Singular solutions, minimal cones and the Hsiang problem

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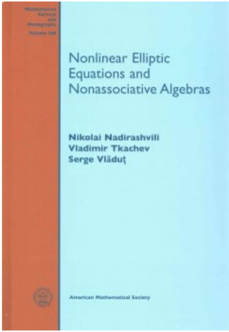
Motivations

- Non-classical and singular solutions to nonlinear uniformly elliptic PDEs
- Cubic minimal cones: Hsiang's problems
- Why nonassociative algebras



Fully nonlinear elliptic PDEs

Let us consider the Dirichlet problem

$$\begin{cases} F(D^2u) & = 0 \text{ in } \quad B_1 \\ \qquad u & = \phi \text{ on } \quad \partial B_1 \end{cases}$$

with F uniformly elliptic:

$$\lambda\|N\| \le F(M+N) - F(M) \le \Lambda\|N\|, \quad \forall N \ge 0 \text{ and } M, N \text{ symmetric}$$

Examples: Hessian eqs. (Monge-Ampère, Special Lagrangian), Bellman eqs.

Pucci's extremal operators, e.g.

$$\mathcal{M}^+ = \Lambda\,\mathrm{tr}_+(M) + \lambda\,\mathrm{tr}_-(M) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{tr}\, AM$$

where $\mathcal{A}_{\lambda,\Lambda}$ is the set of all symmetric matrices with spectrum $\subset [\lambda, \Lambda]$.

A continuous function u in Ω is a *viscosity subsolution* if for any $x_0 \in \Omega$ and any $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum at x_0 it follows that

$$F(D^2\phi(x_0)) \ge 0.$$

Similarly one defines a viscosity supersolution. Then u is a **viscosity solution** when it both viscosity supersolution and subsolution.

Fully nonlinear elliptic PDEs

The existence and uniqueness (Evans, Crandall, Lions, Jensen, Ishii)
If $\phi \in C^0(\partial B_1)$ then the Dirichlet problem

$$\begin{cases} F(D^2u) & = 0 \text{ in } \quad B_1 \\ \quad u & = \phi \text{ on } \quad \partial B_1 \end{cases}$$

has a unique **viscosity solution** $u \in C(B_1)$

$$\boxed{u(x) \text{ is a viscosity solution}} \quad \overset{?}{\Longrightarrow} \quad \boxed{u(x) \text{ is a classical } C^2 \text{ solution}}$$

Fully nonlinear elliptic PDEs

Let $F(D^2u) = 0$ be uniformly elliptic.

- If $n = 2$ then u is classical ($C^{2,\alpha}$) solution (Nirenberg 1953)
- If $n \geq 2$ and $\Lambda \leq 1 + \epsilon(n)$ then u is $C^{2,\alpha}$ (Cordes 1956)
- Hölder estimates and a Harnack inequality for uniformly elliptic or parabolic equations of second order with measurable coefficients (Krylov and Safonov, 1980)
- $u \in C^{1,\alpha}(B_{1/2})$, $\alpha = \alpha(\Lambda, n)$ (Trudinger, Caffarelli, 1989)
- F convex (concave) $\Rightarrow u \in C^{2,\alpha}(B_{1/2})$ (Krylov, Evans 1983)
- u is $C^{2,\alpha}(B_1 \setminus \Sigma)$, $\dim_H \Sigma < n - \epsilon$, $\epsilon = \epsilon(\Lambda, n)$ (Armstrong-Silvestere-Smart, 2011)

Nonclassical and singular examples

For a general uniformly elliptic F the Krylov-Safonov-Cafarelli-Trudinger regularity is optimal, i.e. there exist truly singular viscosity solutions in all dimensions $n \geq 5$.

