Medial and Isospectral Algebras

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"Операторные алгебры и их приложения" семинар под руководством академика Ш.А.Аюпова

Some natural questions:

1. How much an algebra is determined by its spectrum?
2. Characterize **isospectral** algebras, i.e. those where all idempotents share the same spectrum.

Basic notation and terminology

By an **algebra** $\mathbb{A}$ we shall always mean

- a *commutative*, maybe *nonassociative*
- finite dimensional algebra over a field $\mathbb{K}$ of $\mathrm{char}(\mathbb{K}) \neq 2, 3$.

The (left) multiplication operator $L_x : y \to xy$. A *commutative* algebra is associative if and only if $(xy)z = x(yz)$, or equivalently $L_{xy} = L_x L_y$. A commutative algebra is **Jordan** if $[L_x, L_{x^2}] = 0$.

The **spectrum** of $x \in \mathbb{A}$ is the multiset of eigenvalues of L_x. The characteristic polynomial

$$\chi_x(t) := \det(L_x - tI)$$

The set of nonzero idempotents (i.e. $c^2 = c$) of $\mathbb{A}$ is denoted by $\mathrm{Idm}(\mathbb{A})$. Idempotents and their spectra play a prominent role in nonassociative algebra.

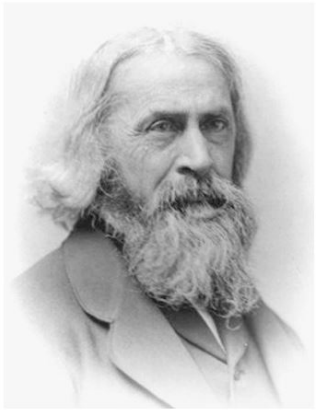
A *quasigroup* is a set Q with a binary operation $\circ$ such that the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions x and y for any given a and b from Q. A set with a binary operation is a quasigroup if and only if its operation table is a Latin square. Two quasigroups $(Q, \circ)$ and $(S, \bullet)$ are called *isotopic* if there exist bijections $f, g, h : Q \to S$ such that $f(x) \bullet g(y) = h(x \circ y)$ for any $x, y \in Q$.

Basic notation and terminology

The **Peirce spectrum of an algebra** $\mathbb{A}$ is the union of all possible distinct eigenvalues of L_c, $c \in \mathrm{Idm}(\mathbb{A})$.

An idempotent c is **semisimple** if $\mathbb{A}$ decomposes into a direct sum of eigenspaces of L_c (Peirce subspaces):

$$\mathbb{A} = \bigoplus_{\lambda \in \sigma(\mathbb{A})} \mathbb{A}_c(\lambda)$$



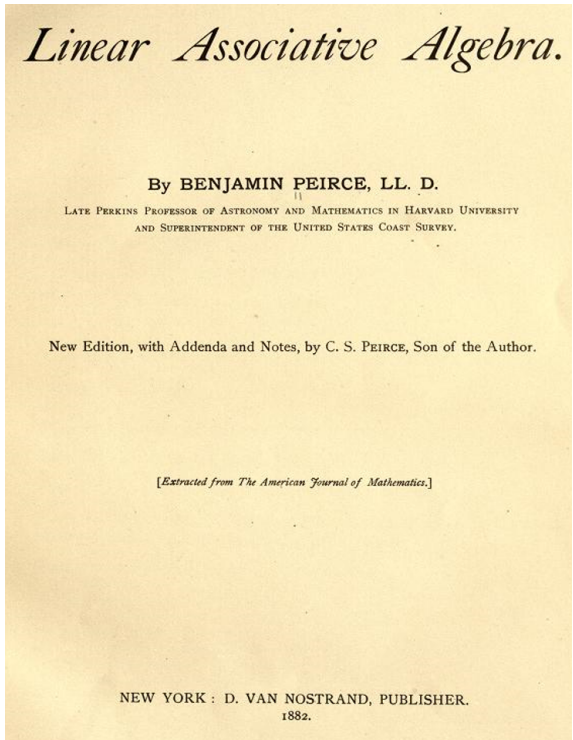
Benjamin Peirce (1809-1880)

Fusion laws: distinguished rules like

$$\mathbb{A}_c(\lambda)\mathbb{A}_c(\mu) \subset \bigoplus_{\tau \in \rho(\lambda,\mu)} \mathbb{A}_c(\tau)$$

For example, the Peirce spectrum of a Jordan algebra is a subset of $\{0, \frac{1}{2}, 1\}$. Fusion laws:

$$\rho(0,0) = \{0\}, \quad \rho(0,\tfrac{1}{2}) = \{\tfrac{1}{2}\}, \quad \rho(1,\tfrac{1}{2}) = \{\tfrac{1}{2}\}, \text{ etc.}$$



Linear Associative Algebra.

By BENJAMIN PEIRCE, LL. D.

Late Perkins Professor of Astronomy and Mathematics in Harvard University and Superintendent of the United States Coast Survey.

New Edition, with Addenda and Notes, by C. S. Peirce, Son of the Author.

[*Extracted from The American Journal of Mathematics.*]

NEW YORK: D. VAN NOSTRAND, PUBLISHER.
1882.

Some starting points and motivations

- Typically one works with the structure constants

$$e_i e_j = \sum_{k=1}^{n} \rho_{ijk} e_k, \qquad \{e_i\} \text{ is a basis of } \mathbb{A}$$

  Compare with linear algebra: how to "diagonalize" the algebra multiplication?

