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SHARP REAL-PART THEOREMS IN THE UPPER HALF-PLANE AND SIMILAR ESTIMATES FOR HARMONIC FUNCTIONS

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Explicit formulas for sharp coefficients in estimates of the modulus of an analytic function and its derivative in the upper half-plane are found. It is assumed that the boundary values of the real part of the function are in L^p. As corollaries, sharp estimates for the modulus of the gradient of a harmonic function in $\mathbb{R}^2_+$ are deduced. Besides, a representation for the best constant in the estimate of the modulus of the gradient of a harmonic function in $\mathbb{R}^n_+$ by the L^p-norm of the boundary normal derivative is given, $1 \leqslant p < \infty$. This representation is formulated in terms of an optimization problem on the unit sphere which is solved for $p \in [1, n]$. Bibliography: 6 titles.

1 Introduction

In this paper, we deal first with a class of analytic functions in the half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ represented by the Schwarz formula

$$f(z) = \frac{1}{\pi i} \int_{\infty}^{\infty} \frac{\operatorname{Re} f(\zeta)}{\zeta - z}\, d\zeta \tag{1.1}$$

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and such that the boundary values on $\partial\mathbb{C}_+$ of the real part of f belong to the space $L^p(-\infty,\infty)$, $1 \leqslant p < \infty$.

In what follows, by $h^p(\mathbb{R}^2_+)$, $1 \leqslant p \leqslant \infty$, we mean the Hardy space of harmonic functions in the upper half-plane $\mathbb{R}^2_+$ which are represented by the Poisson integral with a density in $L^p(-\infty,\infty)$. It is well known (cf., for example, [1, Section 19.3]) that f belongs to the Hardy space $H^p(\mathbb{C}_+)$ of analytic functions in $\mathbb{C}_+$ if $\operatorname{Re} f \in h^p(\mathbb{R}^2_+)$, $1 < p < \infty$. Besides, any function $f \in H^p(\mathbb{C}_+)$, $1 < p < \infty$, admits the representation (1.1) since $\operatorname{Re} f \in h^p(\mathbb{R}^2_+)$.

We consider two inequalities with sharp coefficients

$$|\operatorname{Re}\{e^{i\alpha} f(z)\}| \leqslant \mathscr{C}_p(z;\alpha) ||\operatorname{Re} f||_p \tag{1.2}$$

and

$$|\operatorname{Re}\{e^{i\alpha} f'(z)\}| \leqslant \mathscr{K}_p(z;\alpha) ||\operatorname{Re} f||_p\ , \tag{1.3}$$

where $z \in \mathbb{C}_+$ and $||\cdot||_p$ stands for the norm in $L^p(-\infty,\infty)$. Here and in what follows, we adopt the notation $||\operatorname{Re} f||_p$ for $||\operatorname{Re} f|_{\partial\mathbb{C}_+}||_p$. We find representations for $\mathscr{C}_p(z;\alpha)$ and $\mathscr{K}_p(z;\alpha)$ and also obtain explicit formulas for the sharp coefficients

$$\mathscr{C}_p(z) = \max_{\alpha} \mathscr{C}_p(z;\alpha) \tag{1.4}$$

and

$$\mathscr{K}_p(z) = \max_{\alpha} \mathscr{K}_p(z;\alpha)\ , \tag{1.5}$$

in pointwise estimates for the absolute value of an analytic function and its derivative in $\mathbb{C}_+$. As a corollary, we obtain explicit sharp pointwise estimates for the gradient of a harmonic function in $\mathbb{R}^2_+$ for any $p \in [1,\infty)$. In the second part of the paper, we deal with problems concerning harmonic functions of n variables. They are connected with the following analogue of D. Khavinson's extremal problem [2] stated in [3]. Let $\mathscr{H}$ be a class of harmonic functions in a domain $\mathscr{D} \subset \mathbb{R}^n$, let $x \in \mathscr{D}$ be a fixed point, and let Φ be a positively homogeneous nonnegative functional defined on functions in $\mathscr{H}$ and such that

$$\big|\big(\nabla u(x),\ \boldsymbol{\ell}\big)\big| \leqslant C(x)\Phi(u)$$

for all $|\boldsymbol{\ell}| = 1$ and $u \in \mathscr{H}$. One is looking for a direction $\boldsymbol{\ell}$ for which the sharp coefficient $\mathscr{K}_\Phi(x;\ell)$ in the following inequality:

$$\big|\big(\nabla u(x),\ \boldsymbol{\ell}\big)\big| \leqslant \mathscr{K}_\Phi(x;\boldsymbol{\ell})\Phi(u), \qquad u \in \mathscr{H},$$

attains its largest value, and for the corresponding value of $\mathscr{K}_\Phi(x;\boldsymbol{\ell})$.

An analogue of the above extremal problem for analytic functions is the evaluation of optimal values of α in (1.4) and (1.5). Note that the inequalities (1.2) and (1.3) for analytic functions belong to the class of sharp real-part theorems (cf. [4] and the references therein) which go back to Hadamard's real-part theorem [5].

Now we describe the results of this paper in more detail. Introduction is followed by three sections. The first of them concerns the representation for the sharp coefficient $\mathscr{C}_p(z;\alpha)$. We show that the sharp coefficient in

$$|f(z)| \leqslant \mathscr{C}_p(z) ||\operatorname{Re} f||_p\ , \tag{1.6}$$

with $1 \leqslant p < \infty$ is defined by

$$\mathscr{C}_p(z) = \frac{C_p}{(\operatorname{Im} z)^{1/p}} \ , \tag{1.7}$$

where

$$C_1 = \frac{1}{\pi} \ , \qquad C_2 = \frac{1}{\sqrt{2\pi}} \ ,$$

and

$$C_p = \begin{cases} \dfrac{1}{\pi^{(p+1)/(2p)}} \left\{ \dfrac{\Gamma\left(\frac{p+1}{2p-2}\right)}{\Gamma\left(\frac{p}{p-1}\right)} \right\}^{1-\frac{1}{p}} & \text{for} \quad 1 < p < 2 \ , \\ \\ \dfrac{1}{\pi} \left\{ \dfrac{\Gamma\left(\frac{2p-1}{2p-2}\right)\Gamma\left(\frac{1}{2p-2}\right)}{\Gamma\left(\frac{p}{p-1}\right)} \right\}^{1-\frac{1}{p}} & \text{for} \quad 2 < p < \infty \ . \end{cases}$$

Note that the maximum in (1.4) is attained at $\alpha = 0$ for $1 \leqslant p < 2$ and at $\alpha = \pi/2$ for $p > 2$. The coefficient $\mathscr{C}_2(z;\alpha)$ is independent of α.

As a corollary of (1.6), we arrive at the sharp estimate

$$|\nabla u(z)| \leqslant \frac{C_p}{y^{1/p}} \left|\left| \frac{\partial u}{\partial \boldsymbol{\nu}} \right|\right|_p \ , \tag{1.8}$$

where $z = (x, y) \in \mathbb{R}^2_+$ and u is a solution of the Neumann problem for the Laplace equation in $\mathbb{R}^2_+$ with boundary data in $L^p(-\infty, \infty)$, $1 \leqslant p < \infty$. Here and henceforth, $||\partial u/\partial \boldsymbol{\nu}||_p$ stands for the L_p-norm of the boundary normal derivative of u. It turns out that the largest value of the modulus of the directional derivative of u at a point z, with $||\partial u/\partial \boldsymbol{\nu}||_p \leqslant 1$, is attained at the direction normal to $\partial\mathbb{R}^2_+$ if $1 \leqslant p < 2$, at any direction if $p = 2$, and at the tangential direction if $2 < p < \infty$.

