# Uniform asymptotic approximations of Green's functions in a long rod

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*Dedicated to Professor W.L. Wendland on the occasion of his 70th birthday*

**Abstract**

Asymptotic approximations for Green's function for the operator $-\Delta$ in a long rod are derived. These approximations are uniformly valid over the whole domain including the end regions of the rod. Connections are established between the asymptotic approximations in a long rod and the asymptotics in thin domains. Comparison between the structures of asymptotic approximations in a long rod and a domain with a small hole is also given.

*Key words:* Uniform asymptotic approximations of Green's functions, singular perturbations.

## 1 Introduction

The interest to the asymptotic analysis of Green's functions for domains with perturbed boundaries was initiated by the classical work of Hadamard [1]. The question of uniform asymptotic approximations of Green's functions for boundary value problems in singularly and regularly perturbed domains was addressed in [2], and the detailed analysis for the Dirichlet problems in domains with small holes was presented in [3]. Although some types of asymptotic approximations for Green's functions in singularly perturbed domains (for example, domains with small holes) have already been used in the existing literature (see, for example, [4], [5]), the question of uniformity of such approximations remained open until recently.

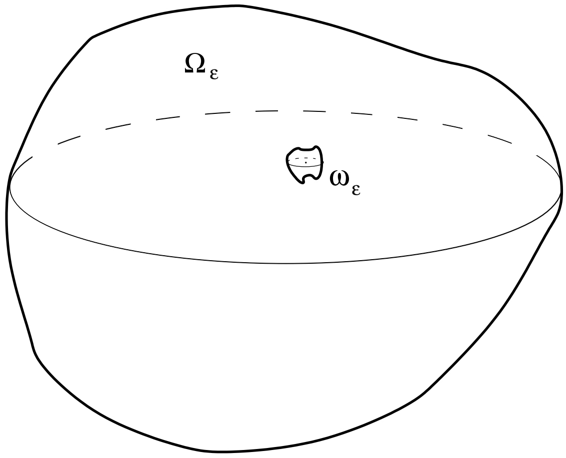


Fig. 1. A domain with a small hole.

To illustrate a concept of uniform asymptotic approximations for Green's functions, we suggest an example of a domain with a small hole. Let $\Omega$ and $\omega$ be bounded domains in $\mathbf{R}^n$, $n > 2$. Assume that $\Omega$ and $\omega$ contain the origin $O$ and introduce the domain $\omega_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in \omega\}$, as shown in Fig. 1. Without loss of generality, we can also assume that the minimum distance between the origin and the points of $\partial\Omega$ as well as the maximum distance between the origin and the points of $\partial\omega$ are equal to 1. Let $G_\varepsilon$ be Green's function of the Dirichlet problem for the Laplace operator in $\Omega_\varepsilon = \Omega \setminus \overline{\omega}_\varepsilon$. We use the notation $|S^{n-1}|$ for the $(n-1)$-dimensional measure of the unit sphere. By $G$ and $\mathfrak{G}$ we denote Green's functions of the Dirichelt problems in $\Omega$ and $\mathbf{R}^n \setminus \overline{\omega}$, respectively. Also the notation $H$ is used for the regular part of $G$, that is $H(\mathbf{x}, \mathbf{y}) = ((n-2)|S^{n-1}|)^{-1}|\mathbf{x}-\mathbf{y}|^{2-n} - G(\mathbf{x}, \mathbf{y})$, and $P$ stands for the harmonic capacitary potential of $\overline{\omega}$. The following asymptotic formula holds (see [2]):

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) + \varepsilon^{2-n}\mathfrak{G}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - ((n-2)|S^{n-1}|)^{-1}|\mathbf{x}-\mathbf{y}|^{2-n}$$
$$+H(0, \mathbf{y})P(\varepsilon^{-1}\mathbf{x}) + H(\mathbf{x}, 0)P(\varepsilon^{-1}\mathbf{y}) - H(0,0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y})$$
$$-\varepsilon^{n-2} \operatorname{cap} \overline{\omega}\; H(\mathbf{x}, 0)H(0, \mathbf{y}) + O\Big(\frac{\varepsilon^{n-1}}{(\min\{|\mathbf{x}|, |\mathbf{y}|\})^{n-2}}\Big) \qquad (1)$$

uniformly with respect to $\mathbf{x}$ and $\mathbf{y}$ in $\Omega_\varepsilon$. (Note that the remainder term in (1) is $O(\varepsilon)$ on $\partial\omega_\varepsilon$ and $O(\varepsilon^{n-1})$ on $\partial\Omega$. )

