From quasi-composition algebras to Hsiang algebras

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Bird(?)'s eye view

Two **equivalent problems**

- (DG/PDE) Characterize algebraic degree three minimal cones $u(x) = 0$, i.e. all homogeneous degree 3 polynomial solutions to the nonlinear PDE (Hsiang's problem, 1967)

$$|\nabla u|^2 \Delta u - \frac{1}{2} \nabla u(x) \cdot \nabla(|\nabla u(x)|^2) = -\frac{3}{2} k \langle x, x \rangle u(x)$$

  Here $\langle x, y \rangle$ is the standard scalar product in $\mathbb{R}^n$, $k \in \mathbb{R}$.

- (NAA) Describe all $\langle , \rangle$-metrized commutative nonassociative algebra (over $\mathbb{R}$) satisfying

$$\langle x^2, x^3 \rangle = k \langle x, x^2 \rangle \langle x, x \rangle, \qquad \operatorname{tr} L_x = 0$$

  and the correspondence is given explicitly by

$$\boxed{u(x) = \tfrac{1}{6} \langle x, x^2 \rangle \qquad \leftrightarrows \qquad \text{algebra multiplication} \quad x \circ y = \operatorname{Hess} u(x) y}$$

- Any Hsiang algebra is either of **Clifford type** (an **infinite** family, well understood, has several nice descriptions) or **exceptional** (a **finite** family related to simple order 3 Euclidean Jordan algebras). All exceptional algebras appear in differential geometry, nonlinear analysis and nonassociative algebra

  One of the main goals in this context is to classify/describe/understand exceptional Hsiang algebras.

It turns out that there exists **another algebraic context**. One can naturally 'discover' to it in an attempt to combain Clifford and exceptional Hsiang algebras together. There two main ingredients are:

- The **tripling construction**: given an involutively metrized (noncommutative nonassociative) algebra $\mathbb{A}$, one associates an H-metrized **commutative** algebra $T(\mathbb{A})$ on $\mathbb{A} \oplus \mathbb{A} \oplus \mathbb{A}$.
- $(T(\mathbb{A}), H)$ is a Hsiang algebra if and only if $(\mathbb{A}, \sigma, h)$ satisfies the **quasicomposition identity**

$$\boxed{L(x)L(x^\sigma)L(x) = h(x,x)L(x).} \qquad \text{(QC)}$$

The QC-property is equivalent to that the quadratic form $q(x) := h(x,x)$ satisfies

$$\boxed{q(x^\sigma(xy)) = q(x^\sigma)q(xy).}$$

It turns out that any QC algebra satisfies the constant rank condition (for example, any Hirwitz algebra is QC with $\operatorname{corank} = 0$) Think of, for example, the standard octonions. Then it is well known that $\bar{x}(xy) = |x|^2 y$ which obviously implies (QC) (with $x^\sigma := \bar{x}$).

Remark. Not any Hsiang algebra can be obtained as the tripling of an algebra in view of the obstruction:

$$\dim \mathbf{T}(\mathbb{A}) = 3 \dim \mathbb{A}.$$

But, the latter perfectly holds for **exceptional** Hsiang algebras. So far we know examples of quasicomposition algebras in dimensions only

$$\dim \mathbb{A} \in \{1, 2, 3, 4', 4'', 5, 6, 7, 8, 9, 10, \underline{12}, \underline{14}, 18\},$$

where the red numbers correspond to the dimensions where exceptional Hsiang algebras can/do exist.

Part I: Minimal cones and Hsiang algebras

Cubic minimal cones

A **minimal surface** (a soap film) is a critical point of the **area functional**. Geometrically, this means that the **mean curvature vanishes totally**. From PDE point of view, a minimal surface is locally a **solution to a certain nonlinear PDE**.

A **cone** is a typical singularity of a minimal surface. All known minimal cones are algebraic, i.e. *zero level sets* of a homogeneous polynomial $u \in \mathbb{R}[x_1, \dots, x_n]$:

- $u = \mathrm{Re}((z_1 z_2) z_3)$, $z_i \in \mathbb{K}_d$, $d = 1, 2, 4, 8$ in $\mathbb{R}^{3d}$ (the **triality polynomials**).
- Many (not all!) examples can be realized as cubic forms on a suitable **Jordan algebra**. For instance, $u(x) = \mathrm{tr}\, x^3$, where x is a hermitean $3 \times 3-$matrix over a Hurwitz (unital composition) algebra.

W.-Y. Hsiang asks in 1967 to classify **cubic minimal cones**, or at least its most interesting family satisfying

$$\Delta u |\nabla u|^2 - \nabla u(x) \cdot \mathrm{Hess}\, u(x) \nabla u(x) = -\frac{3}{2} k \langle x, x \rangle u(x) \tag{1}$$

(ii) Partly due to the lack of "canonical" normal forms for $r < 2$ and partly due to the rapid rate of increase of the dimension of $\mathfrak{H}_n^r$ with respect to r, the little help obtained from the normal forms is not enough to solve the problem of classifying minimal algebraic cones of higher degrees. For example, it is very difficult to solve even the following very special equation: $F(x) = 0$, where $F(x)$ is an irreducible cubic form in n variables such that

$$(\Delta F) \cdot |\nabla F|^2 - \nabla F \cdot \boldsymbol{H} F \cdot \nabla F^t = \pm (x_1^2 + \cdots + x_n^2) \cdot F .$$

Since the above equation is invariant with respect to the orthogonal linear substitutions, we may assume that F is given in some kind of "normal form" which amounts to reduce the number of indeterminant coefficients by $n(n-1)/2$. A systematic attempt to solve the above equation will involve the job of solving over-determined simultaneous algebraic equations of many variables. So far, we have only four non-trivial solutions (cf. §§ 1, 2), but there is no reason why there should be no others.

(W.-Y. Hsiang. Trondheim, Norway, October 2016)

Cubic minimal cones

A direct approach is impossible, see for instance what happens for small dimensions, $n = 3$:

$$u := a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + b_1 x_1^2 x_2 + b_2 x_1^2 x_3 + b_3 x_2^2 x_1 + b_4 x_2^2 x_3 + b_5 x_3^2 x_1 + b_6 x_3^2 x_2 + c_1 x_1 x_2 x_3$$