In Section 2, we consider the multi-dimensional analogue of (1.8):

$$|\nabla u(x)| \leqslant \mathscr{C}_{n,p}(x) \left|\left| \frac{\partial u}{\partial \boldsymbol{\nu}} \right|\right|_p \ , \tag{1.9}$$

where $x \in \mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) : x_n > 0\}$ and u is a solution of the Neumann problem for the Laplace equation in $\mathbb{R}^n_+$ with boundary data in $L^p(\mathbb{R}^{n-1}), 1 \leqslant p < \infty$. With respect to the notation $||\partial u/\partial \boldsymbol{\nu}||_p$, we keep the same meaning as above for the two-dimensional case. We show that

$$\mathscr{C}_{n,p}(x) = \frac{C_{n,p}}{x_n^{(n-1)/p}}$$

and find a representation for $C_{n,p}$ in terms of an extremal problem on the unit sphere in $\mathbb{R}^n$. By solving that problem for $p \in [1, n]$, we find

$$C_{n,p} = \begin{cases} \dfrac{1}{\omega_n} & \text{for} \quad p = 1 \ , \\ \\ \dfrac{2^{1/p}}{\omega_n} \left\{ \dfrac{2\pi^{(n-1)/2}\Gamma\left(\frac{n+p-1}{2p-2}\right)}{\Gamma\left(\frac{np}{2p-2}\right)} \right\}^{1-\frac{1}{p}} & \text{for} \quad 1 < p \leqslant n \ . \end{cases}$$

Note that the maximal value of the sharp coefficient $\mathscr{C}_{n,p}(x;\boldsymbol{\ell})$ in

$$\left|\frac{\partial u(x)}{\partial \ell}\right| \leqslant \mathscr{C}_{n,p}(x;\boldsymbol{\ell})\left|\left|\frac{\partial u}{\partial \boldsymbol{\nu}}\right|\right|_p$$

for a fixed x is attained at the direction normal to $\partial\mathbb{R}^n_+$ if $1 \leqslant p < n$. It turns out that $\mathscr{C}_{n,n}(x;\boldsymbol{\ell})$ is independent of $\boldsymbol{\ell}$.

In Section 3, we find a representation for the sharp coefficient $\mathscr{K}_p(z;\alpha)$ in (1.3). We show that the sharp coefficient $\mathscr{K}_p(z)$ in

$$|f'(z)| \leqslant \mathscr{K}_p(z)||\operatorname{Re} f||_p \tag{1.10}$$

is given by

$$\mathscr{K}_p(z) = \frac{K_p}{(\operatorname{Im} z)^{1+\frac{1}{p}}}\ , \tag{1.11}$$

where

$$K_1 = \frac{1}{\pi}\ , \qquad K_2 = \frac{1}{2\sqrt{\pi}}\ , \qquad K_\infty = \frac{2}{\pi}\ ,$$

$$K_p = \frac{1}{(2\sqrt{\pi^{p+1}})^{1/p}}\left\{\sum_{n=0}^{\infty}\begin{pmatrix} 1/(p-1) \\ 2n \end{pmatrix}\frac{\Gamma\left(\frac{(2n+1)(p-1)+p}{2p-2}\right)}{\Gamma\left(\frac{2(n+1)(p-1)+p}{2p-2}\right)}\right\}^{1-\frac{1}{p}}$$

if $1 < p < 2$, and

$$K_p = \frac{2}{\pi}\left\{\frac{\Gamma\left(\frac{2p+1}{2p-2}\right)\Gamma\left(\frac{2p-1}{2p-2}\right)}{\Gamma\left(\frac{2p}{p-1}\right)}\right\}^{1-\frac{1}{p}}$$

for $2 < p < \infty$. It is shown that the maximum in (1.5) is attained at $\alpha = \pi/2$ if $1 \leqslant p < 2$ and at $\alpha = 0$ if $2 < p < \infty$. For $p = 2$ and for $p = \infty$, the coefficient $\mathscr{K}_p(z;\alpha)$ is independent of α.

Note that the value K_∞ is obtained by passage to the limit of K_p as $p \to \infty$. It was found in [4] by a different method.

As a corollary of (1.10), we arrive at the sharp estimate

$$|\nabla u(z)| \leqslant \frac{K_p}{y^{1+\frac{1}{p}}}\ ||u||_p \tag{1.12}$$

for harmonic functions $u \in h^p(\mathbb{R}^2_+)$.

An analogue of (1.12) for harmonic functions in $\mathbb{R}^n_+$, $n \geqslant 3$, is found in our article [6]. Namely, in [6] a representation for the best coefficient $\mathscr{M}_p(x)$ in the inequality

$$|\nabla u(x)| \leqslant \mathscr{M}_p(x)\big\|u\big\|_p$$

was obtained, where u is harmonic function in $\mathbb{R}^n_+$, represented by the Poisson integral with boundary values in $L^p(\mathbb{R}^{n-1})$, $||\cdot||_p$ is the norm in $L^p(\mathbb{R}^{n-1})$, $1 \leqslant p \leqslant \infty$, $x \in \mathbb{R}^n_+$. It was shown that

$$\mathscr{M}_p(x) = M_p\ x_n^{(1-n-p)/p}$$

and explicit formulas for M_p for the cases $p = 1, 2, \infty$ are given.

2 Best Real-Part Estimates for an Analytic Function

We start with the following assertion.

Proposition 1. *Let* $\operatorname{Re} f \in h^p(\mathbb{R}^2_+), 1 \leqslant p < \infty$, *and let* z *be an arbitrary point in* $\mathbb{C}_+$. *The sharp coefficient* $\mathscr{C}_p(z;\alpha)$ *in the inequality*

$$|\operatorname{Re}\{e^{i\alpha} f(z)\}| \leqslant \mathscr{C}_p(z;\alpha) || \operatorname{Re} f||_p \tag{2.1}$$

is given by

$$\mathscr{C}_p(z;\alpha) = \frac{C_p(\alpha)}{(\operatorname{Im} z)^{1/p}} \ , \tag{2.2}$$

where

$$C_p(\alpha) = \frac{1}{\pi} \left\{ \int_{-\pi/2}^{\pi/2} |\cos(\varphi-\alpha)|^q \cos^{q-2}\varphi d\varphi \right\}^{1/q} , \tag{2.3}$$

and $1/p + 1/q = 1$.

Proof. By (1.2),

$$\operatorname{Re}\{e^{i\alpha} f(z)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{Re}\left\{ \frac{e^{i\left(\alpha - \frac{\pi}{2}\right)}}{\zeta - z} \right\} \operatorname{Re} f(\zeta) d\zeta \ . \tag{2.4}$$

Let $z = x + iy$. We write (2.4) as

$$\begin{aligned} \operatorname{Re}\{e^{i\alpha} f(z)\} &= \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{Re}\left\{ \frac{(\sin\alpha - i\cos\alpha)(\zeta - x + iy)}{(\zeta-x)^2 + y^2} \right\} \operatorname{Re} f(\zeta) d\zeta \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\zeta - x)\sin\alpha + y\cos\alpha}{(\zeta-x)^2 + y^2} \operatorname{Re} f(\zeta) d\zeta \ . \end{aligned} \tag{2.5}$$

Hence the sharp constant $\mathscr{C}_p(z;\alpha)$ in (2.1) has the form

$$\mathscr{C}_p(z;\alpha) = \frac{1}{\pi} \left\{ \int_{-\infty}^{\infty} \left| \frac{(\zeta - x)\sin\alpha + y\cos\alpha}{(\zeta-x)^2 + y^2} \right|^q d\zeta \right\}^{1/q} . \tag{2.6}$$

We introduce the new integration variable $\varphi \in (-\pi/2, \pi/2)$ by the equalities

$$\sin\varphi = \frac{\zeta - x}{\sqrt{(\zeta-x)^2 + y^2}} \ , \qquad \cos\varphi = \frac{y}{\sqrt{(\zeta-x)^2 + y^2}} \ . \tag{2.7}$$

Then

$$\varphi = \arctan\frac{\zeta - x}{y} \tag{2.8}$$

and therefore,

$$d\varphi = \frac{y}{(\zeta-x)^2 + y^2)} \, d\zeta \ . \tag{2.9}$$

Using (2.7)–(2.9) and the equality

$$\frac{1}{[(\zeta - x)^2 + y^2]^{q/2}} = \frac{1}{y^{q-1}} \left(\frac{y}{\sqrt{(\zeta - x)^2 + y^2}} \right)^{q-2} \frac{y}{(\zeta - x)^2 + y^2} , \tag{2.10}$$

we can write (2.6) as

$$C_p(z;\alpha) = \frac{1}{\pi y^{(q-1)/q}} \left\{ \int_{-\pi/2}^{\pi/2} |\cos(\varphi - \alpha)|^q \cos^{q-2} \varphi d\varphi \right\}^{1/q} , \tag{2.11}$$

which implies (2.2) with coefficient (2.3). □

The next assertion concerns the sharp constant in the estimate for the modulus of an analytic function.