Although the above formula is uniformly valid in the whole domain $\Omega_\varepsilon$, it can be simplified if additional constraints are imposed on the positions of the points $\mathbf{x}$ and $\mathbf{y}$. Namely, if $\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon$, then

$$G_\varepsilon(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{y})$$

$$+\varepsilon^{n-2}\text{cap}\,\overline{\omega}\;G(\mathbf{x},0)G(0,\mathbf{y}) = O\Big(\frac{\varepsilon^{n-1}}{(|\mathbf{x}||\mathbf{y}|)^{n-1}\min\{|\mathbf{x}|,|\mathbf{y}|\}}\Big). \tag{2}$$

On the other hand, if $\max\{|\mathbf{x}|, |\mathbf{y}|\} < 1/2$, then

$$G_\varepsilon(\mathbf{x},\mathbf{y}) - \varepsilon^{2-n}\mathfrak{G}\Big(\frac{\mathbf{x}}{\varepsilon},\frac{\mathbf{y}}{\varepsilon}\Big)$$

$$+H(0,0)(P(\varepsilon^{-1}\mathbf{x})-1)(P(\varepsilon^{-1}\mathbf{y})-1) = O(\max\{|\mathbf{x}|,|\mathbf{y}|\}). \tag{3}$$

The asymptotic approximations above employ solutions of model problems defined in $\Omega$ and $\mathbf{R}^n \setminus \overline{\omega}$, independent of the small parameter $\varepsilon$, and such solutions can be efficiently implemented in the numerical algorithms incorporating the asymptotic formulae for Green's functions.

In the present paper we extend the idea of uniform asymptotic approximations of Green's functions to a mixed boundary value problem for the Laplacian in a long (or thin) domain. Dirichlet boundary conditions are set at the end regions of this domain, whereas the Neumann boundary condition are prescribed on the lateral surface.

We use a version of the method of compound asymptotic expansions of solutions to boundary value problems in singularly perturbed domains, developed in [6], in order to construct uniform asymptotic approximations of Green's kernels in elongated domains.

The structure of the paper can be described is as follows. Section 2 presents formulation of the problem and description of the model domains. Section 3 introduces the capacitary potential and its asymptotic approximation in the elongated domain. Asymptotic approximation of Green's function in the long rod is discussed in Sections 4 and 5. Finally, Sections 6 and 7 present the asymptotic formula for Green's function in a thin domain and concluding remarks related to the structures of asymptotic expansions in long (or thin) domains and domains with small holes.

## 2 The Dirichlet-Neumann problem in a long rod.

Let $C$ be the infinite cylinder $\{(\mathbf{x}', x_n) : \mathbf{x}' \in \omega, \ x_n \in \mathbf{R}\}$, where $\omega$ is a bounded domain in $\mathbf{R}^{n-1}$ with smooth boundary; here $n \geq 2$. Also let $C^{\pm}$ denote Lipschitz subdomains of $C$ separated from $\pm\infty$ by surfaces $\gamma^{\pm}$, respectively.

Let us introduce a positive number $M$ and the vector $\mathbf{M} = (\mathbf{O}', M)$, where $\mathbf{O}'$ is the origin of $\mathbf{R}^{n-1}$. It is assumed that the ratio $(\text{diam } \omega)/M$ is small.

A long rod $C_M$ is defined as follows

$$C_M = \{\mathbf{x} : (\mathbf{x} - \mathbf{M}) \in C^+, \ (\mathbf{x} + \mathbf{M}) \in C^-\},$$

the lateral surface of the rod is denoted by $\Gamma$, as shown in Fig. 2.