$$\begin{aligned}\Delta(u) = {} & 54 a_1^2 a_2 x_1^4 x_2 + 54 a_1^2 a_3 x_1^4 x_3 + 18 a_1^2 b_3 x_1^5 + 18 a_1^2 b_4 x_1^4 x_3 + 18 a_1^2 b_5 x_1^5 + 18 a_1^2 b_6 x_1^4 x_2 + 54 a_1 a_2^2 x_1 x_2^4 + 72 a_1 a_2 b_1 x_1^3 x_2^2 + 72 a_1 a_2 b_2 x_1^3 x_2 x_3 + 72 a_1 a_2 b_3 x_1^2 x_2^2 x_3 + 72 a_1 a_2 b_4 x_1 x_2^3 x_3 + 36 a_1 a_2 b_5 x_1^2 x_2 x_3^2 \\ & + 36 a_1 a_2 b_6 x_1 x_2^2 x_3^2 + 54 a_1 a_2 c_1 x_1^2 x_2^2 x_3 + 54 a_1 a_3^2 x_1 x_3^4 + 72 a_1 a_3 b_1 x_1^3 x_2 x_3 + 72 a_1 a_3 b_2 x_1^3 x_3^2 + 36 a_1 a_3 b_3 x_1^2 x_2^2 x_3 + 36 a_1 a_3 b_4 x_1 x_2^2 x_3^2 + 72 a_1 a_3 b_5 x_1^2 x_3^3 + 72 a_1 a_3 b_6 x_1 x_2 x_3^2 + 54 a_1 a_3 c_1 x_1^2 x_2 x_3^2 - 6 a_1 b_1^2 x_1^5 \\ & + 12 a_1 b_1 b_3 x_1^4 x_2 + 24 a_1 b_1 b_4 x_1^3 x_2 x_3 + 24 a_1 b_1 b_5 x_1^4 x_2 + 24 a_1 b_1 b_6 x_1^3 x_2^2 - 6 a_1 b_1 c_1 x_1^4 x_3 - 6 a_1 b_2^2 x_1^5 + 24 a_1 b_2 b_3 x_1^4 x_3 + 24 a_1 b_2 b_4 x_1^3 x_3^2 + 12 a_1 b_2 b_5 x_1^4 x_3 + 24 a_1 b_2 b_6 x_1^3 x_2 x_3 - 6 a_1 b_2 c_1 x_1^4 x_2 + 12 a_1 b_3^2 x_1^3 x_2^2 \\ & + 36 a_1 b_3 b_4 x_1^2 x_2^2 x_3 + 12 a_1 b_3 b_5 x_1^3 x_2^2 + 12 a_1 b_3 b_5 x_1^3 x_3^2 + 12 a_1 b_3 b_6 x_1^2 x_3^2 + 12 a_1 b_3 b_6 x_1^2 x_2 x_3^2 + 12 a_1 b_3 c_1 x_1^3 x_2 x_3 + 6 a_1 b_4^2 x_1 x_2^4 + 24 a_1 b_4^2 x_1 x_2^2 x_3^2 + 12 a_1 b_4 b_5 x_1^2 x_2^2 x_3 + 12 a_1 b_4 b_5 x_1^2 x_3^3 + 24 a_1 b_4 b_6 x_1 x_2^3 x_3 \\ & + 24 a_1 b_4 b_6 x_1 x_2 x_3^3 + 6 a_1 b_4 c_1 x_1^2 x_2^3 + 24 a_1 b_4 c_1 x_1^2 x_2 x_3^2 + 12 a_1 b_5^2 x_1^3 x_3^2 + 36 a_1 b_5 b_6 x_1^2 x_2 x_3^2 + 12 a_1 b_5 c_1 x_1^3 x_2 x_3 + 24 a_1 b_6^2 x_1 x_2^2 x_3^2 + 6 a_1 b_6^2 x_1 x_3^4 + 24 a_1 b_6 c_1 x_1^2 x_2^2 x_3 + 6 a_1 b_6 c_1 x_1^2 x_3^3 + 54 a_2^2 a_3 x_2^4 x_3 + 18 a_2^2 b_1 x_2^5 \\ & + 18 a_2^2 b_2 x_2^4 x_3 + 18 a_2^2 b_5 x_1 x_2^4 + 18 a_2^2 b_6 x_2^5 + 54 a_2 a_3^2 x_2 x_3^4 + 36 a_2 a_3 b_1 x_1^2 x_2^2 x_3 + 36 a_2 a_3 b_2 x_1^2 x_2 x_3^2 + 72 a_2 a_3 b_3 x_1 x_2^2 x_3 + 72 a_2 a_3 b_4 x_2^3 x_3^2 + 72 a_2 a_3 b_5 x_1 x_2 x_3^2 + 72 a_2 a_3 b_6 x_2^2 x_3^2 + 54 a_2 a_3 c_1 x_1 x_2^2 x_3^2 + 12 a_2 b_1^2 x_1^2 x_2^3 \\ & + 36 a_2 b_1 b_2 x_1^2 x_2^2 x_3 + 12 a_2 b_1 b_3 x_1 x_2^4 + 24 a_2 b_1 b_4 x_2^4 x_3 + 12 a_2 b_1 b_5 x_1^2 x_3^2 + 12 a_2 b_1 b_5 x_1 x_2^2 x_3^2 + 12 a_2 b_1 b_6 x_1^2 x_2^3 + 12 a_2 b_1 b_6 x_2^3 x_3^2 + 12 a_2 b_1 c_1 x_1 x_2^3 x_3 + 6 a_2 b_2^2 x_1^2 x_2^4 + 24 a_2 b_2^2 x_1^2 x_2 x_3^2 + 24 a_2 b_2 b_3 x_1 x_2^3 x_3 \\ & + 24 a_2 b_2 b_4 x_2^3 x_3^2 + 24 a_2 b_2 b_5 x_1^2 x_2 x_3 + 24 a_2 b_2 b_5 x_1 x_2 x_3^2 + 12 a_2 b_2 b_6 x_1^2 x_2^2 x_3 + 12 a_2 b_2 b_6 x_2^2 x_3^2 + 6 a_2 b_2 c_1 x_1^2 x_2^2 + 24 a_2 b_2 c_1 x_1 x_2^2 x_3^2 - 6 a_2 b_3^2 x_2^5 + 24 a_2 b_3 b_5 x_1^2 x_2^2 + 24 a_2 b_3 b_6 x_1 x_2^4 - 6 a_2 b_3 c_1 x_2^4 x_3 - 6 a_2 b_4^2 x_2^5 \\ & + 24 a_2 b_4 b_5 x_1 x_2^3 x_3 + 12 a_2 b_4 b_6 x_2^4 x_3 - 6 a_2 b_4 c_1 x_1 x_2^4 + 24 a_2 b_5^2 x_1^2 x_2 x_3^2 + 6 a_2 b_5^2 x_2 x_3^4 + 36 a_2 b_5 b_6 x_1 x_2^2 x_3^2 + 24 a_2 b_5 c_1 x_1^2 x_2^2 x_3 + 6 a_2 b_5 c_1 x_2^2 x_3^2 + 12 a_2 b_6^2 x_2^3 x_3^2 + 12 a_2 b_6 c_1 x_1 x_2^3 x_3 + 18 a_3^2 b_1 x_2 x_3^4 + 18 a_3^2 b_2 x_3^5 \\ & + 18 a_3^2 b_3 x_1 x_3^4 + 18 a_3^2 b_4 x_3^5 + 6 a_3 b_1^2 x_1^4 x_3 + 24 a_3 b_1^2 x_1^2 x_2^2 x_3 + 36 a_3 b_1 b_2 x_1^2 x_2 x_3^2 + 24 a_3 b_1 b_3 x_1^3 x_2 x_3 + 24 a_3 b_1 b_3 x_1 x_2^2 x_3^2 + 12 a_3 b_1 b_4 x_1^2 x_2 x_3^2 + 12 a_3 b_1 b_4 x_2^2 x_3^2 + 24 a_3 b_1 b_5 x_1 x_2 x_3^2 + 24 a_3 b_1 b_6 x_2^2 x_3^2 + 6 a_3 b_1 c_1 x_1^3 \\ & x_3^2 + 24 a_3 b_1 c_1 x_1 x_2^2 x_3^2 + 12 a_3 b_2^2 x_1^2 x_3^3 + 12 a_3 b_2 b_3 x_1^3 x_3^2 + 12 a_3 b_2 b_3 x_1 x_2^2 x_3^2 + 12 a_3 b_2 b_4 x_1^2 x_3^3 + 12 a_3 b_2 b_4 x_2^2 x_3^3 + 12 a_3 b_2 b_5 x_1 x_3^4 + 24 a_3 b_2 b_6 x_2 x_3^4 + 12 a_3 b_2 c_1 x_1 x_2 x_3^3 + 24 a_3 b_3^2 x_1^2 x_2^2 x_3 + 6 a_3 b_3^2 x_2^4 x_3 \\ & + 36 a_3 b_3 b_4 x_1 x_2^2 x_3^2 + 24 a_3 b_3 b_5 x_1^2 x_3^3 + 24 a_3 b_3 b_6 x_1 x_2 x_3^3 + 24 a_3 b_3 c_1 x_1^2 x_2 x_3^2 + 6 a_3 b_3 c_1 x_2^2 x_3^3 + 12 a_3 b_4^2 x_2^2 x_3^3 + 24 a_3 b_4 b_5 x_1 x_3^4 + 12 a_3 b_4 b_6 x_2 x_3^4 + 12 a_3 b_4 c_1 x_1 x_2 x_3^3 - 6 a_3 b_5^2 x_3^5 - 6 a_3 b_5 c_1 x_2 x_3^4 - 6 a_3 b_6^2 x_3^5 \\ & - 6 a_3 b_6 c_1 x_1 x_3^4 - 6 b_1^3 x_1^4 x_2 - 6 b_1^2 b_2 x_1^4 x_3 - 12 b_1^2 b_3 x_1^3 x_2^2 + 2 b_1^2 b_5 x_1^5 + 8 b_1^2 b_5 x_1^3 x_2^2 - 4 b_1^2 b_5 x_1^3 x_3^2 + 2 b_1^2 b_6 x_1^4 x_2 + 8 b_1^2 b_6 x_1^3 x_2^2 - 4 b_1^2 b_6 x_1^3 x_2 x_3^2 - 12 b_1^2 c_1 x_1^3 x_2 x_3 - 6 b_1 b_2^2 x_1^4 x_2 - 4 b_1 b_2 b_4 x_1^4 x_2 - 4 b_1 b_2 b_4 x_1^3 x_2^2 \\ & + 8 b_1 b_2 b_4 x_1^2 x_2 x_3^2 - 4 b_1 b_2 b_6 x_1^4 x_3 + 8 b_1 b_2 b_6 x_1^3 x_2^2 x_3 - 4 b_1 b_2 b_6 x_1^2 x_3^3 - 2 b_1 b_2 c_1 x_1^5 - 8 b_1 b_2 c_1 x_1^3 x_2^2 - 8 b_1 b_2 c_1 x_1^3 x_3^2 - 12 b_1 b_3^2 x_1^3 x_2^2 + 8 b_1 b_3 b_5 x_1^4 x_2 + 8 b_1 b_3 b_5 x_1^3 x_2^2 - 4 b_1 b_3 b_5 x_1^3 x_2 x_3^2 + 8 b_1 b_3 b_6 x_1^3 x_2^2 \\ & + 8 b_1 b_3 b_6 x_1 x_2^4 - 4 b_1 b_3 b_6 x_1 x_2^2 x_3^2 - 18 b_1 b_3 c_1 x_1^2 x_2^2 x_3 - 4 b_1 b_4^2 x_1^2 x_3^2 + 2 b_1 b_4^2 x_2^3 + 8 b_1 b_4^2 x_2^2 x_3^2 - 4 b_1 b_4 b_6 x_1^2 x_2^2 x_3 + 8 b_1 b_4 b_6 x_2^4 x_3 + 8 b_1 b_4 b_6 x_2^2 x_3^2 - 6 b_1 b_4 c_1 x_1^2 x_2^2 - 4 b_1 b_5 b_6 x_1^3 x_2^2 + 8 b_1 b_5 b_6 x_1 x_2^2 x_3^2 \\ & - 4 b_1 b_5 b_6 x_1 x_3^4 - 6 b_1 b_5 c_1 x_1^2 x_3^2 - 4 b_1 b_6^2 x_1^2 x_2 x_3^2 + 8 b_1 b_6^2 x_2^2 x_3^2 + 2 b_1 b_6^2 x_2 x_3^4 - 4 b_1 b_6 c_1 x_1^2 x_2 x_3 + 8 b_1 b_6 c_1 x_1 x_2^2 x_3^2 - 4 b_1 b_6 c_1 x_1 x_2 x_3^2 - 2 b_1 c_1^2 x_1^4 x_2 - 2 b_1 c_1^2 x_1^2 x_2^2 - 8 b_1 c_1^2 x_1^2 x_2 x_3^2 - 6 b_2^2 x_1^4 x_3 + 2 b_2^2 b_3 x_1^5 - 4 b_2^2 b_3 \\ & x_1^3 x_2^2 + 8 b_2^2 b_3 x_1^3 x_2 x_3^2 + 2 b_2^2 b_4 x_1^4 x_3 - 4 b_2^2 b_4 x_1^3 x_2^2 x_3 + 8 b_2^2 b_4 x_1^2 x_2 x_3^2 - 12 b_2^2 b_5 x_1^3 x_3^2 - 12 b_2^2 c_1 x_1^3 x_2 x_3 - 4 b_2 b_3 b_4 x_1^3 x_2^2 - 4 b_2 b_3 b_4 x_1 x_2^4 + 8 b_2 b_3 b_4 x_1 x_2^2 x_3^2 + 8 b_2 b_3 b_5 x_1^4 x_3 - 4 b_2 b_3 b_5 x_1^3 x_2^2 x_3 + 8 b_2 b_3 b_5 x_1^2 x_3^3 \\ & - 6 b_2 b_3 c_1 x_1^3 x_2^2 - 4 b_2 b_4^2 x_1^3 x_2^2 x_3 + 2 b_2 b_4^2 x_2^4 x_3 + 8 b_2 b_4^2 x_2^2 x_3^3 + 8 b_2 b_4 b_5 x_1^3 x_2^2 - 4 b_2 b_4 b_5 x_1 x_2^2 x_3^2 + 8 b_2 b_4 b_5 x_1 x_3^4 - 4 b_2 b_4 b_6 x_1^2 x_2 x_3^2 + 8 b_2 b_4 b_6 x_2^3 x_3^2 + 8 b_2 b_4 b_6 x_2 x_3^4 - 4 b_2 b_4 c_1 x_1^3 x_2 x_3 - 4 b_2 b_4 c_1 x_1 x_2^3 x_3 \\ & + 8 b_2 b_4 c_1 x_1 x_2 x_3^3 - 12 b_2 b_5^2 x_1^2 x_3^3 - 18 b_2 b_5 c_1 x_1^2 x_2 x_3^2 - 4 b_2 b_6^2 x_1^2 x_3^3 + 8 b_2 b_6^2 x_2^2 x_3^3 + 2 b_2 b_6^2 x_3^5 - 6 b_2 b_6 c_1 x_1^3 x_3^2 - 2 b_2 c_1^2 x_1^4 x_3 - 8 b_2 c_1^2 x_1^2 x_2^2 x_3 - 2 b_2 c_1^2 x_1^2 x_3^3 - 6 b_3^2 x_1^3 x_2^2 - 6 b_3^2 b_4 x_2^4 x_3 + 8 b_3^2 b_5 x_1^3 x_2^2 + 2 b_3^2 b_5 x_1 x_2^4 \\ & - 4 b_3^2 b_5 x_1 x_2^2 x_3^2 + 8 b_3^2 b_6 x_1^2 x_2^2 + 2 b_3^2 b_6 x_2^5 - 4 b_3^2 b_6 x_2^3 x_3^2 - 12 b_3^2 c_1 x_1 x_2^3 x_3 - 6 b_3 b_4^2 x_1 x_2^4 + 8 b_3 b_4 b_5 x_1^2 x_2^2 x_3 - 4 b_3 b_4 b_5 x_2^4 x_3 - 4 b_3 b_4 b_5 x_2^2 x_3^3 - 8 b_3 b_4 c_1 x_1^2 x_2^2 - 2 b_3 b_4 c_1 x_2^5 - 8 b_3 b_4 c_1 x_2^3 x_3^2 + 8 b_3 b_5^2 x_1^3 x_3^2 - 4 b_3 \\ & b_5^2 x_1 x_2^2 x_3^2 + 2 b_3 b_5^2 x_1 x_3^4 + 8 b_3 b_5 b_6 x_1^2 x_2 x_3^2 - 4 b_3 b_5 b_6 x_2^3 x_3^2 - 4 b_3 b_5 b_6 x_2 x_3^4 + 8 b_3 b_5 c_1 x_1^3 x_2 x_3 - 4 b_3 b_5 c_1 x_1 x_2^3 x_3 - 6 b_3 b_6 c_1 x_2^2 x_3^2 - 2 b_3 c_1^2 x_1^3 x_2^2 - 2 b_3 c_1^2 x_1 x_2^4 - 8 b_3 c_1^2 x_1 x_2^2 x_3^2 - 6 b_4^3 x_2^4 x_3 - 12 \\ & b_4^2 b_6 x_2^3 x_3^2 - 12 b_4^2 c_1 x_1 x_2^3 x_3 + 8 b_4 b_5^2 x_1^2 x_3^3 - 4 b_4 b_5^2 x_2^2 x_3^3 + 2 b_4 b_5^2 x_3^5 - 6 b_4 b_5 c_1 x_2^3 x_3^2 - 12 b_4 b_6^2 x_2^2 x_3^3 - 18 b_4 b_6 c_1 x_1 x_2^2 x_3^2 - 8 b_4 c_1^2 x_1^2 x_2^2 x_3 - 2 b_4 c_1^2 x_2^4 x_3 - 2 b_4 c_1^2 x_2^2 x_3^3 - 6 b_5^3 x_1 x_3^4 - 6 b_5^2 b_6 x_2 x_3^4 - 12 b_5^2 c_1 x_1 x_2 x_3^3 \\ & - 6 b_5 b_6^2 x_1 x_3^4 - 8 b_5 b_6 c_1 x_1^2 x_3^3 - 8 b_5 b_6 c_1 x_2^2 x_3^3 - 2 b_5 b_6 c_1 x_3^5 - 2 b_5 c_1^2 x_1^3 x_3^2 - 8 b_5 c_1^2 x_1 x_2^2 x_3^2 - 2 b_5 c_1^2 x_1 x_3^4 - 6 b_6^3 x_2 x_3^4 - 12 b_6^2 c_1 x_1 x_2 x_3^3 - 8 b_6 c_1^2 x_1^2 x_2 x_3^2 - 2 b_6 c_1^2 x_2^3 x_3^2 - 2 b_6 c_1^2 x_2 x_3^4 - 2 c_1^3 x_1^3 x_2 x_3 - 2 c_1^3 x_1 x_2^3 x_3 - 2 \\ & c_1^3 x_1 x_2 x_3^3\end{aligned}$$