Theorem 1. *Let* $\operatorname{Re} f \in h^p(\mathbb{R}^2_+), 1 \leqslant p < \infty$, *and let* z *be an arbitrary point in* $\mathbb{C}_+$. *The sharp coefficient* $\mathscr{C}_p(z)$ *in the inequality*

$$|f(z)| \leqslant \mathscr{C}_p(z) ||\operatorname{Re} f||_p \tag{2.12}$$

is given by

$$\mathscr{C}_p(z) = \frac{C_p}{(\operatorname{Im} z)^{1/p}} , \tag{2.13}$$

where

$$C_1 = \frac{1}{\pi} , \qquad C_2 = \frac{1}{\sqrt{2\pi}} , \tag{2.14}$$

and

$$C_p = \begin{cases} \dfrac{1}{\pi^{(p+1)/(2p)}} \left\{ \dfrac{\Gamma\left(\frac{p+1}{2p-2}\right)}{\Gamma\left(\frac{p}{p-1}\right)} \right\}^{1-\frac{1}{p}} & \text{for} \quad 1 < p < 2 , \\[3ex] \dfrac{1}{\pi} \left\{ \dfrac{\Gamma\left(\frac{2p-1}{2p-2}\right) \Gamma\left(\frac{1}{2p-2}\right)}{\Gamma\left(\frac{p}{p-1}\right)} \right\}^{1-\frac{1}{p}} & \text{for} \quad 2 < p < \infty . \end{cases} \tag{2.15}$$

The maximum value of $C_p(\alpha)$, *defined by* (2.3), *is attained at* $\alpha = 0$ *if* $1 \leqslant p < 2$ *and at* $\alpha = \pi/2$ *if* $2 < p < \infty$. *The coefficient* $C_2(\alpha)$ *is independent of* α.

Proof. It follows from Proposition 1 that the sharp coefficient in (2.12) is defined by

$$\mathscr{C}_p(z) = \max_{\alpha} \mathscr{C}_p(z;\alpha),$$

where $\mathscr{C}_p(z;\alpha)$ is given by (2.2) and (2.3). Hence (2.13) is valid with

$$C_p = \max_{\alpha} C_p(\alpha) = \frac{1}{\pi} \max_{\alpha} \left\{ \int_{-\pi/2}^{\pi/2} |\cos(\varphi - \alpha)|^q \cos^{q-2} \varphi d\varphi \right\}^{1/q} . \tag{2.16}$$

We adopt the notation

$$F_q(\alpha) = \int\limits_{-\pi/2}^{\pi/2} |\cos(\varphi - \alpha)|^q \cos^{q-2}\varphi d\varphi \, . \tag{2.17}$$

While calculating the maximum in (2.16) we assume that $\alpha \in [-\pi/2, \pi/2]$ since $F_p(\alpha)$ is a π-periodic function. Besides, we can limit our consideration to the interval $[0, \pi/2]$ since $F_p(\alpha)$ is even, which is easy to check.

(i) *Cases* $p = 1$ *and* $p = 2$. By (2.16), we have

$$C_1 = \frac{1}{\pi} \max_{\alpha \in [0, \frac{\pi}{2}]} \max_{\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]} |\cos(\varphi - \alpha)| \cos\varphi = \frac{1}{\pi} \, , \tag{2.18}$$

and the maximum in α is attained at $\alpha = 0$.

Further, we find

$$F_2(\alpha) = \int\limits_{-\pi/2}^{\pi/2} \cos^2(\varphi - \alpha)\varphi d\varphi = \frac{\pi}{2} \, .$$

Since $F_2(\alpha)$ is independent of α, it follows from (2.16) and (2.17) that

$$C_2 = \frac{1}{\sqrt{2\pi}} \, . \tag{2.19}$$

Now, (2.18) and (2.19) prove (2.14).

(ii) *Cases* $1 < p < 2$ *and* $2 < p < \infty$. We write (2.17) in the form

$$\begin{aligned} F_q(\alpha) &= \int\limits_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+\alpha} \frac{[-\cos(\varphi - \alpha)]^q}{\cos^{2-q}\varphi} d\varphi + \int\limits_{-\frac{\pi}{2}+\alpha}^{\frac{\pi}{2}} \frac{\cos^q(\varphi - \alpha)}{\cos^{2-q}\varphi} d\varphi \\ &= \int\limits_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+\alpha} \frac{\cos^q(\varphi + \pi - \alpha)}{|\cos\varphi|^{2-q}} d\varphi + \int\limits_{-\frac{\pi}{2}+\alpha}^{\frac{\pi}{2}} \frac{\cos^q(\varphi - \alpha)}{|\cos\varphi|^{2-q}} d\varphi \, . \end{aligned}$$

Hence

$$\begin{aligned} \frac{dF_q}{d\alpha} &= q \int\limits_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+\alpha} \frac{\cos^{q-1}(\varphi + \pi - \alpha)\sin(\varphi + \pi - \alpha)}{|\cos\varphi|^{2-q}} d\varphi \\ &+ q \int\limits_{-\frac{\pi}{2}+\alpha}^{\frac{\pi}{2}} \frac{\cos^{q-1}(\varphi - \alpha)\sin(\varphi - \alpha)}{|\cos\varphi|^{2-q}} d\varphi \, . \end{aligned}$$

We make the change of variable $\psi = \varphi + \pi$ in the first integral on the right-hand side to obtain

$$\frac{dF_q}{d\alpha} = q \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} \frac{\cos^{q-1}(\psi - \alpha)\sin(\psi - \alpha)}{|\cos\psi|^{2-q}} d\psi + q \int_{-\frac{\pi}{2}+\alpha}^{\frac{\pi}{2}} \frac{\cos^{q-1}(\varphi - \alpha)\sin(\varphi - \alpha)}{|\cos\varphi|^{2-q}} d\varphi$$

$$= q \int_{-\frac{\pi}{2}+\alpha}^{\frac{\pi}{2}+\alpha} \frac{\cos^{q-1}(\varphi - \alpha)\sin(\varphi - \alpha)}{|\cos\phi|^{2-q}} d\varphi \,.$$

The change of variable $\varphi - \alpha = \vartheta$ in the last integral leads to

$$\frac{dF_q}{d\alpha} = q \int_{-\pi/2}^{\pi/2} \frac{\cos^{q-1}\vartheta \sin\vartheta}{|\cos(\vartheta + \alpha)|^{2-q}} d\vartheta \,. \tag{2.20}$$

This implies $F'_q(0) = F'_q(\pi/2) = 0$. Let us show that $F'_q(\alpha) > 0$ for $1 < q < 2$ and $F'_q(\alpha) < 0$ for $q > 2$ in $(0, \pi/2)$. For this purpose, we write (2.20) as the sum

$$\frac{dF_q}{d\alpha} = q \int_{-\pi/2}^{0} \frac{\cos^{q-1}\vartheta \sin\vartheta}{|\cos(\vartheta + \alpha)|^{2-q}} d\vartheta + q \int_{0}^{\pi/2} \frac{\cos^{q-1}\vartheta \sin\vartheta}{|\cos(\vartheta + \alpha)|^{2-q}} d\vartheta \,,$$

and replace ϑ by $-\vartheta$ in the second integral. This gives

$$\frac{dF_q}{d\alpha} = q \int_{0}^{\pi/2} \frac{\cos^{q-1}\vartheta \sin\vartheta}{|\cos(\vartheta + \alpha)|^{2-q}} d\vartheta - q \int_{0}^{\pi/2} \frac{\cos^{q-1}\vartheta \sin\vartheta}{|\cos(\vartheta - \alpha)|^{2-q}} d\vartheta \,. \tag{2.21}$$