Theorem (Nadirashvili, Vlăduț, 2007-2011). There a cubic form $w_{12}(x)$ such that for any $\delta \in [1, 2)$

$$u_\delta(x) := \frac{w_{12}(x)}{|x|^\delta}$$

is a viscosity solution to a certain uniformally elliptic equation $F(D^2u(x)) = 0$ (depending on δ) in $B_1 \subset \mathbb{R}^{12}$.

Theorem (Nadirashvili, V.T., Vlăduț, 2012). There a cubic form $w_5(x)$ such that $u = \frac{w_5(x)}{|x|} \notin C^2$ is a viscosity solution in $B_1 \subset \mathbb{R}^5$. Furthermore,

$$F(D^2u) = (\Delta u)^5 + 2^8 3^2 (\Delta u)^3 + 2^{12} 3^5 \Delta u + 2^{15} \det D^2(u).$$

The existence of truly viscosity solutions for $n = 3, 4$ is an **open problem**. There are no *homogeneous* order 2 solutions ($\mathbb{R}^3$: Alexandrov, 1939; $\mathbb{R}^4$: Nadirashvili, Vlăduț, 2012).

We consider singular ($C^{1,1}$) solutions (the $C^{1,\alpha}$-case is much more delicate).

A family $\mathcal{A} \subset S^2(\mathbf{R}^n)$ of symmetric matrices A is **uniformly hyperbolic** if $\exists M > 1$:

$$\frac{1}{M} < -\frac{\lambda_n(A)}{\lambda_1(A)} < M, \qquad \forall A \in \mathcal{A}, \quad \operatorname{spec}(A) = \{\lambda_1(A) \leq ... \leq \lambda_n(A)\}.$$

The Ellipticity Criterium: Let $w(x)$ be an *odd homogeneous* function of order 2 defined $B_1 \subset \mathbf{R}^n$ and smooth in $B_1 \setminus \{0\}$. If the family

$$\{D^2w(a) - O^{-1}D^2w(b)O \;:\quad a, b \in B_1 \setminus \{0\} \text{ and } O \in O(n)\}$$

is uniformly hyperbolic then w is a viscosity solution in B_1 of a uniformly elliptic Hessian equation.

Key steps of the proof

Let $u(x)$ be the Cartan isoparametric cubic form in $\mathbb{R}^5$ and $w(x) = u(x)/|x|$. Then

$$\frac{1}{12} \leq -\frac{\Lambda_5}{\Lambda_1} \leq 12, \tag{1}$$

where $\Lambda_1 \leq \cdots \leq \Lambda_5$ are the eigenvalues of $M(a, b, O) = D^2 w(a) - O^{-1} D^2 w(b) O$.

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Sketch of the **proof**. We have for any $|a| = |b| = 1$

$$\operatorname{tr} M(a,b,O) = \Delta w(a) - \Delta w(b) = -8\omega,$$

where $\omega := u(a) - u(b)$.

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- If $\omega = 0$ then (1) holds with $c = n - 1 = 4$ as for any traceless matrix in dimension n.
- Let $\omega \neq 0$, say $\omega > 0$. Then

$$0 \geq -8\omega = \operatorname{tr} M(a, b, O) \geq \Lambda_5 + (5-1)\Lambda_1 = \Lambda_5 - 4|\Lambda_1|, \tag{2}$$

$$\Rightarrow \quad \Lambda_5 \leq 4|\Lambda_1|.$$

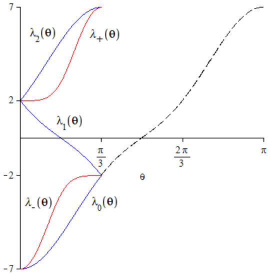
Key steps of the proof

A lower estimate of Λ_5 relies on the spectrum of $D^2 w(x)$:

- $\lambda_j(\theta) = \frac{3\sin(\frac{\pi j}{3} - 2\theta)}{\sin(\frac{\pi j}{3} + \theta)} - \cos 3\theta$, $j = 1, 2, 3$,
- $\lambda_\pm(\theta) = \frac{-5\cos 3\theta \pm 3\sqrt{5\cos^2 3\theta + 4}}{2}$

where

$$u(x) = \cos 3\theta(x), \quad \theta : S^4 \to [0, \frac{\pi}{3}].$$



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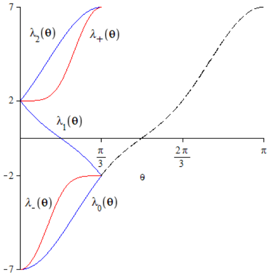


Then $0 < \theta(a) < \theta(b) < \frac{\pi}{3}$, and by Weyl's inequality,

$$\Lambda_5 \ge \max_{1\le i\le n} (\Lambda_i(D^2w(a)) - \Lambda_i(D^2w(b))) \ge \lambda_1(\theta(a)) - \lambda_1(\theta(b))$$

where $\lambda_1(\theta)$ is a decreasing function and

$$\frac{d\lambda_1}{du} = 2\,\frac{\sin(\theta + \frac{\pi}{3})}{\sin 3\theta} - 1 \ge 1, \qquad \theta \in (0, \frac{\pi}{3}).$$

Integrating yields $\Lambda_5 \ge \omega$, thus,

$$-8\Lambda_5 \le -8\omega = \operatorname{tr} M(a, b, O) \le (n-1)\Lambda_1 + \Lambda_5 = 4\Lambda_5 - |\Lambda_1|,$$

yields $|\Lambda_1| \le 12\Lambda_5$, as desired.

Isoparametric cubics

Recall that a submanifold of the Euclidean sphere $S^{n-1} \subset \mathbb{R}^n$ is called **isoparametric** if it has constant principal curvatures.

A celebrated result due to H.F. Münzner (1987) asserts that *any isoparametric hypersurface is algebraic and its defining polynomial* u *is homogeneous of degree* $g = 1, 2, 3, 4$ *or* 6, *where* g *is the number of distinct principal curvatures*.

É. Cartan (1938)

The only cubic polynomial solutions of

$$|Du(x)|^2 = 9|x|^4, \qquad \Delta u(x) = 0$$

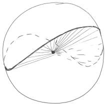
are the cubic forma in $\mathbb{R}^5, \mathbb{R}^8, \mathbb{R}^{14}, \mathbb{R}^{26}$:

$$u_d(x) := \frac{3\sqrt{3}}{2} \det \begin{pmatrix} x_2 - \frac{1}{\sqrt{3}}x_1 & \bar{z}_1 & \bar{z}_2 \\ z_1 & -x_2 - \frac{1}{\sqrt{3}}x_1 & \bar{z}_3 \\ z_2 & z_3 & \frac{2}{\sqrt{3}}x_1 \end{pmatrix}, \quad x \in \mathbb{R}^{3d+2},$$

Here $z_k \in \mathbb{R}^d \cong \mathbb{F}_d$ is the real division algebra of dimension $d \in \{1, 2, 4, 8\}$.

How do minimal cones enter?

In *all* examples of truly viscosity solutions $u(x) = \frac{w(x)}{|x|}$, their zero-locus is a minimal cone:



- $w_{3d} = \mathrm{Re}(z_1 z_2) z_3$, $z_i \in \mathbb{H}$ or $\mathbb{O}$, the triality polynomials in $\mathbb{R}^{12}$ and $\mathbb{R}^{24}$
- $w_5(x) = \begin{vmatrix} \frac{1}{\sqrt{3}}x_1 + x_2 & x_3 & x_4 \\ x_2 & \frac{-2}{\sqrt{3}}x_1 & x_5 \\ x_4 & x_5 & \frac{1}{\sqrt{3}}x_1 - x_2 \end{vmatrix}$ = a Cartan isoparametric cubic in $\mathbb{R}^5$
- $w_9(x) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{vmatrix}$ in $\mathbb{R}^9$ (C. Smart, unpublished)

Remarks:

- all $w_k(x)$ are Hsiang minimal cubics;
- all $w_k(x)$ are generic norms on a suitable cubic Jordan algebra.