- Alternatively, one may decompose an algebra into bigger 'blocks' $\mathbb{A} = \bigoplus_{\alpha \in I} \mathbb{A}_\alpha$ and prescribe (recover) the multiplication (**fusion**) rules between the blocks

$$\mathbb{A}_\alpha \mathbb{A}_\beta \subset \bigoplus_{\gamma \in \rho(\alpha,\beta)} \mathbb{A}_\gamma, \qquad \rho(\alpha,\beta) \subset \operatorname{spectrum}(L_c).$$

- A distinguished example: the Peirce decomposition relative to an idempotent c. Then $\mathbb{A}_\alpha$ are invariant subspaces of L_c. For example, fusion rules of a Jordan algebra

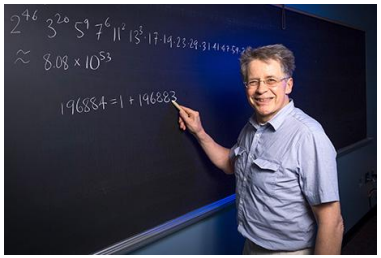
$\star$	1	0	$\frac{1}{2}$
1	1	$\emptyset$	$\frac{1}{2}$
0	$\emptyset$	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1, 0$

Some starting points and motivations

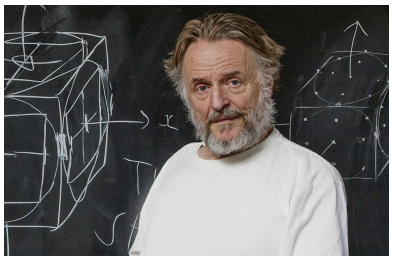
Another example of fusion rules is the 196884-dimensional Griess-Conway-Norton algebra (its automorphism group is the monster sporadic simple group) **unital commutative nonassociative metrized algebra** with fusion rules:

$\star$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	$\emptyset$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\emptyset$	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Only a part of idempotents ('**axes**') share the spectrum and fusion rules. To identify the Monster group some further axioms are needed (Majorana algebras, axel algebras, code algebras).



Robert Griess, born 1945



John Conway (1937-2020)

Simon P. Norton (1952-2019)

Some starting points and motivations

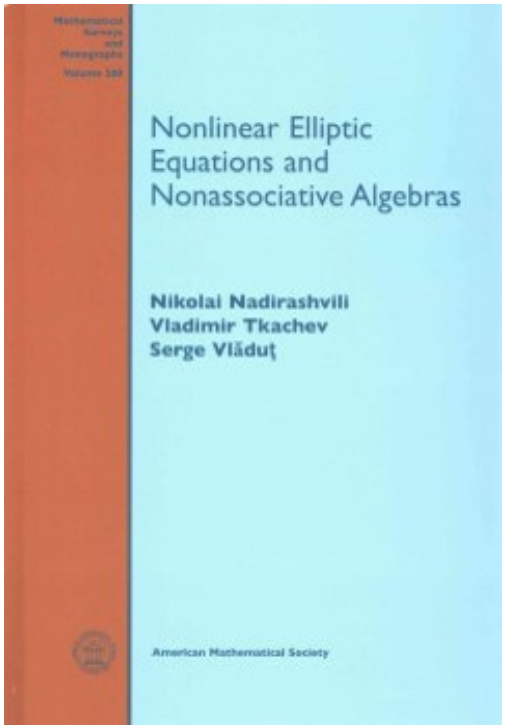
A completely different context coming from **PDEs and Calculus of Variations**: classification of commutative nonassociative algebras of minimal cones (*soap films given by zero level sets of a cubic polynomial*) gives rise to a class of commutative nonassociative metrized algebras satisfying

$$4z^4 + z^2z^2 - 3b(z,z)z^2 - 2b(z,z^2)z = 0,$$

b is an invariant bilinear form: $b(xy,z) = b(x,yz)$ for all $x,y,z \in \mathbb{A}$. One can show that in dimensions $n \geq 3$, there are infinitely many idempotents and any idempotent has the same Peirce spectrum $1, -1, -\frac{1}{2}, \frac{1}{2}$ and the same fusion laws

$\star$	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
1	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
-1	-1	1	$\frac{1}{2}$	$-\frac{1}{2}, \frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$1, -\frac{1}{2}$	$-1, \frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}, \frac{1}{2}$	$-1, \frac{1}{2}$	$1, -1, -\frac{1}{2}$

A recent book *Nonlinear Elliptic Equations and Nonassociative Algebras* Vol.200, Math. Surveys and Monographs, AMS, 2015 by N. Nadirashvili, V.T., S. Vlăduţ



Generic vs Non-generic

Generic algebras

Think of a *generic* polynomial, i.e. a polynomial of degree n with exactly n **different** roots, i.e. a line intersects an nth order curve exactly in n points.

Following to B. Segre (1938), one can interpret idempotents in commutative algebras over $\mathbb{C}$ as solutions of a quadratic system

$$\Psi(x) := x^2 = x \qquad \Leftrightarrow \qquad \sum_{i,j=1}^{n} \theta_{ijk} x_i x_j = x_k, \ 1 \leq k \leq n.$$

where $x = \sum_i x_i e_i$ is a basis decomposition.

Beniamino Segre (1903-1977)

Since a generic (in the Zariski sense) polynomial system has always *Bézout's number* of solutions, a *generic* algebra must have exactly $2^{\dim A}$ **distinct** idempotents (including 0). The genericity here should be understood in the sense that the subset of nonassociative algebra structures on a vector space V is an open Zariski subset in $V^* \otimes V^* \otimes V$.

Definition

An algebra over $\mathbb{C}$ is called a *generic* if it contains exactly $2^{\dim \mathbb{A}}$ *distinct* idempotents.

Remark. The majority of algebras are generic, but the most important algebras (for examples, satisfying identities) are *not* generic, for instance, Jordan algebras in dimension ≥ 3.

Syzygies in generic algebras

Theorem (Krasnov Y., V.Tkachev, 2018)

For a generic algebra $\mathbb{A}$ over $\mathbb{C}$

$$\sum_{c\in \mathrm{Idm}(\mathbb{A})} \frac{\chi_c(t)}{\chi_c(\frac{1}{2})} = 2^n(1-t^n), \quad \forall t \in \mathbb{C}, \tag{1}$$

where $\chi_c(t) = \det(L_c - \mathbf{1}t)$. Furthermore, if $H(x) : \mathbb{K}^n \to \mathbb{K}^s$ is a vector-valued polynomial map such that $\deg H_i \leq n-1$ then

$$\sum_{c\in \mathrm{Idm}(\mathbb{A})\cup\{0\}} \frac{H(c)}{\chi_c(\frac{1}{2})} = 0.$$

Proof. Since $\mathbb{A}$ is commutative, the multiplication is uniquely determined from the square mapping $\Psi(x) = x^2$ by polarization $L_x y = xy = \frac{1}{2}(\Psi(x+y) - \Psi(x) - \Psi(y)) = \frac{1}{2} D\Psi(x)\, y$, hence

$$D\Psi(x) = 2L_x.$$

Idempotents $c \in \mathbb{A}$ are exactly the zeroes of $F(x) = \Psi(x) - x$. $\mathbb{A}$ is generic iff all roots of $F(x) = 0$ are nonsingular, i.e. $DF(c) = 2L_c - \mathbf{1}$ has the maximal rank. Then the Euler-Jacobi formula yields

$$0 = \sum_{c:F(c)=0} \frac{H(c)}{\det[DF(c)]} = \sum_{c\in \mathrm{Idm}(\mathbb{A})\cup\{0\}} \frac{H(c)}{2^n \det[L_c - \frac{1}{2}]} = \sum_{c\in \mathrm{Idm}(\mathbb{A})\cup\{0\}} \frac{H(c)}{2^n \chi_c(\frac{1}{2})}$$

which proves the claims.