For $1 < p < 2$ $(2 < q < \infty)$, we write this equality as

$$\frac{dF_q}{d\alpha} = q \int_{0}^{\pi/2} \Big(|\cos(\vartheta + \alpha)|^{q-2} - |\cos(\vartheta - \alpha)|^{q-2} \Big) \cos^{q-1}\vartheta \sin\vartheta d\vartheta \,, \tag{2.22}$$

while for $2 < p < \infty$ $(1 < q < 2)$ we express (2.21) as

$$\frac{dF_q}{d\alpha} = q \int_{0}^{\pi/2} \frac{|\cos(\vartheta - \alpha)|^{2-q} - |\cos(\vartheta + \alpha)|^{2-q}}{|\cos(\vartheta - \alpha)\cos(\vartheta + \alpha)|^{2-q}} \cos^{q-1}\vartheta \sin\vartheta d\vartheta \,. \tag{2.23}$$

Further, since $\vartheta, \alpha \in [0, \pi/2]$, it follows that

$$|\cos(\vartheta - \alpha)| - |\cos(\vartheta + \alpha)| = \begin{cases} \cos(\vartheta - \alpha) - \cos(\vartheta + \alpha) & \text{for} \quad \vartheta \in \left[\, 0, \frac{\pi}{2} - \alpha \right] \,, \\ \cos(\vartheta - \alpha) + \cos(\vartheta + \alpha) & \text{for} \quad \vartheta \in \left(\frac{\pi}{2} - \alpha, \frac{\pi}{2} \right] \,, \end{cases}$$

i.e.,

$$|\cos(\vartheta - \alpha)| - |\cos(\vartheta + \alpha)| = \begin{cases} 2\sin\vartheta \sin\alpha & \text{for} \quad \vartheta \in \left[\, 0, \frac{\pi}{2} - \alpha \right] \,, \\ 2\cos\vartheta \cos\alpha & \text{for} \quad \vartheta \in \left(\frac{\pi}{2} - \alpha, \frac{\pi}{2} \right] \,. \end{cases}$$

Hence

$$|\cos(\vartheta-\alpha)| \geqslant |\cos(\vartheta+\alpha)| \tag{2.24}$$

for $\alpha, \vartheta \in [0, \pi/2]$ and the equality sign holds for $\alpha = 0$ or for $\alpha = \pi/2$ provided that $\vartheta \in (0, \pi/2)$.

Thus, (2.22)–(2.24) imply that $F_q'(\alpha) < 0$ for $1 < p < 2$ and $\alpha \in (0, \pi/2)$, and $F_q'(\alpha) > 0$ for $2 < p < \infty$ and $\alpha \in (0, \pi/2)$. Therefore, the maximum of $F_q(\alpha)$ is attained for $\alpha = 0$ if $1 < p < 2$ and for $\alpha = \pi/2$ if $2 < p < \infty$. Now, by (2.16) and (2.17),

$$C_p = \frac{1}{\pi} \begin{cases} \left\{ \int\limits_{-\pi/2}^{\pi/2} \cos^{2(q-2)} \varphi d\varphi \right\}^{1/q} & \text{for} \quad 1 < p < 2\,, \\ \left\{ \int\limits_{-\pi/2}^{\pi/2} |\sin\varphi|^q \cos^{q-2} \varphi d\varphi \right\}^{1/q} & \text{for} \quad 2 < p < \infty\,. \end{cases} \tag{2.25}$$

Evaluating the integrals in (2.25), we arrive at (2.15). □

Let u be a harmonic function in $\mathbb{R}^2_+$ given as the single layer potential

$$u(x,y) = \frac{1}{2\pi} \int\limits_{\infty}^{\infty} \log \frac{1}{(\xi-x)^2+y^2} \frac{\partial u}{\partial \boldsymbol{\nu}}(\xi, 0) d\xi \tag{2.26}$$

with the boundary value of $\partial u/\partial \boldsymbol{\nu}$ on $\partial\mathbb{R}^2_+$ in $L^p(-\infty, \infty), 1 \leqslant p < \infty$. Then $\partial u/\partial \boldsymbol{\nu} \in h^p(\mathbb{R}^2_+)$.

The next assertion follows from Theorem 1.

Corollary 1. *Let u be a solution of the Neumann problem for the Laplace equation in $\mathbb{R}^2_+$ with $\partial u/\partial \boldsymbol{\nu} \in h^p(\mathbb{R}^2_+), 1 \leqslant p < \infty$. For any point $z = (x, y) \in \mathbb{R}^2_+$ the inequality*

$$|\nabla u(z)| \leqslant \frac{C_p}{y^{1/p}} \left|\left| \frac{\partial u}{\partial \boldsymbol{\nu}} \right|\right|_p \tag{2.27}$$

is valid with the sharp coefficient and the same C_p as in Theorem 1.

The maximum absolute value of the directional derivative of u at z with $||\partial u/\partial \boldsymbol{\nu}||_p \leqslant 1$ is attained at the direction which is normal to $\partial\mathbb{R}^2_+$ if $1 \leqslant p < 2$, at any direction if $p = 2$, and at the tangential direction if $2 < p < \infty$.

Proof. By (2.26),

$$\nabla u(z) = \frac{1}{\pi} \int\limits_{\infty}^{\infty} \frac{(\xi-x)\boldsymbol{i} - y\boldsymbol{j}}{(\xi-x)^2+y^2} \frac{\partial u}{\partial \boldsymbol{\nu}}(\xi, 0) d\xi\,. \tag{2.28}$$

Let $\boldsymbol{\ell}_\alpha$ be the unit vector at an angle α with respect to the x-axis. In view of (2.28),

$$\frac{\partial u}{\partial \boldsymbol{\ell}_\alpha} = \frac{1}{\pi} \int\limits_{\infty}^{\infty} \frac{(\xi-x)\cos\alpha - y\sin\alpha}{(\xi-x)^2+y^2} \frac{\partial u}{\partial \boldsymbol{\nu}}(\xi, 0) d\xi\,. \tag{2.29}$$

Setting $z = x + iy$ and $\zeta = \xi + i0$, we can write (1.1) in the form

$$f(z) = \frac{1}{\pi} \int\limits_{\infty}^{\infty} \frac{y - (\xi - x)i}{(\xi - x)^2 + y^2} \ \mathrm{Re}\, f(\xi, 0) d\xi \ .$$

Hence

$$\mathrm{Re}\left\{ e^{i\left(\frac{\pi}{2}+\alpha\right)} f(z) \right\} = \frac{1}{\pi} \int\limits_{\infty}^{\infty} \frac{(\xi - x)\cos\alpha - y \sin\alpha}{(\xi - x)^2 + y^2} \ \mathrm{Re}\, f(\xi, 0) d\xi \ . \tag{2.30}$$

Comparing (2.29) and (2.30), we conclude that the sharp constants in

$$\left| \frac{\partial u}{\partial \boldsymbol{\ell}_\alpha} \right| \leqslant \frac{S_p(\alpha)}{y^{1/p}} \left|\left| \frac{\partial u}{\partial \boldsymbol{\nu}} \right|\right|_p \tag{2.31}$$

and

$$|\, \mathrm{Re}\{e^{i\alpha} f(z)\}| \leqslant \frac{C_p(\alpha)}{y^{1/p}} \, ||\mathrm{Re}\, f||_p$$

are related as follows:

$$S_p(\alpha) = C_p \left(\frac{\pi}{2} + \alpha \right) \ . \tag{2.32}$$

Therefore,

$$\max_\alpha S_p(\alpha) = C_p \ , \tag{2.33}$$

with the same C_p as in Theorem 1. This proves the inequality (2.27).

It follows from (2.32), (2.33) and Theorem 1, that the maximum value of $S_p(\alpha)$ is attained for $\alpha = \pi/2$ if $1 \leqslant p < 2$, for $\alpha = 0$ if $2 < p < \infty$ and $S_2(\alpha)$ is independent of α. The corollary is proved. □

3 Sharp Estimates for the Gradient of a Solution of the Neumann Problem

We introduce some notation used in the present section. Let $\mathbb{R}^n_+ = \big\{ x = (x', x_n) : x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0 \big\}$, $\mathbb{S}^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \}$, and $\mathbb{S}^{n-1}_- = \{ x \in \mathbb{R}^n : |x| = 1, \ x_n < 0 \}$. Let $\boldsymbol{e}_\sigma$ stand for the n-dimensional unit vector joining the origin to a point σ on the sphere $\mathbb{S}^{n-1}$. We suppose that $n \geqslant 3$.