Metrized algebras with involution

We shall go another way

Let $\mathbb{A}$ be an algebra over a field $\mathbf{K}$ of char $\neq 2, 3$. An ednomorphism σ of $\mathbb{A}$ is an involution if

$$\boxed{(xy)^\sigma = y^\sigma x^\sigma \qquad \Leftrightarrow \qquad L(x) = \sigma \circ R(x^\sigma) \circ \sigma.}$$

If $\mathbb{A}$ is commutative (resp. anti-commutative) then $\sigma = \mathrm{id}_{\mathbb{A}}$ (resp. $\sigma = -\mathrm{id}_{\mathbb{A}}$) is the standard involutions. For example, the anticommutative case:

$$(x \times y)^\sigma = -x \times y = y \times x = (-y) \times (-x) = y^\sigma \times x^\sigma .$$

A symmetric bilinear form h is called **involutively invariant** if it satisfies

$$\boxed{h(x^\sigma, y^\sigma) = h(x, y), \qquad h(xy, z) = h(y, x^\sigma z),}$$

in other words, if the adjoint $\boxed{L(x)^* = L(x^\sigma)}$. It follows also that

$$h(xy, z) = h(y, x^\sigma z) = h(y^\sigma, z^\sigma x) = h(zy^\sigma, x) = h(x, zy^\sigma) \quad \Rightarrow \quad \boxed{R(y)^* = R(y^\sigma)}$$

Definition. An algebra $(\mathbb{A}, \sigma, h)$ with nondegenerate involutively invariant h is called **metrized**.

Remark. The latter definition is compatible with the standard definition for a *commutative metrized algebra*.

Metrized algebras with involution

An important application.

Let us consider a metrized *commutative* algebra $(\mathbb{A}, h)$ over $\mathbf{K}$ with the standard involution $\sigma = \mathrm{id}_{\mathbb{A}}$. Then the gradient of $u(x) := \frac{1}{6}h(x^2, x)$ can explicitly be found by linearization:

$$\partial_y u(x) = \tfrac{1}{6}(h(x^2, y) + 2h(xy, x)) = (\text{by invariance of } h) = \tfrac{1}{2}h(x^2, y) \quad \Rightarrow \quad \boxed{\nabla u(x) = \tfrac{1}{2}x^2.}$$

Similarly, the algebra multiplication is expressed by the Hessian of u:

$$\operatorname{Hess} u(y) \cdot x = \operatorname{Hess} u(x) \cdot y = xy.$$

For example, if $\mathbf{K} = \mathbb{R}$, $\mathbb{A} = \mathbb{R}^n$ with the standard Euclidean scalar product $h(x, y) = \langle x, y\rangle$ and $\sigma = \mathrm{id}_{\mathbb{A}}$ then $u(x) = \frac{1}{6}\langle xx, x\rangle$ satisfies the Hsiang minimal cone PDE (1) then the **commutative metrized** algebra $\mathbb{A}$ on $\mathbb{R}^n$ with and multiplication $xy := \operatorname{Hess} u(x) \cdot y$ satisfies

$$\tfrac{1}{4}\operatorname{tr} L(x)\langle x^2, x^2\rangle - \tfrac{1}{4}\langle x^2, xx^2\rangle = -\tfrac{3}{12}k\langle x, x\rangle\langle x^2, x\rangle$$

Definition. In general, a commutative metrized algebra is called a **Hsiang algebra** if the latter identity holds.

