Non-associative algebras of minimal cones

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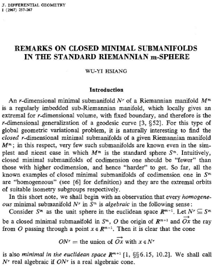
Outline

The problems I will discuss today lie in a common area between:

- differential geometry (minimal and isoparametric submanifolds)
- polynomial solutions to nonlinear PDEs,
- regularity theory fully nonlinear PDEs
- Jordan algebra theory and general metrized algebras
- general commutative metrized algebras

This is why the content is appropriately changed accordingly to a choice of relevant concepts/problems in a given community.

But the original source of my interest here is uniquely determined: it is a paper of Wy-Yi Hsiang of 1967.



J. DIFFERENTIAL GEOMETRY
1 (1967) 257-267

REMARKS ON CLOSED MINIMAL SUBMANIFOLDS IN THE STANDARD RIEMANNIAN m-SPHERE

WU-YI HSIANG

Introduction

An r-dimensional minimal submanifold N^r of a Riemannian manifold M^m is a regularly imbedded sub-Riemannian manifold, which locally gives an extremal for r-dimensional volume, with fixed boundary, and therefore is the r-dimensional generalization of a geodesic curve [3, §52]. For this type of global geometric variational problem, it is naturally interesting to find the *closed* r-dimensional minimal submanifolds of a given Riemannian manifold M^m; in this respect, very few such submanifolds are known even in the simplest and nicest case in which M^m is the standard sphere S^m. Intuitively, closed minimal submanifolds of codimension one should be "fewer" than those with higher codimension, and hence "harder" to get. So far, all the known examples of closed minimal submanifolds of codimension one in S^m are "homogeneous" (see [6] for definition) and they are the extremal orbits of suitable isometry subgroups respectively.

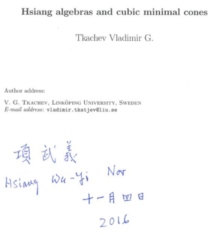
In this short note, we shall begin with an observation that every *homogeneous* minimal submanifold N^r in S^m is *algebraic* in the following sense:

Consider S^m as the unit sphere in the euclidean space R^{m+1}. Let $N^r \subseteq S^m$ be a closed minimal submanifold in S^m, O the origin of R^{m+1} and $\overrightarrow{Ox}$ the ray from O passing through a point $x \in R^{m+1}$. Then it is clear that the cone

$$ON^r = \text{the union of } \overrightarrow{Ox} \text{ with } x \in N^r$$

is also *minimal in the euclidean space* R^{m+1} [1, §§6.15, 10.2]. We shall call N^r real algebraic if ON^r is a real algebraic cone.

Wu-yi Hsiang (Trondheim, Norway, October 2016)



Hsiang algebras and cubic minimal cones

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十一月四日
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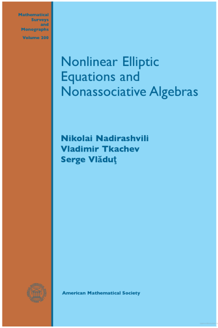
How it started: motivations

Любой математик, неравнодушный к теории чисел, испытал на себе очарование теоремы Ферма о сумме двух натуральных квадратов. Психолог юнговской школы нашел бы, вероятно, что такие диофантовы задачи в высшей степени архитипичны.

(Cubic forms, Yu. I. Manin)



AMS Math. Surv. Monographs, 2014; Vol.200

- Interest in this topic is partly due to Irina Nikolaevna Sosnovtseva in 1980-1981, postgraduate student prof. Isabella Bashmakova (Moscow State University).
- Later, in 1981, when my future supervisor Vladimir Mikhailovich Miklyukov came to VolSU, I began to study the global geometry of minimal submanifolds.
- One of Miklyukov's favorite themes was to achieve a conceptual understanding of Bernstein's theorem in the higher-dimensional case. He liked to mention that it would be interesting to find something in common between different "**phenomenona of eight**", for example, between Milnor's exotic spheres and the termination of Bernstein's theorem in dimensions starting from eight.
- There is an interesting collection of about 100 references on this topic much later in Eight in algebra, topology and mathematical physics on Andrew Ranicki's page
- In 2008, I became interested in the problem of algebraic minimal cones, a topic that turned out to be almost unexplored, except for one paper by Wy Yi Hisang in 1967. From about that time, my research was in the *geometric theory* of non-associative algebras, in particular together with S. Vladuts, N. Nadirashvili, Ya. Krasnov, D.J. Fox. Some papers and my talks are available at this page

Theorem (V.T., 2008) Let $s(x) = \mathrm{sn}(x, \sqrt{-1})$ be the Jacobi lemniscatic sinus, $s(t+\omega) = -s(t)$, $\omega = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}}$. Then

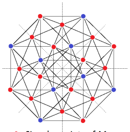
$$M = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ F(x) := s(x_1)s(x_2) - s(x_3)s(x_4) = 0\},$$

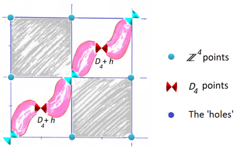
is a connected minimal 4-fold periodic embedded hypersurface in $\mathbb{R}^4$ with isolated singular points at the lattice $\mathbb{Z}^4 \sqcup (\mathbf{h} + D_4)$.

The **Lipschitz integers** $\mathbb{Z}^4 = \{m \in H : m_i \in \mathbb{Z}\}$, the **checkerboard lattice** $D_4 = \{m \in \mathbb{Z}^4 : \sum_{i=1}^4 m_i \equiv 0 \mod 2\}$, the **Hurwitz integers** $\mathcal{H} = \mathbb{Z}^4 \sqcup (\mathbf{h} + \mathbb{Z}^4)$ (the densest possible lattice packing of balls in $\mathbb{R}^4$), where $\mathbf{h} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

M have isolated **Clifford cone type singularities** (note that the cones are given by the **split quaternion norms**)

- Singularities of $\mathbb{Z}^4$-type: if $a \in \mathbb{Z}^4$ then $F(a+x) = \pm x_1x_2 \pm x_3x_4 + O(|x|^4)$, as $x \to 0$.
- Singularities of D_4-type: if $a \in \mathbf{h} + D_4$ then $F(a+x) = \pm(x_3^2 + x_4^2 - x_1^2 - x_2^2) + O(|x|^4)$, as $x \to 0$,





A natural question: **given a family $\mathcal{F}$ of algebraic minimal cones in $\mathbb{R}^n$, can one find an embedded minimal hypersurface with singularities in $\mathcal{F}$?** When I started looking for available algebraic minimal cones it turned out that there were only few known...

S.N. Bernstein (1880 – 1968)

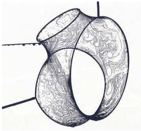
A Russian and Soviet mathematician, known for contributions to partial differential equations, differential geometry, probability theory, and approximation theory:

- 1904 solved Hilbert's 19th problem (a C^3-solution of a nonlinear elliptic analytic equation in 2 variables is analytic)
- 1910s introduced a priori estimates for Dirichlet's boundary problem for non-linear equations of elliptic type
- 1912 laid the foundations of constructive function theory (Bernstein's theorem in approximation theory, Bernstein's polynomial).
- 1915 the famous 'Bernstein's Theorem' on entire solutions of minimal surface equation.
- 1917 the first axiomatic foundation of probability theory, based on the underlying algebraic structure
- 1924 introduced a method for proving limit theorems for sums of dependent random variables
- 1923 axiomatic foundation of a theory of heredity: genetic algebras (Bernstein algebras)

Minimal surface equation and minimal cones

The following definitions of a regular **minimal hypersurface** in $\mathbb{R}^n$ are equivalent:

- A critical point of the area functional
- The mean curvature $= 0$
- If $x_{n+1} = u(x)$ is a graph over $\mathbb{R}^n$ then $\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 0$



Theorem (S. Bernstein, 1915). If u is an **entire** *analytic* solution on $\mathbb{R}^2$ then $u = ax + by + c$.

REMARK. The **regularity assumptions** become important starting with higher dimensions. The **topological/geometrical assumption** is also important: if one removes a tiny disk from a plane, there is a function defined everywhere outside that disk whose graph is a minimal surface (a half-catenoid).

The Bernstein result for C^2- entire solutions holds true for all $2 \le n \le 7$. A key ingredient: blowing-down entire graphs yields *area minimizing cones* Fleming (1962), De Giorgi (1965), Almgren (1966), Simons (1968). But for $n \ge 8$ there appear counterexamples:

Theorem (Bombieri-De Giorgi-Giusti, 1969). *The Clifford-Simons cone*

$$\{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 - |y|^2 = 0\}$$

is area-minimizing in $\mathbb{R}^8$. In particular, Bernstein theorem fails for $n \ge 8$

A.T. Fomenko, Multidimensional variational methods in the topology of extremals, Russ. Math. Surv., 36:6 (1981), 127–165

ноид) с границей $S^1 \times S^0$. И в этом случае конус с вершиной в нуле не минимален, так как существует «сокращающая деформация». Однако сравнение катеноида с «одномерным цилиндром» (рис. 6) показывает, что горловина катеноида расположена ближе к началу координат, чем горловина «цилиндра». Рассмотрим в S^3 двумерный «контур» — $S^1 \times S^1$ (тор), стандартно вложенный в $\mathbb{R}^4 = \mathbb{C}^2$ так: $\{|z| = |w|\} \cap \{|z|^2 + |w|^2 = 1\}$. Прямое вычисление показывает (см. ниже), что трехмерная ГМ-поверхность с границей $S^1 \times S^1$ имеет еще более «узкую горловину» (см. условный рис. 8), провисая больше, чем в двумерном случае. Тем самым обнаруживается интересный эффект: с ростом размерности ГМ-поверхности (указанного типа) она провисает все больше и больше, стремясь к началу координат. Интуитивно ясно, что с ростом размерности наступит момент, когда минимальная пленка с границей $S^p \times S^q$ (вложенной в S^{p+q+1}) провиснет настолько, что «схлопнется» и превратится в конус с вершиной в точке 0 (начало координат). Эта вершина будет особой точкой ГМ-поверхности. Вопрос: в какой размерности впервые появляются глобально минимальные конусы, отличные от стандартного диска (т. е. вершина которых — существенно особая точка)? Оказывается, от решения этой задачи зависят некоторые вопросы в теории дифференциальных уравнений, геометрии групп Ли.

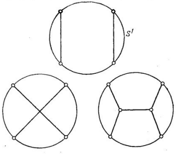


Рис. 6.

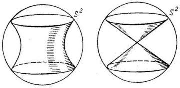


Рис. 7.