By $|| \cdot ||_p$ we denote the norm in the space $L^p(\mathbb{R}^{n-1})$, i.e.,

$$||f||_p = \left\{ \int\limits_{\mathbb{R}^{n-1}} |f(x')|^p \ dx' \right\}^{1/p}$$

if $1 \leqslant p < \infty$ and $||f||_\infty = \mathrm{ess\ sup}\{|f(x')| : x' \in \mathbb{R}^{n-1}\}$.

Next, we denote by $h^p(\mathbb{R}^n_+), 1 \leqslant p \leqslant \infty$, the Hardy space of harmonic functions on $\mathbb{R}^n_+$, which can be represented as the Poisson integral

$$u(x) = \frac{2}{\omega_n} \int\limits_{\mathbb{R}^{n-1}} \frac{x_n}{|y - x|^n} \ u(y', 0) dy' \tag{3.1}$$

with boundary values in $L^p(\mathbb{R}^{n-1})$, where $y = (y', 0)$, $y' \in \mathbb{R}^{n-1}$, and ω_n is the area of the unit sphere in $\mathbb{R}^n$.

Let u be a harmonic function in $\mathbb{R}^n_+$ given as the single layer potential

$$u(x) = \frac{2}{(n-2)\omega_n} \int\limits_{\mathbb{R}^{n-1}} \frac{1}{|y-x|^{n-2}} \frac{\partial u}{\partial \boldsymbol{\nu}}(y', 0)\ )dy' \tag{3.2}$$

with the boundary values of $\partial u/\partial \boldsymbol{\nu}$ on $\partial \mathbb{R}^n_+$ in $L^p(\mathbb{R}^{n-1}), 1 \leqslant p < \infty$. Then $\partial u/\partial \boldsymbol{\nu} \in h^p(\mathbb{R}^n_+)$.

Now, we find a representation for the best coefficient $\mathscr{C}_{n,p}(x; \boldsymbol{\ell})$ in the inequality for the absolute value of derivative $\partial u/\partial \boldsymbol{\ell}$ in an arbitrary direction $\boldsymbol{\ell} \in \mathbb{S}^{n-1}$ with $||\partial u/\partial \boldsymbol{\nu}||_p$ on the right-hand side.

Proposition 2. *Let u be a solution of the Neumann problem for the Laplace equation in $\mathbb{R}^n_+$ with $\partial u/\partial \boldsymbol{\nu} \in h^p(\mathbb{R}^n_+), 1 \leqslant p < \infty$, and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathscr{C}_{n,p}(x; \boldsymbol{\ell})$ in the inequality*

$$|(\nabla u(x), \boldsymbol{\ell})| \leqslant \mathscr{C}_{n,p}(x; \boldsymbol{\ell}) \left|\left| \frac{\partial u}{\partial \boldsymbol{\nu}} \right|\right|_p \tag{3.3}$$

is given by

$$\mathscr{C}_{n,p}(x; \boldsymbol{\ell}) = \frac{C_{n,p}(\boldsymbol{\ell})}{x_n^{(n-1)/p}}\ , \tag{3.4}$$

where

$$C_{n,1}(\boldsymbol{\ell}) = \frac{2}{\omega_n} \max_{\sigma \in \mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{\ell})||(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{n-1}\ , \tag{3.5}$$

and

$$C_{n,p}(\boldsymbol{\ell}) = \frac{2^{1/p}}{\omega_n} \left\{ \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{\ell})|^{p/(p-1)} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{(n-p)/(p-1)} d\sigma \right\}^{1-\frac{1}{p}} \tag{3.6}$$

for $1 < p < \infty$.

In particular, the sharp coefficient $\mathscr{C}_{n,p}(x)$ in the inequality

$$|\nabla u(x)| \leqslant \mathscr{C}_{n,p}(x) \left|\left| \frac{\partial u}{\partial \boldsymbol{\nu}} \right|\right|_p \tag{3.7}$$

is given by

$$\mathscr{C}_{n,p}(x) = \frac{C_{n,p}}{x_n^{(n-1)/p}}\ , \tag{3.8}$$

where

$$C_{n,p} = \max_{|\boldsymbol{\ell}|=1} C_{n,p}(\boldsymbol{\ell})\ . \tag{3.9}$$

Proof. By (3.2),

$$\nabla u(x) = \frac{2}{\omega_n} \int\limits_{\mathbb{R}^{n-1}} \frac{\boldsymbol{e}_{xy}}{|y-x|^{n-1}} \frac{\partial u}{\partial \boldsymbol{\nu}}(y', 0)\ )dy'\ ,$$

where $\boldsymbol{e}_{xy} = (y-x)/|y-x|$. Hence the sharp coefficient $\mathscr{C}_{n,p}(x; \boldsymbol{\ell})$ in (3.3) has the form

$$\mathscr{C}_{n,p}(x; \boldsymbol{\ell}) = \frac{2}{\omega_n} \left\{ \int\limits_{\mathbb{R}^{n-1}} \frac{|(\boldsymbol{e}_{xy}, \boldsymbol{\ell})|^q}{|y-x|^{(n-1)q}}\, dy' \right\}^{1/q}, \tag{3.10}$$

where $1/p+1/q=1$. Since

$$\frac{1}{|y-x|^{(n-1)q}}=\frac{1}{x_n^{(n-1)(q-1)}}\left(\frac{x_n}{|y-x|}\right)^{(n-1)q-n}\frac{x_n}{|y-x|^n}\ ,$$

it follows that (3.10) can be written in the form (3.4), where

$$C_{n,p}(\boldsymbol{\ell})=\frac{2}{\omega_n}\left\{\int\limits_{\mathbb{S}_-^{n-1}}|(\boldsymbol{e}_\sigma,\boldsymbol{\ell})|^q(\boldsymbol{e}_\sigma,-\boldsymbol{e}_n)^{(n-1)q-n}d\sigma\right\}^{1/q}$$

$$=\frac{2}{\omega_n}\left\{\frac{1}{2}\int\limits_{\mathbb{S}^{n-1}}|(\boldsymbol{e}_\sigma,\boldsymbol{\ell})|^q|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{(n-1)q-n}d\sigma\right\}^{1/q}. \tag{3.11}$$

The last equality is equivalent to (3.6). Formula (3.5) is the limit case of (3.11) as $q\to\infty$.

Formulas (3.8) and (3.9) for the best coefficient in (3.7) are direct consequences of (3.3) and (3.4). □

The next assertion provides a solution of the optimization problem (3.9) on the unit sphere in $\mathbb{R}^n$ for $p\in[1,n]$.

Theorem 2. *Let u be a solution of the Neumann problem for the Laplace equation in $\mathbb{R}^n_+$ with $\partial u/\partial\boldsymbol{\nu}\in h^p(\mathbb{R}^n_+)$, $p\in[1,n]$, and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathscr{C}_{n,p}$ in the inequality*

$$|(\nabla u(x)|\leqslant\frac{C_{n,p}}{x_n^{(n-1)/p}}\left|\left|\frac{\partial u}{\partial\boldsymbol{\nu}}\right|\right|_p \tag{3.12}$$

is given by

$$\mathscr{C}_{n,p}(x;\boldsymbol{\ell})=\frac{C_{n,p}(\boldsymbol{\ell})}{x_n^{(n-1)/p}}\ , \tag{3.13}$$

where

$$C_{n,1}=\frac{2}{\omega_n}\ , \tag{3.14}$$

and

$$C_{n,p}=\frac{2^{1/p}}{\omega_n}\left\{\frac{2\pi^{(n-1)/2}\Gamma\left(\frac{n+p-1}{2p-2}\right)}{\Gamma\left(\frac{np}{2p-2}\right)}\right\}^{1-\frac{1}{p}} \tag{3.15}$$

for $1<p\leqslant n$.

The maximum in $\boldsymbol{\ell}$ in (3.9) is attained for $\boldsymbol{\ell}=\boldsymbol{e}_n$ or for $\boldsymbol{\ell}=-\boldsymbol{e}_n$ if $1\leqslant p<n$. The coefficient $C_{n,n}(\boldsymbol{\ell})$ is independent of $\boldsymbol{\ell}$.