Syzygies in generic algebras

An example in dimension 2. Let $\mathbb{A}$ be an arbitrary two-dimensional (commutative nonassociative) algebra over a field and such that it contains at least three **distinct** nonzero idempotents c_1, c_2, c_3 such that the corresponding characteristic polynomials $\chi_{c_i}(t) = (t-1)(t-\lambda_i)$ satisfy $\lambda_i \neq \frac{1}{2}$. Then $\mathbb{A}$ is generic, c_1, c_2, c_3 are the only possible idempotents and the following *equivalent* conditions hold:

$$4\lambda_1\lambda_2\lambda_3 - \lambda_1 - \lambda_2 - \lambda_3 + 1 = 0,$$

$$\frac{1}{1-2\lambda_1} + \frac{1}{1-2\lambda_2} + \frac{1}{1-2\lambda_3} = 1,$$

Furthermore, the idempotents satisfy the 'weighted mean value' type identity:

$$\sum_{i=1}^{3} \frac{c_i}{\chi_{c_i}(\frac{1}{2})} = 0 \qquad \Leftrightarrow \qquad \sum_{i=1}^{3} \frac{c_i}{1-2\lambda_i} = 0. \tag{2}$$

Remark. These identities were appeared in the paper of S. Walcher (1999) *On algebras of rank three*

It is easily verified that one also has $\alpha_1 + \alpha_2 + \alpha_3 \neq 0$; otherwise A is of rank two. The above relation also shows for the left multiplication $L(c_1)$ that $2L(c_1)$ has eigenvalues 2 and $\beta_1 := -(\alpha_2 + \alpha_3)/\alpha_1$. Similarly, $2L(c_2)$ has eigenvalues 2 and $\beta_2 := -(\alpha_3 + \alpha_1)/\alpha_2$, and $2L(c_3)$ has eigenvalues 2 and $\beta_3 := -(\alpha_1+\alpha_2)/\alpha_3$. It follows that $1-\beta_i = (\alpha_1+\alpha_2+\alpha_3)/\alpha_i$ for all i, and hence

$$\frac{1}{1-\beta_1} + \frac{1}{1-\beta_2} + \frac{1}{1-\beta_3} = 1.$$

These relations also show that the α_i can be recovered from the β_i: If one specifies that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ then $\alpha_i = 1/(1-\beta_i)$.

Isospectral algebras

In this talk I discuss the case when *all idempotents have the same spectrum*. One has to distinguish two different principal cases:

- **generic** algebras (the syzygy relations hold),
- **non-generic** algebras (normally $\frac{1}{2}$ presents in the algebras spectrum). The non-generic case is much more subtle. We know so far only two classes of non-generic isospectral algebras.

Proposition 1 (Krasnov Y., V.Tkachev, 2018)

*Let $\mathbb{A}$ be an **isospectral generic** algebra. Then the spectrum of any nonzero idempotent c is **cyclic**: $\{\exp\frac{2\pi ik}{n} : 1 \leq k \leq n\}$. In particular, any idempotent is semi-simple.*

Proof. Since all $\chi_c(t)$ are equal, we have

$$\sum_{c\in \mathrm{Idm}(\mathbb{A})} \frac{\chi_c(t)}{\chi_c(\frac{1}{2})} = 2^n(1-t^n) \quad \Rightarrow \quad (2^n-1)\frac{\chi_c(t)}{\chi_c(\frac{1}{2})} = 2^n(1-t^n)$$

implying $\chi_c(t) = (-1)^n(t^n-1)$. In particular, all eigenvalues are **simple**, thus implying the claims. □

Remark. Any **isospectral generic** algebra is non-unital. Indeed, the unit is an idempotent with the only eigenvalue $= 1$.

Example 1. Non-generic 3D isospectral algebra

Let $\mathbb{T}$ be an algebra over a $\mathbb{K}$ of characteristic $\neq 2, 3$, generated by three idempotents e_1, e_2, e_3 subject to

$$e_i \circ e_j = -\tfrac{1}{2} e_k, \quad \text{where } i, j, k \text{ is any permutation of } 1, 2, 3.$$

Proposition 2

The algebra $\mathbb{T}$ is isospectral and non-generic. Any idempotent $x \in \mathrm{Idm}(\mathbb{T})$ has spectrum $\{1, -\frac{1}{2}, \frac{1}{2}\}$. The set of nonzero idempotents of $\mathbb{T}$ is the circle given by

$$\mathrm{Idm}(\mathbb{T}) = \{x_1 e_1 + x_2 e_2 + x_3 e_3 : \sum_{i=1}^{3} x_i^2 = \sum_{i=1}^{3} x_i = 1\}. \tag{3}$$

Proof. There are $> 2^3 - 1 = 7$ nonzero idempotents, hence non-generic. For any $x \in \mathrm{Idm}(\mathbb{T})$ one has $(e - x) \circ x = -\frac{1}{2}(e - x)$, where $e = (1, 1, 1)$, i.e. $(e - x)$ is an eigenvector of L_x° with eigenvalue $-\frac{1}{2}$. Also if $u = e \times x$ then $x \circ u = \frac{1}{2} u$, hence u is an eigenvector of L_x° with eigenvalue $\frac{1}{2}$. □