One can prove that if $\mathbb{A}$ is a nontrivial Hsiang algebra (i.e. $\dim \mathbb{A}\mathbb{A} > 1$) then $\operatorname{tr} L(x) = 0$. In whar follows we always assume that a Hsiang algebra is nontrivial.

Basic facts on Hsiang algebras

Theorem 1

For any **nontrivial** *solution* $u(x)$ *of* (1) *there exists a commutative metrized algebra on* $\mathbb{R}^n$ *(= Hsiang) with*

1. $u(x) = \frac{1}{6}\langle x, x^2\rangle$ *(the recovering property)*
2. $\langle x^2, x^3\rangle = \langle x, x\rangle\langle x^2, x\rangle$ *(the defining identity)*
3. $x^3 x + \frac{1}{4}x^2 x^2 - |x|^2 x^2 - \frac{2}{3}\langle x^2, x\rangle x = 0$ *(the 1st linearization)*
4. $\operatorname{tr} L_x = 0$ *(the harmonicity property)*

In the converse direction, given a commutative metrized algebra with properties (2) and (4) (or (3) and (4)) above, the cubic form u *defined by (1) is a nontrivial solution* $u(x)$ *of* (1).

Example 1. Gradl, Meyberg and Walcher (1992) studied commutative **pseudo-composition algebras** $x^3 = h(x,x)x$ with invariant bilinear form h. If h is additionally positive definite then by the invariance property $h(x^3, x^2) = h(x,x)h(x,x^2)$ which immediately implies that any such an algebra **with** $\operatorname{tr} L(x) = 0$ is Hsiang.

Example 2. Elduque and Okubo (2000) studied commutative algebras satisfying $x^2x^2 = N(x)x$ and proved the existence of an invariant bilinear form h such that $N(x) = h(x,x)$. Then $h(x^3, x^2) = h(x,x)h(x,x^2)$, hence any such an algebra **with** $\operatorname{tr} L(x) = 0$ is Hsiang.

Example 3. The contraction of the Jordan algebra structure $\mathrm{Herm}_3(\mathbf{F_d})$ on the trace free subspace is a Hsiang algebra with $h(x,y) = \operatorname{tr} xy$, where $\mathbf{F_d}$ is a $\mathbf{d}$-dimensional Hurwitz algebra.

Basic facts on Hsiang algebras

Definition. A commutative metrised $\mathbb{Z}_2$-graded algebra $\mathbb{A} = \mathbb{A}_0 \oplus \mathbb{A}_1$ is called **polar** if $\mathbb{A}_0\mathbb{A}_0 = \{0\}$ and

$$x_0(x_0x_1) = h(x_0, x_0)x_1 \text{ for all } x_i \in \mathbb{A}_i, i = 0, 1.$$

Theorem 2

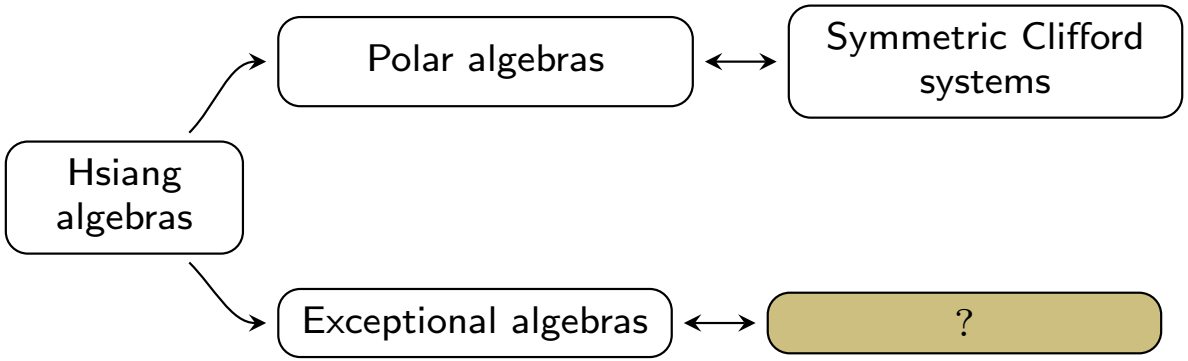
Any polar algebra is Hsiang. Any polar algebra is canonically associated with a symmetric Clifford system.

Symmetric Clifford systems are well-known and completely classified. A Hsiang algebra isomorphic to a polar algebra is said to be of **Clifford type**. A Hsiang algebra which is not of Clifford type is called **exceptional**. The above three examples are exceptional Hsiang algebras.

Mutants. There is a borderline case between Clifford and exceptional Hsian algebras, the so-called mutants: they are of Clifford type but they also share all important properties of exceptional algebras. Any mutant appears as a **polar algebra over a Hurwitz composition algebra** as follows. Let $\mathbf{F_d}$ be a Hurwitz composition algebra over $\mathbb{R}$ with involution $x^\sigma = \bar{x}$ and define an algebra $\mathbb{A} = \mathbf{F_d} \times \mathbf{F_d} \times \mathbf{F_d}$ with multiplication

$$\begin{aligned}
(x_1, 0, 0) \cdot (y_1, 0, 0) &= (0, 0, 0), \\
(x_1, 0, 0) \cdot (0, y_2, y_3) &= (0, x_1y_3^\sigma, y_2^\sigma x_1), \\
(0, x_2, x_3) \cdot (0, y_2, y_3) &= (x_2y_3 + y_2x_3, 0, 0).
\end{aligned}$$

Using the fact that $x(x^\sigma y) = h(x, x)y$ hold in a Hurwitz algebra with invariant trace form $h(x, y)$, one can show that $\mathbb{A}$ is a **polar algebra**, and thus is Hsiang. Its dimension is $3\mathbf{d}$, where $\mathbf{d} \in \{1, 2, 4, 8\}$.


Hsiang algebras
Polar algebras
Symmetric Clifford systems
Exceptional algebras
?

In summary, what is known about exceptional Hsiang algebras.