Proof. First, let $p=1$. It follows from (3.5) and (3.9) that

$$C_{n,1}=\frac{2}{\omega_n}\max_{|\boldsymbol{\ell}|=1}\max_{\sigma\in\mathbb{S}^{n-1}}|(\boldsymbol{e}_\sigma,\boldsymbol{\ell})||(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{n-1}\ . \tag{3.16}$$

Interchanging the order of maxima, we obtain

$$C_{n,1}=\frac{2}{\omega_n}\max_{\sigma\in\mathbb{S}^{n-1}}\max_{|\boldsymbol{\ell}|=1}|(\boldsymbol{e}_\sigma,\boldsymbol{\ell})||(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{n-1}=\frac{2}{\omega_n}\max_{\sigma\in\mathbb{S}^{n-1}}|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{n-1}=\frac{2}{\omega_n}\ ,$$

which proves (3.14). The maximum in $\boldsymbol{\ell}$ in (3.16) is attained at $\boldsymbol{\ell} = \pm \boldsymbol{e}_n$.

The equality (3.6) with $p = n$ takes the form

$$C_{n,n}(\boldsymbol{\ell}) = \frac{2^{1/n}}{\omega_n} \left\{ \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{\ell})|^{n/(n-1)} d\sigma \right\}^{1-\frac{1}{n}} . \tag{3.17}$$

Thus, $C_{n,n}(\boldsymbol{\ell})$ is independent of $\boldsymbol{\ell}$.

Next, let $p \in (1, n)$. By (3.6) and (3.9),

$$C_{n,p} = \max_{|\boldsymbol{\ell}|=1} C_{n,p}(\boldsymbol{\ell}) \geqslant C_{n,p}(\boldsymbol{e}_n) = \frac{2^{1/p}}{\omega_n} \left\{ \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{n/(p-1)} d\sigma \right\}^{1-\frac{1}{p}} . \tag{3.18}$$

We show the estimate opposite to (3.18) also holds.

Let

$$s = \frac{n}{p} , \quad t = \frac{n}{n-p} . \tag{3.19}$$

Then

$$\frac{1}{s} + \frac{1}{t} = \frac{p}{n} + \frac{n-p}{n} = 1 . \tag{3.20}$$

Using the Hölder inequality and (3.19), we obtain

$$\begin{aligned} \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{\ell})|^{\frac{p}{p-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{n-p}{p-1}} d\sigma &\leqslant \left\{ \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{\ell})|^{\frac{ps}{p-1}} d\sigma \right\}^{1/s} \left\{ \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{(n-p)t}{p-1}} d\sigma \right\}^{1/t} \\ &= \left\{ \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{\ell})|^{\frac{n}{p-1}} d\sigma \right\}^{1/s} \left\{ \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{n}{p-1}} d\sigma \right\}^{1/t} . \end{aligned} \tag{3.21}$$

Since the first integral in the last equality is independent of $\boldsymbol{\ell}$, it follows that

$$\int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{\ell})|^{\frac{n}{p-1}} d\sigma = \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{n}{p-1}} d\sigma ,$$

which, in view of (3.20), allows us to write (3.21) as

$$\int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{\ell})|^{\frac{p}{p-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{n-p}{p-1}} d\sigma \leqslant \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{\frac{n}{p-1}} |d\sigma .$$

This, together with (3.6) and (3.9), implies

$$C_{n,p} \leqslant \frac{2^{1/p}}{\omega_n} \left\{ \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{n/(p-1)} d\sigma \right\}^{1-\frac{1}{p}} .$$

This estimate and (3.18) imply

$$C_{n,p} = \frac{2^{1/p}}{\omega_n} \left\{ \int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{n/(p-1)} d\sigma \right\}^{1-\frac{1}{p}} . \tag{3.22}$$

Besides, the maximum in $\boldsymbol{\ell}$ in (3.9) is attained at $\boldsymbol{\ell} = \pm \boldsymbol{e}_n$.

Evaluating the integral in (3.22), we find

$$\int\limits_{\mathbb{S}^{n-1}} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{n/(p-1)} d\sigma = 2\omega_{n-1} \int\limits_0^{\pi/2} \cos^{n/(p-1)} \vartheta \sin^{n-2} \vartheta d\vartheta$$

$$= \omega_{n-1} B\left(\frac{n+p-1}{2(p-1)}\, , \frac{n-1}{2} \right) = \frac{2\pi^{(n-1)/2}\Gamma\left(\frac{n+p-1}{2(p-1)}\right)}{\Gamma\left(\frac{np}{2(p-1)}\right)} \, ,$$

which, together with (3.22) and (3.17), proves (3.15). □

4 Best Real-Part Estimates for the Derivative of an Analytic Function

By (1.1),

$$f'(z) = \frac{1}{\pi i} \int\limits_{\infty}^{\infty} \frac{\operatorname{Re} f(\zeta)}{(\zeta - z)^2} \, d\zeta \, , \tag{4.1}$$

where $z \in \mathbb{C}_+$.

Proposition 3. *Let* $\operatorname{Re} f \in h^p(\mathbb{R}^2_+), 1 \leqslant p < \infty$, *and let* z *be an arbitrary point in* $\mathbb{C}_+$. *The sharp coefficient* $\mathscr{K}_p(z;\alpha)$ *in the inequality*

$$|\operatorname{Re}\{e^{i\alpha} f'(z)\}| \leqslant \mathscr{K}_p(z;\alpha) || \operatorname{Re} f||_p \tag{4.2}$$

is given by

$$\mathscr{K}_p(z;\alpha) = \frac{K_p(\alpha)}{(\operatorname{Im} z)^{(p+1)/p}} \, , \tag{4.3}$$

where

$$K_p(\alpha) = \frac{1}{\pi} \left\{ \int\limits_{-\pi/2}^{\pi/2} |\sin(2\varphi - \alpha)|^q \cos^{2(q-1)} \varphi d\varphi \right\}^{1/q} \tag{4.4}$$

and $1/p + 1/q = 1$.

Proof. It follows from (4.1) that

$$\operatorname{Re}\{e^{i\alpha} f'(z)\} = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \operatorname{Re}\left\{ \frac{e^{i\left(\alpha - \frac{\pi}{2}\right)}}{(\zeta - z)^2} \right\} \operatorname{Re} f(\zeta) d\zeta \, .$$

Putting here $z = x + iy$, we obtain

$$\operatorname{Re}\{e^{i\alpha} f'(z)\} = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \operatorname{Re}\left\{ \frac{e^{i\left(\alpha - \frac{\pi}{2}\right)}[(\zeta - x) + iy]^2}{[(\zeta - x)^2 + y^2]^2} \right\} \operatorname{Re} f(\zeta) d\zeta \, .$$

Hence the sharp constant $\mathscr{K}_p(z;\alpha)$ in (4.2) takes the form

$$\mathscr{K}_p(z;\alpha) = \frac{1}{\pi}\left\{\int_{-\infty}^{\infty}\left|\mathrm{Re}\left\{\frac{e^{i(\alpha-\frac{\pi}{2})}[(\zeta-x)+iy]^2}{[(\zeta-x)^2+y^2]^2}\right\}\right|^q d\zeta\right\}^{1/q}. \tag{4.5}$$

We introduce a new variable $\varphi \in (-\pi/2, \pi/2)$ by the equalities (2.7). Since

$$e^{i(\alpha-\frac{\pi}{2})}\left\{\frac{(\zeta-x)+iy}{\sqrt{(\zeta-x)^2+y^2}}\right\}^2 = e^{i(\alpha-\frac{\pi}{2})}\big(\sin\varphi + i\cos\varphi\big)^2 = e^{i(\alpha+\frac{\pi}{2}-2\varphi)}$$

and

$$\frac{1}{[(\zeta-x)^2+y^2]^q} = \frac{1}{y^{2q-1}}\left(\frac{y}{\sqrt{(\zeta-x)^2+y^2}}\right)^{2(q-1)}\frac{y}{(\zeta-x)^2+y^2}\,,$$

one can, using (2.7)–(2.9), write (4.5) as

$$K_p(z;\alpha) = \frac{1}{\pi y^{(2q-1)/q}}\left\{\int_{-\pi/2}^{\pi/2}|\sin(2\varphi-\alpha)|^q\cos^{2(q-1)}\varphi d\varphi\right\}^{1/q}.$$

The last representation is equivalent to (4.3) and (4.4). □

The next assertion contains the sharp constant in the estimate of the modulus of the derivative of an analytic function in $\mathbb{C}_+$. Here, K_∞ is understood as the limit of K_p as $p \to \infty$.