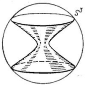


Рис. 8.

Теорема 5.2.2. *Единственными глобально минимальными поверхностями с границей A^{n-2}, где A^{n-2} — орбита действия компактной связной группы Ли $G \subset \mathrm{SO}_n$ на $\mathbb{R}^n$, $A^{n-2} \subset S^{n-1}$, являются конусы с вершиной в начале координат, имеющие в качестве границы следующие многообразия $A = G/H$:* 1) $S^{r-1} \times S^{s-1} = (\mathrm{SO}_r \times \mathrm{SO}_s)/(\mathrm{SO}_{r-1} \times \mathrm{SO}_{s-1})$ *в* $\mathbb{R}^{r+s}$ *для* $r + s \geqslant 8$; 2) $(\mathrm{SO}_2 \times \mathrm{SO}_k)/(\mathbb{Z}_2 \times \mathrm{SO}_{k-2})$ *в* $\mathbb{R}^{2k}$ *для* $k \geqslant 8$; 3) $(\mathrm{SU}_2 \times \mathrm{SU}_k)/(T^1 \times \mathrm{SU}_{k-2})$ *в* $\mathbb{R}^{4k}$ *для* $k \geqslant 4$; 4) $(\mathrm{Sp}_2 \times \mathrm{Sp}_k)/(\mathrm{Sp}_1^2 \times \mathrm{Sp}_{k-2})$ *в* $\mathbb{R}^{8k}$ *для* $k \geqslant 2$; 5) $U_5/(\mathrm{SU}_2^2 \times T^1)$ *в* $\mathbb{R}^{20}$; 6) $\mathrm{Sp}_3/\mathrm{Sp}_1^3$ *в* $\mathbb{R}^{14}$; 7) F_4/Spin_8 *в* $\mathbb{R}^{26}$; 8) $(\mathrm{Spin}_{10} \times U_1)/(\mathrm{SU}_4 \times T^1)$ *в* $\mathbb{R}^{32}$. *Здесь через T^1 обозначена подгруппа S^1 в H. Для всех остальных многообразий*

$A = G/H$, указанных в списке СП, соответствующие конусы над ними не являются минимальными, т. е. существует вариация, уменьшающая их объем. Во всех этих случаях индекс этих конусов, рассматриваемых как экстремали функционала объема, всегда равен бесконечности. Перечисленные выше ГМ-конусы являются G-инвариантными относительно действия соответствующих групп G, указанных выше.

По сравнению с [58] новыми здесь являются утверждения о глобальной минимальности конусов над многообразиями: $S^4 \times S^2$ и $S^5 \times S^1$ в $\mathbb{R}^8$; $S^6 \times S^1$ в $\mathbb{R}^9$; $(\mathrm{SO}_2 \times \mathrm{SO}_8)/(\mathbb{Z}_2 \times \mathrm{SO}_6)$ в $\mathbb{R}^{16}$; $(\mathrm{SO}_2 \times \mathrm{SO}_9)/(\mathbb{Z}_2 \times \mathrm{SO}_7)$ в $\mathbb{R}^{18}$; $(\mathrm{SU}_2 \times \mathrm{SU}_4)/(T^1 \times \mathrm{SU}_2)$ в $\mathbb{R}^{16}$. Если же конус не минимален, он тем не менее является экстремалью функционала объема, так как аннулирует оператор Эйлера. Следовательно, для такого конуса определен его индекс — число отрицательных собственных чисел, стоящих на диагонали бесконечной матрицы оператора второй вариации. Этот индекс указывает, сколько есть независимых вариаций, уменьшающих объем поверхности. Ввиду редукции задачи к проблеме нахождения геодезических на двумерной плоскости каждой «сопряженной границе» на конусе (возникающей при бесконечно малом возмущении конуса с помощью векторного якобиева поля [60]) одно-

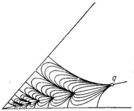
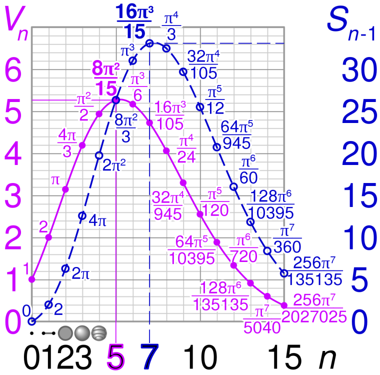


Рис. 12.

All known explicitly given minimal cones are **algebraic**, i.e. zero level sets of homogeneous polynomials. In fact, many of them cones over the so-called **isoparametric hyperesurfaces** in Euclidean spheres, see below.

A curious remark on the n-dimensional spheres and balls

Graphs of volumes V_n and surface areas S_{n-1} of n-balls of radius 1 (Wiki).



Isoparametric hypersurfaces

A hypersurface M is called **isoparametric** if its principal curvatures are constant; by g one denotes the number of **distinct principal curvatures**.

Münzner (1980) used algebraic topology to prove that

- the number g of distinct principal curvatures: $g \in \{1, 2, 3, 4, 6\}$
- $M = u^{-1}(0) \cap S^{n-1}$, $u(x)$ is a **homogeneous polynomial** in $\mathbb{R}^n$:

$$\begin{aligned} \deg u(x) &= g \\ |\nabla u(x)|^2 &= g^2 \langle x, x \rangle^{g-1}, \\ \Delta u(x) &= C \langle x, x \rangle^{(3g-4)/2}. \end{aligned} \tag{1}$$

- there are **at most two** multiplicities (m_1, m_2) for the g principal curvatures, $\dim M = \frac{1}{2}(m_1 + m_2)g$

E. Cartan (1938) completely characterized the cases $g = 1, 2, 3$. In particular, he proved that for $g = 3$, there are exactly four isoparametric hypersurfaces

$$M^{3d} \subset S^{3d+1} \subset \mathbb{R}^{3d+2}, \qquad d = 1, 2, 4, 8,$$

given by the zero level set of the degree 3 homogeneous polynomials. For example, for $d = 1$, the Cartan isoparametric cubic in $\mathbb{R}^5$ given by the zero-level set of

$$x_5^3 + \frac{3}{2} x_5 (x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \frac{3\sqrt{3}}{2} x_4 (x_2^2 - x_1^2) + 3\sqrt{3} x_1 x_2 x_3$$

Remarks.

1) Note that for $g = 4$ there are infinitely many **inhomogeneous** isoparametric hypersurfaces.

2)The following table (by Quo-Shin Chi) is the collection of all symmetric spaces G/K whose isotropy representations give **homogeneous** isoparametric hypersurfaces M:

G	K	$\dim M$	g	(m_1, m_2)
$S^1 \times SO(n+1)$	$SO(n)$	n	1	$(1,1)$
$SO(p+1) \times SO(n+1-p)$	$SO(p) \times SO(n-p)$	n	2	$(p, n-p)$
$SU(3)$	$SO(3)$	3	3	$(1,1)$
$SU(3) \times SU(3)$	$SU(3)$	6	3	$(2,2)$
$SU(6)$	$Sp(3)$	12	3	$(4,4)$
E_6	F_4	24	3	$(8,8)$
$SO(5) \times SO(5)$	$SO(5)$	8	4	$(2,2)$
$SO(10)$	$U(5)$	18	4	$(4,5)$
$SO(m+2), m \geq 3$	$SO(m) \times SO(2)$	$2m-2$	4	$(1, m-2)$
$SU(m+2), m \geq 2$	$S(U(m) \times U(2))$	$4m-2$	4	$(2, 2m-2)$
$Sp(m+2), m \geq 2$	$Sp(m) \times Sp(2)$	$8m-2$	4	$(4, 4m-5)$
E_6	$(Spin(10) \times SO(2))/\mathbb{Z}_4$	30	4	$(6,9)$
G_2	$SO(4)$	6	6	$(1,1)$
$G_2 \times G_2$	G_2	12	6	$(2,2)$

2) **Chern's conjecture for isoparametric hypersurfaces in a sphere**: Any closed, minimally immersed hypersurface of a sphere with *constant scalar curvature* is isoparametric.

Some explicit examples of algebraic minimal cones in $\mathbb{R}^n$

- In $\mathbb{R}^8$: $u = (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$ (**the norm of split octonions**).
- In $\mathbb{R}^5$ (a Cartan **isoparametric** cubic)

$$u(x) = x_5^3 + \frac{3}{2}x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \frac{3\sqrt{3}}{2}x_4(x_2^2 - x_1^2) + 3\sqrt{3}x_1x_2x_3$$

In general, in $\mathbb{R}^{3d+2}$:

$$u(X) = \mathrm{tr}\, X^3, \qquad\qquad X \in \mathrm{Herm}_3(\mathbb{F}_d), \quad \mathrm{tr}\, X = 0,$$

Here $\mathrm{Herm}_3(\mathbb{F}_d)$ should be understood as a hermitian rank 3 **Jordan algebra** over a Hurwitz algebra $\mathbb{K}_d$.

- In $\mathbb{R}^{3d}$: $u = \mathrm{Re}((z_1z_2)z_3)$, $z_i \in \mathbb{K}_d$, $d = 1, 2, 4, 8$ (the **triality polynomials**).
- In $\mathbb{R}^9$:

$$u(x) = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$$

and in the general case for $\det X$ in $\mathbb{R}^{n\times n}$ in (Tkachev 2010). Recently generalized for various determinantal and Pfaffian varieties producing minimal cones: Bordemann, Choe, Hoppe, Heaton, Kozhasov, Venturello.

Observe that all the above solutions can be written in a very compact form using *norms, determinants, traces*. A **natural question arises**: what is a **natural context** for all these examples? Isoparametric? Determinantal? Jordan algebraic?...

Truly $C^{1,\alpha}$ viscosity solutions

Somewhat **very strange appearance** of minimal cones in the context of 'truly' $C^{1,\alpha}$-viscosity solutions constructed in 2007-2011 by Nikolai Nadirashvili and Serge Vlăduț (2010 ICM Invited Speakers in PDEs)

There are important classes of fully nonlinear Dirichlet problems for which the viscosity solution is in fact a classical one, e.g. due to Krylov–Evans regularity theory, in the case when the function F is convex, (see [CC], [K]). However, for the general F the problem of the coincidence of viscosity solutions with the classical solutions remained open.

The central result of this paper is the existence of a nonclassical viscosity solution of (1) in dimension 12. More precisely we prove

Theorem. *The function*

$$w(x) := \frac{Re(\omega_1\omega_2\omega_3)}{|x|},$$

where $\omega_i \in \mathbf{H}$, $i = 1, 2, 3$, *are Hamiltonian quaternions,* $x = (\omega_1, \omega_2, \omega_3) \in \mathbf{H}^3 = \mathbf{R}^{12}$ *is a viscosity solution in* $\mathbf{R}^{12}$ *of a uniformly elliptic equation (1) with a smooth* F.