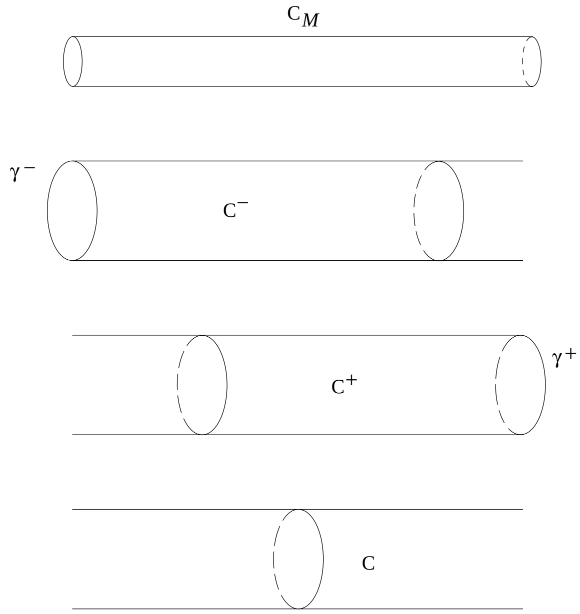


Fig. 2. A long rod $C_M$ and associated unbounded model domains.

Let $G_M(\mathbf{x}, \mathbf{y})$ denote the fundamental solution for $-\Delta$ in the domain $C_M$ subject to zero Neumann condition on the lateral surface $\Gamma$ and zero Dirichlet conditions on the end parts $\gamma^{\pm}$ of the boundary of the long rod:

$$\Delta_x G_M(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in C_M,$$

$$\frac{\partial G_M}{\partial n_x}(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \Gamma, \ \mathbf{y} \in C_M,$$
$$G_M(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \gamma^{\pm}, \ \mathbf{y} \in C_M.$$

In order to obtain an approximation of $G_M$ we also introduce several model problems independent of the cylinder length $M$.

By $G_\infty(\mathbf{x},\mathbf{y})$ we denote Green's function of the Neumann problem in $C$

$$\Delta_x G_\infty(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y} \in C,$$

$$\frac{\partial G_\infty}{\partial n_x}(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x} \in \Gamma, \mathbf{y} \in C,$$

$$G_\infty(\mathbf{x},\mathbf{y}) = -(2|\omega|)^{-1}|x_n - y_n| + O(\exp(-\alpha|x_n - y_n|)) \text{ as } |x_n| \to \infty,$$

where $\alpha$ is a positive constant, and $|\omega|$ is the $(n-1)$-dimensional measure of $\omega$.

Similarly, $G^+$ and $G^-$ stand for the fundamental solutions for $-\Delta$ in the domains $C^\pm$, with the homogeneous boundary conditions defined as follows

$$\Delta G^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm) + \delta(\mathbf{x}^\pm - \mathbf{y}^\pm) = 0, \quad \mathbf{x}^\pm, \mathbf{y}^\pm \in C^\pm,$$

$$G^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm) = 0, \quad \mathbf{x}^\pm \in \gamma^\pm, \mathbf{y}^\pm \in C^\pm,$$

$$\frac{\partial G^\pm}{\partial n}(\mathbf{x}^\pm,\mathbf{y}^\pm) = 0, \quad \mathbf{x}^\pm \in \Gamma, \mathbf{y}^\pm \in C^\pm,$$

and it is also assumed that $G^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm)$ are bounded as $x_n^\pm \to \mp\infty$.

## 3 Capacitary potential.

The capacitary potential $P_M$ is defined as a solution of the Dirichlet-Neumann boundary value problem in $C_M$:

$$\Delta P_M(\mathbf{x}) = 0, \quad \mathbf{x} \in C_M, \tag{4}$$

$$\frac{\partial P_M}{\partial n}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma, \tag{5}$$

$$P_M(\mathbf{x}) = 1, \quad \mathbf{x} \in \gamma^- \text{ and } P_M(\mathbf{x}) = 0, \quad \mathbf{x} \in \gamma^+. \tag{6}$$