Remark. The algebra $\mathbb{T}$ satisfies the so-called 'adjoint identity' (in the sense of A. Elduque and S. Okubo)

$$(x \circ x) \circ (x \circ x) = N(x)x, \qquad N(x) = x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 = \langle x; x \circ x \rangle,$$

where the bilinear form $\langle x; y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$ is invariant. In particular,

$$\langle (x \circ x) \circ (x \circ x); x \rangle = \langle x \circ x; x \circ (x \circ x) \rangle = \langle x; x \circ x \rangle \langle x; x \rangle$$

Peirce number $\frac{1}{2}$ is universal in algebras with identities

Let a commutative algebra $\mathbb{A}$ satisfy an identity of the kind

$$P := \sum_{\alpha} \phi_\alpha(x) x^\alpha = 0, \qquad x^\alpha \in N(x), \tag{4}$$

where $N(x) = \{x^\alpha : x,\ x^2,\ x^3,\ x^2x^2,\ xx^3,\ x^2x^3,\ x(x^2x^2), \ldots\}$ is the commutative groupoid generated by x and $\phi_\alpha(x)$ are polynomial function.

Theorem (V.T., *The universality of one half ...*, J. Algebra, 2021)

Let $\mathbb{A}$ be a finite dimensional commutative nonassociative algebra over a field of characteristic $\neq 2, 3$ and let A satisfy a **nontrivial** identity $P = 0$ in one nonassociative indeterminate x. Then the following holds:

(A) $\frac{1}{2}$ is the Peirce number for *any* idempotent $c \neq 0$;

(B) if additionally c is semi-simple and λ is a single root of the Peirce polynomial then

$$\mathbb{A}_c(\lambda)\mathbb{A}_c(\tfrac{1}{2}) \subset \bigoplus_{\nu \in \sigma(c), \nu \neq \lambda} \mathbb{A}_c(\nu). \tag{5}$$

Remark. For example, if $\mathbb{A}$ is Jordan, its Peirce polynomial $t(t-1)(t-\frac{1}{2})$, hence

$$\mathbb{A}_c(\tfrac{1}{2})\mathbb{A}_c(\tfrac{1}{2}) \subset \mathbb{A}_c(0) \oplus \mathbb{A}_c(1),$$

Now I will exhibit a couple of examples of **generic** isospectral algebras.

The 2D Harada algebra

Let $\mathbb{A}_2$ be the commutative algebra generated by idemotents c_1 and c_2 with

$$c_1 c_2 = -c_1 - c_2.$$

This immediately implies for $c_3 = -c_1 - c_2$ that

$$c_3^2 = c_1^2 + 2c_1c_2 + c_2^2 = c_1 + 2(-c_1 - c_2) + c_2 = c_3.$$

One can show that c_1, c_2, c_3 are the only nonzero idempotents in $\mathbb{A}_2$ and

$$c_i c_j = c_k, \quad \text{where } \{i, j, k\} = \{1, 2, 3\}.$$

In particular,

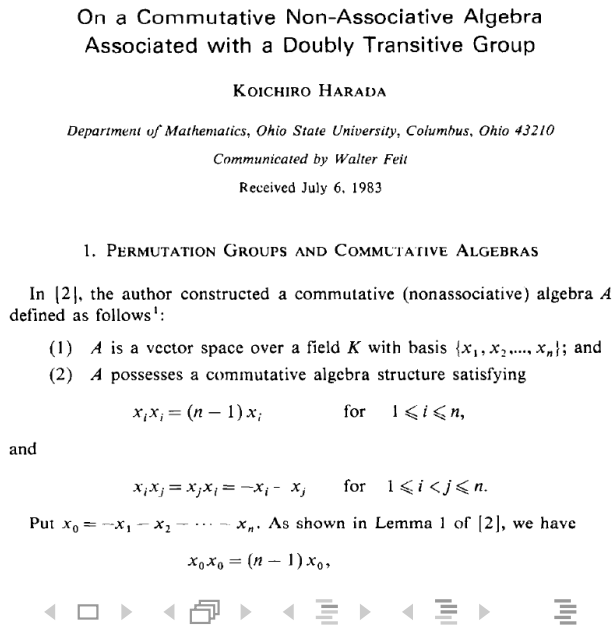
$$c_i(c_j \pm c_k) = \pm 1 \cdot (c_j \pm c_k).$$

Thus the spectrum of each c_i is $\{1, -1\}$, i.e. $\mathbb{A}_2$ is an **isospectral algebra**.

A family algebras including $\mathbb{A}_2$ has been defined and studied by Koichiro Harada in 1981 and

$$\mathrm{Aut}(\mathbb{A}_2) = \mathrm{Perm}(\mathrm{Idm}(\mathbb{A}_2)) \cong S_3,$$

Remark. The general Harada algebras in dimensions $n \geq 3$ are **not** isospectral.



On a Commutative Non-Associative Algebra Associated with a Doubly Transitive Group

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1. PERMUTATION GROUPS AND COMMUTATIVE ALGEBRAS

In [2], the author constructed a commutative (nonassociative) algebra A defined as follows[1]:

(1) A is a vector space over a field K with basis $\{x_1, x_2, \ldots, x_n\}$; and

(2) A possesses a commutative algebra structure satisfying

$$x_i x_i = (n-1)\, x_i \qquad \text{for} \quad 1 \leqslant i \leqslant n,$$

and

$$x_i x_j = x_j x_i = -x_i - x_j \qquad \text{for} \quad 1 \leqslant i < j \leqslant n.$$

Put $x_0 = -x_1 - x_2 - \cdots - x_n$. As shown in Lemma 1 of [2], we have

$$x_0 x_0 = (n-1)\, x_0,$$

The 2D Harada algebra

One can prove that $\mathbb{A}_2$ satisfies the identity

$$x^3 = \beta_2(x)x, \qquad \forall x \in \mathbb{A}_2,$$

where the positive definite quadratic form

$$\beta_2(x) = x_1^2 - x_1x_2 + x_2^2, \quad \text{where } x = x_1c_1 + x_2c_2.$$

is a *multiplicative* homomorphism:

$$\beta_2(xy) = \beta_2(x)\beta_2(y).$$

Hence $\mathbb{A}_2$ is a commutative **composition nonunital** algebra.