Minimal cones

Blowing-down entire graphs yields area minimizing cones (Fleming, De Giorgi). In 1969, Bombieri-De Giorgi-Giusti have shown that the quadratic minimal cone

$$\{(x,y)\in\mathbb{R}^4\times\mathbb{R}^4 : |x|^2-|y|^2=0\}$$

is area-minimizing in $\mathbb{R}^8$. In particular, Bernstein theorem fails for $n\geq 8$

In general, if a minimal hypersurface is given implicitly by $u(x)=0$ then

$$\Delta_1 u(x) := |Du(x)|^2\Delta u(x) - \tfrac{1}{2}\langle Du(x), D|Du(x)|^2\rangle = 0 \quad \text{whenever } u(x)=0$$

If u is a homogeneous polynomial then the cone $u^{-1}(0)$ is minimal iff there exists a polynomial f, $\deg f = 2\deg u - 4$, such that

$$\Delta_1 u(x) = f(x)u(x).$$

W.Y. Hsiang (J. Diff. Geometry, 1967): Classify all solutions of $\deg u = 3$.

Hsiang's Problems

(ii) Partly due to the lack of "canonical" normal forms for $r < 2$ and partly due to the rapid rate of increase of the dimension of $\mathfrak{H}_n^r$ with respect to r, the little help obtained from the normal forms is not enough to solve the problem of classifying minimal algebraic cones of higher degrees. For example, it is very difficult to solve even the following very special equation: $F(x) = 0$, where $F(x)$ is an irreducible cubic form in n variables such that

$$(\Delta F)\cdot|\nabla F|^2 - \nabla F\cdot HF\cdot\nabla F^t = \pm(x_1^2 + \cdots + x_n^2)\cdot F .$$

Since the above equation is invariant with respect to the orthogonal linear substitutions, we may assume that F is given in some kind of "normal form" which amounts to reduce the number of indeterminant coefficients by $n(n-1)/2$. A systematic attempt to solve the above equation will involve the job of solving over-determined simultaneous algebraic equations of many variables. So far, we have only four non-trivial solutions (cf. §§ 1, 2), but there is no reason why there should be no others.

A homogeneous cubic form $u(x)$ is called a *Hsiang cubic* if

$$\Delta_1 u = \lambda|x|^2 u(x), \quad \lambda \in \mathbb{R}.$$

Jordan algebras

An algebra V with a **commutative** product $\bullet$ is called Jordan if

$$[L_x, L_{x^2}] = 0 \qquad \forall x \in V.$$

Main examples

1) The Jordan algebra $\mathscr{H}_n(\mathbb{A}_d)$ of Hermitian matrices of order n, $d = 1, 2, 4$ with

$$x \bullet y = \tfrac{1}{2}(xy + yx)$$

2) The spin factor $\mathscr{S}(\mathbb{R}^{n+1})$ with $(x_0, x) \bullet (y_0, y) = (x_0y_0 + \langle x, y\rangle;\, x_0y + y_0x)$

Theorem (Jordan-von Neumann-Wigner, 1934)

Any finite-dimensional *formally real* Jordan algebra is a direct sum of the simple ones:

- the spin factors $\mathscr{S}(\mathbb{R}^{n+1})$;
- the Jordan algebras $\mathscr{H}_n(\mathbb{A}_d)$, $n \geq 3$, $d = 1, 2, 4$;
- the Albert algebra $\mathscr{H}_3(\mathbb{A}_8)$.

What about the four Hsiang solutions?