Theorem 3

The set of nonzero idempotents in any Hsiang algebra $\mathbb{A}$ *is nonempty and all idempotents* **have the same length and the same fusion laws**. *In particular, for any idempotent* c, *the associated* **Peirce decomposition** *is*

$$\mathbb{A} = \mathbb{A}_c(1) \oplus \mathbb{A}_c(-1) \oplus \mathbb{A}_c(-\tfrac{1}{2}) \oplus \mathbb{A}_c(\tfrac{1}{2}) \quad \textit{and} \quad \dim \mathbb{A}_c(1) = 1$$

- $\mathbb{A}_c(1) \oplus \mathbb{A}_c(-1)$ *is a subalgebra. It carries a* **hidden Clifford algebra structure**, $n_1 = \dim \mathbb{A}_c(-1)$
- $\mathbb{A}_c(1) \oplus \mathbb{A}_c(-\frac{1}{2})$ *is a subalgebra. It carries a* **hidden rank 3 Jordan algebra structure**, $n_2 = \dim \mathbb{A}_c(-\frac{1}{2})$.
- $\mathbb{A}$ *is* **exceptional if and only if** $\mathbb{A}_c(1) \oplus \mathbb{A}_c(-\frac{1}{2})$ *is (isotopy of) a simple Jordan algebra. In this case, either* $n_2 = 0$ *or* $n_2 = 3\mathbf{d} + 2$ *and the hidden simple Jordan algebra is* $\mathrm{Herm}_3(\mathbf{F}_\mathbf{d})$, $\mathbf{d} \in \{1, 2, 4, 8\}$.
- $\mathbb{A}$ *is* **mutant iff** $n_2 = 2$, *this corresponds to* $\mathbf{d} = 0$.
- $\mathbb{A}$ *is* **exceptional or mutant iff** $\operatorname{tr} L(x)^2 = m\langle x, x\rangle$ *for some real* m. *In this case,* $m = 2(n_1 + \mathbf{d} + 1)$.
- *There are* **finitely many dimensions** n *of* $\mathbb{A}$ *where exceptional Hsiang algebras can exist. Except the case* $n_2 = 0$, *in all other cases,* $\dim \mathbb{A} = \mathbf{3}(n_1 + 2\mathbf{d} + 1)$, *where* $\dim \mathbb{A}_c(-\frac{1}{2}) = 3\mathbf{d} + 2$, $\mathbf{d} \in \{0, 1, 2, 4, 8\}$.

n	2	5	8	14	26	3	6	12	24	9	12	21	15	18	30	42	27	30	54
n_1	1	2	3	5	9	0	1	3	7	0	1	4	0	1	5	9	0	1	1
n_2	0	0	0	0	0	2	2	2	2	5	5	5	8	8	8	8	14	14	26
$\mathbf{d}$	–	–	–	–	–	0	0	0	0	1	1	1	2	2	2	2	4	4	8

The colours correspond to pseudo-composition, mutants and two unsettled cases.

Part II: Quasi-composition algebras and the triple construction

The tripling construction

Given an algebra $\mathbb{A}$ over a field $\mathbf{K}$ of char $\neq 2, 3$ with an involution σ. Define the **triple** of $(\mathbb{A}, \sigma)$ as the commutative algebra on $T(\mathbb{A}) = \mathbb{A} \times \mathbb{A} \times \mathbb{A}$ with multiplication given by

$$xy \equiv x \cdot y = (x_1, x_2, x_3) \cdot (y_1, y_2, y_3) := (x_2y_3 + y_2x_3, x_1y_3^\sigma + y_1x_3^\sigma, y_2^\sigma x_1 + x_2^\sigma y_1).$$

Remark. It is natural to compare this with Cayley-Dickson doubling construction.

If $\mathbb{A}$ carries an involutively invariant metric h then $T(\mathbb{A})$ is H-metrized with $H(x, y) := \sum_{i=1}^{3} h(x_i, y_i)$:

$$\begin{aligned} H(x \cdot y, z) &= h(x_2y_3 + y_2x_3, z_1) + h(x_1y_3^\sigma + y_1x_3^\sigma, z_2) + h(y_2^\sigma x_1 + x_2^\sigma y_1, z_3) \\ &= h(x_1, y_2z_3 + z_2y_3) + h(x_2, z_1y_3^\sigma + y_1z_3^\sigma) + h(x_3, z_2^\sigma y_1 + y_2^\sigma z_1) = H(x, y \cdot z) \end{aligned}$$

Now suppose that $(\mathbb{A}, \sigma, h)$ is a **(QC) quasicomposition algebra**, i.e. $L(x)L(x^\sigma)L(x) = h(x, x)L(x)$ holds for any $x \in \mathbb{A}$, or equivalently

$$x(x^\sigma(xy)) = h(x, x)(xy), \qquad \forall x, y \in \mathbb{A}.$$

Applying the latter identity to x^σ and y^σ followed by involution σ yields

$$(yx)x^\sigma)x = h(x, x)(xy), \qquad \forall x, y \in \mathbb{A}.$$

I.e. $L(x)L(x^\sigma)L(x) = h(x, x)L(x)$ holds on $(\mathbb{A}, \sigma, h)$ if and only $R(x)R(x^\sigma)R(x) = h(x, x)R(x)$ does.

Notice that $T(\mathbb{A})$ is naturally decomposed into an orthogonal sum

$$\boxed{T(\mathbb{A}) = \mathbb{T}_1 \oplus \mathbb{T}_2 \oplus \mathbb{T}_3, \qquad \text{where } \mathbb{T}_i\mathbb{T}_i = 0, \qquad \mathbb{T}_i\mathbb{T}_j = \mathbb{T}_k, \quad i,j,k \sim 1,2,3.}$$

If $v_1 = (x_1, 0, 0) \in \mathbb{T}_1$, $v_2 = (0, x_2, 0) \in \mathbb{T}_2$, $v_3 = (0, 0, x_3) \in \mathbb{T}_3$, then

$$v_1v_1 = 0, \quad v_1v_2 = (0, 0, x_2^\sigma x_1), \quad v_1v_3 = (0, x_1x_3^\sigma, 0) \quad \text{etc.}$$

hence $v_1(v_1v_2) = (0, x_1(x_2^\sigma x_1)^\sigma, 0) = (0, x_1(x_1^\sigma x_2), 0)$ and similarly

$$L(v_1)^3v_2 = v_1(v_1(v_1v_2) = (0, 0, (x_1(x_1^\sigma x_2))^\sigma x_1) = (0, 0, R(x_1)R(x_1^\sigma)R(x_1)x_2^\sigma). \qquad (*)$$

For any $x = v_1 \oplus v_2 \oplus v_3$, $x^2 = 2(v_2v_3 \oplus v_3v_1 \oplus v_1v_2)$. Hence $H(x^2, x) = 6H(v_1v_2, v_3)$ and also

$$x^3 = 2(v_2(v_2v_1) + v_3(v_3v_1),\ v_1(v_1v_2) + v_3(v_3v_2),\ v_1(v_1v_3) + v_2(v_2v_3)).$$

$$\Rightarrow \quad H(x^3, x^2) = 4 \sum_{\text{permutations}} H(L(v_i)^3v_k, v_j) = 4 \sum_{\text{permutations}} h(R(x_i)R(x_i^\sigma)R(x_i)x_k^\sigma,\ x_j)$$

If $(\mathbb{A}, \sigma, h)$ is a QC-algebra then $R(x_i)R(x_i^\sigma)R(x_i)x_j^\sigma = h(x_i, x_i)R(x_i)x_j^\sigma$

$$H(x^3, x^2) = 8H(v_1v_2, v_3)\sum\nolimits_{i=1}^{3} h(x_i, x_i) = \tfrac{4}{3}H(x^2, x)H(x, x)$$

Theorem 4

Given $(\mathbb{A}, \sigma, h)$, $T(\mathbb{A})$ is a Hsiang algebra if and only if $(\mathbb{A}, \sigma, h)$ is a quasi-composition (QC) algebra.

