\alpha_{x_i} u^\beta_{x_j} u^\beta_{x_j x_i} \chi_{\{\nabla \mathbf{u} \neq 0\}} - a(|\nabla \mathbf{u}|)\Delta u^\alpha \Big|^2 dx\Big)^{\frac12} \\
&\quad \leq \Big(\int_\Omega \big|a(|\nabla \mathbf{u_k}|)(\Delta u^\alpha_k - \Delta u^\alpha)\big|^2 dx\Big)^{\frac12} + \Big(\int_\Omega \big|(a(|\nabla \mathbf{u_k}|) - a(|\nabla \mathbf{u}|))\Delta u^\alpha\big|^2 dx\Big)^{\frac12} \\
&\qquad + \Big(\int_\Omega \Big| \frac{a'(|\nabla \mathbf{u_k}|)}{|\nabla \mathbf{u_k}|} \sum_{\beta=1}^{N} \sum_{i,j=1}^{n} u^\alpha_{k\,x_i} u^\beta_{k\,x_j} \chi_{\{\nabla \mathbf{u}_k \neq 0\}} (u^\beta_{k\,x_j x_i} - u^\beta_{x_j x_i}) \Big|^2 dx\Big)^{\frac12} \\
&\qquad + \Big(\int_\Omega \Big| \sum_{\beta=1}^{N} \Big(\frac{a'(|\nabla \mathbf{u_k}|)}{|\nabla \mathbf{u_k}|} \sum_{i,j=1}^{n} u^\alpha_{k\,x_i} u^\beta_{k\,x_j} \chi_{\{\nabla \mathbf{u}_k \neq 0\}} - \frac{a'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \sum_{i,j=1}^{n} u^\alpha_{x_i} u^\beta_{x_j} \chi_{\{\nabla \mathbf{u} \neq 0\}} \Big) u^\beta_{x_j x_i} \Big|^2 dx\Big)^{\frac12}.
\end{aligned}$$

Since the functions $a(t)$ and $a'(t)t$ are bounded, the first and the third addend on the rightmost side of (6.14) converge to 0 as $k \to \infty$, inasmuch as $u^\beta_{k\,x_j x_i} \to u^\beta_{x_j x_i}$ in $L^2(\Omega)$ as $k \to \infty$, for $\beta = 1, \dots N$ and $i, j = 1, \dots n$. The boundedness of the functions $a(t)$ and $a'(t)t$, and the convergence of $\nabla \mathbf{u}_k$ to $\nabla \mathbf{u}$ a.e. in Ω implies that the second and the fourth addend also converge to 0 by the dominated convergence theorem for integrals. Hence, (6.12) follows.

Step 2. Let $\{\mathbf{u}_k\}$ be the sequence considered in Step 1. For each $k \in \mathbb{N}$, the function $\mathbf{u}_k$ satisfies the same assumptions as the function $\mathbf{v}$ in Lemma 5.4. Hence, inequality (5.15) holds with $\mathbf{v}$ replaced with $\mathbf{u}_k$, namely

(6.15)
$$\begin{aligned} Cb(t) \int_{\{|\nabla \mathbf{u}_k|=t\}} |\nabla|\nabla \mathbf{u}_k||\, d\mathcal{H}^{n-1}(x) \leq t \int_{\{|\nabla \mathbf{u}_k|=t\}} &|\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)| d\mathcal{H}^{n-1}(x) \\ &+ \int_{\{|\nabla \mathbf{u}_k|>t\}} \frac{1}{a(|\nabla \mathbf{u}_k|)} |\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)|^2 \\ &+ a(t)t^2 \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{u}_k|>t\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } t > t_{\mathbf{u}_k}, \end{aligned}$$

where $C = \frac{(1+\min\{i_a,0\})^2}{2}$, and $t_{\mathbf{u}_k}$ is defined analogously to $t_{\mathbf{v}}$, with $\mathbf{v}$ replaced with $\mathbf{u}_k$. We claim that inequality (6.15) continues to hold with $\mathbf{u}_k$ replaced with $\mathbf{u}$, namely that

(6.16)
$$\begin{aligned} Cb(t) \int_{\{|\nabla \mathbf{u}|=t\}} |\nabla|\nabla \mathbf{u}||\, d\mathcal{H}^{n-1}(x) \leq t \int_{\{|\nabla \mathbf{u}|=t\}} &|\mathbf{f}(x)| d\mathcal{H}^{n-1}(x) \\ &+ \int_{\{|\nabla \mathbf{u}|>t\}} \frac{1}{a(|\nabla \mathbf{u}|)} |\mathbf{f}(x)|^2 dx \\ &+ a(t)t^2 \int_{\partial\Omega \cap \partial^M\{|\nabla \mathbf{u}|>t\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \qquad \text{for a.e. } t > t_{\mathbf{u}}. \end{aligned}$$

To verify this claim, observe that $t_{\mathbf{u}_k} \to t_{\mathbf{u}}$ as $k \to \infty$, fix any $t > t_{\mathbf{u}}$ and $h > 0$, and, for sufficiently large k, integrate inequality (6.15) over the interval $(t, t+h)$, and make use of the coarea formula (5.5) to obtain

(6.17)
$$\begin{aligned} C \int_{\{t<|\nabla \mathbf{u}_k|<t+h\}} b(|\nabla \mathbf{u}_k|) |\nabla|\nabla \mathbf{u}_k||^2\, dx \leq \int_{\{t<|\nabla \mathbf{u}_k|<t+h\}} &|\nabla \mathbf{u}_k| |\nabla|\nabla \mathbf{u}_k|| |\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)|\, dx \\ &+ \int_t^{t+h} \int_{\{|\nabla \mathbf{u}_k|>\tau\}} \frac{1}{a(|\nabla \mathbf{u}_k|)} |\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)|^2 dx\, d\tau \\ &+ \int_t^{t+h} a(\tau)\tau^2 \int_{\partial\Omega \cap \partial\{|\nabla \mathbf{u}_k|>\tau\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\, d\tau\, . \end{aligned}$$

We have that

(6.18)
$$\int_{\{t<|\nabla \mathbf{u}_k|<t+h\}} b(|\nabla \mathbf{u}_k|)|\nabla|\nabla \mathbf{u}_k||^2\,dx = \int_\Omega \chi_{\{t<|\nabla \mathbf{u}_k|<t+h\}}(x) b(|\nabla \mathbf{u}_k|)\sum_{i=1}^n \Big(\sum_{\beta=1}^N\sum_{j=1}^n \frac{u^\beta_{k\,x_j}}{|\nabla \mathbf{u}_k|}u^\beta_{k\,x_jx_i}\Big)^2\,dx$$
$$= \int_\Omega \chi_{\{t<|\nabla \mathbf{u}_k|<t+h\}}(x) b(|\nabla \mathbf{u}_k|)\sum_{i=1}^n \Big(\sum_{\beta=1}^N\sum_{j=1}^n \frac{u^\beta_{k\,x_j}}{|\nabla \mathbf{u}_k|}(u^\beta_{k\,x_jx_i} - u^\beta_{x_jx_i}) + \sum_{\beta=1}^N\sum_{j=1}^n \frac{u^\beta_{k\,x_j}}{|\nabla \mathbf{u}_k|}u^\beta_{x_jx_i}\Big)^2\,dx$$
$$= \int_\Omega \chi_{\{t<|\nabla \mathbf{u}_k|<t+h\}}(x) b(|\nabla \mathbf{u}_k|)\sum_{i=1}^n \Big(\sum_{\beta=1}^N\sum_{j=1}^n \frac{u^\beta_{k\,x_j}}{|\nabla \mathbf{u}_k|}(u^\beta_{k\,x_jx_i} - u^\beta_{x_jx_i})\Big)^2\,dx$$
$$+ 2\int_\Omega \chi_{\{t<|\nabla \mathbf{u}_k|<t+h\}}(x) b(|\nabla \mathbf{u}_k|)\sum_{i=1}^n \Big(\sum_{\beta=1}^N\sum_{j=1}^n \frac{u^\beta_{k\,x_j}}{|\nabla \mathbf{u}_k|}(u^\beta_{k\,x_jx_i} - u^\beta_{x_jx_i})\Big)\Big(\sum_{\beta=1}^N\sum_{j=1}^n \frac{u^\beta_{k\,x_j}}{|\nabla \mathbf{u}_k|}u^\beta_{x_jx_i}\Big)\,dx$$
$$+ \int_\Omega \chi_{\{t<|\nabla \mathbf{u}_k|<t+h\}}(x) b(|\nabla \mathbf{u}_k|)\sum_{i=1}^n \Big(\sum_{\beta=1}^N\sum_{j=1}^n \frac{u^\beta_{k\,x_j}}{|\nabla \mathbf{u}_k|}u^\beta_{x_jx_i}\Big)^2\,dx.$$