Let $X \in \mathscr{H}'_k(\mathbb{A})$ = trace free hermitian $k \times k$-matrices over $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$

- Δ_1 is an $O(n)$-invariant $\Rightarrow \Delta_1(\operatorname{tr} X^3) =$ is a polynomial in $\operatorname{tr} X^2, \ldots, \operatorname{tr} X^k$
- $\deg(\Delta_1 \operatorname{tr} X^3) = 5$
- if $3 \le k \le 4$ then $\Delta_1 u(X) = c_1 \operatorname{tr} X^2 \operatorname{tr} X^3 = c_1 |X|^2 u(X)$.

$\Rightarrow$ $u = \operatorname{tr} X^3$ is a Hsiang cubic!

This yields the four Hsiang examples u in

$$\mathscr{H}'_3(\mathbb{R}) \cong \mathbb{R}^5, \quad \mathscr{H}'_3(\mathbb{C}) \cong \mathbb{R}^8, \quad \mathscr{H}'_4(\mathbb{R}) \cong \mathbb{R}^9, \quad \mathscr{H}'_4(\mathbb{C}) \cong \mathbb{R}^{15}$$

The Hsiang cubics in $\mathbb{R}^5$ and $\mathbb{R}^9$ are used in construction of the viscosity solutions.

An important observation: $\deg u = 3$ implies

$\operatorname{tr}(D^2 u) = 0$	the harmonicity
$\operatorname{tr}(D^2 u)^2 = C_1 \lvert x\rvert^2$	the quadratic trace identity
$\operatorname{tr}(D^2 u)^3 = C_2 u$	the cubic trace identity

Clifford type examples of Hsiang cubics

Example. The **Lawson cubic cone** in $\mathbb{R}^4$ with the defining polynomial

$$u(z) = (x_1^2 - x_2^2)y_1 + 2x_1x_2y_2 = \langle x, A_1x\rangle y_1 + \langle x, A_2x\rangle y_1, \qquad z = (x, y) \in \mathbb{R}^4$$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Theorem (V.T. 2010) Let $\{A_i\}_{1\le 1\le q}$ be a symmetric Clifford system, i.e.

$$A_i^2 = I \quad \text{and} \quad A_iA_j + A_jA_i = 0, \ \forall i \ne j.$$

Then

$$u_A(z) = \sum_{i=1}^{q} \langle x, A_ix\rangle y_i, \quad z = (x, y) \in \mathbb{R}^{2p} \times \mathbb{R}^q$$

is a Hsiang cubic.

The existence of a symmetric Clifford system in $\mathbb{R}^{2p}$ is equivalent to

$$q - 1 \le \rho(p),$$

where $\rho(p)$ is the Hurwitz-Radon function ($= 1+$ the number of vector fields on $\mathbb{S}^{p-1}$)

The dichotomy of Hisang cubics

Definition. A Hsiang cubic u is said to be of **Clifford type** if $u \cong u_A$ up to an orthogonal transformation; otherwise, it is called **exceptional**.

Representation theory of Clifford algebras yields a complete classification of Hsiang cubics of Clifford type.

How to determine all exceptional Hsiang cubics?

The dichotomy of Hisang cubics

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How to determine all exceptional Hsiang cubics?

Proposition. *The Hsiang examples w in $\mathbb{R}^5, \mathbb{R}^8, \mathbb{R}^9, \mathbb{R}^{15}$ are exceptional Hsiang cubics.*

Proof. Indeed, assume by contradiction that w is of Clifford type. Since Δ_1 is $O(n)$-invariant and for $u_A(z)$ in $\mathbb{R}^n = \mathbb{R}^{2p} \times \mathbb{R}^q$ one has

$$\mathrm{tr}(D^2 u_A)^2 = 2q|x|^2 + 2p|y|^2,$$

the **quadratic trace identity** $\mathrm{tr}(D^2 w)^2 = C_1|x|^2$ implies $q = p$. Combining with $q - 1 \leq \rho(p)$ yields $p \in \{1, 2, 4, 8\}$, thus, $n = q + 2p \in \{3, 6, 12, 24\}$, a contradiction. $\square$