One can find the explicit expression for w in the coordinates of $\mathbf{R}^{12}$ in sections 3 and 4. The elliptic operator F will be defined in a constructive way in section 2, and its ellipticity constant $\Lambda < 10^8$.

As an immediate consequence of the theorem we have

but loses the uniform ellipticity in a neighborhood of the subset of S_1^5 formed by the points with $|z_1| = |z_2| = |z_3|$, $Re(z_1z_2z_3) = 0$.

To explain why P_6 does not work and P_{12} does work we give in the next section a short excursion in the area of division algebras and exceptional Lie groups. That will lead us also to various extensions of Theorem 3.1.

4. Trialities, Quaternions, Octonions and Hessian Equations

As we have seen in the previous section, cubic forms for which the quadratic form P_d verifies the inequalities (3.2) or (3.3) should be rather exceptional. In fact all examples of such forms known to us come from trialities, which in turn are intimately related to division algebras and exceptional Lie groups. Let us recall some of their elementary properties [1, 3].

Duality is ubiquous in algebra; triality is similar, but subtler. For two real vector spaces V_1 and V_2, a duality is simply a nondegenerate bilinear map

$$f : V_1 \times V_2 \longrightarrow \mathbf{R}.$$

Similarly, for three real vector spaces V_1, V_2, and V_3, a triality is a trilinear map

$$t : V_1 \times V_2 \times V_3 \longrightarrow \mathbf{R}$$

Proceedings of the International Congress of Mathematicians
Hyderabad, India, 2010

Weak Solutions of Nonvariational Elliptic Equations

Nikolai Nadirashvili* and Serge Vlăduţ†

equation in those dimensions [28, 30]. Moreover, one can formulate a test similar to the second part of Lemma 4.2 which garanties that w_δ is a solution to an Isaacs equation.

In this way we get the following:

1). For any δ, $1 \le \delta < 2$ and any plane $H' \subset \mathbf{R}^{24}$, $\dim H' = 21$ the function

$$(P_{24}(x)/|x|^\delta)_{|H'}$$

is a viscosity solution to a uniformly elliptic Hessian (1.2) in the unit ball $B \subset \mathbf{R}^{21}$.

2). For any δ, $1 \le \delta < 2$ the function

$$w_{12,\delta} = P_{12}(x)/|x|^\delta$$

is a viscosity solution to a uniformly elliptic Hessian equation (1.2) in the unit ball $B \subset \mathbf{R}^{12}$.

3). For any hyperplane $H \subset \mathbf{R}^{12}$ the function

$$(P_{12}(x)/|x|)_{|H}$$

is a viscosity solution to a uniformly elliptic Hessian equation (1.2) in the unit ball $B \subset \mathbf{R}^{11}$.

4). For any δ, $1 \le \delta < 2$ the function

$$w_{12,\delta} = P_{12}(x)/|x|^\delta$$

is a viscosity solution to Isaacs equation (1.8) in the unit ball $B \subset \mathbf{R}^{12}$.

Truly $C^{1,\alpha}$ viscosity solutions

- Evans, Crandall, Lions, Jensen, Ishii: If $\Omega \subset \mathbb{R}^n$ is bounded with C^1-boundary, ϕ continuous on $\partial\Omega$, F uniformly elliptic operator then the Dirichlet problem

$$F(D^2u) = 0, \quad \text{in} \quad \Omega$$
$$u = \phi \quad \text{on} \quad \partial\Omega$$

  has a unique weak (the so-called **viscosity**) solution $u \in C(\Omega)$;
- Krylov, Safonov, Trudinger, Caffarelli, early 80's: the solution is always $C^{1,\varepsilon}$
- Nirenberg, 50's: if $n = 2$ then u is classical (C^2) solution
- Nadirashvili, Vlăduț, 2007-2011: if $n \geq 12$ then there are solutions which are not C^2.

Theorem (Nadirashvili-Vlăduț-V.T, 2012) The function $w(x) := \frac{u_1(x)}{|x|}$ where

$$u_1(x) = x_5^3 + \tfrac{3}{2}x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \tfrac{3\sqrt{3}}{2}x_4(x_2^2 - x_1^2) + 3\sqrt{3}x_1x_2x_3,$$

is a **singular viscosity solution** of a uniformly elliptic Hessian equation. This also gives the best possible dimension ($n = 5$) where homogeneous order 2 real analytic functions in $\mathbb{R}^n \setminus \{0\}$.

- Some further comments can be found in Tkachev V., Spectral properties of nonassociative algebras and breaking regularity for nonlinear elliptic type PDEs. Algebra i Analiz, 31(2019)

Outline

Hsiang's 1967 paper

In the 1st volume of *J. of Diff. Geom.*, 1967, W.-Y. Hsiang published a paper *"Remarks on closed minimal submanifolds in the standard Riemannian m-sphere"*, where he remarks:

> *The classically known examples of closed minimal submanifolds of condimension one in S^m, namely, those of the type S^{m-1}, $S^p \times S^q$, are exactly the real algebraic minimal submanifolds of degree 1 and 2. In addition to the homogeneous examples given in the list of observation 2, §1, we produce some new algebraic minimal submanifolds which are no longer homogeneous. Hence the set of algebraic minimal submanifolds in S^m is essentially larger than that of homogeneous ones. It is then quite interesting to classify real algebraic minimal submanifolds of degree higher than two up to equivalence under the orthogonal transformations. It turns out that the algebraic difficulties involved in such a problem are rather formidable. As a by-product, we derive the existence of some kind of a normal form for homogeneous polynomials of arbitrary degree and arbitrary number of variables over real closed fields with respect to the orthogonal linear substitutions...*

Any algebraic cone comes from a polynomial solution $u \in \mathbb{R}[x_1, \ldots, x_n]$ of $\deg u = m$ to

$$\Delta_1 u(x) := |\nabla u|^2 \Delta u - \tfrac{1}{2}\langle \nabla u, \nabla |\nabla u|^2 \rangle = P(x)u(x) \qquad (2)$$

where $P(x)$ is **undefined** homogeneous polynomial of degree $3m-4$. In particular, if $m = 3$ then

$$\Delta_1 u(x) = \text{a quadratic form} \cdot u(x) \qquad (3)$$

Explicitly, in dimension $n = 2$ one has a quasilinear PDE:

$$u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = (ax^2 + 2bxy + cy^2)u, \qquad \text{where } a, b, c \text{ are unknown constants.} \qquad (4)$$

It is difficult, but a possible task to solve (4). But already for $m = 3$ the corresponding LHS contains almost 300 terms!

Hsiang's 1967 paper

$$u := a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + b_1 x_1^2 x_2 + b_2 x_1^2 x_3 + b_3 x_2^2 x_1 + b_4 x_2^2 x_3 + b_5 x_3^2 x_1 + b_6 x_3^2 x_2 + c_1 x_1 x_2 x_3$$