We shall also use the solutions $\zeta^\pm$ of the homogeneous Dirichelt-Neumann problems in semi-infinite domains $C^\pm$, as follows:

$$\Delta \zeta^\pm(\mathbf{x}^\pm) = 0, \quad \mathbf{x} \in C^\pm, \tag{7}$$

$$\frac{\partial \zeta^\pm}{\partial n}(\mathbf{x}^\pm) = 0, \quad \mathbf{x}^\pm \in \Gamma, \tag{8}$$

$$\zeta^\pm(\mathbf{x}^\pm) = 0, \quad \mathbf{x} \in \gamma^\pm, \tag{9}$$

and

$$\zeta^\pm(\mathbf{x}^\pm) = \mp x_n^\pm + \zeta_\infty^\pm + O(\exp(-\alpha|x_n^\pm|)) \text{ as } |x_n^\pm| \to \infty, \tag{10}$$

where $\alpha$ is a positive constant, $\mathbf{x}^{\pm} = (\mathbf{x}', x_n \mp M)$ are local coordinates at the ends of the long rod $C_M$, and $\zeta_\infty^{\pm}$ are constant terms that depend on the geometry of the cross-section $\omega$ and the end parts $\gamma^{\pm}$ of the boundary of the long rod.

**Theorem 1.** *The following asymptotic formula, uniform with respect to* $\mathbf{x} \in C_M$*, for the capacitary potential* $P_M(\mathbf{x})$ *holds:*

$$P_M(\mathbf{x}) = \frac{M + x_n + \zeta_\infty^- - \zeta^-(\mathbf{x}^-) + \zeta^+(\mathbf{x}^+)}{2M + \zeta_\infty^+ + \zeta_\infty^-} + O(\exp(-\alpha M)). \tag{11}$$

*Here, the functions* $\zeta^{\pm}$*, variables* $\mathbf{x}^{\pm}$ *and the constants* $\zeta_\infty^{\pm}$ *are the same as in (7)–(10),* $\alpha$ *is a positive constant.*

The proof of this statement is achieved by the direct substitution of (11) into (5)–(6) and applying the energy estimate to the function $P_M$.

## 4 Asymptotic approximation of Green's function.

Let $H^{\pm}(\mathbf{x}^{\pm}, \mathbf{y}^{\pm})$ be functions defined in semi-infinite domains $C^{\pm}$, and assume that they also satisfy the Dirichlet-Neumann boundary value problems

$$\Delta_x H^{\pm}(\mathbf{x}^{\pm}, \mathbf{y}^{\pm}) = 0, \quad \mathbf{x}^{\pm}, \mathbf{y}^{\pm} \in C^{\pm}, \tag{12}$$

$$\frac{\partial H^{\pm}}{\partial n_x}(\mathbf{x}^{\pm}, \mathbf{y}^{\pm}) = 0, \quad \mathbf{x}^{\pm} \in \Gamma, \ \mathbf{y}^{\pm} \in C^{\pm}, \tag{13}$$

$$H^{\pm}(\mathbf{x}^{\pm}, \mathbf{y}^{\pm}) = G_\infty(\mathbf{x}, \mathbf{y}) + (2|\omega|)^{-1}\zeta^{\pm}(\mathbf{y}^{\pm}), \quad \mathbf{x} \in \gamma^{\pm}, \ \mathbf{y}^{\pm} \in C^{\pm}, \tag{14}$$

and

$$H^{\pm}(\mathbf{x}^{\pm}, \mathbf{y}^{\pm}) \to 0 \quad \text{as} \quad x_n^{\pm} \to \mp\infty. \tag{15}$$

The asymptotic approximation is given by the following statement.

**Theorem 2.** *Green's function* $G_M(\mathbf{x}, \mathbf{y})$ *is approximated by the asymptotic formula, uniform with respect to* $\mathbf{x}, \mathbf{y} \in C_M$

$$G_M(\mathbf{x}, \mathbf{y}) = G_\infty(\mathbf{x}, \mathbf{y}) - H^+(\mathbf{x}^+, \mathbf{y}^+) - H^-(\mathbf{x}^-, \mathbf{y}^-)$$

$$-\frac{\mathfrak{A}_M}{|\omega|}(\frac{1}{2} - P_M(\mathbf{x}))(\frac{1}{2} - P_M(\mathbf{y})) + \frac{\mathfrak{A}_M}{4|\omega|} + O(-\alpha M), \tag{16}$$

*where* $\mathfrak{A}_M = 2M + \zeta_\infty^+ + \zeta_\infty^-$, *and* $\alpha$ *is a positive constant.*

In the text below we present a formal argument that leads to the asymptotic formula (16).