The main results

Main Theorem, Part I

If u is a cubic homogeneous polynomial solution of

$$|Du(x)|^2\Delta u(x) - \tfrac{1}{2}\langle Du(x), D|Du(x)|^2\rangle = \lambda |x|^2 u(x)$$

then

- either $\Delta u(x) = 0$ or u is trivial (depends on one variable, $\sim x_1^3$)
- the cubic trace identity holds:

$$\operatorname{tr}(D^2u)^3 = 3\lambda(n_1 - 1)u, \qquad n_1 \in \mathbb{Z}^+$$

- $n_2 = \frac{1}{2}(n + 1 - 3n_1) \in \mathbb{Z}^+$
- $u(x)$ is exceptional Hsiang cubic iff $n_2 \neq 2$ and the quadratic trace identity holds

$$\operatorname{tr}(D^2u)^2 = C|x|^2, \quad C \in \mathbb{R}$$

Main Theorem, Part II

There exists finitely many isomorphy classes of exceptional Hsiang algebras.

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

In the realizable cases (uncolored):

- If $n_2 = 0$ then $u = \frac{1}{6}\langle z, z^2\rangle$, $z \in \mathscr{H}_3'(\mathbb{A}_d)$, $d = 0, 1, 2, 4, 8$.
- If $n_1 = 0$ then $u(z) = \frac{1}{12}\langle z^2, 3\bar{z} - z\rangle$, $z \in \mathscr{H}_3(\mathbb{A}_d)$, $d = 2, 4, 8$.
- If $n_1 = 1$ then $u(z) = \mathrm{Re}\langle z, z^2\rangle$, $z \in \mathscr{H}_3(\mathbb{A}_d) \otimes \mathbb{C}$, $d = 1, 2, 4, 8$.
- If $(n_1, n_2) = (4, 5)$ then $u = \frac{1}{6}\langle z, z^2\rangle$, $z \in \mathscr{H}_3(\mathbb{O}) \ominus \mathscr{H}_3(\mathbb{R})$

$\mathscr{H}_3(\mathbb{A}_d)$ is the Jordan algebra of 3×3-hermitian matrices over the Hurwitz algebra $\mathbb{A}_d$

Key steps of the proof

u is a solution of a PDE	$\Rightarrow$	a metrized algebra $V(u)$ with an identity

A commutative nonassociative algebra V with an inner product $\langle , \rangle$ is called **metrized** if the multiplication operator $L_x y := xy$ is **self-adjoint**, i.e.

$$\langle xy, z\rangle = \langle x, yz\rangle, \qquad \forall x, y, z \in V.$$

The Freudenthal-Springer construction: given a cubic form u, define an algebra by

$$u(x) = \tfrac{1}{6}\langle x, x^2\rangle \quad \Leftrightarrow \quad x \cdot y := (D^2 u(x))y$$

In this setting,

- the algebra $V = V(u)$ is metrized
- $Du(x) = \frac{1}{2}x^2$
- $L_x = D^2 u(x)$, i.e. the multiplication operator by x is the *Hessian* of u at x

Key steps of the proof

Let $u(x)$ be a Hsiang cubic, i.e.

$$|Du(x)|^2\Delta u(x) - \tfrac{1}{2}\langle Du(x), D|Du(x)|^2\rangle = \lambda|x|^2u(x)$$

and let $V = V(u)$ be the corresponding Freudenthal-Springer algebra. Then

$$\langle x^2, x^2\rangle \operatorname{tr} L_x - \langle x^2, x^3\rangle = \tfrac{2}{3}\lambda\langle x, x\rangle\langle x^2, x\rangle$$

Definition. A metrized algebra is called Hsiang if the latter identity satisfied.