Note that the first equality holds by the chain rule for vector-valued functions. Since b is an increasing function,

$$\chi_{\{t<|\nabla \mathbf{u}_k|<t+h\}}(x) b(|\nabla \mathbf{u}_k|) \le b(t+h) \quad \text{for } x \in \Omega,$$

for every $k \in \mathbb{N}$. Moreover, $|u^\beta_{k\,x_j}|/|\nabla \mathbf{u}_k| \le 1$ for every $k \in \mathbb{N}$, $\beta = 1, \dots, N$, $j = 1, \dots, n$. Thus, the first integral on the rightmost side of (6.18) converges to 0 as $k \to \infty$, since $u^\beta_{k\,x_jx_i} \to u^\beta_{x_jx_i}$ in $L^2(\Omega)$ as $k \to \infty$, for $\beta = 1, \dots N$ and $i, j = 1, \dots n$. The same observation, combined with Hölder's inequality, ensures that also the second integral converges to 0 as $k \to \infty$. Since $\nabla \mathbf{u}_k \to \nabla \mathbf{u}$ a.e. in Ω, the last integral in (6.18) tends to

$$\int_{\{t<|\nabla \mathbf{u}|<t+h\}} b(|\nabla \mathbf{u}|)\sum_{i=1}^n \Big(\sum_{\beta=1}^N\sum_{j} \frac{u^\beta_{x_j}}{|\nabla \mathbf{u}|}u^\beta_{x_jx_i}\Big)^2\,dx, \tag{6.19}$$

by to the dominated convergence theorem for integrals, and the expression (6.19) agrees with $\int_{\{t<|\nabla \mathbf{u}|<t+h\}} b(|\nabla \mathbf{u}|)\,|\nabla|\nabla \mathbf{u}||^2\,dx$. Thus, we have shown that

$$\int_{\{t<|\nabla \mathbf{u}_k|<t+h\}} b(|\nabla \mathbf{u}_k|)|\nabla|\nabla \mathbf{u}_k||^2\,dx \to \int_{\{t<|\nabla \mathbf{u}|<t+h\}} b(|\nabla \mathbf{u}|)\,|\nabla|\nabla \mathbf{u}||^2\,dx \tag{6.20}$$

as $k \to \infty$. A similar argument implies, via (6.12), that

(6.21)
$$\int_{\{t<|\nabla \mathbf{u}_k|<t+h\}} |\nabla \mathbf{u}_k|\,|\nabla|\nabla \mathbf{u}_k||\,|\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)|\,dx \to \int_{\{t<|\nabla \mathbf{u}|<t+h\}} |\nabla \mathbf{u}|\,|\nabla|\nabla \mathbf{u}||\,|\mathbf{f}(x)|\,dx$$

as $k \to \infty$. Moreover, equation (6.12) and the boundedness of $\frac{1}{a}$ entail that the sequence

$$\int_{\{|\nabla \mathbf{u}_k|>\tau\}} \frac{1}{a(|\nabla \mathbf{u}_k|)}|\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)|^2 dx$$

is uniformly bounded for $\tau > 0$, and that, for every $\tau > 0$,

$$\int_{\{|\nabla \mathbf{u}_k|>\tau\}} \frac{1}{a(|\nabla \mathbf{u}_k|)} |\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)|^2 dx \to \int_{\{|\nabla \mathbf{u}|>\tau\}} \frac{1}{a(|\nabla \mathbf{u}|)} |\mathbf{f}(x)|^2 dx \tag{6.22}$$

as $k \to \infty$. Consequently,

$$\int_t^{t+h} \int_{\{|\nabla \mathbf{u}_k|>\tau\}} \frac{1}{a(|\nabla \mathbf{u}_k|)} |\mathbf{div}(a(|\nabla \mathbf{u}_k|)\nabla \mathbf{u}_k)|^2 dx\, d\tau \to \int_t^{t+h} \int_{\{|\nabla \mathbf{u}|>\tau\}} \frac{1}{a(|\nabla \mathbf{u}|)} |\mathbf{f}(x)|^2 dx\, d\tau \tag{6.23}$$

as $k \to \infty$. Let us finally focus on the last integral on the right-hand side of (6.15). For a.e. $\tau > 0$,

$$\partial\Omega \cap \partial\{|\nabla \mathbf{u}_k| > t\} = \{\mathrm{Tr}\, |\nabla \mathbf{u}_k| > \tau\} \quad \text{up to subsets of } \partial\Omega \text{ of } \mathcal{H}^{n-1} \text{ measure zero,} \tag{6.24}$$

and

$$\partial\Omega \cap \partial^M\{|\nabla \mathbf{u}| > t\} = \{\mathrm{Tr}\, |\nabla \mathbf{u}| > \tau\} \quad \text{up to subsets of } \partial\Omega \text{ of } \mathcal{H}^{n-1} \text{ measure zero.} \tag{6.25}$$

Equations (6.24) follow, for instance, from a close inspection of the proof of [Ma5, Lemma 6.5.1/2]. By (6.11), for a.e. $\tau > 0$,

$$\chi_{\{\mathrm{Tr}\, |\nabla \mathbf{u}_k|>\tau\}}(x) |\mathrm{tr}\mathcal{B}(x)| \to \chi_{\{\mathrm{Tr}\, |\nabla \mathbf{u}|>\tau\}} |\mathrm{tr}\mathcal{B}(x)| \quad \mathcal{H}^{n-1} \text{ a.e. on } \partial\Omega. \tag{6.26}$$

Hence, by the dominated convergence theorem for integrals,

$$\int_{\partial\Omega} \chi_{\{\mathrm{Tr}\, |\nabla \mathbf{u}_k|>\tau\}}(x) |\mathrm{tr}\mathcal{B}(x)|\, d\mathcal{H}^{n-1}(x) \to \int_{\partial\Omega} \chi_{\{\mathrm{Tr}\, |\nabla \mathbf{u}|>\tau\}}(x) |\mathrm{tr}\mathcal{B}(x)|\, d\mathcal{H}^{n-1}(x), \tag{6.27}$$

and the first integral in (6.27) is uniformly bounded for $\tau > 0$. Thus,

$$\begin{aligned} &\int_t^{t+h} a(\tau)\tau^2 \int_{\partial\Omega} \chi_{\{\mathrm{Tr}\, |\nabla \mathbf{u}_k|>\tau\}}(x)\, |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\, d\tau \\ &\qquad \to \int_t^{t+h} a(\tau)\tau^2 \int_{\partial\Omega} \chi_{\{\mathrm{Tr}\, |\nabla \mathbf{u}|>\tau\}}(x)\, |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\, d\tau \end{aligned} \tag{6.28}$$

as $k \to \infty$, whence, by (6.24) and (6.25),

$$\int_t^{t+h} a(\tau)\tau^2 \int_{\partial\Omega\cap\partial\{|\nabla \mathbf{u}_k|>\tau\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\, d\tau \to \int_t^{t+h} a(\tau)\tau^2 \int_{\partial\Omega\cap\partial^M\{|\nabla \mathbf{u}|>\tau\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\, d\tau \tag{6.29}$$

as $k \to \infty$. Combining (6.17), (6.20), (6.21), (6.23) and (6.29) tells that

$$\begin{aligned} C \int_{\{t<|\nabla \mathbf{u}|<t+h\}} b(|\nabla \mathbf{u}|) |\nabla |\nabla \mathbf{u}||^2\, dx \le &\int_{\{t<|\nabla \mathbf{u}|<t+h\}} |\nabla \mathbf{u}| |\nabla |\nabla \mathbf{u}|| |\mathbf{f}(x)|\, dx \\ &+ \int_t^{t+h} \int_{\{|\nabla \mathbf{u}|>\tau\}} \frac{1}{a(|\nabla \mathbf{u}|)} |\mathbf{f}(x)|^2 dx\, d\tau \\ &+ \int_t^{t+h} a(\tau)\tau^2 \int_{\partial\Omega\cap\partial^M\{|\nabla \mathbf{u}|>\tau\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\, d\tau. \end{aligned} \tag{6.30}$$

Dividing through by h in (6.30), making use of the coarea formula again, and passing to the limit as $h \to 0^+$ yields (6.16).