Let

$$G_M(\mathbf{x}, \mathbf{y}) = G_\infty(\mathbf{x}, \mathbf{y}) - H_M^+(\mathbf{x}, \mathbf{y}) - H_M^-(\mathbf{x}, \mathbf{y}), \tag{17}$$

where the functions $H_M^\pm$ are defined as solutions of the boundary value problems

$$\Delta_x H_M^\pm(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in C_M,$$
$$\frac{\partial H_M^\pm}{\partial n}(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \Gamma, \mathbf{y} \in C_M,$$
$$H_M^\pm(\mathbf{x}, \mathbf{y}) = G_\infty(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \gamma^\pm, \mathbf{y} \in C_M,$$
$$H_M^\pm(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \gamma^\mp, \ \mathbf{y} \in C_M.$$

We note that the sum $\sum_\pm H_M^\pm$ is symmetric, i.e.

$$H_M^+(\mathbf{x}, \mathbf{y}) + H_M^-(\mathbf{x}, \mathbf{y}) = H_M^+(\mathbf{y}, \mathbf{x}) + H_M^-(\mathbf{y}, \mathbf{x}).$$

The functions $H_M^\pm$ can be approximated by the formulae

$$H_M^+(\mathbf{x}, \mathbf{y}) = H^+(\mathbf{x}^+, \mathbf{y}^+) - \frac{1}{2|\omega|}\zeta^+(\mathbf{y}^+)$$

$$-P_M(\mathbf{x})\Big(H^+(\mathbf{x}^{+\prime}, -\infty, \mathbf{y}^+) - \frac{1}{2|\omega|}\zeta^+(\mathbf{y}^+)\Big) + h_M^+,$$

and

$$H_M^-(\mathbf{x}, \mathbf{y}) = H^-(\mathbf{x}^-, \mathbf{y}^-) - \frac{1}{2|\omega|}\zeta^-(\mathbf{y}^-)$$
$$-P_M(\mathbf{x})\Big(H^-(\mathbf{x}^{-\prime}, +\infty, \mathbf{y}^-) - \frac{1}{2|\omega|}\zeta^-(\mathbf{y}^-)\Big) + h_M^-,$$

with exponentially small remainder terms $h_M^\pm$. Applying Green's formula to the functions $H^\pm$ and $\zeta^\pm$ in the domains $C^\pm$, respectively, we deduce that

$$H^-(\mathbf{x}^{-\prime}, +\infty, \mathbf{y}^-) = -\frac{1}{2|\omega|}\{\zeta^-(\mathbf{y}^-) - (M + \mathbf{y}_n + \zeta_-^\infty)\},$$

and

$$H^+(\mathbf{x}^{+\prime}, -\infty, \mathbf{y}^+) = -\frac{1}{2|\omega|}\{\zeta^+(\mathbf{y}^+) - (M - \mathbf{y}_n + \zeta_+^\infty)\}.$$

The condition (10) yields

$$\lim_{\mathbf{y}_n^- \to +\infty} H^-(\mathbf{y}^{-\prime}, +\infty, \mathbf{y}^-) = 0,$$

and

$$\lim_{\mathbf{y}_n^+ \to -\infty} H^+(\mathbf{y}^{+\prime}, -\infty, \mathbf{y}^+) = 0.$$

If $\mathfrak{A} = 2M + \zeta_\infty^+ + \zeta_\infty^-$, then the following identity holds

$$H_M^+(\mathbf{x}, \mathbf{y}) + H_M^-(\mathbf{x}, \mathbf{y}) = H^+(\mathbf{x}^+, \mathbf{y}^+) + H^-(\mathbf{x}^-, \mathbf{y}^-)$$

$$+\frac{\mathfrak{A}}{|\omega|}\Big(\frac{1}{2} - P_M(\mathbf{x})\Big)\Big(\frac{1}{2} - P_M(\mathbf{y})\Big) - \frac{\mathfrak{A}_M}{4|\omega|}. \qquad (18)$$

Combining the formulae (17) and (18) we deduce (16).

