# Metrized algebras with involutions and their applications 

(a true story based on a joint work with Daniel Fox, Universidad Politécnica de Madrid)

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Dedicated to Yakov Krasnov (1952-2023)

## Outline

(1) Introduction and motivations
(2) Algebras with involution and associative bilinear form
(3) Quasicomposition algebras and triple algebras

4 The degeneracy index and the triad principle
(5) A hidden Hurwitz algebra structure

6 The Petterson autotopy and the classification of 3D QC-algebras

## Outline

（1）Introduction and motivations

2）Algebras with involution and associative bilinear form
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（5）A hidden Hurwitz algebra structure
（5）The Petterson autotopy and the classification of 3D QC－algebras

Irregular objects: do not follow a simple (regular) rule but very relevant (frequently used).

The most languages (not all! only exceptional or regular?) contain two group of verbs: regular and irregular (with an unpredictable tense forms).

$$
\#(\text { irregular verbs })=o(\# \text { regular verbs }), \quad \text { BUT! }
$$

$$
\text { frequency of irregular verbs } \gg \text { that of regular verbs }
$$

Compare, for example with "exceptional mathematical results" vs "general mathematical results" etc.

There are many irregular (exceptional or sporadic) objects, among others (in historical order):

- Platonic solids in $\mathbb{R}^{3}$ (absent in $\mathbb{R}^{2}$ ): the tetrahedron, cube, octahedron, dodecahedron, and icosahedron.
- Real (normed) division algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$
- Exceptional simple Lie algebras over $\mathbb{C}: \mathfrak{g}_{2}, \mathfrak{f}_{4}, \mathfrak{e}_{6}, \mathfrak{e}_{7}$ and $\mathfrak{e}_{8}$
- Exceptional Jordan algebra $\mathbb{R}$ (the 27-dimensional Albert algebra).
- Cartan isoparametric hypersurfaces in Euclidean spheres (only for an ambient curvature +1 )
- Sporadic finite simple groups (totally 26 )

Remarks.
a) Sometimes, there are relevant borderline cases, or mutants, i.e. regular objects which share some important properties with irregular ones (for example $\mathfrak{s o}(3), \mathfrak{s u}(3), \mathfrak{s o}(12)$ or $\mathfrak{s u}(6)$ )
b) There is no a unified conceptual explanation/classification for irregular objects, their construction is individual or archaic (Yu.Manin).

## Episode 1: Isoparametric hypersurfaces

Start with a classical variational problem: the light propagation on a sphere (the case of Euclidean space is trivial). This leads to a PDE system for the front-hypersurfaces. E. Cartan (1938) proved that the shape operator of each regular level hypersurface has constant principal eigenvalues (curvatures) counting multiplicities. Moreover, if $m \geq 1$ is the number of distinct curvatures then $M$ is a level-set of a homogeneous polynomial solution in $\mathbb{R}^{n}$

$$
\begin{aligned}
\|\nabla P(x)\|^{2} & =n^{2}\langle x ; x\rangle^{2 m-2}, \\
\Delta P(x) & =C\langle x ; x\rangle^{3 m-4}, \quad \operatorname{deg} P(x)=m .
\end{aligned}
$$

Cartan completely characterized the cases $\operatorname{deg} P(x) \leq 3$. In fact, he proves that the degree 3 homogeneous polynomial solutions are exactly the triality polynomials:

$$
P\left(x_{1}, x_{2}, x_{3}\right)=\operatorname{Re}\left(x_{1} \cdot\left(x_{2} \cdot x_{3}\right)\right),
$$

where $x_{i} \in \mathbf{K}_{d}, \mathbf{K}_{d}$ is a normed real division algebra, $d=1,2,4,8$.
Münzner (1980) used algebraic topology to express certain $\mathbb{Z}_{2^{-}}$ cohomology ring of $M$ to prove that
$m=\operatorname{deg} P(x) \in\{1,2,3,4,6\} \quad$ cf. the "plane regular tessellations"