Step 3. Here, we show that, given $r > n-1$,

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \leq C b^{-1}\big(\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}\big) \tag{6.31}$$

for some constant C depending on i_a, s_a, n, r, $\|\mathrm{tr}\mathcal{B}\|_{L^r(\partial\Omega)}$, $|\Omega$, and on the constants in (5.1) and (5.2). More precisely, hereafter dependence on i_a and $|\Omega$ will mean just through a lower bound, and dependence on s_a, $\|\mathrm{tr}\mathcal{B}\|_{L^r(\partial\Omega)}$, and on the constants in (5.1) and (5.2) is just through an upper bound.
By the Hardy-Littlewood inequality (3.3),

$$\int_{\partial\Omega\cap\partial^M\{|\nabla \mathbf{u}|>t\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x) \leq \int_0^{\mathcal{H}^{n-1}(\partial\Omega\cap\partial^M\{|\nabla \mathbf{u}|>t\})} (\mathrm{tr}\mathcal{B})^*(r)dr \qquad \text{for a.e. } t>0, \tag{6.32}$$

where $(\mathrm{tr}\mathcal{B})^*$ denotes the decreasing rearrangement of $\mathrm{tr}\mathcal{B}$ with respect to the measure $\mathcal{H}^{n-1}$ on $\partial\Omega$. Since $|\nabla \mathbf{u}|$ is a Sobolev function, for a.e. $t > 0$,

$$\Omega \cap \partial^M\{|\nabla \mathbf{u}| > t\} = \{|\nabla \mathbf{u}| = t\}, \quad \text{up to sets of } \mathcal{H}^{n-1} \text{ measure zero}$$

(see e.g. [BZ].) Thus, by (5.2), there exists a constant $C = C(\Omega)$ such that

$$\mathcal{H}^{n-1}(\partial\Omega\cap\partial^M\{|\nabla \mathbf{u}|>t\}) \leq C\mathcal{H}^{n-1}(\{|\nabla \mathbf{u}|=t\}) \qquad \text{for a.e. } t \geq |\nabla \mathbf{u}|^*(|\Omega|/2). \tag{6.33}$$

Denote the distribution function $\mu_{|\nabla \mathbf{u}|}$ of $|\nabla \mathbf{u}|$, defined as in (3.1), simply by μ. By (5.1), there exists a constant $C = C(\Omega)$ such that

$$\mu(t)^{1/n'} \leq C\mathcal{H}^{n-1}(\{|\nabla \mathbf{u}|=t\}) \qquad \text{for a.e. } t \geq |\nabla \mathbf{u}|^*(|\Omega|/2). \tag{6.34}$$

From (6.33) and (6.34), we obtain that

$$\begin{aligned} \int_0^{\mathcal{H}^{n-1}(\partial\Omega\cap\partial^M\{|\nabla \mathbf{u}|>t\})} (\mathrm{tr}\mathcal{B})^*(r)dr &\leq \int_0^{C\mathcal{H}^{n-1}(\{|\nabla \mathbf{u}|=t\})} (\mathrm{tr}\mathcal{B})^*(r)dr \\ &= C\mathcal{H}^{n-1}(\{|\nabla \mathbf{u}|=t\})\, (\mathrm{tr}\mathcal{B})^{**}(C\mathcal{H}^{n-1}(\{|\nabla \mathbf{u}|=t\})) \\ &\leq C\mathcal{H}^{n-1}(\{|\nabla \mathbf{u}|=t\})\, (\mathrm{tr}\mathcal{B})^{**}\big(C'\mu(t)^{1/n'}\big) \\ &\qquad \text{for a.e. } t \geq |\nabla \mathbf{u}|^*(|\Omega|/2), \end{aligned} \tag{6.35}$$

for some constants $C = C(\Omega)$ and $C' = C'(\Omega)$. Observe that the last inequality holds since $(\mathrm{tr}\mathcal{B})^{**}$ is a non-increasing function. Coupling (6.16) with (6.35) tells us that

$$\begin{aligned} Cb(t)\int_{\{|\nabla \mathbf{u}|=t\}} |\nabla|\nabla \mathbf{u}||\, d\mathcal{H}^{n-1}(x) &\leq t\int_{\{|\nabla \mathbf{u}|=t\}} |\mathbf{f}(x)| d\mathcal{H}^{n-1}(x) + \int_{\{|\nabla \mathbf{u}|>t\}} \frac{1}{a(|\nabla \mathbf{u}|)}|\mathbf{f}(x)|^2 dx \\ &+ a(t)t^2\mathcal{H}^{n-1}(\{|\nabla \mathbf{u}|=t\})\, (\mathrm{tr}\mathcal{B})^{**}\big(C'\mu(t)^{1/n'}\big) \text{ for a.e. } t > t_{\mathbf{u}}, \end{aligned} \tag{6.36}$$

for some constants $C = C(\Omega, \min\{i_a, 0\})$ and $C' = C'(\Omega)$.
Now, we distinguish into the cases when a is non-decreasing or non-increasing.

First, assume that a is non-decreasing. Then we infer from (6.36) that

(6.37)
$$
\begin{aligned}
Cb(t)\int_{\{|\nabla \mathbf{u}|=t\}} |\nabla|\nabla \mathbf{u}||\, d\mathcal{H}^{n-1}(x) \le\ & t\int_{\{|\nabla \mathbf{u}|=t\}} |\mathbf{f}(x)| d\mathcal{H}^{n-1}(x)\\
&+\frac{1}{a(t)}\int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx\\
&+a(t)t^2\mathcal{H}^{n-1}(\{|\nabla \mathbf{u}|=t\})(\mathrm{tr}\mathcal{B})^{**}\big(C'\mu(t)^{1/n'}\big) \quad \text{for a.e. } t>t_{\mathbf{u}}.
\end{aligned}
$$

By Hölder's inequality and by (5.8) and (5.6),

(6.38)
$$
\begin{aligned}
\int_{\{|\nabla \mathbf{u}|=t\}} |\mathbf{f}(x)|\, d\mathcal{H}^{n-1}(x) &\le \bigg(\int_{\{|\nabla \mathbf{u}|=t\}} \frac{|\mathbf{f}(x)|^2}{|\nabla|\nabla \mathbf{u}||} d\mathcal{H}^{n-1}(x)\bigg)^{1/2}\bigg(\int_{\{|\nabla \mathbf{u}|=t\}} |\nabla|\nabla \mathbf{u}|| d\mathcal{H}^{n-1}(x)\bigg)^{1/2}\\
&\le \bigg(-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx\bigg)^{1/2}\bigg(-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\nabla|\nabla \mathbf{u}||^2 dx\bigg)^{1/2}
\end{aligned}
$$

for a.e. $t>0$. An analogous chain as in (6.38), with $|\mathbf{f}(x)|$ replaced with 1 yields

$$
\mathcal{H}^{n-1}(\{|\nabla \mathbf{u}|=t\}) \le (-\mu'(t))^{1/2}\bigg(-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\nabla|\nabla \mathbf{u}||^2 dx\bigg)^{1/2} \qquad \text{for a.e. } t>0. \tag{6.39}
$$

By the Hardy-Littlewwod inequality (3.3),

$$
\int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx \le \int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr \qquad \text{for } t>0. \tag{6.40}
$$