$$\begin{aligned}
\Delta(u) = {} & 54 a_1^2 a_2 x_1^4 x_2 + 54 a_1^2 a_3 x_1^4 x_3 + 18 a_1^2 b_3 x_1^5 + 18 a_1^2 b_4 x_1^4 x_3 + 18 a_1^2 b_5 x_1^5 + 18 a_1^2 b_6 x_1^4 x_2 + 54 a_1 a_2^2 x_1 x_2^4 + 72 a_1 a_2 b_1 x_1^3 x_2^2 + 72 a_1 a_2 b_2 x_1^3 x_2 x_3 + 72 a_1 a_2 b_3 x_1^2 x_2^3 + 72 a_1 a_2 b_4 x_1 x_2^3 x_3 + 36 a_1 a_2 b_5 x_1^2 x_2 x_3^2 \\
& + 36 a_1 a_2 b_6 x_1 x_2^2 x_3^2 + 54 a_1 a_2 c_1 x_1^2 x_2^2 x_3 + 54 a_1 a_3^2 x_1 x_3^4 + 72 a_1 a_3 b_1 x_1^3 x_2 x_3 + 72 a_1 a_3 b_2 x_1^3 x_3^2 + 36 a_1 a_3 b_3 x_1^2 x_2^2 x_3 + 36 a_1 a_3 b_4 x_1 x_2^2 x_3^2 + 72 a_1 a_3 b_5 x_1^2 x_3^3 + 72 a_1 a_3 b_6 x_1 x_2 x_3^3 + 54 a_1 a_3 c_1 x_1^2 x_2 x_3^2 - 6 a_1 b_1^2 x_1^5 \\
& + 12 a_1 b_1 b_3 x_1^4 x_2 + 24 a_1 b_1 b_4 x_1^3 x_2 x_3 + 24 a_1 b_1 b_5 x_1^4 x_2 + 24 a_1 b_1 b_6 x_1^3 x_2^2 - 6 a_1 b_1 c_1 x_1^4 x_3 - 6 a_1 b_2^2 x_1^5 + 24 a_1 b_2 b_3 x_1^4 x_3 + 24 a_1 b_2 b_4 x_1^3 x_3^2 + 12 a_1 b_2 b_5 x_1^4 x_3 + 24 a_1 b_2 b_6 x_1^3 x_2 x_3 - 6 a_1 b_2 c_1 x_1^4 x_2 + 12 a_1 b_3^2 x_1^3 x_2^2 \\
& + 36 a_1 b_3 b_4 x_1^2 x_2^2 x_3 + 12 a_1 b_3 b_5 x_1^3 x_2^2 + 12 a_1 b_3 b_5 x_1^3 x_3^2 + 12 a_1 b_3 b_6 x_1^2 x_2^3 + 12 a_1 b_3 b_6 x_1^2 x_2 x_3^2 + 12 a_1 b_3 c_1 x_1^3 x_2 x_3 + 6 a_1 b_4^2 x_1 x_2^4 + 24 a_1 b_4^2 x_1 x_2^2 x_3^2 + 12 a_1 b_4 b_5 x_1^2 x_2^2 x_3 + 12 a_1 b_4 b_5 x_1^2 x_3^3 + 24 a_1 b_4 b_6 x_1 x_2^3 x_3 \\
& + 24 a_1 b_4 b_6 x_1 x_2 x_3^3 + 6 a_1 b_4 c_1 x_1^2 x_2^3 + 24 a_1 b_4 c_1 x_1^2 x_2 x_3^2 + 12 a_1 b_5^2 x_1^3 x_3^2 + 36 a_1 b_5 b_6 x_1^2 x_2 x_3^2 + 12 a_1 b_5 c_1 x_1^3 x_2 x_3 + 24 a_1 b_6^2 x_1 x_2^2 x_3^2 + 6 a_1 b_6^2 x_1 x_3^4 + 24 a_1 b_6 c_1 x_1^2 x_2^2 x_3 + 6 a_1 b_6 c_1 x_1^2 x_3^3 + 54 a_2^2 a_3 x_2^4 x_3 + 18 a_2^2 b_1 x_2^5 \\
& + 18 a_2^2 b_2 x_2^4 x_3 + 18 a_2^2 b_5 x_1 x_2^4 + 18 a_2^2 b_6 x_2^5 + 54 a_2 a_3^2 x_2 x_3^4 + 36 a_2 a_3 b_1 x_1^2 x_2^2 x_3 + 36 a_2 a_3 b_2 x_1^2 x_2 x_3^2 + 72 a_2 a_3 b_3 x_1 x_2^3 x_3 + 72 a_2 a_3 b_4 x_2^3 x_3^2 + 72 a_2 a_3 b_5 x_1 x_2 x_3^3 + 72 a_2 a_3 b_6 x_2^2 x_3^3 + 54 a_2 a_3 c_1 x_1 x_2^2 x_3^2 + 12 a_2 b_1^2 x_1^2 x_2^3 \\
& + 36 a_2 b_1 b_2 x_1^2 x_2^2 x_3 + 12 a_2 b_1 b_3 x_1 x_2^4 + 24 a_2 b_1 b_4 x_2^4 x_3 + 12 a_2 b_1 b_5 x_1^3 x_2^2 + 12 a_2 b_1 b_5 x_1 x_2^2 x_3^2 + 12 a_2 b_1 b_6 x_1^2 x_2^3 + 12 a_2 b_1 b_6 x_2^3 x_3^2 + 12 a_2 b_1 c_1 x_1 x_2^3 x_3 + 6 a_2 b_2^2 x_1^4 x_2 + 24 a_2 b_2^2 x_1^2 x_2 x_3^2 + 24 a_2 b_2 b_3 x_1 x_2^3 x_3 \\
& + 24 a_2 b_2 b_4 x_2^3 x_3^2 + 24 a_2 b_2 b_5 x_1^3 x_2 x_3 + 24 a_2 b_2 b_5 x_1 x_2 x_3^3 + 12 a_2 b_2 b_6 x_1^2 x_2^2 x_3 + 12 a_2 b_2 b_6 x_2^2 x_3^3 + 6 a_2 b_2 c_1 x_1^3 x_2^2 + 24 a_2 b_2 c_1 x_1 x_2^2 x_3^2 - 6 a_2 b_3^2 x_2^5 + 24 a_2 b_3 b_5 x_1^2 x_2^3 + 24 a_2 b_3 b_6 x_1 x_2^4 - 6 a_2 b_3 c_1 x_2^4 x_3 - 6 a_2 b_4^2 x_2^5 \\
& + 24 a_2 b_4 b_5 x_1 x_2^3 x_3 + 12 a_2 b_4 b_6 x_2^4 x_3 - 6 a_2 b_4 c_1 x_1 x_2^4 + 24 a_2 b_5^2 x_1^2 x_2 x_3^2 + 6 a_2 b_5^2 x_2 x_3^4 + 36 a_2 b_5 b_6 x_1 x_2^2 x_3^2 + 24 a_2 b_5 c_1 x_1^2 x_2^2 x_3 + 6 a_2 b_5 c_1 x_2^2 x_3^3 + 12 a_2 b_6^2 x_2^3 x_3^2 + 12 a_2 b_6 c_1 x_1 x_2^3 x_3 + 18 a_3^2 b_1 x_2 x_3^4 + 18 a_3^2 b_2 x_3^5 \\
& + 18 a_3^2 b_3 x_1 x_3^4 + 18 a_3^2 b_4 x_3^5 + 6 a_3 b_1^2 x_1^4 x_3 + 24 a_3 b_1^2 x_1^2 x_2^2 x_3 + 36 a_3 b_1 b_2 x_1^2 x_2 x_3^2 + 24 a_3 b_1 b_3 x_1^3 x_2 x_3 + 24 a_3 b_1 b_3 x_1 x_2^3 x_3 + 12 a_3 b_1 b_4 x_1^2 x_2 x_3^2 + 12 a_3 b_1 b_4 x_2^3 x_3^2 + 24 a_3 b_1 b_5 x_1 x_2 x_3^3 + 24 a_3 b_1 b_6 x_2^2 x_3^3 + 6 a_3 b_1 c_1 x_1^3 \\
& x_3^2 + 24 a_3 b_1 c_1 x_1 x_2^2 x_3^2 + 12 a_3 b_2^2 x_1^2 x_3^3 + 12 a_3 b_2 b_3 x_1^3 x_3^2 + 12 a_3 b_2 b_3 x_1 x_2^2 x_3^2 + 12 a_3 b_2 b_4 x_1^2 x_3^3 + 12 a_3 b_2 b_4 x_2^2 x_3^3 + 12 a_3 b_2 b_5 x_1 x_3^4 + 24 a_3 b_2 b_6 x_2 x_3^4 + 12 a_3 b_2 c_1 x_1 x_2 x_3^3 + 24 a_3 b_3^2 x_1^2 x_2^2 x_3 + 6 a_3 b_3^2 x_2^4 x_3 \\
& + 36 a_3 b_3 b_4 x_1 x_2^2 x_3^2 + 24 a_3 b_3 b_5 x_1^2 x_3^3 + 24 a_3 b_3 b_6 x_1 x_2 x_3^3 + 24 a_3 b_3 c_1 x_1^2 x_2 x_3^2 + 6 a_3 b_3 c_1 x_2^3 x_3^2 + 12 a_3 b_4^2 x_2^2 x_3^3 + 24 a_3 b_4 b_5 x_1 x_3^4 + 12 a_3 b_4 b_6 x_2 x_3^4 + 12 a_3 b_4 c_1 x_1 x_2 x_3^3 - 6 a_3 b_5^2 x_3^5 - 6 a_3 b_5 c_1 x_2 x_3^4 - 6 a_3 b_6^2 x_3^5 \\
& - 6 a_3 b_6 c_1 x_1 x_3^4 - 6 b_1^3 x_1^4 x_2 - 6 b_1^2 b_2 x_1^4 x_3 - 12 b_1^2 b_3 x_1^3 x_2^2 + 2 b_1^2 b_5 x_1^5 + 8 b_1^2 b_5 x_1^3 x_2^2 - 4 b_1^2 b_5 x_1^3 x_3^2 + 2 b_1^2 b_6 x_1^4 x_2 + 8 b_1^2 b_6 x_1^2 x_2^3 - 4 b_1^2 b_6 x_1^2 x_2 x_3^2 - 12 b_1^2 c_1 x_1^3 x_2 x_3 - 6 b_1 b_2^2 x_1^4 x_2 - 4 b_1 b_2 b_4 x_1^4 x_2 - 4 b_1 b_2 b_4 x_1^2 x_2^3 \\
& + 8 b_1 b_2 b_4 x_1^2 x_2 x_3^2 - 4 b_1 b_2 b_6 x_1^4 x_3 + 8 b_1 b_2 b_6 x_1^2 x_2^2 x_3 - 4 b_1 b_2 b_6 x_1^2 x_3^3 - 2 b_1 b_2 c_1 x_1^5 - 8 b_1 b_2 c_1 x_1^3 x_2^2 - 8 b_1 b_2 c_1 x_1^3 x_3^2 - 12 b_1 b_3^2 x_1^2 x_2^3 + 8 b_1 b_3 b_5 x_1^4 x_2 + 8 b_1 b_3 b_5 x_1^2 x_2^3 - 4 b_1 b_3 b_5 x_1^2 x_2 x_3^2 + 8 b_1 b_3 b_6 x_1^3 x_2^2 \\
& + 8 b_1 b_3 b_6 x_1 x_2^4 - 4 b_1 b_3 b_6 x_1 x_2^2 x_3^2 - 18 b_1 b_3 c_1 x_1^2 x_2^2 x_3 - 4 b_1 b_4^2 x_1^2 x_2^3 + 2 b_1 b_4^2 x_2^5 + 8 b_1 b_4^2 x_2^3 x_3^2 - 4 b_1 b_4 b_6 x_1^2 x_2^2 x_3 + 8 b_1 b_4 b_6 x_2^4 x_3 + 8 b_1 b_4 b_6 x_2^2 x_3^3 - 6 b_1 b_4 c_1 x_1^3 x_2^2 - 4 b_1 b_5 b_6 x_1^3 x_3^2 + 8 b_1 b_5 b_6 x_1 x_2^2 x_3^2 \\
& - 4 b_1 b_5 b_6 x_1 x_3^4 - 6 b_1 b_5 c_1 x_1^2 x_3^3 - 4 b_1 b_6^2 x_1^2 x_2 x_3^2 + 8 b_1 b_6^2 x_2^3 x_3^2 + 2 b_1 b_6^2 x_2 x_3^4 - 4 b_1 b_6 c_1 x_1^3 x_2 x_3 + 8 b_1 b_6 c_1 x_1 x_2^3 x_3 - 4 b_1 b_6 c_1 x_1 x_2 x_3^3 - 2 b_1 c_1^2 x_1^4 x_2 - 2 b_1 c_1^2 x_1^2 x_2^3 - 8 b_1 c_1^2 x_1^2 x_2 x_3^2 - 6 b_2^3 x_1^4 x_3 + 2 b_2^2 b_3 x_1^5 - 4 b_2^2 b_3 \\
& x_1^3 x_2^2 + 8 b_2^2 b_3 x_1^3 x_3^2 + 2 b_2^2 b_4 x_1^4 x_3 - 4 b_2^2 b_4 x_1^2 x_2^2 x_3 + 8 b_2^2 b_4 x_1^2 x_3^3 - 12 b_2^2 b_5 x_1^3 x_3^2 - 12 b_2^2 c_1 x_1^3 x_2 x_3 - 4 b_2 b_3 b_4 x_1^3 x_2^2 - 4 b_2 b_3 b_4 x_1 x_2^4 + 8 b_2 b_3 b_4 x_1 x_2^2 x_3^2 + 8 b_2 b_3 b_5 x_1^4 x_3 - 4 b_2 b_3 b_5 x_1^2 x_2^2 x_3 + 8 b_2 b_3 b_5 x_1^2 x_3^3 \\
& - 6 b_2 b_3 c_1 x_1^2 x_2^3 - 4 b_2 b_4^2 x_1^2 x_2^2 x_3 + 2 b_2 b_4^2 x_2^4 x_3 + 8 b_2 b_4^2 x_2^2 x_3^3 + 8 b_2 b_4 b_5 x_1^3 x_3^2 - 4 b_2 b_4 b_5 x_1 x_2^2 x_3^2 + 8 b_2 b_4 b_5 x_1 x_3^4 - 4 b_2 b_4 b_6 x_1^2 x_2 x_3^2 + 8 b_2 b_4 b_6 x_2^3 x_3^2 + 8 b_2 b_4 b_6 x_2 x_3^4 - 4 b_2 b_4 c_1 x_1^3 x_2 x_3 - 4 b_2 b_4 c_1 x_1 x_2^3 x_3 \\
& + 8 b_2 b_4 c_1 x_1 x_2 x_3^3 - 12 b_2 b_5^2 x_1^2 x_3^3 - 18 b_2 b_5 c_1 x_1^2 x_2 x_3^2 - 4 b_2 b_6^2 x_1^2 x_3^3 + 8 b_2 b_6^2 x_2^2 x_3^3 + 2 b_2 b_6^2 x_3^5 - 6 b_2 b_6 c_1 x_1^3 x_3^2 - 2 b_2 c_1^2 x_1^4 x_3 - 8 b_2 c_1^2 x_1^2 x_2^2 x_3 - 2 b_2 c_1^2 x_1^2 x_3^3 - 6 b_3^3 x_1 x_2^4 - 6 b_3^2 b_4 x_2^4 x_3 + 8 b_3^2 b_5 x_1^3 x_2^2 + 2 b_3^2 b_5 x_1 x_2^4 \\
& - 4 b_3^2 b_5 x_1 x_2^2 x_3^2 + 8 b_3^2 b_6 x_1^2 x_2^3 + 2 b_3^2 b_6 x_2^5 - 4 b_3^2 b_6 x_2^3 x_3^2 - 12 b_3^2 c_1 x_1 x_2^3 x_3 - 6 b_3 b_4^2 x_1 x_2^4 + 8 b_3 b_4 b_5 x_1^2 x_2^2 x_3 - 4 b_3 b_4 b_5 x_2^4 x_3 - 4 b_3 b_4 b_5 x_2^2 x_3^3 - 8 b_3 b_4 c_1 x_1^2 x_2^3 - 2 b_3 b_4 c_1 x_2^5 - 8 b_3 b_4 c_1 x_2^3 x_3^2 + 8 b_3 b_5^2 x_1^3 x_3^2 - 4 b_3 \\
& b_5^2 x_1 x_2^2 x_3^2 + 2 b_3 b_5^2 x_1 x_3^4 + 8 b_3 b_5 b_6 x_1^2 x_2 x_3^2 - 4 b_3 b_5 b_6 x_2^3 x_3^2 - 4 b_3 b_5 b_6 x_2 x_3^4 + 8 b_3 b_5 c_1 x_1^3 x_2 x_3 - 4 b_3 b_5 c_1 x_1 x_2^3 x_3 - 4 b_3 b_5 c_1 x_1 x_2 x_3^3 - 6 b_3 b_6 c_1 x_2^2 x_3^3 - 2 b_3 c_1^2 x_1^3 x_2^2 - 2 b_3 c_1^2 x_1 x_2^4 - 8 b_3 c_1^2 x_1 x_2^2 x_3^2 - 6 b_4^3 x_2^4 x_3 - 12 \\
& b_4^2 b_6 x_2^3 x_3^2 - 12 b_4^2 c_1 x_1 x_2^3 x_3 + 8 b_4 b_5^2 x_1^2 x_3^3 - 4 b_4 b_5^2 x_2^2 x_3^3 + 2 b_4 b_5^2 x_3^5 - 6 b_4 b_5 c_1 x_2^3 x_3^2 - 12 b_4 b_6^2 x_2^2 x_3^3 - 18 b_4 b_6 c_1 x_1 x_2^2 x_3^2 - 8 b_4 c_1^2 x_1^2 x_2^2 x_3 - 2 b_4 c_1^2 x_2^4 x_3 - 2 b_4 c_1^2 x_2^2 x_3^3 - 6 b_5^3 x_1 x_3^4 - 6 b_5^2 b_6 x_2 x_3^4 - 12 b_5^2 c_1 x_1 x_2 x_3^3 \\
& - 6 b_5 b_6^2 x_1 x_3^4 - 8 b_5 b_6 c_1 x_1^2 x_3^3 - 8 b_5 b_6 c_1 x_2^2 x_3^3 - 2 b_5 b_6 c_1 x_3^5 - 2 b_5 c_1^2 x_1^3 x_3^2 - 8 b_5 c_1^2 x_1 x_2^2 x_3^2 - 2 b_5 c_1^2 x_1 x_3^4 - 6 b_6^3 x_2 x_3^4 - 12 b_6^2 c_1 x_1 x_2 x_3^3 - 8 b_6 c_1^2 x_1^2 x_2 x_3^2 - 2 b_6 c_1^2 x_2^3 x_3^2 - 2 b_6 c_1^2 x_2 x_3^4 - 2 c_1^3 x_1^3 x_2 x_3 - 2 c_1^3 x_1 x_2^3 x_3 - 2 \\
& c_1^3 x_1 x_2 x_3^3
\end{aligned}$$