The direct substitution of (16) into (13), (14) and application of the energy estimate completes the proof of the theorem.

**Example of Green's functions in model domains.** In some cases, Green's functions for model problems required for the above asymptotic approximation can be constructed in a simple form. As an illustration, we suggest an example involving a long rectangular strip. In this case, the function $G_\infty(\mathbf{x}, \mathbf{y})$ is the Neumann function for the Laplacian in the infinite strip $\Pi = \{(x_1, x_2) : -\infty < x_1 < \infty, |x_2| < 1/2\}$, given in the form

$$G_\infty(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k, x_2, y_2)\exp(-ik(x_1 - y_1))dk,$$

where

$$\tilde{G}(k, x_2, y_2) = -\frac{1}{2k}\cosh(kx_2)/\sinh(k/2)$$

$$-\begin{cases} \frac{1}{2}(x_2 - y_2), \ x_2 > y_2 \\ -\frac{1}{2}(x_2 - y_2), \ x_2 < y_2. \end{cases}$$

Assuming that the end regions of the rectangular domain are "flat", i.e. they are located on the vertical straight lines $x_1 = \pm M$, we can construct Green's functions $G_\pm$ for semi-infinite strips as follows:

$$G_\pm(\mathbf{x}^\pm, \mathbf{y}^\pm) = G_\infty(\mathbf{x}^\pm, y_1^\pm, y_2^\pm) - G_\infty(\mathbf{x}^\pm, -y_1^\pm, y_2^\pm).$$

These model fields are readily applicable in the asymptotic formula of Theorem 2.

## 5 Connection between Green's function $G_M$ and Green's functions for unbounded domains.

The result of Section 4 together with definitions of functions $G_\infty$ and $G^\pm$ lead to the following

**Theorem 3.** *The Green's function $G_M(\mathbf{x},\mathbf{y})$ and the functions $G^\pm$, $G_\infty$ are related by the asymptotic formula*

$$G_M(\mathbf{x},\mathbf{y}) = \sum_{\pm} G^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm) - G_\infty(\mathbf{x},\mathbf{y}) - \frac{1}{2|\omega|}\sum_{\pm}\Big(\zeta^\pm(\mathbf{x}^\pm)+\zeta^\pm(\mathbf{y}^\pm)\Big)$$
$$-\frac{\mathfrak{A}_M}{|\omega|}\Big(\frac{1}{2}-P_M(\mathbf{x})\Big)\Big(\frac{1}{2}-P_M(\mathbf{y})\Big)+\frac{\mathfrak{A}_M}{4|\omega|}+O(-\alpha M,) \tag{19}$$

*where $\alpha$ is a positive constant independent of $M$.*

**Corollary 1.** *The formula* (19) *allows for an equivalent representation involving the model fields $\zeta^\pm$ defined as solutions of the boundary value problems* (7)–(10)*:*

$$G_M(\mathbf{x},\mathbf{y}) = \sum_{\pm} G^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm) - G_\infty(\mathbf{x},\mathbf{y}) + \frac{1}{4|\omega|}\Big\{\mathfrak{A}_M - 2\sum_{\pm}\Big(\zeta^\pm(\mathbf{x}^\pm)+\zeta^\pm(\mathbf{y}^\pm)\Big)\Big\}$$
$$-\Big(|\omega|\mathfrak{A}_M\Big)^{-1}\Big(\mathbf{x}_n-\tfrac{1}{2}(\zeta^+_\infty-\zeta^-_\infty)+\zeta^+(\mathbf{x}^+)-\zeta^-(\mathbf{x}^-)\Big) \tag{20}$$
$$\times\Big(\mathbf{y}_n-\tfrac{1}{2}(\zeta^+_\infty-\zeta^-_\infty)+\zeta^+(\mathbf{y}^+)-\zeta^-(\mathbf{y}^-)\Big)+O(\exp(-\alpha M)),$$

*where $\alpha$ is a positive constant independent of $M$.*

The above formulae can be simplified if we introduce additional constraints on the positions of the points $\mathbf{x}$ and $\mathbf{y}$ within $C_M$.