Inequalities (6.37) – (6.40), and inequality (5.9) applied with $v=|\nabla \mathbf{u}|$ entail that

$$
\begin{aligned}
Cb(t)&\bigg(-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\nabla|\nabla \mathbf{u}||^2 dx\bigg)\\
&\le t\bigg(-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx\bigg)^{1/2}\bigg(-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\nabla|\nabla \mathbf{u}||^2 dx\bigg)^{1/2}\\
&\quad+\frac{1}{a(t)}(-\mu'(t))^{1/2}\mu(t)^{-1/n'}\int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr\bigg(-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\nabla|\nabla \mathbf{u}||^2 dx\bigg)^{1/2}\\
&\quad+a(t)t^2(-\mu'(t))^{1/2}(\mathrm{tr}\mathcal{B})^{**}\big(C'\mu(t)^{1/n'}\big)\bigg(-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\nabla|\nabla \mathbf{u}||^2 dx\bigg)^{1/2}
\end{aligned} \tag{6.41}
$$

for a.e. $t>t_{\mathbf{u}}$. By (5.9) with $v=|\nabla \mathbf{u}|$, we have that $-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\nabla|\nabla \mathbf{u}||^2 dx>0$ for a.e. $t>t_{\mathbf{u}}$. Hence, we can may divide through by $-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\nabla|\nabla \mathbf{u}||^2 dx$ in (6.41), and exploit (5.9) with $v=|\nabla \mathbf{u}|$ again to obtain

$$
\begin{aligned}
Cb(t) \le\ & t(-\mu'(t))^{1/2}\mu(t)^{-1/n'}\bigg(-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx\bigg)^{1/2}\\
&+\frac{1}{a(t)}(-\mu'(t))\mu(t)^{-2/n'}\int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr\\
&+a(t)t^2(-\mu'(t))\mu(t)^{-1/n'}(\mathrm{tr}\mathcal{B})^{**}\big(C'\mu(t)^{1/n'}\big) \quad \text{for a.e. } t>t_{\mathbf{u}}.
\end{aligned} \tag{6.42}
$$

Since $|\nabla \mathbf{u}|$ is a Sobolev function, the function $|\nabla \mathbf{u}|^*$ is (locally absolutely) continuous [CEG, Lemma 6.6], and $|\nabla \mathbf{u}|^*(\mu(t)) = t$ for $t > 0$. Define the function $\phi_{\mathbf{f}} : (0, |\Omega|) \to [0, \infty)$ as

$$\phi_{\mathbf{f}}(s) = \bigg(\frac{d}{ds}\int_{\{|\nabla \mathbf{u}|>|\nabla \mathbf{u}|^*(s)\}} |\mathbf{f}(x)|^2 dx\bigg)^{1/2} \qquad \text{for a.e. } s \in (0, |\Omega|). \tag{6.43}$$

As a consequence,

$$\bigg(-\frac{d}{dt}\int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx\bigg)^{1/2} = (-\mu'(t))^{1/2}\phi_{\mathbf{f}}(\mu(t)) \qquad \text{for a.e. } t > 0, \tag{6.44}$$

and, by [CM1, Proposition 3.4],

$$\int_0^s \phi_{\mathbf{f}}^*(r)^2 dr \le \int_0^s |\mathbf{f}|^*(r)^2 dr \qquad \text{for } s \in (0, |\Omega|). \tag{6.45}$$

We thus deduce from inequality (6.42) that

$$\begin{aligned} Ca(t)b(t) \le\ & b(t)(-\mu'(t))\mu(t)^{-1/n'}\phi_{\mathbf{f}}(\mu(t)) \\ & + (-\mu'(t))\mu(t)^{-2/n'}\int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr + b(t)^2(-\mu'(t))\mu(t)^{-1/n'}(\mathrm{tr}\mathcal{B})^{**}\big(C'\mu(t)^{1/n'}\big) \end{aligned} \tag{6.46}$$

for a.e. $t > t_{\mathbf{u}}$. Let $t_{\mathbf{u}} \le t_0 < T < \|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}$, and let H be the function defined by (4.2). On estimating $b(t)$ by $b(T)$ for $t \in (t_0, T)$ on the right-hand side of (6.46), and integrating the resulting inequality over (t_0, T) yields

(6.47)

$$\begin{aligned} CH(T) \le\ & CH(t_0) + b(T)\int_{t_0}^T (-\mu'(t))\mu(t)^{-1/n'}\phi_{\mathbf{f}}(\mu(t))dt \\ & + \int_{t_0}^T (-\mu'(t))\mu(t)^{-2/n'}\int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr\, dt + b(T)^2\int_{t_0}^T (-\mu'(t))\mu(t)^{-1/n'}(\mathrm{tr}\mathcal{B})^{**}\big(C'\mu(t)^{1/n'}\big)dt \\ \le\ & CH(t_0) + b(T)\int_{\mu(T)}^{\mu(t_0)} s^{-1/n'}\phi_{\mathbf{f}}(s)\, ds \\ & + \int_{\mu(T)}^{\mu(t_0)} s^{-2/n'}\int_0^s |\mathbf{f}|^*(r)^2 dr\, ds + b(T)^2\int_{\mu(T)^{1/n'}}^{\mu(t_0)^{1/n'}} (\mathrm{tr}\mathcal{B})^{**}(C's)s^{\frac{1}{n-1}}\frac{ds}{s}. \end{aligned}$$

Hence, owing to (4.13),

$$\begin{aligned} b(T)^2 \le\ & Cb(t_0)^2 + Cb(T)\int_0^{\mu(t_0)} s^{-1/n'}\phi_{\mathbf{f}}(s)\, ds \\ & + C\int_0^{\mu(t_0)} s^{-2/n'}\int_0^s |\mathbf{f}|^*(r)^2 dr\, ds + Cb(T)^2\int_0^{\mu(t_0)^{1/n'}} (\mathrm{tr}\mathcal{B})^{**}(C's)s^{\frac{1}{n-1}}\frac{ds}{s} \end{aligned} \tag{6.48}$$

for some constants $C = C(\Omega, i_a, s_a)$ and $C' = C'(\Omega)$. Note that the last integral is actually finite, since $\mathrm{tr}\mathcal{B} \in L^r(\partial\Omega) \to L^{n-1,1}(\partial\Omega)$ for $r > n-1$, by (3.7). Define the function $G : [0, \infty) \to [0, \infty)$ as

$$G(s) = C\int_0^{s^{1/n'}} (\mathrm{tr}\mathcal{B})^{**}(C'r)r^{\frac{1}{n-1}}\frac{dr}{r} \qquad \text{for } s \ge 0, \tag{6.49}$$

where C and C' are as in (6.48). Set $s_0 = \min\{\alpha|\Omega|, G^{-1}(\frac{1}{2C})\}$, where α is given by (5.31), with $\mathbf{v}$ replaced with $\mathbf{u}$, and choose

$$t_0 = |\nabla \mathbf{u}|^*(s_0).$$

One has that $t_0 \geq t_{\mathbf{u}}$, inasmuch as $s_0 \leq \alpha|\Omega|$. Moreover, since $\mu(t_0) \leq G^{-1}(\frac{1}{2C})$,

$$C \int_0^{\mu(t_0)^{1/n'}} (\mathrm{tr}\mathcal{B})^{**}(C'r) r^{\frac{1}{n-1}} \frac{dr}{r} \leq \tfrac{1}{2}.$$

From (6.48) we thus infer that

$$b(T)^2 \leq Cb(t_0)^2 + Cb(T) \int_0^{|\Omega|} s^{-1/n'} \phi_{\mathbf{f}}(s)\, ds + C \int_0^{|\Omega|} s^{-2/n'} \int_0^s |\mathbf{f}|^*(r)^2 dr\, ds \tag{6.50}$$

for some constant $C = C(\Omega, i_a, s_a)$. By (6.45) and [CM1, Lemma 3.5], there exists a constant $C = C(n)$ such that

$$\int_0^{|\Omega|} s^{-1/n'} \phi_{\mathbf{f}}(s) ds \leq C \|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}. \tag{6.51}$$

Moreover, by [CM1, Lemma 3.6], there exists a constant $C = C(n)$ such that

$$\int_0^{|\Omega|} s^{-2/n'} \int_0^s |\mathbf{f}|^*(r)^2 dr\, ds \leq C \|\mathbf{f}\|^2_{L^{n,1}(\Omega,\mathbb{R}^N)}. \tag{6.52}$$

Owing to (6.50)–(6.52), there exists a constant $C = C(\Omega, i_a, s_a)$ such that

$$b(T)^2 \leq Cb(t_0)^2 + Cb(T)\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)} + C\|\mathbf{f}\|^2_{L^{n,1}(\Omega,\mathbb{R}^N)}. \tag{6.53}$$