Theorem 3. *Let* $\mathrm{Re}\, f \in h^p(\mathbb{R}^2_+), 1 \leqslant p \leqslant \infty$, *and let* z *be an arbitrary point in* $\mathbb{C}_+$. *The sharp coefficient* $\mathscr{K}_p(z)$ *in the inequality*

$$|f'(z)| \leqslant \mathscr{K}_p(z)||\,\mathrm{Re}\, f||_p \tag{4.6}$$

is given by

$$\mathscr{K}_p(z) = \frac{K_p}{(\mathrm{Im}\, z)^{(p+1)/p}}\,, \tag{4.7}$$

where

$$K_1 = \frac{1}{\pi}\,, \qquad K_2 = \frac{1}{2\sqrt{\pi}}\,, \qquad K_\infty = \frac{2}{\pi}\,, \tag{4.8}$$

$$K_p = \frac{1}{(2\sqrt{\pi^{p+1}})^{1/p}}\left\{\sum_{n=0}^{\infty}\binom{1/(p-1)}{2n}\frac{\Gamma\left(\frac{(2n+1)(p-1)+p}{2p-2}\right)}{\Gamma\left(\frac{2(n+1)(p-1)+p}{2p-2}\right)}\right\}^{1-\frac{1}{p}} \tag{4.9}$$

for $1 < p < 2$ *and*

$$K_p = \frac{2}{\pi}\left\{\frac{\Gamma\left(\frac{2p+1}{2p-2}\right)\Gamma\left(\frac{2p-1}{2p-2}\right)}{\Gamma\left(\frac{2p}{p-1}\right)}\right\}^{1-\frac{1}{p}} \tag{4.10}$$

for $2 < p < \infty$.

The maximum of the function $K_p(\alpha)$ *defined by* (4.4) *is attained for* $\alpha = \pi/2$ *if* $1 \leqslant p < 2$ *and for* $\alpha = 0$ *if* $2 < p < \infty$. *The coefficients* $K_2(\alpha)$ *and* $K_\infty(\alpha)$ *are independent of* α.

Proof. It follows from Proposition 3 that the sharp constant $\mathscr{K}_p(z)$ in (4.6) is defined by (4.7), where

$$K_p = \max_{\alpha} K_p(\alpha) \ , \tag{4.11}$$

and $K_p(\alpha)$ is given by (4.4).

(i) *Cases* $p = 1, 2,$ *and* ∞. By (4.4) and (4.11),

$$K_1 = \frac{1}{\pi} \max_{\alpha} \max_{\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} |\sin(2\varphi - \alpha)| \cos^2 \varphi \ .$$

The maxima in φ and α are attained at $2\varphi - \alpha = \pm\pi/2$ and $\varphi = 0$, i.e., for $\alpha = \pm\pi/2$, $\varphi = 0$. Thus, $K_1 = 1/\pi$ which proves the first formula in (4.8).

Evaluating the integral (4.4) for $p = 2$, we find

$$K_2(\alpha) = \frac{1}{2\sqrt{\pi}} \ , \tag{4.12}$$

i.e., $K_2(\alpha)$ is independent of α. Now, the second equality in (4.8) follows from (4.11) and (4.12).

The value $K_\infty = \lim_{p\to\infty} K_p$ admits a straightforward calculation. By (4.4),

$$K_\infty(\alpha) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} |\sin(2\varphi - \alpha)| d\varphi \ .$$

The change of variable $\psi = 2\varphi - \alpha$ implies

$$K_\infty(\alpha) = \frac{1}{2\pi} \int_{-\pi-\alpha}^{\pi-\alpha} |\sin\psi| \ d\psi = \frac{1}{2\pi} \int_{0}^{2\pi} |\sin\psi| \ d\psi = \frac{2}{\pi} \ . \tag{4.13}$$

Thus, $K_\infty(\alpha)$ is independent of α. Now, the third formula in (4.8) results from (4.11) and (4.13).

(ii) *Cases* $1 < p < 2$ *and* $2 < p < \infty$. Making the change of variable $\psi = 2\varphi - \alpha$ in the integral

$$G_q(\alpha) = \int_{-\pi/2}^{\pi/2} |\sin(2\varphi - \alpha)|^q \cos^{2(q-1)} \varphi d\varphi \ , \tag{4.14}$$

which is involved in (4.4), we find

$$G_q(\alpha) = \frac{1}{2} \int_{-\pi-\alpha}^{\pi-\alpha} |\sin\psi|^q \cos^{2(q-1)} \frac{\psi + \alpha}{2} \ d\psi \ .$$

Hence

$$\frac{\partial G_q}{\partial \alpha} = -\frac{q-1}{2} \int_{-\pi-\alpha}^{\pi-\alpha} |\sin\psi|^q \cos^{2q-3} \frac{\psi + \alpha}{2} \sin \frac{\psi + \alpha}{2} \ d\psi \ .$$

Returning back to the variable $\varphi = (\psi + \alpha)/2$, we obtain

$$\frac{\partial G_q}{\partial \alpha} = (1 - q) \int_{-\pi/2}^{\pi/2} |\sin(2\varphi - \alpha)|^q \cos^{2q-3} \varphi \sin\varphi \ d\varphi \ . \tag{4.15}$$

We divide the integration interval into two: over $(0, \pi/2)$ and over $(-\pi/2, 0)$, and make the change of variable $\vartheta = -\varphi$ in the integral over $(-\pi/2, 0)$. As a result, (4.15) can be written in the form

$$\frac{\partial G_q}{\partial \alpha} = (1-q) \int\limits_0^{\pi/2} \left[\, |\sin(2\varphi+\alpha)|^q - |\sin(2\varphi-\alpha)|^q \,\right] \cos^{2q-3}\varphi \sin\varphi \, d\varphi \; .$$

Next we make the change of variable $\theta = 2\varphi$ in the last integral to obtain

$$\frac{\partial G_q}{\partial \alpha} = \frac{q-1}{2} \int\limits_0^{\pi} \left[\, |\sin(\theta+\alpha)|^q - |\sin(\theta-\alpha)|^q \,\right] \cos^{2q-3}\frac{\theta}{2} \sin\frac{\theta}{2} \, d\theta$$

$$= \frac{q-1}{2} \sum_{k=1}^{2} \int\limits_{\pi(k-1)/2}^{\pi k/2} \left[\, |\sin(\theta+\alpha)|^q - |\sin(\theta-\alpha)|^q \,\right] \cos^{2q-3}\frac{\theta}{2} \sin\frac{\theta}{2} \, d\theta \; . \tag{4.16}$$

The change of variable $\eta = \pi - \theta$ in the integral

$$I = \int\limits_{\pi/2}^{\pi} \left[\, |\sin(\theta+\alpha)|^q - |\sin(\theta-\alpha)|^q \,\right] \cos^{2q-3}\frac{\theta}{2} \sin\frac{\theta}{2} \, d\theta$$

gives

$$I = -\int\limits_0^{\pi/2} \left[\, |\sin(\eta+\alpha)|^q - |\sin(\eta-\alpha)|^q \,\right] \cos^{2q-3}\frac{\pi-\eta}{2} \sin\frac{\pi-\eta}{2} \, d\eta \; ,$$

which enables one to write (4.16) as

$$\frac{\partial G_q}{\partial \alpha} = \frac{q-1}{2} \int\limits_0^{\pi/2} \left[\, |\sin(\theta+\alpha)|^q - |\sin(\theta-\alpha)|^q \,\right] \Phi_q(\theta) d\theta \; , \tag{4.17}$$

where

$$\Phi_q(\theta) = \cos^{2q-3}\frac{\theta}{2} \sin\frac{\theta}{2} \; - \; \cos^{2q-3}\frac{\pi-\theta}{2} \sin\frac{\pi-\theta}{2} \; . \tag{4.18}$$

We find the signs of the factors under the integral in (4.17).