Remark. Even if one is able to find an explicit solution, it will still be a nontrivial task to identify a **relevant algebraic context** where this solution appears in?... Hsiang mentions a "normal" form (similar to the diagonal form of a matrix) and suggested an elegant method which does not work in general, but is good enough to produce at least some further solutions.

Hsiang's 1967 paper

W.-Y. Hsiang suggests the following problem:

(ii) Partly due to the lack of "canonical" normal forms for $r < 2$ and partly due to the rapid rate of increase of the dimension of $\mathfrak{H}_n^r$ with respect to r, the little help obtained from the normal forms is not enough to solve the problem of classifying minimal algebraic cones of higher degrees. For example, it is very difficult to solve even the following very special equation: $F(x) = 0$, where $F(x)$ is an irreducible cubic form in n variables such that

$$(\Delta F) \cdot |\nabla F|^2 - \nabla F \cdot HF \cdot \nabla F^t = \pm (x_1^2 + \cdots + x_n^2) \cdot F .$$

Since the above equation is invariant with respect to the orthogonal linear substitutions, we may assume that F is given in some kind of "normal form" which amounts to reduce the number of indeterminant coefficients by $n(n-1)/2$. A systematic attempt to solve the above equation will involve the job of solving over-determined simultaneous algebraic equations of many variables. So far, we have only four non-trivial solutions (cf. §§ 1, 2), but there is no reason why there should be no others.

On other words, Hsiang asked to classify all cubic polynomial solutions of the following PDE in $\mathbb{R}^n$:

$$\Delta_1 u(x) = |\nabla u|^2 \Delta u - \tfrac{1}{2} \langle \nabla u, \nabla |\nabla u|^2 \rangle = \lambda \langle x, x \rangle u(x)$$

Such a solution $u(x)$ is called a *Hsiang eigencubic*.

Hsiang's trick

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $\mathrm{Herm}'_N(\mathbb{K})$ be the real vector space of **trace free** hermitian matrices of order $N \geq 3$ with the inner product $\langle X, Y\rangle := \mathrm{tr}\, XY$. Define

$$u(X) := \mathrm{tr}\, X^3, \qquad X \in \mathrm{Herm}'_N(\mathbb{K}).$$

Hsiang shows that $\Delta_1 u = |\nabla u|^2 \Delta u - \frac{1}{2}\langle \nabla u, \nabla |\nabla u|^2\rangle$ is an $O(N)$-**invariant** operator, which implies

$$\Delta_1 u \in \mathbb{K}[\cancel{\mathrm{tr}\, X}, \mathrm{tr}\, X^2, \ldots, \mathrm{tr}\, X^N].$$

Note that $\deg \Delta_1 u(x) = 5$ therefore

$$\Delta_1 u \in \mathbb{K}[\mathrm{tr}\, X^2, \mathrm{tr}\, X^3, \mathrm{tr}\, X^4, \mathrm{tr}\, X^5] = c_1 \mathrm{tr}\, X^2\, \mathrm{tr}\, X^3 + c_2 \mathrm{tr}\, X^5.$$

Therefore if we additionally assume that $N \leq 4$ then $c_2 = 0$ implying that

$$\Delta_1 u(X) = c_1\, \mathrm{tr}\, X^2\, \mathrm{tr}\, X^3 = c_1 \langle X, X\rangle\, u(X).$$

This yields the **four** Hsiang examples in $\mathbb{R}^{(k-1)(2+k\dim \mathbb{K})/2}$, i.e.

$$k = 3: \quad \mathrm{Herm}'_3(\mathbb{R}) \cong \mathbb{R}^5 \quad \text{and} \quad \mathrm{Herm}'_3(\mathbb{C}) \cong \mathbb{R}^8$$
$$k = 4: \quad \mathrm{Herm}'_4(\mathbb{R}) \cong \mathbb{R}^9 \quad \text{and} \quad \mathrm{Herm}'_4(\mathbb{C}) \cong \mathbb{R}^{15}$$

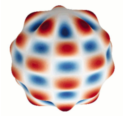
If $k = 3$, $\mathbb{K} = \mathbb{R}$ then $u(x) = C_5(x) := \begin{vmatrix} \frac{x_1}{\sqrt{3}} + x_2 & x_3 & x_4 \\ x_2 & \frac{-2x_1}{\sqrt{3}} & x_5 \\ x_4 & x_5 & \frac{x_1}{\sqrt{3}} - x_2 \end{vmatrix}$ = a Cartan isoparametric cubic in $\mathbb{R}^5$

Outline

How Cartan approached isoparametric equations?