Thus,

$$b(T) \leq Cb(t_0) + C\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)} \tag{6.54}$$

for some constant $C = C(\Omega, i_a, s_a)$. Next, let $\beta, \psi : [0, \infty) \to [0, \infty)$ be the functions defined by $\beta(t) = b(t)t$ for $t \geq 0$ and $\psi(s) = sb^{-1}(s)$ for $s \geq 0$. Proposition 6.2 and inequality (4.10) ensure that

$$C\psi\big(\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}\big) \geq \int_\Omega \beta(|\nabla \mathbf{u}|)dx \geq \int_{\{|\nabla \mathbf{u}| \geq t_0\}} \beta(|\nabla \mathbf{u}|)dx \geq \beta(t_0) \lim_{t \to t_0^-} \mu(t) \geq \beta(t_0)s_0, \tag{6.55}$$

for some constant $C = C(\Omega, i_a, s_a)$, whence, by (4.8),

$$\beta(t_0) \leq \psi\big(C\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}\big), \tag{6.56}$$

for some constant $C = C(\Omega, r, i_a, s_a)$. Since $b(\beta^{-1}(\psi(s))) = s$ for $s \geq 0$, inequality (6.56) implies that

$$b(t_0) \leq C\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}. \tag{6.57}$$

Hence, by (6.54),

$$b(T) \leq C\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)} \tag{6.58}$$

for some constant $C = C(\Omega, r, i_a, s_a)$. Taking the limit as $T \to \|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}$ in (6.58), and making use of (4.9), yields inequality (6.31). An inspection of the proof shows that the constant in (6.31) actually depends on the specified quantities.

Assume next that a is non-increasing. From (6.36) we deduce that

(6.59)

$$Cb(t)\int_{\{|\nabla \mathbf{u}|=t\}} |\nabla|\nabla \mathbf{u}||\, d\mathcal{H}^{n-1}(x) \le t\int_{\{|\nabla \mathbf{u}|=t\}} |\mathbf{f}(x)| d\mathcal{H}^{n-1}(x) + \frac{1}{a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega)})}\int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{f}(x)|^2 dx$$
$$+ a(t)t^2\, \mathcal{H}^{n-1}(\{|\nabla \mathbf{u}| = t\})\, (\mathrm{tr}\mathcal{B})^{**}\big(C'\mu(t)^{1/n'}\big) \text{ for a.e. } t > t_{\mathbf{u}}.$$

Observe that, although $\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}$ is not known to be finite at this stage yet, the quantity $\frac{1}{a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})}$ is finite, since assumption (6.9) is still in force. Starting from (6.59), and arguing as in the proof of (6.46), one can now show that

(6.60)

$$\begin{aligned} Ca(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})b(t) \le\; & ta(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})(-\mu'(t))\mu(t)^{-1/n'}\phi_{\mathbf{f}}(\mu(t)) \\ & + (-\mu'(t))\mu(t)^{-2/n'}\int_0^{\mu(t)} |\mathbf{f}|^*(r)^2 dr \\ & + a(t)t^2 a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})(-\mu'(t))\mu(t)^{-1/n'}(\mathrm{tr}\mathcal{B})^{**}\big(C'\mu(t)^{1/n'}\big) \end{aligned}$$

for a.e. $t > t_{\mathbf{u}}$. Let us fix t_0 and T as above. For every $t \in (t_0, T)$, the expression $ta(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})$ on the right-hand side of (6.60) can be estimated from above by $Ta(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})$. Also, owing to (4.10), the quantity $a(t)t^2$ can be bounded by $CB(T)$ for some constant $C = C(s_a)$. Integrating the resulting inequality over (t_0, T) tells us that

(6.61)

$$B(T)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}) \le CB(t_0)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}) + CTa(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})\int_0^{\mu(t_0)} s^{-1/n'}\phi_{\mathbf{f}}(s)\, ds$$
$$+ C\int_0^{\mu(t_0)} s^{-2/n'}\int_0^s |\mathbf{f}|^*(r)^2 dr\, ds + Ca(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})B(T)\int_0^{\mu(t_0)^{1/n'}} (\mathrm{tr}\mathcal{B})^{**}(C's)s^{\frac{1}{n-1}}\frac{ds}{s}$$

for some constants $C = C(\Omega, i_a, s_a)$ and $C' = C'(\Omega)$. Exploiting inequality (6.61) instead of (6.48), and arguing as in the proof of (6.53) lead to

(6.62)
$$\begin{aligned} & B(T)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}) \\ & \quad \le CB(t_0)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}) + CTa(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)} + C\|\mathbf{f}\|^2_{L^{n,1}(\Omega,\mathbb{R}^N)} \end{aligned}$$

for some constant $C = C(\Omega, i_a, s_a)$. Dividing through by T, and recalling (4.10) entail that

(6.63)
$$\begin{aligned} & b(T)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}) \\ & \quad \le C\frac{B(t_0)}{T}a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}) + Ca(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)} + \frac{C}{T}\|\mathbf{f}\|^2_{L^{n,1}(\Omega,\mathbb{R}^N)} \end{aligned}$$

for some constant $C = C(\Omega, i_a, s_a)$. The limit as $T \to \|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}$ of the right-hand side of (6.63) is obviously finite, and hence the limit of the left-hand side is finite as well. Thus,

$\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} < \infty$, since $\lim_{T\to\infty} b(T) = \infty$. Taking $T = \|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}$ in (6.63), and multiplying through by $\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}$ yields

$$\begin{aligned}(6.64)\quad & b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})^2 \\ &\le CB(t_0)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}) + Cb(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)} + C\|\mathbf{f}\|^2_{L^{n,1}(\Omega,\mathbb{R}^N)}.\end{aligned}$$

Observe that, by (4.10) and (6.57),

$$\begin{aligned}(6.65)\quad B(t_0)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}) &\le Ct_0b(t_0)a(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}) \\ &\le Cb(t_0)b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}) \le C'b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)},\end{aligned}$$

for some constants $C = C(s_a)$ and $C' = C'(\Omega, r, i_a, s_a)$. Coupling (6.64) with (6.65) tells us that

$$b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})^2 \le Cb(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)} + C\|\mathbf{f}\|^2_{L^{n,1}(\Omega,\mathbb{R}^N)}$$

for some constant $C = C(\Omega, r, i_a, s_a)$. Hence,

$$b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}) \le C\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)} \tag{6.66}$$

for some constant $C = C(\Omega, r, i_a, s_a)$, and (6.31) follows also in this case, with a constant C depending on the specified quantities.

Step 4. The present step exploits some variants of the arguments of Steps 1-3 in order to show that inequality (6.31) holds with a constant C which only depends on i_a, s_a, n, $|\Omega$, the constants in (5.1) and (5.2), and on $\mathrm{tr}\mathcal{B}$ just through (an upper bound for) the norm $\|\mathrm{tr}B\|_{L^{n-1,1}(\partial\Omega)}$, instead of a stronger norm $\|\mathrm{tr}B\|_{L^r(\partial\Omega)}$ with $r > n-1$. The piece of information that was missing until this stage, and makes this further step possible, is that now the solution $\mathbf{u}$ is known to satisfy

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} < \infty, \tag{6.67}$$

and hence we can exploit inequality (5.14) in the place of (5.15). By (6.67), $\mathbf{u} \in W^{1,\infty}_0(\Omega,\mathbb{R}^N) \cap W^{2,2}(\Omega,\mathbb{R}^N)$. Hence, there exists a sequence $\{\mathbf{u}_k\} \subset C^\infty(\Omega,\mathbb{R}^N) \cap C^2(\overline{\Omega},\mathbb{R}^N)$ fulfilling (6.10)–(6.12), such that $\mathbf{u}_k = 0$ on $\partial\Omega$, and, ij addition,

$$\|\nabla \mathbf{u}_k\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \to \|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \quad \text{as } k \to \infty. \tag{6.68}$$

Inequality (5.14) holds with $\mathbf{v}$ replaced with $\mathbf{u}_k$. An analogous argument as in Step 2 shows that the same inequality continues to hold for $\mathbf{u}$, namely