## Eille Cartan (a Faris).

Dans un article récent ${ }^{1}$ ) j'ai signalé l'existence dans l'espace sphérique $^{\text {a }}$ à quatre dimensions d'une famille d'hypersurfaces isoparamétriques à trois oourbures principales distinctes. Je me propose de rechercher s'il existe, dans un espace sphérique à un nombre queloonque de dimensions, des familles d'hypersurfaces isoparamétriques admettant seulement trois courbures principales distinctes. Dans la première partie de ce Mémoire, je montrerai l'existence de telles familles dans les espaces a $4,7,13$ et 25 dimensions; cette existence sera liée à celle d'un polynome homogène du troisième degré à $n+2$ variables ( $n+1$ étant la dimension de l'espace sphérique) jouissant de la double propriété que son premier paramètre différentiel, calculé dans l'espace euclidien à $n+2$ dimensions de coordonnées rectangulaires $x_{2}, \ldots$, $x_{0+2}$, soit constant sur l'hypersphère de reyon. 1 et que son second paramètre différentiel soit nul (polynome harmonique). De tels polynomes n'existent que pour $n-3,6,12,24$ ou $n=3.2^{k}(k-0,1,2,3)$. Le cas $n=24$ est particulièrement intéressant parce qu'il est lié à différentes théories (théorie des spineurs, principe de trialité dans l'espace elliptique à 7 dimensions) et qu'il fournit la première apparition dans un problème de Géométrie (et même d'Analyse) du groupe simple à 52 paramètres qui ne rentre dans aucune des grandes classes de groupes simples.

## E. Cartan.

Dans l'espace euclidien à 26 dimensions l'espace euclidien à 24 dimensions tangent au point $P(u=\cos t, v=\sin t, X=Y=Z=0)$ à l'hypersurface de paramètre $t$ situé dans l'hypersphère de rayon 1 se décompose en trois sous-espaces à 8 dimensions, celui des vecteurs $X$, des vecteurs $Y$ et des vecteurs $Z$. Si nous portons notre attention sur ce dernier, les vecteurs $X$ du premier peuvent être regardés comme les semi-spineurs de première espèce réels et les vecteurs $Y$ du second comme les semi-spineurs de seconde espèce réels de l'espace des $Z^{15}$ ). Le groupe $G_{2}$ indique comment le groupe des rotations de l'espace à 8 dimensions transforme les vecteurs réels $Z$, les semi-spineurs de première espèce réels $X$ et les semi-spineurs de seconde espèce $Y$. Le principe de trialite ${ }^{10}$ ) de $l$ 'espace elliptique d 7 dimensions est ainsi mis en évidence d'une manilere concrète.
E. Cartan, Sur des familles remarquables d'hypersurfaces isoparamétriques ....

Math. Z. (1939)

## Episode 2: Algebraic minimal cones



Wu-yi Hsiang (born 1937)
(ii) Partly due to the lack of "canonical" normal forms for $r<2$ and partly due to the rapid rate of increase of the dimension of $\oint_{n}^{r}$ with respect to $r$, the little help obtained from the normal forms is not enough to solve the problem of classifying minimal algebraic cones of higher degrees. For example, it is very difficult to solve even the following very special equation: $F(x)=0$, where $F(x)$ is an irreducible cubic form in $n$ variables such that

$$
\left.(\Delta F) \cdot \nabla F\right|^{2}-\nabla F \cdot \boldsymbol{H F} \cdot \nabla F^{\prime}= \pm\left(x_{i}^{2}+\cdots+x_{2}^{2}\right) \cdot F .
$$

Since the above equation is invariant with respect to the orthogonal linear substitutions, we may assume that $F$ is given in some kind of "normal form" which amounts to reduce the number of indeterminant coefficients by $n(n-1) / 2$. A systematic attempt to solve the above equation will involve the job of solving over-determined simultaneous algebraic equations of many variables. So far, we have only four non-trivial solutions (cf. $\S \S 1,2$ ), but there is no reason why there should be no others.

- W.-Y. Hsiang (J. Diff. Geometry, 1967): Given a homogeneous polynomial $u(x), x \in \mathbb{R}^{n}$, the cone $u^{-1}(0)$ is a minimal hypersurface in $\mathbb{R}^{n}$ iff

$$
\begin{equation*}
\left.\Delta_{1} u:=|\nabla u|^{2} \Delta u-\left.\frac{1}{2}\langle\nabla u ; \nabla| \nabla u\right|^{2}\right\rangle=Q(x) u(x) \tag{1}
\end{equation*}
$$

- The first non-trivial case: $\operatorname{deg} u=3$. All known irreducible cubic minimal cones satisfy

$$
\begin{equation*}
\Delta_{1} u=\|x\|^{2} \cdot u(x) \tag{*}
\end{equation*}
$$