It follows from (4.14) that $G_q(\alpha)$ is π-periodic in α. Hence one can assume that $\alpha \in [-\pi/2, \pi/2]$. Besides, (4.14) implies that $G_q(\alpha)$ is even in α:

$$G_q(-\alpha) = \int\limits_{-\pi/2}^{\pi/2} |\sin(2\varphi+\alpha)|^q \cos^{2(q-1)}\varphi \, d\varphi$$

$$= \int\limits_{\pi/2}^{-\pi/2} |\sin(-2\vartheta+\alpha)|^q \cos^{2(q-1)}(-\vartheta) \, (-d\vartheta)$$

$$= \int\limits_{-\pi/2}^{\pi/2} |\sin(2\vartheta-\alpha)|^q \cos^{2(q-1)}\vartheta \, d\vartheta = G_q(\alpha) \; .$$

Therefore, when looking for the maximum of $G_q(\alpha)$ one cane assume that $\alpha \in [0, \pi/2]$. Next, consider the function

$$g(\theta, \alpha) = |\sin(\theta+\alpha)| - |\sin(\theta-\alpha)| \ ,$$

with $\theta, \alpha \in [0, \pi/2]$. Since $0 \leqslant \theta + \alpha \leqslant \pi$ and $-\pi/2 \leqslant \theta - \alpha \leqslant \pi/2$, we have

$$g(\theta, \alpha) = \begin{cases} \sin(\theta+\alpha) - \sin(\theta-\alpha) & \text{for} \quad 0 \leqslant \theta - \alpha \leqslant \pi/2 \ , \\ \sin(\theta+\alpha) + \sin(\theta-\alpha) & \text{for} \quad -\pi/2 \leqslant \theta - \alpha \leqslant 0 \ . \end{cases}$$

Thus, $g(\theta, \alpha) = 2\cos\theta\sin\alpha$ for $0 \leqslant \theta - \alpha \leqslant \pi/2$ and $g(\theta, \alpha) = 2\sin\theta\cos\alpha$ if $-\pi/2 \leqslant \theta - \alpha \leqslant 0$. Therefore, $g(\theta, \alpha) \geqslant 0$ for $\theta, \alpha \in [0, \pi/2]$. This implies

$$|\sin(\theta+\alpha)|^q \geqslant |\sin(\theta-\alpha)|^q \tag{4.19}$$

if $\theta, \alpha \in [0, \pi/2]$ and the equality sign in (4.19) is attained for $\alpha = 0, \pi/2$ or $\theta = 0, \pi/2$.

Next, consider the function $\Phi_q(\theta)$ defined by (4.18). We have

$$\Phi_q(\theta) = \sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(\cos^{2(q-2)}\frac{\theta}{2} - \sin^{2(q-2)}\frac{\theta}{2}\right) \ . \tag{4.20}$$

Since $\cos(\theta/2) > \sin(\theta/2)$ for $0 < \theta < \pi/2$, it follows from (4.20) that $\Phi_q(\theta) > 0$ for $2 < q < \infty$ $(1 < p < 2)$ and $\Phi_q(\theta) < 0$ for $1 < q < 2$ $(2 < p < \infty)$ as $\theta \in (0, \pi/2)$. Hence, by (4.17) and (4.19), for $\alpha \in (0, \pi/2)$ one has

$$\frac{\partial G_q}{\partial \alpha} > 0 \quad \text{for} \quad 1 < p < 2 \ ,$$

and

$$\frac{\partial G_q}{\partial \alpha} < 0 \quad \text{for} \quad 2 < p < \infty \ .$$

In view of (4.4) and (4.14), we have $K_p(\alpha) = \pi^{-1}\{G_q(\alpha)\}^{1/q}$. Hence

$$K_p = \max_\alpha K_p(\alpha) = \begin{cases} K_p(\pi/2) & \text{for} \quad 1 < p < 2 \ , \\ K_p(0) & \text{for} \quad 2 < p < \infty \ . \end{cases} \tag{4.21}$$

By (4.4),

$$K_p(0) = \frac{1}{\pi}\left\{\int_{-\pi/2}^{\pi/2} |\sin(2\varphi)|^q \cos^{2(q-1)}\varphi d\varphi\right\}^{1/q} = \frac{2^{(q+1)/q}}{\pi}\left\{\int_0^{\pi/2} \sin^q\varphi\cos^{3q-2}\varphi d\varphi\right\}^{1/q} ,$$

which, together with (4.21), leads to (4.10).

Now, by (4.4),

$$K_p\left(\frac{\pi}{2}\right) = \frac{1}{\pi}\left\{\int_{-\pi/2}^{\pi/2} |\cos 2\varphi|^q \cos^{2(q-1)}\varphi d\varphi\right\}^{1/q} ,$$

or, equivalently,

$$K_p\left(\frac{\pi}{2}\right)=\frac{2^{1/q}}{\pi}\left\{\int\limits_0^{\pi/4}\cos^q 2\varphi\cos^{2(q-1)}\varphi d\varphi+\int\limits_{\pi/4}^{\pi/2}(-\cos 2\varphi)^q\cos^{2(q-1)}\varphi d\varphi\right\}^{1/q}.$$

We put $\psi=(\pi/2)-\varphi$ in the second integral on the right-hand side to obtain

$$\begin{aligned}K_p\left(\frac{\pi}{2}\right)&=\frac{2^{1/q}}{\pi}\left\{\int\limits_0^{\pi/4}\cos^q 2\varphi\cos^{2(q-1)}\varphi d\varphi+\int\limits_0^{\pi/4}\cos^q 2\varphi\sin^{2(q-1)}\varphi d\varphi\right\}^{1/q}\\&=\frac{2^{1/q}}{\pi}\left\{\int\limits_0^{\pi/4}\cos^q 2\varphi\left(\cos^{2(q-1)}\varphi+\sin^{2(q-1)}\varphi\right)d\varphi\right\}^{1/q}.\end{aligned}$$

Letting $2\varphi=\vartheta$, we find

$$K_p\left(\frac{\pi}{2}\right)=\frac{2^{1/q}}{2\pi}\left\{\int\limits_0^{\pi/2}\left[\,(1+\cos\vartheta)^{q-1}+(1-\cos\vartheta)^{q-1}\,\right]\cos^q\vartheta d\vartheta\right\}^{1/q}.$$

The use of the binomial series gives

$$K_p\left(\frac{\pi}{2}\right)=\frac{1}{2^{1/p}\pi}\left\{2\sum_{n=0}^{\infty}\binom{q-1}{2n}\int\limits_0^{\pi/2}\cos^{2n+q}\vartheta\, d\vartheta\right\}^{1/q},$$

which leads to (4.9) because of (4.21). □

Note that the series in (4.9) is a finite sum if $(p-1)^{-1}$ is a natural number.

Concluding this section, we give an explicit estimate for the gradient of a harmonic function in $\mathbb{R}^2_+$. Let u be a harmonic function in $h^p(\mathbb{R}^2_+)$, and let f be an analytic function in $\mathbb{C}_+$ with $\mathrm{Re}\, f=u$. Setting

$$f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$$

in Theorem 3, we arrive at the following assertion.

Corollary 2. *Let $u\in h^p(\mathbb{R}^2_+), 1\leqslant p\leqslant\infty$, and let $z=(x,y)$ be an arbitrary point in $\mathbb{R}^2_+$. The sharp coefficient $\mathscr{K}_p(z)$ in the inequality*

$$|\nabla u(z)|\leqslant\mathscr{K}_p(z)\,||u||_p \tag{4.22}$$

is given by

$$\mathscr{K}_p(z)=\frac{K_p}{y^{(p+1)/p}}\,, \tag{4.23}$$

where the constant K_p is defined by formulas (4.8)–(4.10).

The maximum value of the modulus of the directional derivative of u at a point z with $||u||_p\leqslant 1$ is attained at the direction of the normal to $\partial\mathbb{R}^2_+$ if $1\leqslant p<2$, at any direction if $p=2$ and $p=\infty$, and at the tangential direction if $2<p<\infty$.

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