It is naturally to ask the following questions:

- Which of Hsiang algebras are QC-**generated**, i.e. obtained from QC-algebras?
- How to classify all QC-algebras?

Notice that **the dimension of a QC-generated Hsiang algebra is always divisible by 3.**

$\dim \mathbb{T}(\mathbb{A})$	2	5	8	14	26	3	6	12	24	9	12	21	15	18	30	42	27	30	54
n_1	1	2	3	5	9	0	1	3	7	0	1	4	0	1	5	9	0	1	1
n_2	0	0	0	0	0	2	2	2	2	5	5	5	8	8	8	8	14	14	26
$\mathbf{d}$	–	–	–	–	–	0	0	0	0	1	1	1	2	2	2	2	4	4	8
$\dim \mathbb{A}$	–	–	–	–	–	1	2	4	8	3	4	7	5	6	10	14	9	10	18
$\operatorname{corank}(\mathbb{A})$	–	–	–	–	–	0	0	0	0	1	1	1	?	2	?	?	?	?	?

Table: The blue color corresponds to the **quasi-composition** algebras.

Here $n_2 = 3\mathbf{d} + 2$.

One can guess if the equality $\mathbf{d}(T(\mathbb{A})) = \delta(\mathbb{A})$ above a coincidence? The answer as we shall se NO.

Examples of QC algebras

Example 1. The complex numbers $(\mathbb{C}, \sigma, h)$, with $x^\sigma = \bar{x}$ and $h(x,y) = \mathrm{Re}(x\bar{y})$ satisfies $x(x^\sigma(xy)) = h(x,x)xy$. In general, any Hurwitz (= unital composition) algebra $\mathbb{A}$ with norm $q(x)$, the associated invariant bilinear form $h(x,y) = q(x+y) - q(x) - q(y)$, unity e and the standard involution $x^\sigma = h(x,e)e = x$ is a **quasicomposition** algebra. Possible dimensions $1, 2, 4, 8$.

Example 2. A **cross product algebra** is a metrized anticommutative algebra $(\mathbb{A}, \times, \sigma, h)$, such that $x \times (x \times y) = -h(x,x)y + h(x,y)x$, $\sigma(x) = -x$ $\forall x, y \in \mathbb{A}$. Then the latter implies

$$x^\sigma \times (x \times y) = h(x,x)y - h(x,y)x \quad \Rightarrow x \times (x^\sigma \times (x \times y)) = h(x,x)x \times y,$$

which together with $h(x \times y, z) = -h(y, x \times z) = h(y, x^\sigma \times z)$ shows that $(\mathbb{A}, \times, \sigma, h)$ is a **quasicomposition** algebra. There nontrivial cross algebras can exist only in dimensions 3 and 7.

Example 3. Domokos and Kövesi-Domokos (1978) introduced the color algebra Col as a unique 7D algebra (over $\mathbb{C}$) with unity e and a basis $u_i, i = \pm 1, \pm 2, \pm 3$ with multiplication $u_{\pm i} \circ u_{\pm j} = \epsilon_{ijk} u_{\mp k}$, $u_{\pm i} \circ u_{\mp j} = \delta_{ij} e$, where ϵ_{ijk} is the totally skew-symmetric tensor with $\epsilon_{123} = 1$. The algebra Col is metrized with respect to a natural h. Elduque (1988) studied in particular the 'imaginary' subspace $\mathrm{Col}_0 = e^\perp$ (the **contraction**, i.e. $x \times y = \mathbf{pr}_{\mathrm{Col}_0}(x \circ y)$) and proved that it satisfies the QC identity which implies an example of **a 6D quasicomposition** algebra. Explicitly, $\mathrm{Col}_0 \cong \mathbf{K}^3 \oplus \mathbf{K}^3$ with

$$(x', x'') \times (y', y'') = (-x' \times y' + x'' \times y'', x' \times y'' + x'' \times y')$$

Observation. All the known so far examples of QC algebras are either Hurwitz composition algebras $(n = 1, 2, 4, 8)$ or the **anti-commutative contractions** of $\mathbf{F_D}$ on the orthogonal complement to a one-dimensional (real) or two-dimensional (complex) commutative subalgebra $(n = 3, 6, 7)$.

Theorem 5

Suppose $(\mathbb{A}, \sigma, h)$ *is a QC algebra of dimension* n*. Then for any anisotropic* $x \in \mathbb{A}$*, the operator* $P(x) := \frac{1}{h(x,x)} L(x)L(x^\sigma)$ *is a* **projection** *onto* $\operatorname{Im} L(x)$*. Suppose additionally the field* $\mathbf{K}$ *is infinite and* $\mathrm{char}\mathbf{K} \neq 2$*. Then there exists* $k \in \{0, 1, 2, \ldots, n\}$ *such that for any anisotropic* x

$$\operatorname{tr} L(x)L(x^\sigma) = \operatorname{tr} R(x)R(x^\sigma) = rh(x,x), \qquad \forall x \in \mathbb{A}. \tag{$\triangle$}$$

Furthermore, $\ker L(x^\sigma)L(x) = \ker L(x)$ *and* $\mathrm{rank} L(x) = \mathrm{rank} R(x) = r$ *for any anisotropic* $x \in \mathbb{A}$*.*

Definition. If $(\triangle)$ there holds for all $x \in \mathbb{A}$, the QC algebra $(\mathbb{A}, \sigma, h)$ has **degeneracy index**

$$\delta(\mathbb{A}) := \dim \mathbb{A} - r = \dim \ker L(x) \text{ for a generic } x \in \mathbb{A}$$

Corollary 6

In a QC-algebra with anisotropic h*,* $L(x)$ *and* $R(x)$ *have* **the same constant rank** *for any* $x \neq 0$*.*

Corollary 7

A unital quasicomposition algebra over infinite field of $\mathrm{char}\mathbf{K} \neq 2$ *is a* **Hurwitz algebra***.*

Proof. If e is a unit then e^σ is also. Therefore $e^\sigma e = e^\sigma = e$. Also, $e(e(ee)) = h(e,e)e$ implies that $h(e,e) = 1$. Since $\operatorname{tr} L(e)L(e^\sigma) = \operatorname{tr} \mathrm{id}_{\mathbb{A}} = \dim \mathbb{A}$, by Theorem B $L(x)L(x^\sigma) = nh(x,x)$ for any anisotropic x, hence by Zariski argument, the latter holds $\forall x$. Therefore $\mathbb{A}$ is a unital composition algebra.