The corresponding bilinear form

$$b(x, y) = \beta_2(x + y) - \beta_2(x) - \beta_2(y)$$

is *invariant*, therefore $\mathbb{A}_2$ Euclidean metrized.

Furthermore, one can show that $\mathbb{A}_2$ is a Hsiang algebra, i.e.

$$b(x^2, x^3) = Cb(x, x)b(x, x^2)$$

An isospectral algebra in 2D

Let $\mathbb{A}$ be an isospectral generic algebra of dimension 2. Let c_1, c_2, c_3 be (necessarily distinct) its nonzero idempotents. Then L_{c_i} has spectrum $\{-1, 1\}$ and by the syzygy relation (2)

$$c_1 + c_2 + c_3 = 0.$$

It follows that

$$c_k = c_k^2 = (-c_i - c_j)^2 = c_i + c_j + 2c_ic_j = -c_k + 2c_ic_j$$

i.e. $c_ic_j = c_k$ for any permutation of $1, 2, 3$. This readily implies

Proposition 3 (Uniqueness)

*Any **isospectral generic** algebra of dimension 2 $\mathbb{A}$ is isomorphic to the 2D Harada algebra.*

An isospectral algebra in 3D

One can appropriately generalize the above in dimension $n = 3$: the corresponding isospectral algebra $\mathbb{A}_3$ over the field of complex numbers $\mathbb{C}$ has been announced in (Ya. Krasnov, V. Tkachev, *Trends in Math.*, 2019): let $\mathbb{A}_3$ be the free commutative algebra over $\mathbb{C}$ spanned by three **idempotents** c_1, c_2, c_3 subject to

$$\begin{aligned} c_1 c_2 &= \quad (\gamma - 1)c_1 - \gamma c_2 + \gamma c_3 =: c_5, \\ c_2 c_3 &= \quad \gamma c_1 + (\gamma - 1)c_2 - \gamma c_3 =: c_6, \\ c_3 c_1 &= -\gamma c_1 + \gamma c_2 + (\gamma - 1)c_3 =: c_7, \end{aligned}$$

where γ is the *Kleinian integer unit*, i.e. a root of $2\gamma^2 - \gamma + 1 = 0$. Define also

$$c_4 := -\gamma(c_1 + c_2 + c_3).$$

Then each of $I := \{0, c_1, c_2, \dots, c_7\}$ is an idempotent and I are the only idempotents of $\mathbb{A}_3$. They have the same (cyclotomic) spectrum: for any $1 \leq i \leq 7$

$$\sigma(c_i) = \{1, \epsilon, \epsilon^2\}, \qquad \epsilon = \tfrac{-1-\sqrt{-3}}{2}, \quad \epsilon^3 = 1.$$

In particular, the algebra $\mathbb{A}_3$ is **isospectral and generic**, and $\sum_{i=1}^{7} c_i = 0$.

One can show that the nonzero idempotents in $\mathbb{A}_3$ are closed under multiplication:

$$\boxed{\forall c_i, c_j \in \mathrm{Idm}(\mathbb{A}_3) \Rightarrow c_i c_j \in \mathrm{Idm}(\mathbb{A}_3).}$$

Since the eigenvalue 1 is single, $c_i c_j \neq c_j, c_i$. Thus, $\mathbb{A}_3$ is a **quasigroup**.

An isospectral algebra in 3D over $\mathbb{F}_7$

Let us assume in the previous example the ground field $\mathbb{F}_7$ instead of $\mathbb{C}$. Then

$$\gamma = \frac{1+\sqrt{-7}}{4} = \frac{1}{4} = \frac{8}{4} = 2, \qquad \epsilon = \frac{-1+\sqrt{-3}}{2} = \frac{1}{2} = \frac{8}{2} = 4.$$

Let $\mathbb{A}_3$ be the free commutative algebra over $\mathbb{C}$ spanned by three **idempotents** c_1, c_2, c_3 subject to

$$\begin{aligned} c_1c_2 &= c_1 - 2c_2 + 2c_3 =: c_5, \\ c_2c_3 &= 2c_1 + c_2 - 2c_3 =: c_6, \\ c_3c_1 &= -2c_1 + 2c_2 + c_3 =: c_7, \\ & \ -2c_1 - 2c_2 - 2c_3 =: c_4. \end{aligned}$$

Then each of $I := \{0, c_1, c_2, \ldots, c_7\}$ is an idempotent with the same 'cyclotomic' spectrum $\{1, 2, 4\}$. In particular, the algebra $\mathbb{A}_3$ is **generic and isospectral**, and $\sum_{i=1}^{7} c_i = 0$.

The set of idempotents in $\mathbb{A}_3$ is closed under multiplication, for example

$$c_1c_5 = c_1(c_1 - 2c_2 + 2c_3) = c_1 - 2(c_1 - 2c_2 + 2c_3) + 2(-2c_1 + 2c_2 + c_3) = 2c_1 + c_2 - 2c_3 = c_6.$$

It is natural to ask *What is the structure of the latter quasigroup?*..

An isospectral algebra in 3D

The multiplication rules between c_i follow the pattern in the **left** table below:

	1	2	3	4	5	6	7
1	1	5	7	3	6	2	4
2	5	2	6	1	4	7	3
3	7	6	3	2	1	4	5
4	3	1	2	4	7	5	6
5	6	4	1	7	5	3	2
6	2	7	4	5	3	6	1
7	4	3	5	6	2	1	7

$$\to \sigma := \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,5\,2\,3\,6\,4\,7 \end{pmatrix} \to$$

	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	5	2	6	3	7	4	1
3	2	6	3	7	4	1	5
4	6	3	7	4	1	5	2
5	3	7	4	1	5	2	6
6	7	4	1	5	2	6	3
7	4	1	5	2	6	3	7

Remarkably, $\mathrm{Idm}(\mathbb{A}_3)$ also satisfies the **medial magma identity**

$$(xy)(zw) = (xz)(yw), \qquad \forall x, y, z, w \in \mathrm{Idm}(\mathbb{A}_3). \tag{MMI}$$

For example, $(c_1c_2)(c_4c_7) = c_5c_6 = c_3$, $(c_1c_4)(c_2c_7) = c_3c_3 = c_3$, $(c_1c_7)(c_2c_4) = c_4c_1 = c_3$. The permutation σ (a *isotopy* of the quasigroup) implies the right table which is much more illuminating. In fact, it is easy to see that its multiplication is **medial**

$$i \circledast j = \tfrac{1}{2}(i+j) \mod 7 \quad \Rightarrow \quad (i \circledast j) \circledast (k \circledast l) \equiv \frac{1}{4}(i+j+k+l) \mod 7,$$

Finally, $\mathbb{A}_3$ satisfies a similar principal power identity

$$x^4 = \beta_3(x)x, \qquad \forall x \in \mathbb{A}_3,$$

where $\beta_3(x) : \mathbb{A}_3 \to \mathbb{K}$ is a certain **multiplicative homomorphism**.