When the points $\mathbf{x}$ and $\mathbf{y}$ are "far away" from the ends $\gamma^\pm$ of the long rod the quantities $H^\pm$ become exponentially small, and hence we arrive to the following

**Corollary 2.** When $\min\{(\mathbf{x}\pm M)/M,(\mathbf{x}\pm M)/M\}\geq$ Const, the Green's function $G_M$ is approximated by the formula

$$G_M(\mathbf{x},\mathbf{y}) \sim G_\infty(\mathbf{x},\mathbf{y}) - (|\omega|\mathfrak{A}_M)^{-1}\Big(x_n-\tfrac{1}{2}(\zeta^+_\infty-\zeta^-_\infty)\Big)\Big(y_n-\tfrac{1}{2}(\zeta^+_\infty-\zeta^-_\infty)\Big)$$
$$+\frac{\mathfrak{A}_M}{4|\omega|}, \tag{21}$$

*as $M\to\infty$.*

Another simplified formula for the Green's function can be written for the case when the points $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to one of the ends of the rod.

**Corollary 3.** *Assume that the points $\mathbf{x}$ and $\mathbf{y}$ are close to the left end $\gamma^-$ of the long rod $C_M$, i.e.* $\max\{\mathbf{x}+M,\mathbf{y}+M\}\leq$ *Const. Then the function $G_M$ is* *approximated by the formula*

$$G_M(\mathbf{x},\mathbf{y}) \sim G^-(\mathbf{x}^-,\mathbf{y}^-) - |\omega| \frac{G^-(\mathbf{x}^{-\prime},+\infty,\mathbf{y}^-)G^-(\mathbf{x}^-,\mathbf{y}^{-\prime},+\infty)}{\mathfrak{A}_M}, \tag{22}$$

*as* $M \to \infty$.

Similar approximation is valid near the other end $\gamma^+$ of the long rod.

## 6 The Dirichlet-Neumann problem in a thin rod.

By rescaling, the above results can be used to find an asymptotic approximation for Green's function $G^{(\varepsilon)}$ in a thin rod rather than the long rod. Let a thin domain be defined by

$$C_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}(\mathbf{x}-\mathbf{a}) \in C^+,\ \varepsilon^{-1}(\mathbf{x}+\mathbf{a}) \in C^-\},$$

where the notations $C^\pm$ are the same as in Section 2, $2a$ is the length of the rod, and now $\varepsilon$ is a positive small parameter. As above, it is assumed that Green's function is subject to zero Neumann condition on the cylindrical part of $C_\varepsilon$ and zero Dirichlet condition on the remaining part of $\partial C_\varepsilon$.

**Theorem 4.** *The following asymptotic formula for* $G^{(\varepsilon)}(\mathbf{x},\mathbf{y})$*, uniform with respect to* $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$*, holds*

$$\begin{aligned} G^{(\varepsilon)}(\mathbf{x},\mathbf{y}) = \varepsilon^{2-n}\Big\{ & G^+(\varepsilon^{-1}(\mathbf{x}-\mathbf{a}),\varepsilon^{-1}(\mathbf{y}-\mathbf{a})) + G^-(\varepsilon^{-1}(\mathbf{x}+\mathbf{a}),\varepsilon^{-1}(\mathbf{y}+\mathbf{a})) \\ & -G_\infty(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\ & -\varepsilon\{2|\omega|^{-1}a + \varepsilon(\zeta_\infty^+ + \zeta_\infty^-)\}^{-1}\Big(\frac{x_n}{\varepsilon|\omega|} - \tfrac{1}{2}(\zeta_\infty^- - \zeta_\infty^+) + \zeta^+\big(\frac{\mathbf{x}-\mathbf{a}}{\varepsilon}\big) - \zeta^-\big(\frac{\mathbf{x}+\mathbf{a}}{\varepsilon}\big)\Big) \\ & \times\Big(\frac{y_n}{\varepsilon|\omega|} - \tfrac{1}{2}(\zeta_\infty^- - \zeta_\infty^+) + \zeta^+\big(\frac{\mathbf{y}-\mathbf{a}}{\varepsilon}\big) - \zeta^-\big(\frac{\mathbf{y}+\mathbf{a}}{\varepsilon}\big)\Big) \\ & +\frac{1}{4}\Big((\varepsilon|\omega|)^{-1}2a + \zeta_\infty^- + \zeta_\infty^+ - 2\sum_\pm \big(\zeta^\pm(\varepsilon^{-1}(\mathbf{x}\mp\mathbf{a})) + \zeta^\pm(\varepsilon^{-1}(\mathbf{y}\mp\mathbf{a}))\big)\Big) \\ & +O(\exp(-\beta/\varepsilon))\Big\}, \end{aligned} \tag{23}$$