(6.69)
$$\begin{aligned}\frac{(1+\min\{i_a,0\})^2}{2} b(t) \int_{\{|\nabla \mathbf{u}|=t\}} |\nabla|\nabla \mathbf{u}||\, d\mathcal{H}^{n-1}(x) &\le t\int_{\{|\nabla \mathbf{u}|=t\}} |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})| d\mathcal{H}^{n-1}(x) \\ &+ \frac{\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}}{b(t)} \int_{\{|\nabla \mathbf{u}|>t\}} |\mathbf{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u})|^2 dx \\ &+ a\big(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\big)\|\nabla \mathbf{u}\|^2_{L^\infty(\Omega,\mathbb{R}^{Nn})} \int_{\partial\Omega\cap\partial\{|\nabla \mathbf{u}|>t\}} |\mathrm{tr}\mathcal{B}(x)| d\mathcal{H}^{n-1}(x)\end{aligned}$$

for a.e. $t>0$. We now start from (6.69), make use of arguments similar to – and even simpler than – those which lead to either (6.46) or (6.60) from (6.16) (in particular, now we do not need to distinguish into the cases when a is non-decreasing or non-increasing), to show that

(6.70)

$$\begin{aligned}Cb(t)^2 \leq\, & b\big(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\big)\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}(-\mu'(t))\mu(t)^{-1/n'}\phi_{\mathbf{f}}(\mu(t))\\ &+ \|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}(-\mu'(t))\mu(t)^{-2/n'}\int_0^{\mu(t)} |\mathbf{f}|^*(\mathbf{r})^{\mathbf{2}}\mathbf{dr}\\ &+ b\big(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\big)^2\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}(-\mu'(t))\mu(t)^{-1/n'}(\mathrm{tr}\mathcal{B})^{**}\big(C'\mu(t)^{1/n'}\big)\\ &\qquad\qquad \text{for a.e. } t\in[|\nabla \mathbf{u}|^*(|\Omega|/2), \|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}],\end{aligned}$$

for some positive constants $C=C(\Omega,\min\{i_a,0\})$, and $C'=C'(\Omega)$. Moreover, the dependence on Ω is only through the constants in (5.1) and (5.2).
Let F be the function defined by (4.3). Given $t_1\in[|\nabla u|^*(|\Omega|/2), \|\nabla u\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}]$, an integration in (6.70) yields, via (4.12),

(6.71)

$$\begin{aligned}F(|\nabla \mathbf{u}|^*(s)) \leq\, & CF(t_1) + Cb(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})\|\nabla\ u\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\int_0^{\mu(t_1)} r^{-1/n'}\phi(r)dr\\ &+ C\|\nabla\ u\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\int_0^{\mu(t_1)} r^{-2/n'}\int_0^r f^*(\rho)^2 d\rho\, dr\\ &+ CF\big(\|\nabla\ u\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\big)\int_0^{\mu(t_1)^{1/n'}}(\mathrm{tr}\mathcal{B})^{**}(C'r)r^{\frac{1}{n-1}}\frac{dr}{r} \quad \text{for } s\in[0,\mu(t_1)),\end{aligned}$$

for some constants $C=C(\Omega,i_a,s_a)$ and $C'=C'(\Omega)$. Let G be the function defined as in (6.49), save that now C and C' are the constants appearing in (6.71). Set $s_1=\min\{\frac{|\Omega|}{2}, G^{-1}\big(\frac{1}{2C}\big)\}$, and choose

$$t_1 = |\nabla\ u|^*(s_1).$$

Since $\mu(t_1)\leq G^{-1}\big(\frac{1}{2C}\big)$, we have that

$$C\int_0^{\mu(t_1)^{1/n'}}(\mathrm{tr}\mathcal{B})^{**}(C'r)r^{\frac{1}{n-1}}\frac{dr}{r}\leq 1/2.$$

From the resulting inequality with $s=0$ we infer that

(6.72)

$$\begin{aligned}F\big(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\big) \leq\, & CF(t_1) + C\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})\int_0^{\mu(t_1)} r^{-1/n'}\phi(r)dr\\ &+ C\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\int_0^{\mu(t_1)} r^{-2/n'}\int_0^r |\mathbf{f}|^*(\rho)^2 d\rho\, dr\end{aligned}$$

for some constant $C=C(\Omega,i_a,s_a)$. From (6.72), via (6.51), (6.52) and (4.12), we deduce that there exists a constant $C=C(\Omega,i_a,s_a)$ such that

$$b(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})^2 \leq Cb(t_1)^2 + Cb(\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})})\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)} + C\|\mathbf{f}\|^2_{L^{n,1}(\Omega,\mathbb{R}^N)}. \tag{6.73}$$

An inspection of the proof shows that, in fact, the dependence of C on Ω is only through $|\Omega|$ and on the constants in (5.1) and (5.2). Starting from (6.73) instead of (6.53), and arguing as in Step 3, yield (6.31) with a constant C depending on i_a, s_a, n, $|\Omega|$, $\|\mathrm{tr}\mathcal{B}\|_{L^{n-1,1}(\partial\Omega)}$ and on the constants in (5.1) and (5.2).

Step 5. Here we remove the additional assumption (6.8). CHECK
Since smooth functions are dense in $W^2L^{n-1,1}(U)$ for every open set $U \subset \mathbb{R}^{n-1}$, there exists a sequence $\{\Omega_m\}_{m\in\mathbb{N}}$ of domains $\Omega_m \supset \Omega$ such that $\partial\Omega_m \in C^\infty$, $|\Omega_m \setminus \Omega| \to 0$, $\Omega_m \to \Omega$ with respect to the Hausdorff distance, and $\|\mathrm{tr}\mathcal{B}_m\|_{L^{n-1,1}(\partial\Omega_m)} \le C$ for some constant $C = C(\Omega)$, where $\mathrm{tr}\mathcal{B}_m$ denotes the trace of the second fundamental form on $\partial\Omega_m$. The sequence $\{\Omega_m\}_{m\in\mathbb{N}}$ can be chosen in such a way that the constants appearing in (5.1) and (5.2), and hence in (5.3) and (5.9), with Ω replaced with Ω_m are bounded, up to a multiplicative constant independent of m, by the corresponding constants for Ω. Let $\mathbf{f}$ be continued by 0 in $\Omega_m \setminus \Omega$, and let $\mathbf{u}_m$ be the solution to (2.5) with Ω replaced with Ω_m. Owing to the estimates for $\mathbf{u}_m$ in $W^{2,2}_{\mathrm{loc}}(\Omega_m)$ [BF, Theorem 8.1], for every open set Ω' such that $\overline{\Omega'} \subset \Omega$, there exists a constant C such that

$$\|\mathbf{u}_m\|_{W^{2,2}(\Omega')} \le C \tag{6.74}$$

for $m \in \mathbb{N}$, and by estimate (6.31) (established in Step 4) with Ω replaced with Ω_m and $\mathbf{u}$ replaced with $\mathbf{u}_m$, there exists a a constant C such that

$$\|\nabla\mathbf{u}_m\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \le C \tag{6.75}$$

for $m \in \mathbb{N}$. Note that the constant C in (6.74) and (6.75) is independent of m. Let $s \in [1, \frac{2n}{n-2})$. If $\partial\Omega'$ is smooth, then the embedding $W^{2,2}(\Omega',\mathbb{R}^N) \to W^{1,s}(\Omega',\mathbb{R}^N)$ is compact. Thus, by (6.74) and (6.75), there exists a function $\mathbf{u} \in W^{1,\infty}(\Omega,\mathbb{R}^N)$, and a subsequence of $\{\mathbf{u}_m\}$, still denoted by $\{\mathbf{u}_m\}$, such that

$$\mathbf{u}_m \to \mathbf{u} \qquad \text{in } W^{1,s}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$$

and

$$\nabla\mathbf{u}_m \to \nabla\mathbf{u} \quad \text{a.e. in } \Omega. \tag{6.76}$$