Some natural questions

- In which math contexts **isospectral algebras** are relevant?
- Given $n \geq 2$, do there exist **isospectral algebras** in dimension n? How many?
- Describe all **generic isospectral algebras**. What can be said at least in dimension 3?
- What is about non-generic isospectral algebras? (for example, Hsiang algebras)

Medial algebras

Definition. A commutative algebra $\mathbb{A}$ is said to be *medial* if it satisfies

$$(xy)(zw) = (xz)(yw)$$

If the characteristic of the field not 2 or 3 then the definition is equivalent to

$$x^2y^2 = (xy)(xy).$$

Proposition 4

Any associative commutative algebra is medial. Any ***unital*** *medial algebra is associative.*

Proof. If e is the algebra unit then $x(yz) = (ex)(yz) = (ez)(xy) = z(xy) = (xy)z$. □

Remark 1. In the above context, a natural question arises: Does a unitalization preserve the medial property? The answer is 'no'. Indeed, let $\mathbb{A}$ be any nonassociative medial algebra. Then its unitalization $\mathbb{A}^{\mathrm{uni}}$ contains $\mathbb{A}$ as a subalgebra, therefore $\mathbb{A}^{\mathrm{uni}}$ is also nonassociative. But by Proposition 4, any unital medial algebra is associative, hence $\mathbb{A}^{\mathrm{uni}}$ in this case is *not* medial.

Remark 2. The above observations show that medial algebras are 'nearly associative'. Of course, a general medial algebra is not associative.

Idempotents in medial algebrs

Proposition 5

In a medial algebra idempotents form a multiplicative magma.
The map L_c is an algebra homomorphism.

Proof. Indeed, $(cc_1)(cc_1) = (cc)(c_1c_1) = cc_1$ and $(cx)(cy) = (cc)(xy) = c(xy)$.

The subset of idempotents with invertible adjoint operator is denoted by

$$\mathrm{Idm}^{\times}(\mathbb{A}) := \{c \in \mathrm{Idm}(A) : \exists L_c^{-1}\}$$

Remark. For any $c \in \mathrm{Idm}^{\times}(\mathbb{A})$, both L_c and L_c^{-1} is an **algebra isomorphism**.

Proposition 6

Let $\mathbb{A}$ be a medial algebra, $c \in \mathrm{Idm}^{\times}(\mathbb{A})$. Then a strong isotopy $(\mathbb{A}, \diamond)$ with new multiplication $x \diamond y = L_c^{-1}(xy)$ is a unital commutative associative algebra.

Proof. The algebra $(\mathbb{A}, \diamond)$ is commutative and c is a unit in $(\mathbb{A}, \diamond)$. Since L_c^{-1} is $\mathbb{A}$-isomorphism,

$$(x \diamond y) \diamond (z \diamond w) = L_c^{-1}(L_c^{-1}(xy)\, L_c^{-1}(zw)) = L_c^{-2}((xy)(zw))$$

$(\mathbb{A}, \diamond)$ is medial. Since $(\mathbb{A}, \diamond)$ is unital medial it is associative.

Medial algebras and quasigroups

Useful identities: since $L_{c_2}L_{c_1}x = c_2(c_1x) = (c_2c_2)(c_1x) = (c_2c_1)(c_2x) = L_{c_2c_1}L_{c_2}x$, we have

$$L_{c_2}L_{c_1} = L_{c_2c_1}L_{c_2}. \tag{6}$$

Definition. A *quasigroup* is a set Q with a binary operation $\circ$ in which the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions x and y for any given a and b from Q.

Proposition 7

$\mathrm{Idm}^{\times}(\mathbb{A})$ *is a* ***commutative medial idempotent quasigroup****. Furthermore, all idempotents in* $\mathrm{Idm}^{\times}(\mathbb{A})$ *have the same characteristic polynomial.*

Proof. Clearly $\mathrm{Idm}^{\times}(\mathbb{A})\,\mathrm{Idm}^{\times}(\mathbb{A}) \subset \mathrm{Idm}(\mathbb{A})$. Furthermore, if L_{c_1} and L_{c_2} are invertible, by (6) $L_{c_1c_2} = L_{c_1}L_{c_2}L_{c_1}^{-1}$ is invertible too, hence in fact $\mathrm{Idm}^{\times}(\mathbb{A})\,\mathrm{Idm}^{\times}(\mathbb{A}) \subset \mathrm{Idm}^{\times}(\mathbb{A})$ i.e. $\mathrm{Idm}^{\times}(\mathbb{A})$ is a multiplicative magma. Equation $c_1x = c_2$ is **uniquely** solvable because L_{c_1} is invertible and

$$xx = L_{c_1}^{-1}(c_2)L_{c_1}^{-1}(c_2) = L_{c_1}^{-1}(c_2c_2) = L_{c_1}^{-1}(c_2) = x \quad \Rightarrow x \in \mathrm{Idm}(\mathbb{A}),$$

and moreover $L_{c_1}L_x = L_{c_1x}L_{c_1} = L_{c_2}L_{c_1}$, hence $L_x = L_{c_1}^{-1}L_{c_2}L_{c_1}$ is also invertible, thus $x \in \mathrm{Idm}^{\times}(\mathbb{A})$. This implies that $\mathrm{Idm}^{\times}(\mathbb{A})$ is a commutative medial idempotent quasigroup. Next, it follows from (6) and $c_1c_2 = c_2c_1$ that $L_{c_1}L_{c_2}L_{c_1}^{-1} = L_{c_2}L_{c_1}L_{c_2}^{-1}$, i.e. L_{c_1} and L_{c_2} are conjugate linear endomorphisms, thus have the same characteristic polynomials. □

From medial to isospectral algebras

Let $N(x) = \det L_x$. If $N(x) \neq 0$ then the principal powers $x^{(1)} := x$, $x^{(k)} := x(x^{(k-1)})$ are nonzero.