*where* $\beta$ *is a positive constant independent of* $\varepsilon$.

## 7 Concluding remarks and comparison of asymptotic formulae for long domains and domains with small holes

Although the domains shown in Figures 1 and 2 are very different, we can note a similarity in the structure of the asymptotic approximations of Green's functions (see, for example, simplified asymptotic formulae (2), (3) and (21), (22)). This similarity becomes even more explicit if we consider the two-dimensional cases of a long strip $C_M$ and a domain with a small hole (the latter case was discussed in [3]).

In particular, the capacitary potential $P_\varepsilon$ for a two-dimensional domain $\Omega_\varepsilon$ with a small hole $\omega_\varepsilon$ is defined as a solution of the boundary value problem:

$$\Delta P_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{24}$$
$$P_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega, \tag{25}$$
$$P_\varepsilon(\mathbf{x}) = 1, \quad \mathbf{x} \in \partial\omega_\varepsilon. \tag{26}$$

Its uniform asymptotic approximation, as $\varepsilon \to 0$, is given by the formula

$$P_\varepsilon(\mathbf{x}) \sim \frac{-G(\mathbf{x},0) + \zeta(\frac{\mathbf{x}}{\varepsilon}) - \frac{1}{2\pi}\log\frac{|\mathbf{x}|}{\varepsilon} - \zeta_\infty}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty}, \tag{27}$$

where $H(\mathbf{x},\mathbf{y})$ is the regular part of Green's function $G(\mathbf{x},\mathbf{y})$ in the limit domain $\Omega$ without the hole, and the quantities $\zeta$ and $\zeta_\infty$ are defined as follows

$$\zeta(\boldsymbol{\eta}) = \lim_{|\boldsymbol{\xi}|\to\infty} g(\boldsymbol{\xi},\boldsymbol{\eta}), \tag{28}$$

and

$$\zeta_\infty = \lim_{|\boldsymbol{\eta}|\to\infty} \{\zeta(\boldsymbol{\eta}) - (2\pi)^{-1}\log|\boldsymbol{\eta}|\}. \tag{29}$$

Here $g(\boldsymbol{\xi},\boldsymbol{\eta})$ stands for Green's function in the unbounded model domain $\mathbf{R}^2 \setminus \overline{\omega}$.

The structure of the asymptotic approximation (27) is similar to (11), with the linear terms (growing at infinity) being replaced by the corresponding logarithmic terms. Also, the uniform asymptotic approximation (as $\varepsilon \to 0$) of Green's function $G_\varepsilon$ in the two-dimensional domain $\Omega_\varepsilon$ with the small hole has the form

$$G_\varepsilon(\mathbf{x},\mathbf{y}) \sim G(\mathbf{x},\mathbf{y}) + g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + (2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|)$$

$$+\frac{\Big((2\pi)^{-1}\log\varepsilon+\zeta(\frac{\mathbf{x}}{\varepsilon})-\zeta_\infty+H(\mathbf{x},0)\Big)\Big((2\pi)^{-1}\log\varepsilon+\zeta(\frac{\mathbf{y}}{\varepsilon})-\zeta_\infty+H(0,\mathbf{y})\Big)}{(2\pi)^{-1}\log\varepsilon+H(0,0)-\zeta_\infty}$$

$$-\zeta(\varepsilon^{-1}\mathbf{x})-\zeta(\varepsilon^{-1}\mathbf{y})+\zeta_\infty, \tag{30}$$

whose structure resembles the one of formula (19) (and formula (20)).

## References


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