Since $\mathbf{u}_m = 0$ on $\partial\Omega_m$, and $\Omega_m \to \Omega$ in the Hausdorff distance, it is easily seen from (6.75) that $\mathbf{u} = 0$ on $\partial\Omega$. Thus, in particular, $\mathbf{u} \in W^{1,B}_0(\Omega,\mathbb{R}^N)$. The function $\mathbf{u}$ is the weak solution to the Dirichlet problem (2.5). Indeed, inasmuch as $\Omega \subset \Omega_m$ for each $m \in \mathbb{N}$,

$$\int_{\Omega_m} a(|\nabla\mathbf{u}_m|)\nabla\mathbf{u}_m \cdot \nabla\phi\, dx = \int_{\Omega_m} \mathbf{f}\cdot\phi\, dx \tag{6.77}$$

for every $\phi \in C^\infty_0(\Omega,\mathbb{R}^N)$. Passing to the limit as $m \to \infty$ in (6.77) yields

$$\int_\Omega a(|\nabla\mathbf{u}|)\nabla\mathbf{u} \cdot \nabla\phi\, dx = \int_\Omega \mathbf{f}\cdot\phi\, dx \tag{6.78}$$

for every $\phi \in C^\infty_0(\Omega,\mathbb{R}^N)$, owing to (6.76) and (6.75), via the dominated convergence theorem for integrals. Since $B \in \Delta_2$, the space $C^\infty_0(\Omega,\mathbb{R}^N)$ is dense in $W^{1,B}_0(\Omega,\mathbb{R}^N)$. Thus, (6.78) holds, in fact, for every $\phi \in W^{1,B}_0(\Omega,\mathbb{R}^N)$. Note that, by (6.76), the solution u satisfies

$$\|\nabla\mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \le Cb^{-1}\big(\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}\big), \tag{6.79}$$

since such an estimate is fulfilled, with $\mathbf{u}$ replaced with $\mathbf{u}_m$, by (6.31). Here, the constant C depends on i_a, s_a, n, $|\Omega|$, $\|\mathrm{tr}\mathcal{B}\|_{L^{n-1,1}(\partial\Omega)}$ and on the constants in (5.1) and (5.2).

Step 6 We conclude by removing assumption (6.9).
Let $\{a_\varepsilon\}_{\varepsilon\in(0,1)}$ be the family of functions approximating the function a, defined in Lemma 4.5, and let b_ε and B_ε be as in its statement. Let $\mathbf{u}$ be the weak solution in $W^{1,B}_0(\Omega,\mathbb{R}^N)$ to problem (2.5), and let $\mathbf{u}_\varepsilon$ denote the solution in $W^{1,B_\varepsilon}_0(\Omega,\mathbb{R}^N)$ to the problem

$$\begin{cases} -\mathrm{div}(a_\varepsilon(|\nabla\mathbf{u}_\varepsilon|)\nabla\mathbf{u}_\varepsilon) = \mathbf{f}(x) & \text{in } \Omega\,, \\ \mathbf{u}_\varepsilon = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{6.80}$$

We claim that

$$\lim_{\varepsilon\to 0^+} \nabla\mathbf{u}_\varepsilon \to \nabla\mathbf{u} \qquad \text{in measure,} \tag{6.81}$$

and hence there exists a sequence $\varepsilon_k \to 0$ such that

$$\lim_{k\to\infty} \nabla\mathbf{u}_{\varepsilon_k} \to \nabla\mathbf{u} \qquad \text{a.e. in } \Omega. \tag{6.82}$$

By the previous steps, there exists a constant $C = C(\Omega, i_a, s_a)$ (in particular, independent of ε, owing to (4.32)), such that

$$b_\varepsilon\big(C\|\nabla\mathbf{u}_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^{Nn})}\big) \le \|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}. \tag{6.83}$$

Thus, by the definition of b_ε and by (4.34), it is easily seen that there exists a constant C, independent of ε, such that

$$\|\nabla\mathbf{u}_\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \le C, \tag{6.84}$$

whence

$$\int_\Omega B(|\nabla\mathbf{u}_\varepsilon|)\,dx \le C \tag{6.85}$$

for some constant C independent of ε.
We preliminarily observe that, although the function $\mathbf{u}$ need not belong to $W^{1,B_\varepsilon}_0(\Omega,\mathbb{R}^N)$ in the case when a is non-increasing, it can still be used as a test function in the weak formulation of problem (6.80). Indeed, by (6.84), $a_\varepsilon(|\nabla\mathbf{u}_\varepsilon|)\nabla\mathbf{u}_\varepsilon \in L^\infty(\Omega,\mathbb{R}^N)$. Therefore, since $\mathbf{u} \in W^{1,B}_0(\Omega,\mathbb{R}^N)$, and the latter space is embedded into $W^{1,1}_0(\Omega,\mathbb{R}^N)$, the function $\mathbf{u}$ can be approximated by a sequence $\{\mathbf{u}_k\} \in C^\infty_0(\Omega)$ of functions such that $\mathbf{u}_k \to \mathbf{u}$ in $L^{n'}(\Omega,\mathbb{R}^N)$, and hence in $L^{n',\infty}(\Omega,\mathbb{R}^N)$, and $\nabla\mathbf{u}_k \to \nabla\mathbf{u}$ in $L^1(\Omega,\mathbb{R}^N)$. This allows one to employ $\mathbf{u}_k$ as a test function in the weak formulation of problem (6.80), and then pass to the limit as $k\to\infty$.
The test function $\phi = \mathbf{u}_\varepsilon - \mathbf{u}$ can thus be used both in the weak formulation of problem (2.5), and in that of problem (6.80). Subtract the resulting equations yields

$$\begin{aligned} &\int_\Omega [a_\varepsilon(|\nabla\mathbf{u}_\varepsilon|)\nabla\mathbf{u}_\varepsilon - a(|\nabla\mathbf{u}_\varepsilon|)\nabla\mathbf{u}_\varepsilon]\cdot(\nabla\mathbf{u}_\varepsilon - \nabla\mathbf{u})\,dx \\ &\qquad = \int_\Omega [a(|\nabla\mathbf{u}|)\nabla\mathbf{u} - a(|\nabla\mathbf{u}_\varepsilon|)\nabla\mathbf{u}_\varepsilon]\cdot(\nabla\mathbf{u}_\varepsilon - \nabla\mathbf{u})\,dx. \end{aligned} \tag{6.86}$$

Fix any $\sigma > 1$. By Young's inequality,

(6.87)
$$\begin{aligned} &\left|\int_\Omega [a_\varepsilon(|\nabla\mathbf{u}|)\nabla\mathbf{u} - a(|\nabla\mathbf{u}|)\nabla\mathbf{u}]\cdot(\nabla\mathbf{u}_\varepsilon - \nabla\mathbf{u})\,dx\right| \le \int_\Omega |a_\varepsilon(|\nabla\mathbf{u}|)\nabla\mathbf{u} - a(|\nabla\mathbf{u}|)\nabla\mathbf{u}|\,|\nabla\mathbf{u}_\varepsilon - \nabla\mathbf{u}|\,dx \\ &\quad \le \int_\Omega \widetilde{B}\big(\sigma|a_\varepsilon(|\nabla\mathbf{u}_\varepsilon|)\nabla\mathbf{u}_\varepsilon - a(|\nabla\mathbf{u}_\varepsilon|)\nabla\mathbf{u}_\varepsilon|\big)\,dx + \int_\Omega B\big(\tfrac{1}{\sigma}|\nabla\mathbf{u}_\varepsilon - \nabla\mathbf{u}|\big)\,dx. \end{aligned}$$

Since B is a Young function of class Δ_2, there exists a constant C such that $B((t+s)/\sigma) \leq CB(t)/\sigma + CB(s)/\sigma$ for $t \geq 0$ and and $\sigma > 1$. Hence, owing to (6.85), there exist positive constants C and C', independent of ε, such that

$$\int_\Omega B\big(\tfrac{1}{\sigma}|\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}|\big)\,dx \leq \frac{C}{\sigma}\bigg(\int_\Omega B\big(|\nabla \mathbf{u}_\varepsilon|\big)\,dx + \int_\Omega B\big(|\nabla \mathbf{u}|\big)\,dx\bigg) \leq \frac{C'}{\sigma}\,. \tag{6.88}$$

Next, let C be the constant appearing in (6.84), and fix any $t > C$. We have that

$$\begin{aligned}\int_\Omega &\widetilde{B}\big(\sigma|a_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon - a(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon|\big)\,dx \\ &= \int_{\{|\nabla u_\varepsilon| \leq t\}} \widetilde{B}\big(\sigma|a_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon - a(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon|\big)\,dx \\ &\quad + \int_{\{|\nabla u_\varepsilon| > t\}} \widetilde{B}\big(\sigma|a_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon - a(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon|\big)\,dx.\end{aligned} \tag{6.89}$$