Proposition 8

Let $\mathbb{A}$ be a medial algebra, $c \in \mathrm{Idm}(\mathbb{A})$. If λ is an eigenvalue of L_c such that $L_c x = \lambda x$ with $N(x) \neq 0$ then λ is a root of unity.

Proof. Indeed, the principal powers $x^{(k)}$ are nonzero elements and

$$c(xx) = (cc)(xx) = (cx)(cx) = \lambda^2 xx,$$
$$c(x(xx)) = (cc)(x(xx)) = (cx)(c(xx)) = \lambda^3 x(xx),$$

implying by induction that $c(x^{(k)}) = \lambda^k x^{(k)}$. Since $\mathbb{A}$ is finite dimensional, there are integer $k < m$ such that $\lambda^k = \lambda^m$, hence λ is a root of unity. □

Corollary 1

If in a medial algebra $\mathbb{A}$ all idempotents satisfy $N(c) \neq 0$ (i.e. $\mathrm{Idm}^{\times}(\mathbb{A}) = \mathrm{Idm}(\mathbb{A})$) then $\mathbb{A}$ is isospectral.

By Proposition 6 $(\mathbb{A}, \diamond)$ with $x \diamond y = L_c^{-1}(xy)$ is a unital commutative associative algebra, hence

Corollary 2

*Let $c \in \mathrm{Idm}^{\times}(\mathbb{A})$. Then $\mathrm{Idm}^{\times}(\mathbb{A})$ is an **abelian group** with the identity c.*

Medial algebras

We have already seen that medial algebras in dimension 2 and 3 carry a multiplicative homomorphism.

Theorem 3

Let $\mathbb{A}$ be a medial algebra, $c \in \mathrm{Idm}^{\times}(A)$. Then $\phi(x) = \frac{1}{\det L_c} \det L_x$, is $\mathbb{A}$-multiplicative, i.e.

$$\phi(xy) = \phi(x)\phi(y), \qquad \forall x, y \in \mathbb{A}.$$

Proof. Indeed, the algebra $(\mathbb{A}, \diamond)$ with $L_c^{-1}(xy) = x \diamond y$ is commutative associative. Let $M_x : y \to x \diamond y$. Then $M_x = L_c^{-1} L_x$ and $\phi(x) = \det M_x$. By the associativity we have $M_{x \diamond y} = M_x M_y$, hence by the multiplicative property of the determinant

$$\phi(x \diamond y) = \det M_{x \diamond y} = \det(M_x M_y) = \det M_x \det M_y = \phi(x)\phi(y). \tag{7}$$

It remains to show that $\phi(xy) = \phi(x \diamond y)$. Since $L_c^{-1}(xy) = L_c^{-1}(x) L_c^{-1}(y)$, for any $z \in \mathbb{A}$

$$(L_c^{-1} z)x = (L_c^{-1} z)(L_c^{-1} L_c x) = L_c^{-1}(z L_c x) = L_c^{-1} L_z L_c x$$

implying $L_{L_c^{-1} z} = L_c^{-1} L_z L_c$. Setting $z = xy$, awe obtain in view of $L_c^{-1}(xy) = x \diamond y$

$$\det L_{x \diamond y} = \det L_{L_c^{-1}(xy)} = \det L_c^{-1} L_{xy} L_c = \det L_{xy}.$$

i.e. $\det M_{x \diamond y} = \det L_c^{-1} L_{x \diamond y} = \det L_c^{-1} \det L_{xy} = \det L_c^{-1} L_{xy} = \det M_{xy}$, thus $\phi(xy) = \phi(x \diamond y)$.

Given $n \geq 2$, do there exist **isospectral algebras** in dimension n? How many?

The cyclotomic polynomial model

Let $z^n - 1$ be solvable over $\mathbb{K}$ and let $\mathscr{C}_n(z, \circ)$ denote the quotient algebra

$$\mathscr{C}_n = \mathbb{K}[z]/(z^n - 1)$$

with a new multiplication on $\mathscr{C}_n$ defined by

$$p(z) \circ q(z) = p(\epsilon_n z) q(\epsilon_n z) \mod (z^n - 1),$$

where ϵ_n is a primitive root of unity of order n. In other words, $\mathscr{C}_n$ is a **strong isotopy** of the associative commutative algebra $\mathbb{K}[z]/(z^n - 1)$ by a *cyclic linear substitution*.

Example. If $n = 2$ then $\epsilon_2 = -1$, hence $1 \circ z = -z$, $z \circ z = z^2 \equiv 1 \mod (z^2 - 1)$, i.e.

	1	z
1	1	$-z$
z	$-z$	1

One has $(1 \circ z) \circ z = -z \circ z = -1$, while $1 \circ (z \circ z) = 1 \circ 1 = 1$, i.e. $\mathscr{C}_2(z, \circ)$ is **not associative!** In particular, $c_1 = 1$ is an idempotent and z is its eigenvector with eigenvalue -1. Also

$$(a + bz) \circ (a + bz) = (a^2 + b^2) \cdot 1 - 2ab \cdot z$$

implies that $c_{2,3} := -\frac{1}{2} \pm \frac{d}{2} z$ are the only idempotents $\neq c_1$ if $d^2 + 3 = 0$ is solvable in $\mathbb{K}$. The algebra $\mathscr{C}_2(z)$ is generic and thus *it is isomorphic to the 2D Harada algebra*. Indeed, $c_1 c_2 = c_3$, $c_1 c_3 = c_2$, and

$$c_2 c_3 = \frac{1}{4} + \frac{3}{4} z^2 \equiv 1 \mod z^2 - 1 \qquad \Rightarrow c_2 c_3 = c_1.$$

Theorem 4 (Existence)