Consider the isoparametric system, that for $g = 3$ takes the form

$$|\nabla u(x)|^2 = g^2 \langle x, x\rangle^{g-1} = 9\langle x, x\rangle^4, \qquad \Delta u(x) = 0. \tag{5}$$



To simplify analysis of (5), Cartan makes use of the so-called normal coordinates To get it, note that $u(x) \not\equiv 0$ is a cubic form then it has at least one **local maximum** $c \in \mathbb{R}^n$ **at the unit sphere**, which is automatically a **stationary point** of $u(x)/|x|^3$ in $\mathbb{R}^n$, which easily implies by the Lagrange principle that $\nabla u(c)$ is proportional to $\nabla |x|^2$ at c, i.e. $\nabla u(c) = \lambda c$, where by (5) $\lambda = 3$ and by the Euler homogeneous function theorem

$$0 \neq 3 \max_{S^{n-1}} u(x) = 3u(c) = \langle \nabla u(c), c\rangle = \lambda |c|^2 = 3$$

Thus $\lambda \neq 0$. Let $x = (y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ be the new orthonormal coordinates such that $t = \langle x, c\rangle$ and $y \perp c$, then

$$u(y, t) = t^3 - 3tP(y) + Q(y),$$

where $\deg P(y) = 2$, $\deg Q(y) = 3$. Substitution into (5) yields

$$9t^4 + 9t^2(|\nabla P|^2 - 2P) - 6t\langle \nabla P, \nabla Q\rangle + (9P^2 + |\nabla Q|^2) = 9t^4 + 18t^2|y|^2 + 9|y|^4$$

- $|\nabla P|^2 - 2P = 2|y|^2$ implies that $P(y)$ has two eigenvalues: -1 and $\frac{1}{2}$ of multiplicities n_1 and n_2 resp.;
- $\Delta u = 3t(2 - \Delta P(y)) + \Delta Q(y) = 0$ implies that $2n_1 = n_2 + 2$, and $n = 3n_1 - 1$.
- this naturally decomposes the y-subspace: $\mathbb{R}^{n-1} = V_{-1} \oplus V_{1/2}$ and further considerations;
- the remaining part is much more difficult, see my paper [Tka10].

An alternative approach

A commutative algebra satisfying the identity $x^2 \bullet (x \bullet x) = x \bullet (x^2 \bullet y)$ is called a **Jordan alegrba**.

Theorem([Tka14])
There is a natural correspondence between the following categories:

$$\text{cubic solutions of } |\nabla u(x)|^2 = 9|x|^4 \quad \leftrightarrow \quad \text{rank 3 formally real Jordan algebras}$$

Given a cubic form $u : V \to \mathbb{K}$, consider its linearizations

- $u(x, y, z) = u(x + y + z) - u(x + y) - u(x + z) - u(y + z) + u(x) + u(y) + u(z)$
- $u(x; y) = \frac{1}{2} u(x, x, y)$

The Freudenthal-Springer Construction (McCrimmon, 1969). A cubic form $N : V \to \mathbb{K}$ with a distinguished point e, $N(e) = 1$, is called a Jordan cubic form if

- the bilinear form $T(x; y) = N(e; x)N(e; y) - N(e; x; y)$ is a *nondegenerate*
- the map $\# : V \to V$ uniquely determined by $T(x^{\#}; y) = N(x; y)$ satisfies the adjoint identity $(x^{\#})^{\#} = N(x)x$.

If N is Jordan then the multiplication

$$x \bullet y = \frac{1}{2}(x \# y + N(e; x)y + N(e; y)x - N(e; x; y)e)$$

defines a Jordan algebra structure on V, where $x \# y := (x + y)^{\#} - x^{\#} - y^{\#}$.

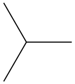
Yet another approach: by metrized algebras

Approach by the *normal* coordinates	Approach by *metrized algebras*
1. A **cubic** homogeneous form $u(x)$ on a **Euclidean** space	A metrized algebra $(\mathbb{A}, \bullet, \langle , \rangle)$: $\langle x \bullet y, z\rangle = \langle \mathrm{Hess} u(x) y, z\rangle$
2. The **isoparametric (eiconal) equation** $\langle \nabla u(x), \nabla u(x)\rangle = 9\langle x, x\rangle^4$	The defining equation of a metrized algebra $\mathbb{A}$ $\frac{1}{4}\langle x \bullet x, x \bullet x\rangle = 9\langle x, x\rangle^4$ The linearized defining equation: $x \bullet (x \bullet x) = 36x\langle x, x\rangle$
3. A **stationary point** $c \in \mathbb{R}^n$ of $u(x)/\lvert x\rvert^3$ and $t = \langle x, c\rangle$.	An idempotent $c \in \mathbb{A}$
4. The **"normal" orthonormal coordinates** $x = (y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ $u(y, t) = t^3 - 3tP(y) + Q(y)$, where $\deg P = 2$, $\deg Q = 3$.	the Peirce decomposition $\mathbb{R}^n = \bigoplus_j \mathbb{A}_c(\lambda_j)$
5. The diagonal form of the **quadratic form** $P(y)$ is $\mathbf{diag}(-1, \frac{1}{2})$.	The Peirce spectrum $\sigma(\mathbb{A}) = \{-1, \frac{1}{2}\}$
6. The source PDE by equating coefficients of t^α reduces to a PDE **system** involving ∇P, ∇Q, for example $9t^4 + 9t^2(\lvert\nabla P\rvert^2 - 2P) - 6t\langle \nabla P, \nabla Q\rangle + \dots$	The "fusion laws" on $\mathbb{A}$: (see table below)
7. There appear some **distinguished** PDE relations	Hidden algebra structures (Clifford, Hurwitz, Jordan etc) inside of $\mathbb{A}$
8. etc...	etc...

$\bullet$	$\mathbb{R}c$	$\mathbb{A}(-1)$	$\mathbb{A}(\frac{1}{2})$
$\mathbb{R}c$	$\mathbb{R}c$	$\mathbb{A}(-1)$	$\mathbb{A}(\frac{1}{2})$
$\mathbb{A}(-1)$	$\mathbb{A}(-1)$	$\mathbb{R}c$	$\mathbb{A}(\frac{1}{2})$
$\mathbb{A}(\frac{1}{2})$	$\mathbb{A}(\frac{1}{2})$	$\mathbb{A}(\frac{1}{2})$	$\mathbb{R}c \oplus \mathbb{A}(-1)$

Summary: three equivalent problems

(DG) Algebraic degree three minimal cones $u(x) = 0$ (Hsiang's problem, 1967). For example, the minimal tree



given by the zero level set of $u(x_1, x_2) = \mathrm{Re}(x_1 + ix_2)^3$

(PDE) Homogeneous degree 3 polynomial solutions to the Hsiang equation:

$$|\nabla u|^2 \Delta u - \frac{1}{2}\langle \nabla u(x), \nabla(|\nabla u(x)|^2)\rangle = Q(x)u(x), \qquad Q(x) \text{ is an } \textit{undefined} \text{ quadratic form.}$$

The above example satisfies the PDE with $Q(x) = -54(x_1^2 + x_2^2)$.

(NA) Commutative nonassociative algebras over $\mathbb{R}$ with an associative bilinear form $\langle , \rangle$ and defining identities

$$\begin{aligned} &\operatorname{tr} L_\bullet(x) = 0 \\ &x^{\bullet 4} + \tfrac{1}{4}(x \bullet x)(x \bullet x) - \langle x, x\rangle (x \bullet x) - \tfrac{2}{3}\langle x \bullet x, x\rangle x = 0. \end{aligned}$$

The corresponding to the above example metrized algebra spanned by three nonzero equilateral idempotents, known also as the two-dimensional Koichiro Harada algebra with automorphism group $\cong S_3$ (1984).

The correspondence between (PDE) and (NA):

$$\boxed{u(x) = \tfrac{1}{6}\langle x, x \bullet x\rangle \qquad \leftrightarrows \qquad \text{algebra multiplication: } x \bullet y = \operatorname{Hess} u(x) y}$$

Outline

Basic notation and terminology

- **Many interesting algebras** (not all!) are defined by/satisfied identities. For example, the standard multiplication is both commutative and associative. But the vector (**cross**) product in $\mathbb{R}^3$ is neither commutative nor associative and in fact, $(\mathbb{R}^3, \times)$ is a **Lie algebra**.
- The left multiplication operator is denoted by: $L_\bullet(x) : y \to x \bullet y$. The (left) **spectrum** of an element $x \in \mathbb{A}$ is the **multiset** of eigenvalues of $L(x)$.
- $c \in \mathbb{A}$ is called an **idempotent** if $c \bullet c = c$. The set of nonzero idempotents is denoted by $\mathrm{Idm}(\mathbb{A})$.
- A commutative algebra is **metrized** if there exists a nondegenerate bilinear form $\langle , \rangle$ such that:

$$\langle x \bullet y, z\rangle = \langle y, x \bullet z\rangle.$$

- A commutative metrized algebra generates the distinguished cubic form $u(x) := \frac{1}{6}\langle x \bullet x, x\rangle$. Conversely, a cubic form $u(x)$ on an inner product vector space induces a structure of commutative nonassociative metrized algebra by

  $x \bullet y := D^2 u(x) y$ (see an example below).

Idempotents and their spectra play a prominent role in nonassociative algebra. An idempotent c is **semisimple** if $\mathbb{A}$ decomposes into a direct sum of eigenspaces of $L_\bullet(c)$ (**Peirce subspaces**):

$$\mathbb{A} = \bigoplus\nolimits_{\lambda \in \sigma(\mathbb{A})} \mathbb{A}_c(\lambda)$$

A fusion law is a set $\mathcal{F} \subset \mathbf{K}$ together with a symmetric binary map $* : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$,

$$\mathbb{A}_c(\lambda) \bullet \mathbb{A}_c(\mu) = \bigoplus\nolimits_{\alpha \in \lambda * \mu} \mathbb{A}_c(\alpha)$$

For example, the **Peirce spectrum** of an **eiconal algebra** is $\{1, -1, \frac{1}{2}\}$ with fusion laws:

$\star$	-1	$-\frac{1}{2}$
-1	1	$-\frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$1, -1$

Now let $u(x)$ be an **arbitrary cubic form** in $\mathbb{R}^n$ and let

$$x \bullet y := (D^2 u(x))y.$$

Recall that any quadratic form comes from a **symmetric** bilinear form. Similarly, any cubic form u comes from a **symmetric** trilinear form U. In other words,

$$u(x) = \tfrac{1}{6}U(x,x,x), \qquad U(x,y,z) = U(x,z,y) = \ldots \quad \text{for any permutation}$$

In fact,

$$U(x,y,z) = u(x+y+z) - u(x+y) - u(x+z) - u(y+z) + u(x) + u(y) + u(z)$$

Example 1. If $u(x) = x_1x_2x_3$ then $U(x,y,z) = \sum_{\sigma\in S_3} x_{\sigma(1)}y_{\sigma(2)}z_{\sigma(3)}$.

The by the definition of the Hessian: $\boxed{\langle (D^2u(x))y, z\rangle = U(x,y,z) = \langle x\bullet y, z\rangle}$.

This implies have the following important properties:

(i) $x \bullet y = y \bullet x$ (the algebra $\mathbb{A}(u)$ is **commutative**)

(ii) $\langle x\bullet y, z\rangle = \langle y, x\bullet z\rangle \quad \Leftrightarrow \quad \langle L(x)y, z\rangle = \langle y, L(x)z\rangle \quad \Leftrightarrow \quad L(x)$ is self-adjoint!