We assume that $(\mathbb{A}, \sigma, h)$ is a QC-algebra with h **anisotropic**, i.e. $h(x,x) \neq 0$ iff $x \neq 0$. Then one can show $(\ker L(x))^\sigma = \ker R(x^\sigma)$ etc. A nontrivial corollary of definitions and Theorem 5 is that for any $x \neq 0$

$$\ker L(x^\sigma) = (\ker R(x))^\sigma = (\operatorname{Im} L(x))^\perp \tag{LR}$$

Also notice the linearization of the QC-identity:

$$(L(x)L(z^\sigma) + L(z^\sigma)L(x))(xy) + L(x)L(x^\sigma)(zy) = 2h(x,z)xy + h(x,x)zy \tag{$\diamondsuit$}$$

Proposition 1

If $x, y \neq 0$, $xy = 0$ *then* $\boxed{\ker L(y) = \ker R(x) \text{ and } \operatorname{Im} R(y) = \operatorname{Im} L(x).}$

Proof. By ($\diamondsuit$) in view of $xy = 0$ we have $\forall z \in \mathbb{A}\ (L(x)L(x^\sigma) - h(x,x)\mathbf{1})zy = 0$. By Theorem 5, $\frac{1}{h(x,x)} L(x)L(x^\sigma)$ is a projection onto $\operatorname{Im} L(x)$, implying $\operatorname{Im} R(y) = R(y)\mathbb{A} \subset \operatorname{Im} L(x)$, and also $\operatorname{Im} R(y)$ and $\operatorname{Im} L(x)$ have the same dimensions, hence $\operatorname{Im} R(y) = \operatorname{Im} L(x)$, in particular,

$$\ker L(y) = (\ker R(y^\sigma))^\sigma = (\operatorname{Im} L(y^\sigma))^\perp = ((\operatorname{Im} R(y))^\sigma)^\perp = ((\operatorname{Im} L(x))^\sigma)^\perp = \ker R(x).$$

Corollary 8

For any $x \neq 0$: $\boxed{\ker L(x) \ker R(x) = 0.}$

Proof. For any $y \in \ker L(x)$, $y \neq 0$, we have $xy = 0$, hence $\ker L(y) = \ker R(x)$, hence $y \ker R(x) = 0$.

The triad principle

Corollary 9 (The triad principle)

If $y \neq 0$ and $xy = yz = 0$ then $zx = 0$.

Proof. Since $z \in \ker L(y)$ and $x \in \ker R(y)$ then $zx = 0$.

Theorem 10

Let $x_1, x_2 \neq 0$ and $x_1 x_2 = 0$. Then there exists $0 \neq x_3 \in \mathbb{A}$ such that

$$x_1 x_2 = x_2 x_3 = x_3 x_1 = 0. \qquad (\clubsuit)$$

Furthermore, for any triple x_1, x_2, x_3 satisfying ($\clubsuit$) there holds ($i \in \mathbb{Z}/3\mathbb{Z}$)

$$\boxed{\ker R(x_i) = \ker L(x_{i+1}), \qquad \operatorname{Im} L(x_i) = \operatorname{Im} R(x_{i+1}).}$$

Proof Since $x_2 \neq 0$, $\ker L(x_1)$ is nontrivial, hence so is $\ker R(x_1)$. Let $x_3 \in \ker R(x_1)$, then $x_3 x_1 = 0$. Since $x_2 \in \ker L(x_1)$ and $x_3 \in \ker R(x_1)$, by Corollary 8, $x_2 x_3 = 0$, this proves ($\clubsuit$). The remained part follows by the dimension argument from Proposition 8 and the constant rank property. □

The triad principle

Theorem 11

Let $u \in \mathbb{A}$ *and* $w \in \mathrm{Im}R(u)$, $h(u,u) = h(w,w) = 1$. *Define* $v = uw^\sigma$. *Then*

$$vw = u, \quad uw^\sigma = v, \quad v^\sigma u = w \tag{2}$$

In particular, $c = \frac{1}{2}(u,v,w)$ *is an* **idempotent** *in* $T(\mathbb{A})$.

Proof. We have $v^\sigma u = (wu^\sigma)u = R(u)R(u^\sigma)w = w$ because $R(u)R(u^\sigma)$ is a projection onto $\mathrm{Im}\, R(u)$. It remains to verify that $s := vw$ is equal to u. To this end, note that

$$v^\sigma s = v^\sigma(vw) = v^\sigma(v(v^\sigma u)) = h(v,v)v^\sigma u = h(v,v)w = w,$$

where

$$h(v,v) = h(uw^\sigma, uw^\sigma) = h(wu^\sigma, wu^\sigma) = h((wu^\sigma)u, w) = h(w,w) = 1.$$

Therefore $v^\sigma(s-u) = 0$. Similarly,

$$sw^\sigma = (vw)w^\sigma = ((uw^\sigma)w)w^\sigma = h(w^\sigma, w^\sigma)uw^\sigma = v,$$

implying $(s-u)w^\sigma = 0$, therefore by the triad principle either $s - u = 0$ or $0 = w^\sigma v^\sigma = (vw)^\sigma = s^\sigma$. The latter is impossible vecause $h(s,s) = h(w, v^\sigma(vw)) = h(w,w) = 1$. This proves that $vw = u$. Finally, $c \cdot c = c$ follows from (2). □

Theorem 12

If $\mathbf{K} = \mathbb{R}$ and h is Euclidean then $T(\mathbb{A})$ is a exceptional or mutant Hsiang algebra with

$$\delta(\mathbb{A}) = \mathbf{d} \in \{0, 1, 2, 4, 8\}.$$

Proof. Notice that $L(x) = \sigma \circ R(x^\sigma) \circ \sigma$ and for $x = (x_1, x_2, x_3) \in T(\mathbb{A})$

$$L_\bullet(x) = \begin{pmatrix} 0 & R(x_3) & L(x_2) \\ R(x_3^\sigma) & 0 & L(x_1)\sigma \\ L(x_2^\sigma) & R(x_1)\sigma & 0 \end{pmatrix} = \begin{pmatrix} 0 & R(x_3) & L(x_2) \\ \sigma L(x_3)\sigma & 0 & L(x_1)\sigma \\ \sigma R(x_3)\sigma & R(x_1)\sigma & 0 \end{pmatrix}$$

The linearization of $\operatorname{tr} L(x)L(x^\sigma) = r\, h(x, x)$ yields $\operatorname{tr}(L(y)L(x^\sigma) + L(x)L(y^\sigma)) = 2r\, h(x, y)$, hence

$$\begin{aligned} \operatorname{tr}(L_\bullet(x)L_\bullet(y)) &= \sum\nolimits_{i=1}^{3} \operatorname{tr}(R(x_i)\sigma L(y_i)\sigma + L(x_i)\sigma R(y_i)\sigma) \\ &= \sum\nolimits_{i=1}^{3} \operatorname{tr}(\sigma R(x_i)\sigma L(y_i) + L(x_i)\sigma R(y_i)\sigma) \\ &= \sum\nolimits_{i=1}^{3} \operatorname{tr}(L(x_i^\sigma)L(y_i) + L(x_i)L(y_i^\sigma)) \\ &= \sum\nolimits_{i=1}^{3} 2r h(x_i, y_i) = 2r\, H(x, y). \end{aligned}$$

In particular, $\operatorname{tr}(L_\bullet(x)^2 = 2r\, h(x, x)$, hence Theorem 3 implies $r = n_1 + \mathbf{d} + 1$, where $n_2 = \mathbf{3d} + 2$. On the other hand,

$$3 \dim \mathbb{A} = \dim(T(\mathbb{A})) = 3n_1 + 2n_2 - 1 = 3(n_1 + \mathbf{2d} + 1),$$

hence $\delta(\mathbb{A}) = \dim \mathbb{A} - r = \mathbf{d}$ as desired.

THANK YOU FOR YOUR ATTENTION!