The $\mathscr{C}_n$ is an isospectral medial generic algebra and $L^n_{\circ p(z)} = \Delta(p)\mathbf{1}$, where the circulant $\Delta(p) := p(1)p(\epsilon_n)p(\epsilon_n^2)\cdots p(\epsilon_n^{n-1}) : \mathscr{C}_n \to \mathbb{K}$ is a multiplicative homomorphism. In particular,

$$x^{\circ(n+1)} = \Delta(x)x$$

Proof. Since

$$(p(z)\circ q(z))\circ(r(z)\circ s(z)) = p(\epsilon_n^2 z)q(\epsilon_n^2 z)r(\epsilon_n^2 z)s(\epsilon_n^2 z) \mod (z^n-1),$$

the algebra $\mathscr{C}_n$ is medial. Also $p(z)\circ q(z) = p(\epsilon_n z)\circ q(\epsilon_n z)$, etc,

$$\begin{aligned} L^n_{\circ p(z)}q(z) &\equiv p(\epsilon_n z)p(\epsilon_n^2 z)\cdots p(\epsilon_n^n z)q(\epsilon_n^n z) \equiv p(\epsilon_n z)p(\epsilon_n^2 z)\cdots p(z)q(z) \mod (z^n-1)\\ &\equiv P(z)q(z) \mod (z^n-1), \end{aligned}$$

where $P(z) = p(z)p(\epsilon_n z)p(\epsilon_n^2 z)\cdots p(\epsilon_n^{n-1} z)$. Since $P(\epsilon_n^i z) = P(z)$ for all i, there exists a polynomial Q such that $P(z) = Q(z^n)$. Therefore

$$P(z) = Q(z^n) \equiv Q(1) \equiv P(1) \equiv \Delta(p(z)) \mod (z^n-1).$$

This shows that $L^n_{\circ p}q = \Delta(p)q$. Finally, $\Delta(pq) = \Delta(p)\Delta(q)$, hence Δ is a multiplicative homomorphism. The proof of genericity, isospectrality requires more care.

Theorem 5 (Uniqueness)

If two medial generic isospectral algebras over an algebraically closed field $\mathbb{K}$ have the same dimension they are isomorphic.

Remark. The proof i nontrivial and make an essential use of the fact that if $\mathbb{A}$ be a medial generic isospectral algebra, c be an idempotents and $\mathbb{A}_c(\epsilon_n) = \langle w_1 \rangle$ its 1st eigenspace then one can choose the Peirce eigenbasis such that $w_m = w_1^m$ for all $0 \leq m \leq n-1$, where $w_1^0 = c$.

Idempotents as a quasigroup

While the algebra structure on $\mathscr{C}_n$ is given very explicitly, the structure of idempotents of $\mathscr{C}_n$ is more sophisticated. In order to understand the relations between idempotents in a satisfactory way, one need to look at $\mathrm{Idm}(\mathbb{A})$ as a **quasigroup**.

A quasigroup Q is called *medial* if the identity $(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w)$ holds.

Theorem (The Bruck-Murdoch-Toyoda)

For any medial quasigroup $(Q, \circ)$, there exists an abelian group $(G, +)$, a fixed element $g \in G$, and commuting automorphisms $\phi, \psi : G \to G$ such that $x \circ y = g + \phi(x) + \psi(y)$.

Hence, *any medial quasigroup is an isotope of a certain abelian group* G.

Question. How to characterize the underlying abelian group in a medial **algebra**?

Theorem 6

Let $\mathbb{A}$ *be a medial generic isospectral algebra of dimension* n. *Given an arbitrary* $c \in \mathrm{Idm}(\mathbb{A})$, *let* $\oplus$ *be a new operation on the set of nonzero idempotents defined by* $u \oplus v := T(uv)$, $u, v \in \mathrm{Idm}(\mathbb{A})$. *Then* $(\mathrm{Idm}(\mathbb{A}), \oplus)$ *is a* ***cyclic*** *group (of order* $2^n - 1$ *with identity* c*).*

The proof

Under the made assumption, $\mathrm{Idm}(\mathbb{A}) = \mathrm{Idm}^{\times}(\mathbb{A})$. Let us write $T := L_c^{-1}$. By Corollary 2, $(Q, \oplus)$ is an abelian group of order $2^n - 1$ with $u \oplus v := T(uv)$ and with c being the identity. We shall prove that $(Q, \oplus)$ is cyclic i.e. $Q \cong \mathbb{Z}_{2^n-1}$. Let $(Q, \oplus)$ be a natural $\mathbb{Z}$-module. We need to show that **there is no** $1 \leq m < 2^n - 1$ such that

$$m \otimes u = c \text{ holds for all } u \in Q. \tag{8}$$

We argue by contradiction and write $m = 2^{m_1} + \ldots + 2^{m_{k-1}} + 2^{m_k}$, $n > m_k > \ldots > m_1 \geq 0$, the binary decomposition of m. Then $k < n$. Notice that $2 \otimes u = u \oplus u = T(uu) = T(u)$ for any $u \in Q$, hence $2^s \otimes u = T^s(u)$ implying

$$m \otimes u = T^{m_k}(u) \oplus \ldots \oplus T^{m_1}(u) = c. \tag{9}$$

Combining the terms into a nonassociative products and using that T is an algebra isomorphism we obtain

$$T^{m_2}(u) \oplus T^{m_1}(u) = T(T^{m_2}(u)T^{m_1}(u)) = T^{m_2+1}(u)T^{m_1+1}(u)$$
$$T^{m_3}(c) \oplus T^{m_2}(c) \oplus T^{m_1}(c) = T^{m_3+1}(u)\big(T^{m_2+2}(u)T^{m_1+2}(u)\big) \quad \text{etc}$$

Define $H(x) = T^{m_k+1}(x)\big(T^{m_{k-1}+2}(x)(\ldots(T^{m_2+k-1}(x)T^{m_1+k-1}(x)\big)$. By the linearity of T, $H(x)$ is a nonassociative polynomial of degree $k \leq n-1$, $H(0) = 0$ and by (9), $H(u) = c$ for all $u \in \mathrm{Idm}(\mathbb{A})$. Hence by the syzygy relation

$$(2^n - 1)c = \sum_{x \in Q} H(x) = 0, \quad \text{(here the addition is on the algebra level)}$$

a contradiction follows.

THANK YOU FOR YOUR ATTENTION!