(in other words, $\mathbb{A}(u)$ is a **Euclidean metrized algebra**)

(iii) $x\bullet x = 2\nabla u(x)$ (by Euler's homogeneous function theorem)

(iv) $\langle x\bullet x, x\rangle = 6u(x)$ (by Euler's homogeneous function theorem)

Important: these formulae and properties **do NOT depend on a concrete represenation of** $u(x)$!
We don't know, for instance, what is $x\bullet x$ *exactly*? In fact, if one computes (iii) and (iv) explicitly by $u(x)$, it will imply the algebraic identities for $\mathbb{A}(u)$ in the spirit of (6).

An example

Consider a cubic form $u(x) = \frac{1}{6}x_1(x_1^2 - 3x_2^2 - 3x_3^2)$.
Notice that the Hessian matrix of $u(x)$ is a *symmetric* matrix with entries *linearly depending* on x:

$$H(x) := D^2u(x) = \begin{pmatrix} x_1 & -x_2 & -x_3 \\ -x_2 & -x_1 & 0 \\ -x_3 & 0 & -x_1 \end{pmatrix}$$

Define on $\mathbb{R}^3$ an algebra $\mathbb{A}(u)$ with multiplication

$$x \bullet y := H(x)y = H(y)x = (x_1y_1 - x_2y_2 - x_3y_3,\ -x_2y_1 - x_1y_2,\ x_3y_1 - x_1y_3)$$

The obtained multiplication is **commutative** and the corresponding **Peirce decomposition** satisfies the following **fusion rules**:

$\bullet$	e_1	e_2	e_3
e_1	e_1	$-e_2$	$-e_3$
e_2	$-e_2$	$-e_1$	0
e_3	$-e_3$	0	e_1

$\bullet$	$\mathbb{A}_1$	$\mathbb{A}_{-1}$
$\mathbb{A}_1$	$\mathbb{A}_1$	$\mathbb{A}_{-1}$
$\mathbb{A}_{-1}$	$\mathbb{A}_{-1}$	$\mathbb{A}_1$

$$\mathbb{A} = \mathbb{A}_1 \oplus \mathbb{A}_{-1}, \qquad \mathbb{A}_\lambda = \{x : e_1 \bullet x = \lambda x\}$$

In particular, $e_1 \in \mathrm{Idm}(\mathbb{A})$, its Peirce spectrum of e_1 is $\{1, -1\}$, the eigenvalue -1 of multiplicity 2.
A simple examination reveals the identities in the spirit of "3D-complex numbers":

$$\begin{aligned} x \bullet x &= (x_1^2 - x_2^2 - x_3^2,\ -2x_1x_2,\ -2x_1x_3) \\ (x \bullet x) \bullet x &= (x_1^2 + x_2^2 + x_3^2)x = \langle x, x\rangle x \end{aligned} \tag{6}$$

Jordan algebras

The most natural in the commutative context are ‘stepsisters’ of the classically known Lie algebras: Jordan algebras.

How to multiply two Hermitian matrices?... unfortunately

$$\mathrm{Matrix} \cdot \mathrm{Matrix} = \mathrm{Matrix}, \qquad \text{but}$$

$$\mathrm{Hermitian} \cdot \mathrm{Hermitian} \neq \mathrm{Hermitian}$$

In 1932 Pascual Jordan proposed a program to discover a new algebraic setting for quantum mechanics such that it is independent on an invisible but all-determining metaphysical matrix structure. By *linearizing* the quadratic squaring operation, Jordan replaces the usual matrix multiplication by the anticommutator product (called also the Jordan product)

$$x \bullet y = \frac{1}{2}(xy + yx);$$

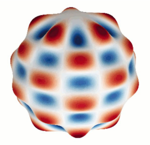
A commutative algebra satisfying the identity $x^2 \bullet (x \bullet x) = x \bullet (x^2 \bullet y)$ is called a **Jordan algebra**.

Example (H. Freudenthal 1954, T. Springer 1961) Let $V = \mathscr{H}_3(\mathbb{F}_d)$ be the vector space of *self-adjoint* 3×3-matrices with coefficients in a Hurwitz algebra $\mathbb{F}_d$ and

$$u(x) = \mathrm{Det}(x) := \tfrac{1}{6}((\mathrm{tr}\, x)^3 - 3\,\mathrm{tr}\, x\, \mathrm{tr}\, x^2 + 2\,\mathrm{tr}\, x^3).$$

Then $V(u)$ is a Jordan algebra w.r.t. the multiplication $x \bullet y = \frac{1}{2}(xy + yx)$.

Idempotents in a Euclidean metrized algebra



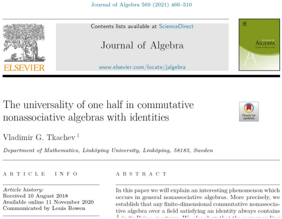
Let S be the sphere $\langle x, x\rangle = 1$. Then

$$
\begin{aligned}
x \text{ is a \textbf{stationary point} of } u(x) \text{ on } S \Leftrightarrow\quad & \nabla u(x) = k\nabla\langle x, x\rangle = 2kx, \quad k \in \mathbb{R} \\
\Leftrightarrow\quad & x \bullet x = 4kx \qquad (\text{by (iv) } 4k = 6u(x)) \\
\Leftrightarrow\quad & \text{either } x^2 = 0 \text{ or } c \bullet c = c, \text{ where } c := x/4k
\end{aligned}
$$

Any **stationary** point of $u(x)$ is proportional to either a 2-**nilpotent** (if $u(x) = 0$) or an **idempotent** (if $u(x) \neq 0$). If $u(x)$ is a point of a local maximum on S then the corresponding idempotent is called **maximal**.

Theorem. A maximal idempotent $c \in \mathbb{A}(u)$ is semisimple, primitive and its Peirce spectrum is subset of $(-\infty, \frac{1}{2}]$ and

$$\mathbb{A}(\tfrac{1}{2})\mathbb{A}(\tfrac{1}{2}) \subset \mathbb{A}(\tfrac{1}{2})^{\perp} \qquad \text{('Jordan type fusion law').}$$



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The universality of one half in commutative nonassociative algebras with identities

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ABSTRACT

In this paper we will explain an interesting phenomenon which occurs in general nonassociative algebras. More precisely, we establish that any finite-dimensional commutative nonassociative algebra over a field satisfying an identity always contains

The Peirce eigenvalue $\frac{1}{2}$ is very special. It turns out that this eigenvalue is universal and it appears in the Peirce spectrum of **any** commutative nonassociative algebra (over an arbitrary field char$\neq 2, 3$) **possessing a polynomial identity** [Tka21].

Outline

How to connect a PDE and a nonassociative algebra?

Example 2. The simplest example is the Laplace operator. To solve a cubic equation in $\mathbb{R}^n$ for a cubic form is not a difficult problem, but on the nonassociative algebra side we obtain

$$\Delta u(x) = \operatorname{tr} D^2 u(x) = \operatorname{tr} L(x) = 0$$

So, in the corresponding metrized algebra $\mathbb{A}(u)$, any multiplication operator is trace-free.

Example 3. A less trivial example. Let us consider with the **eiconal equation**

$$|\nabla u(x)|^2 = \tfrac{1}{4}|x|^4, \quad x \in \mathbb{R}^n. \tag{7}$$

Suppose we are searching a cubic homogeneous polynomial solution (a cubic form). By (iii): $\nabla u(x) = \frac{1}{2} x \bullet x$, hence

$$\langle x \bullet x, x \bullet x \rangle = \langle x, x \rangle^2$$

Now you can work as in calculus: the directional derivative along y (it is called **linearization** in Algebra) gives

$$4\langle x \bullet x, x \bullet y \rangle = 4\langle x, x \rangle \langle x, y \rangle$$
$$\langle (x \bullet x) \bullet x, y \rangle = \langle \langle x, x \rangle x, y \rangle$$

implying that

$$(x \bullet x) \bullet x = \langle x, x \rangle x \qquad \text{(think of } x^3 = |x|^2 x\text{)} \tag{8}$$

The converse is also true (why?), i.e. (8) $\Leftrightarrow$ (7). Compare with (6).

How to connect a PDE and a nonassociative algebra?

Example 3. The Hsiang equations (2) reads as:

$$\Delta_1 u := |\nabla u|^2 \Delta u - \tfrac{1}{2}\langle \nabla u, \nabla |\nabla u|^2\rangle = \alpha \langle x, x\rangle u(x). \tag{9}$$

Recall that

$$\begin{aligned}
\Delta u(x) &= \operatorname{tr} L(x), \\
\nabla u(x) &= \frac{1}{2} x \bullet x, \\
\nabla |\nabla u|^2 = D^2 u(x) \nabla u(x) &= \frac{1}{2} x \bullet (x \bullet x)
\end{aligned}$$

Then

$$\tfrac{1}{4}\langle x \bullet x, x \bullet x\rangle \operatorname{tr} L(x) - \tfrac{1}{4}\langle x \bullet x, x \bullet (x \bullet x)\rangle = \tfrac{1}{6}\alpha \langle x, x\rangle \langle x, x \bullet x\rangle$$

therefore we arrive at the **Hsiang algerba defining relation**:

$$\boxed{\langle x^2, x^2\rangle \operatorname{tr} L_x - \langle x^2, x^3\rangle = \frac{2}{3}\theta \langle x, x\rangle \langle x^2, x\rangle}$$

This is an 'equation of order 5' and therefore it is quite nontrivial (for example, the Jordan algebra identity $x^2(xy) = x(x^2y)$ has an order 4. My next story only starts here.

Main results

A solution u is called a **trivial eigencubic** if $u(x) = \langle b, x \rangle^3$, i.e. the corresponding cone is a hyperplane $\langle b, x \rangle = 0$.

Theorem A

u is a trivial eigencubic if and only if $\Delta u \neq 0$.

In what follows we assume that u is a **normalized** Hsiang eigencubic, i.e. $\Delta_1 u = -2|x|^2 u(x)$, $x \in V = \mathbb{R}^n$.

Theorem B

Given a **nontrivial** normalized u, there exists a commutative metrized algebra such that

1. $u(x) = \frac{1}{6}\langle x, x^2 \rangle$ (the recovering property)
2. $x^3 x + \frac{1}{4}x^2 x^2 - |x|^2 x^2 - \frac{2}{3}\langle x^2, x \rangle x = 0$ (the defining identity)
3. $\operatorname{tr} L_x = 0$ (the harmonicity property)

In the converse direction, given a commutative metrized algebra with properties (2) and (3) above, the cubic form u defined by (1) is a (normalized) Hsiang eigencubic.