By the choice of t, the last integral vanishes for every $\varepsilon \in (0,1)$. On the other hand, by (4.33),

$$\int_{\{|\nabla \mathbf{u}_\varepsilon| \leq t\}} \widetilde{B}\big(\sigma|a_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon - a(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon|\big)\,dx < \delta\,, \tag{6.90}$$

if ε is sufficently small. Thanks to the arbirtrariness of δ, we infer from (6.86)–(6.90) that

$$\lim_{\varepsilon \to 0^+} \int_\Omega [a_\varepsilon(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon - a_\varepsilon(|\nabla \mathbf{u}|)\nabla \mathbf{u}] \cdot (\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u})\,dx = 0. \tag{6.91}$$

We now follow an argument from [BBGGPV]. Fix any $\delta > 0$. Given $t, \tau > 0$, we have that

$$\begin{aligned}&|\{|\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}| > t\}| \\ &\qquad \leq |\{|\nabla \mathbf{u}_\varepsilon| > \tau\}| + |\{|\nabla \mathbf{u}| > \tau\}| + |\{|\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}| > t, |\nabla \mathbf{u}_\varepsilon| \leq \tau, |\nabla \mathbf{u}| \leq \tau\}|.\end{aligned} \tag{6.92}$$

If τ is sufficiently large, then

$$|\{|\nabla \mathbf{u}| > \tau\}| < \delta, \tag{6.93}$$

and, by (6.84),

$$|\{|\nabla \mathbf{u}_\varepsilon| > \tau\}| = 0 \quad \text{for } \varepsilon \in (0,1). \tag{6.94}$$

Next, define

$$\vartheta(t,\tau) = \inf\{[a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) : |\xi - \eta| \geq t, |\xi| \leq \tau, |\eta| \leq \tau\},$$

and observe that, by Lemma 4.4, $\vartheta(t,\tau) > 0$. Thus, since

$$\begin{aligned}\int_\Omega &[a(|\nabla \mathbf{u}_\varepsilon|)\nabla \mathbf{u}_\varepsilon - a(|\nabla \mathbf{u}|)\nabla \mathbf{u}] \cdot (\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u})\,dx \\ &\qquad \geq \vartheta(t,\tau)|\{|\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}| > t, |\nabla \mathbf{u}_\varepsilon| \leq \tau, |\nabla \mathbf{u}| \leq \tau\}|,\end{aligned}$$

by (6.91)

$$|\{|\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}| > t, |\nabla \mathbf{u}_\varepsilon| \leq \tau, |\nabla \mathbf{u}| \leq \tau\}| < \delta$$

if ε is sufficiently small. Consequently, by (6.92), (6.94), and (6.93),

$$|\{|\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}| > t\}| < 2\delta$$

if ε is sufficiently small. This proves (6.81). Inequality (2.6) follows from (6.82) and (6.83).

□

Proof of Theorem 2.2. The proof is the same as that of Theorem 2.1, save that the simplified versions of inequalities (5.14) and (5.15) without the term depending on tr$\mathcal{B}$, described in the last part of Lemma 5.4, have to be exploited in Steps 2 and 4, respectively. Also, a sequence of (smooth) convex approximating domains Ω_m has to be employed in Step 5. □

Proof of Theorem 2.4. CHECK The outline of the proof is analogous to that of Theorem 2.1. Hereafter, we point out some minor required variants.

Step 1. Under assumptions (6.8)–(6.9), the solution $\mathbf{u}$ to the Neumann problem (2.8) belongs to $W^{2,2}(\Omega,\mathbb{R}^N)$. A proof of this fact parallels that of [BF, Theorem 8.2] in the case of homogeneous Dirichlet boundary conditions. One can make use of this piece of information to conclude that $a(|\nabla\mathbf{u}|)\nabla\mathbf{u} \in W^{1,2}(\Omega,\mathbb{R}^N)$, and hence derive from an integration by parts in (6.2) that the boundary condition $\frac{\partial \mathbf{u}}{\partial \nu} = 0$ in fulfilled in the sense of traces.

As a consequence, a sequence $\{\mathbf{u}_k\} \subset C^\infty(\Omega,\mathbb{R}^N) \cap C^2(\overline{\Omega},\mathbb{R}^N)$ can be constructed such that

$$\mathbf{u}_k \to \mathbf{u} \quad \text{in } W^{2,2}(\Omega,\mathbb{R}^N), \quad \nabla\mathbf{u}_k \to \nabla\mathbf{u} \quad \text{a.e. in } \Omega, \text{ and } \frac{\partial \mathbf{u}_k}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{6.95}$$

Such construction can be accomplished as follows. First, one can (locally) reduce the problem in some neighborhood of each point $x_0 \in \partial\Omega$ to the case when $\partial\Omega$ is flat via a change of variables. In order to preserve the boundary condition $\frac{\partial \mathbf{u}}{\partial \nu} = 0$, the new coordinates $(y_1,\dots,y_n)$ can be chosen in such a way that the level surfaces $\{y_n = c_n\}$, with $c_n \in \mathbb{R}$, are level surfaces of the distance function to $\partial\Omega$, and the curves $\{y_1 = c_1,\cdots,y_{n-1} = c_{n-1}\}$, with $c_1 \in \mathbb{R}, \dots, c_{n-1}$, are orthogonal to these level surfaces. Second, the function $\mathbf{u}$ can be extended to a function $\widetilde{\mathbf{u}}$ beyond the flattened boundary of Ω by reflection, so that $\widetilde{\mathbf{u}}$ is symmetric with respect to the boundary. The function $\widetilde{\mathbf{u}}$ is now defined in a complete neighborhood U of x_0. The fact that $\frac{\partial \mathbf{u}}{\partial \nu} = 0$ on the relevant boundary ensures that such an extension is twice weakly differentiable, and hence belongs to $W^{2,2}(U)$. Standard mollification of $\widetilde{\mathbf{u}}$ by a symmetric kernel provides an approximation of $\widetilde{\mathbf{u}}$ in $W^{2,2}(U)$ by a sequence smooth functions $\widetilde{\mathbf{u}}_k$ which satisfy

$$\widetilde{\mathbf{u}}_k \to \mathbf{u} \quad \text{in } W^{2,2}(U,\mathbb{R}^N), \quad \nabla\widetilde{\mathbf{u}}_k \to \nabla\mathbf{u} \quad \text{a.e. in } U,$$

and are symmetric about the boundary of Ω. The latter property ensures that $\frac{\partial \widetilde{\mathbf{u}}_k}{\partial \nu} = 0$ on the boundary of Ω. The function $\mathbf{u}_k$ is then just defined as the restriction of $\widetilde{\mathbf{u}}_k$ to Ω.

As in the case of the Dirichlet problem, the sequence $\mathbf{u}_k$ so obtained also fulfills (6.11) and (6.12).

Step 2. Here one shows that inequality (5.34) is fulfilled when $\mathbf{v}$ equals the solution $\mathbf{u}$ to (2.8). This follows on applying (5.34) of Lemma 5.5 with $\mathbf{v} = \mathbf{u}_k$ (defined in Step 1), and passing to the limit as $k \to \infty$ via the same argument as in the Dirichlet case.

Step 3. This step is exactly the same as in the Dirichlet case, save that $|\mathcal{B}|$ replaces tr$\mathcal{B}$ everywhere.

Step 4. Here one applies inequality (5.33) with $\mathbf{v} = \mathbf{u}_k$, and obtains the same inequality for $\mathbf{u}$ on passing to the limit as $k \to \infty$ as in the case of the Dirichlet problem.

Step 5. We obtain equation (6.77) for every function $\phi \in \mathrm{Lip}(\Omega,\mathbb{R}^N)$, since any such function can be continued to a function $\phi \in \mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^N)$. Passing to the limit as $m \to \infty$ yields (6.78) for every $\phi \in \mathrm{Lip}(\Omega,\mathbb{R}^N)$, and by the density of the space $\mathrm{Lip}(\Omega,\mathbb{R}^N)$ in $W^{1,B}(\Omega,\mathbb{R}^N)$, we deduce that (6.78) holds for every $\phi \in W^{1,B}(\Omega,\mathbb{R}^N)$.

Step 6. This step is the same as in the case of the Dirichlet problem. □

Proof of Theorem 2.5. The proof consists in a slight modification of that of Theorem 2.4. Specifically, the versions of inequalities (5.33) and (5.34) where the term depending on $|\mathcal{B}|$ is dropped, described in the last part of Lemma 5.5, play a role in Steps 2 and 4, respectively. Moreover, the approximating domains Ω_m in Step 5 have to be chosen convex. $\square$

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