The correspondence: if V is a Hsiang algebra then $u(x) = \frac{1}{6}\langle x, x^2\rangle$ is a Hsiang cubic. In the converse direction, if $u(x)$ is a Hsiang cubic then $V(u)$ is a Hsiang algebra.

Theorem A (The Dichotomy)

- Any nontrivial Hsiang algebra is harmonic: $\operatorname{tr} L_x = 0$.
- u is a Hsiang cubic of Clifford type iff $V(u)$ admits a non-trivial $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$ such that $V_0V_0 = 0$.

Key steps of the proof

The set of idempotents of $V(u)$ is nonempty: any maximum point of $u(x)$ on $\mathbb{S}^{n-1}$ gives rise to an idempotent:

$$Du(x_0) = kx_0 \quad \Leftrightarrow \quad \tfrac{1}{2}x_0^2 = kx_0 \quad \Leftrightarrow \quad c^2 = c \text{ for } c = x_0/2k$$

Given an idempotent $c \in V$, L_c is a self-adjoint. Consider the **Peirce decomposition**

$$V = \bigoplus_{\alpha=1}^{k} V_c(t_\alpha), \qquad V_c(t_\alpha) := \ker(L_c - t_\alpha)$$

A key point is by using the original PDE, to determine the multiplicative properties of the Peirce decomposition:

$$V_c(t_\alpha)V_c(t_\beta) \subset \bigoplus_{\gamma} V_c(t_\gamma)$$

If the PDE is 'good enough', there are some hidden (e.g., Clifford or Jordan) algebra structures inside V.

Key steps of the proof

Theorem B (The hidden Clifford algebra structure)

Let V be a Hisang algebra. Then

(i) given an idempotent $c \in V$, the associated Peirce decomposition is

$$V = V_c(1) \oplus V_c(-1) \oplus V_c(-\tfrac{1}{2}) \oplus V_c(\tfrac{1}{2}), \quad \dim V_c(1) = 1;$$

(ii) the Peirce dimensions $n_1 = \dim V_c(-1)$, $n_2 = \dim V_c(-\frac{1}{2})$ and $n_3 = \dim V_c(\frac{1}{2})$ do not depend on a particular choice of c and

$$n_3 = 2n_1 + n_2 - 2;$$

(iii) the following obstruction holds:

$$n_1 - 1 \le \rho(n_1 + n_2 - 1),$$

where ρ is the Hurwitz-Radon function.

Key steps of the proof

The Peirce decomposition

Setting $V_0 = V_c(1), \quad V_1 = V_c(-1), \quad V_2 = V_c(-\frac{1}{2}), \quad V_3 = V_c(\frac{1}{2})$ we have

	V_0	V_1	V_2	V_3
V_0	V_0	V_1	V_2	V_3
V_1	V_1	V_0	V_3	$V_2 \oplus V_3$
V_2	V_2	V_3	$V_0 \oplus V_2$	$V_1 \oplus V_2$
V_3	V_3	$V_2 \oplus V_3$	$V_1 \oplus V_2$	$V_0 \oplus V_1 \oplus V_2$

In particular, $V_0 \oplus V_1$ and $V_0 \oplus V_2$ are subalgebras of V.

Key steps of the proof

Theorem C (The hidden Jordan algebra structure)

Let V be a Hisang algebra. For any idempotent $c \in V$, the subspace

$$J_c := V_c(1) \oplus V_c(-\tfrac{1}{2})$$

carries a structure of a formally real rank 3 Jordan algebra, and the following conditions are equivalent:

(i) the Hsiang algebra V is *exceptional*;

(ii) J_c is a *simple* Jordan algebra;

(iii) $n_2 \neq 2$ and the *quadratic trace identity* $\operatorname{tr} L_x^2 = c|x|^2$ holds for some $c \in \mathbb{R}$.

The proof of the first part of the theorem is heavily based on the McCrimmon-Springer construction of a cubic Jordan algebra.

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THANK YOU FOR YOUR ATTENTION!