A commutative metrized algebra with (2) and (3) above is called **Hsiang algebra**.

Theorem C

Let V be a Hisang algebra. Then

(i) For any nonzero idempotent c, the associated **Peirce decomposition** is

$$V = V_c(1) \oplus V_c(-1) \oplus V_c(-\tfrac{1}{2}) \oplus V_c(\tfrac{1}{2}) \quad \text{and} \quad \dim V_c(1) = 1;$$

and the following **fusion laws** hold:

	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
1	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
-1	-1	1	$\frac{1}{2}$	$-\frac{1}{2} \oplus \frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$1 \oplus -\frac{1}{2}$	$-1 \oplus -\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2} \oplus \frac{1}{2}$	$-1 \oplus -\frac{1}{2}$	$1 \oplus -1 \oplus -\frac{1}{2}$

(ii) The Peirce dimensions $n_1 = \dim V_c(-1)$, $n_2 = \dim V_c(-\frac{1}{2})$ and $n_3 = \dim V_c(\frac{1}{2})$ do not depend on a particular choice of c and $n_3 = 2n_1 + n_2 - 2$;

(iii) All idempotents have the same length and the same fusion law

(iv) $V_c(1) \oplus V_c(-1)$ is a subalgebra. It carries a **hidden Clifford algebra structure**.

(v) $V_c(1) \oplus V_c(-\frac{1}{2})$ is a subalgebra. It carries a **hidden rank 3 Jordan algebra structure**.

Any polar algebra is Hsiang

Definition. A commutative algebra V with associating form $\langle x, y\rangle$ is called polar if

(i) there is a $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$ (the orthogonal sum),

(ii) $V_0V_0 = 0$,

(iii) $x(xy) = \langle x, x\rangle y$ for $x \in V_0$ and $y \in V_1$.

Definition. (X, Y, A), is called a *symmetric Clifford system*, or $A \in \mathrm{Cliff}(X, Y)$, if A is a linear map $X \to \mathrm{End}_{\mathrm{sym}}(Y)$ and $A(x)^2 = |x|^2\mathbf{1}_Y$, $\forall x \in X$.

Example. Let us consider $A(x) = x_1A_1 + x_2A_2 \in \mathrm{End}_{\mathrm{sym}}(\mathbb{R}^2)$, where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Theorem (Radon, Hurwitz). The set $\mathrm{Cliff}(X, Y)$ is nonempty if and only if $\dim X \le 1 + \rho(\frac{1}{2}\dim Y)$, where the Hurwitz-Radon function ρ is defined by $\rho(m) = 8a + 2^b$, if $m = 2^{4a+b}\cdot \mathrm{odd},\ 0 \le b \le 3$.

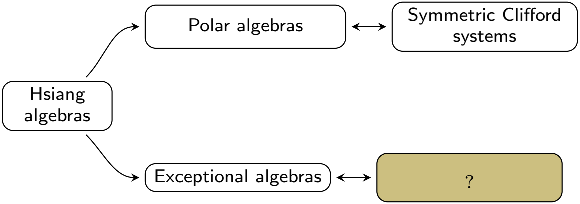
For example, $\rho(2) = 2$, $\rho(3) = 1$, $\rho(4) = 4$, $\rho(6) = 2$, $\rho(7) = 1$, $\rho(8) = 4$, but $\rho(16) = 9$.

The Correspondence If $A \in \mathrm{Cliff}(X, Y)$ then $X \oplus Y$ is a polar algebra of the cubic $u(z) = \frac{1}{2}\langle y, A(x)y\rangle$, $z = x \oplus y$. Conversely, if $V = V_0 \oplus V_1$ is a polar algebra then $L_x \in \mathrm{Cliff}(V_0, V_1)$, $x \in V_0$.

The classification of symmetric Clifford system is well-known (D. Husemöller *Fibre bundles*, 1994.)

Polar vs exceptional

Definition. A Hsiang algebra V isomorphic to a polar algebra is of **Clifford type**; otherwise it is **exceptional**.



Proposition. Any commutative **pseudocomposition algebra** V, i.e. an algebra with $x^3 = |x|^2 x$ and $\operatorname{tr} L_x = 0$ is an *exceptional* Hsiang algebra.

Proof. The scalar product by x^2: $\langle x^3, x^2\rangle = |x|^2\langle x, x^2\rangle$, therefore V is Hsiang. Let us assume by contradiction that V is polar, and let $V = V_0 \oplus V_1$. If $x \in V_0$ then $x^2 = 0$, hence $0 = x^3 = |x|^2 x$, i.e. $x = 0$, this proves that $V_0 = \{0\}$ is trivial. Similarly $V_1 V_1 \subset V_0$ implies that $V_1 = \{0\}$, a contradiction. $\square$

Example. The trace free subspace of the Jordan algebra $(M_3, \circ)$ of all 3×3 real symmetric matrices with the Jordan multiplication $x \circ y = \frac{1}{2}(xy + yx)$ is a pseudocomposition algebra.

The hidden Jordan algebra structure

Theorem D

Let V be a Hsiang algebra and $c \in \mathrm{Idm}(V)$. Let $\Lambda_c = \mathbb{R}c \oplus V_c(-\frac{1}{2})$ be an **isotope** with the new multiplication

$$x \bullet y = \frac{1}{2}xy + \langle x, c\rangle y + \langle y, c\rangle x - 2\langle xy, c\rangle c, \qquad x, y \in \mathbb{R}c \oplus V_c(-\tfrac{1}{2}), \tag{10}$$

Then Λ_c is **a rank 3 Jordan algebra** with e_c being the unit and the trace bilinear trace form $T(x; y) = \langle x, y\rangle$ (associative with respect to $\bullet$). In particular, the Jordan algebra Λ_c is Euclidean and any element $x \in \Lambda_c$ satisfies the cubic identity

$$x^{\bullet 3} - 2\langle x, c\rangle x^{\bullet 2} + (2\langle x, c\rangle^2 - \tfrac{1}{2}|x|^2)x - \tfrac{1}{3}\langle x, x^2\rangle c = 0. \tag{11}$$

Furthermore, $n_1 - 1 \le \rho(n_1 + n_2 - 1)$, where ρ is the Hurwitz-Radon function. In particular, for each n_2 **there exist only finitely many possible values of** n_1.

Theorem E

The following conditions are equivalent:

1. A Hsiang algebra V is exceptional
2. The Jordan algebra Λ_c is simple for some c
3. The Jordan algebra Λ_c is simple for all c
4. The quadratic form $x \to \operatorname{tr} L_x^2$ has a single eigenvalue and $n_2(V) \neq 2$

The finiteness of exceptional Hsiang algebras

Theorem F

There are at most 24 classes of exceptional Hsiang algebras. For any such an algebras $n_2 \in \{0, 5, 8, 14, 26\}$ and the possible corresponding Peirce dimensions are

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

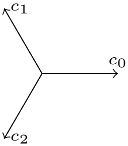
The cells in blue color represent non-realizable Peirce dimensions and the cells in gold color represent unsettled cases

- If $n_2 = 0$ then $n_2 \in \{2, 5, 8, 14, 26\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $u = \frac{1}{6}\langle z, z^2\rangle$, $V = \mathscr{H}_3(\mathbb{K}_d) \ominus \mathbb{R}e$, $d = 0, 1, 2, 4, 8$.
- If $n_1 = 0$ then $n_2 \in \{5, 8, 14\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $\frac{1}{12}\langle z^2, 3\bar{z} - z\rangle$, where $z \to \bar{z}$ is the natural involution on $V = \mathscr{H}_3(\mathbb{K}_d)$, $d = 2, 4, 8$.
- If $n_1 = 1$ then $n_2 \in \{5, 8, 14, 26\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $u(z) = \mathrm{Re}\langle z, z^2\rangle$, where $z \in V = \mathscr{H}_3(\mathbb{K}_d) \otimes \mathbb{C}$, $d = 1, 2, 4, 8$.
- If $(n_1, n_2) = (4, 5)$ then $V = V^{\mathrm{FS}}(u)$, $u = \frac{1}{6}\langle z, z^2\rangle$ on $\mathscr{H}_3(\mathbb{K}_8) \ominus \mathscr{H}_3(\mathbb{K}_1)$

Two basic examples in dimensions 2 and 3

Example 1. Let V be the 2 dimensional algebra generated by three **idempotents** c_i, $i = 0, 1, 2$ which can be realized as unit vectors in $\mathbb{R}^2$ subject to the conditions:

- $\langle c_i, c_j \rangle = -\frac{1}{2}$, $i \neq j$,
- $c_0 + c_1 + c_2 = 0$



Then for any triple $\{i, j, k\} = \{1, 2, 3\}$ we have

$$c_k = c_k^2 = (-c_i - c_j)^2 = c_i + c_j + 2c_i c_j = -c_k + 2c_i c_j$$

hence $c_i c_j = c_k$ and $c_k(c_i - c_j) = -(c_i - c_j)$. This implies $V = V_{c_i}(1) \oplus V_{c_i}(-1)$, the both Peirce subspaces being 1-dimensional. The corresponding fusion rules are

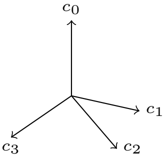
$\star$	1	-1
1	1	-1
-1		1

The Peirce dimensions are $n_1 = 1, n_2 = n_3 = 0$, the ambient dimension $n = 2$.
The minimal cone is given by $x_1^3 - 3x_1 x_2^2 = 0$.

Two basic examples in dimensions 2 and 3

Example 2. Similarly, let V be the 3 dimensional algebra generated by four **idempotents** c_i, $i = 0, 1, 2, 3$ realized as unit vectors in $\mathbb{R}^3$ subject to the conditions:

- $c_i + c_j$ is a 2-nilpotent, i.e. $(c_i + c_j)^2 = 0$ $(i \neq j)$



Then similarly to the above, one easily verifies that

$$V = V_{c_i}(1) \oplus V_{c_i}(-\tfrac{1}{2}),$$

where $\dim V_{c_i}(1) = 1$ and $\dim V_{c_i}(-\frac{1}{2}) = n_2 = 2$.

The corresponding fusion rules are

$\star$	1	$-\frac{1}{2}$
1	1	$-\frac{1}{2}$
$-\frac{1}{2}$		$1, -\frac{1}{2}$

The underlying algebra structure after a 1-rank perturbation becomes a Jordan algebra of Clifford type. The minimal cone is given by $x_1 x_2 x_3 = 0$, i.e. the triple of coordinate planes in $\mathbb{R}^3$.

Спасибо за внимание!

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