- Hisang's Problem (ii): Classify all cubic solutions of (*).
- (V.T.,2010-2014): There are three distinguished families of solutions:
(a) of Clifford type (infinitely many, almost in any dimension $n \geq 3$ )
(b) Cartan's isosparametric solutions in dimensions $5,8,14,26$ (i.e. $3 d+2)$
(c) exceptional eigencubics which may exist only in the following dimensions:

| Hsi.dim $=3 n$ | 3 | 6 | 12 | 24 | 9 | 12 | 15 | 21 | 15 | 18 | 21 | 24 | 30 | 42 | 27 | 30 | 33 | 36 | 51 | 54 | 57 | 60 | 72 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n$ | 1 | 2 | 4 | 8 | 3 | 4 | 5 | 7 | 5 | 6 | 7 | 8 | 10 | 14 | 9 | 10 | 11 | 12 | 17 | 18 | 19 | 20 | 24 |
| $n_{1}=n-2 d-1$ | 0 | 1 | 3 | 7 | 0 | 1 | 2 | 4 | 0 | 1 | 2 | 3 | 5 | 9 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 7 |
| $d$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 4 | 4 | 4 | 4 | 8 | 8 | 8 | 8 | 8 |

The last line: $d=1,2,4,8$ are the dimensions of real division algebras

## Episode 3: Truly $C^{1, \alpha}$ viscosity solutions

## Another very strange appearance:

There are important classes of fully nonlinear Dirichlet problems for which the viscosity solution is in fact a classical one, e.g. due to KrylovEvans regularity theory, in the case when the function $F$ is convex. (see $[\mathrm{CC}],[\mathrm{K}]$ ). However, for the general $F$ the problem of the coincidence of viscosity solutions with the classical solutions remained open.

The central result of this paper is the existence of a nonclassical viscosity solution of (1) in dimension 12. More precisely we prove
Theorem. The function

$$
w(x):=\frac{\operatorname{Re}\left(\omega_{1} \omega_{2} \omega_{3}\right)}{|x|}
$$

where $\omega_{j} \in \mathbf{H}, i=1,2,3$, are Hamiltonian quateraions, $x=\left(\omega_{1}, \omega_{2}, \omega_{3}\right) \in \mathbf{H}^{3}$ $=\mathbf{R}^{12}$ is a viscosity solution in $\mathbf{R}^{12}$ of a uniformly elliptic equation (1) with a smooth $F$.

One can find the explicit expression for $w$ in the coordinates of $\mathbf{R}^{12}$ in sections 3 and 4. The elliptic operator $F$ will be defined in a constructive way in section 2 , and its ellipticity constant $\mathrm{A}<10^{8}$.

As an immediate consequence of the theorem we have
but loses the uniform ellipticlty in a neighborhood of the subset of $S_{1}^{5}$ formed by the points with $\left|z_{1}\right|=\left|z_{2}\right|=\left|z_{3}\right|, \operatorname{Re}\left(z_{1} z_{2} z_{3}\right)=0$,

To explain why $P_{6}$ does not work and $P_{12}$ does work we give in the next section a short excursion in the area of division algebras and exceptional Lie groups. That will lead us also to varlous extensions of Theorem 3.1.
4. Trialities, Quaternions, Octonions and

## Hessian Equations

As we have seen in the previous section, cubic forms for which the quadratic form $P_{d}$ verifies the inequalities (3.2) or (3.3) should be rather exceptional. In fact all examples of such forms known to us come from trialities, which in turn are intimately related to division algebras and exceptional Lie groups. Let us recall some of their elementary properties $[1,3]$.

Duality is ubiquons in algebra; triality is similar, but subtler. For two real vector spaces $V_{1}$ and $V_{2}$, a duality is simply a nondegenerate bilinear map

$$
f: V_{1} \times V_{2} \longrightarrow \mathbf{R}
$$

Similarly, for three real vector spaces $V_{1}, V_{2}$, and $V_{3}$, a triality is a trilinear map

$$
t: V_{1} \times V_{2} \times V_{3} \longrightarrow \mathbf{R}
$$

Proceedings of the International Congress of Mathematicians
Hyderabad, India, 2010

## Weak Solutions of Nonvariational Elliptic Equations

Nikolai Nadirashvili* and Serge Vlǎduţ ${ }^{\dagger}$
equation in those dimensions [28,30]. Moreover, one can formulate a test similar to the second part of Lemma 4.2 which garanties that $w_{6}$ is a solution to an Isaacs equation.

In this way we get the following
1). For any $\delta, 1 \leq \delta<2$ and any plane $H^{\prime} \subset \mathrm{R}^{24}$, $\mathrm{dim} H^{\prime}=21$ the function

$$
\left(P_{24}(x) /|x|^{\delta}\right)_{\mid B^{\prime}}
$$

is a viscosity solution to a uniformly elliptic Hessian (1.2) in the unit ball $B \subset \mathbf{R}^{21}$.
2). For any $\delta, 1 \leq \delta<2$ the function

$$
w_{12, \delta}=P_{12}(x) /|x|^{\delta}
$$

is a viscosity solution to a uniformly elliptic Hessian equation (1.2) in the unit ball $B \subset \mathbf{R}^{12}$.
3). For any hyperplane $H \subset \mathbf{R}^{12}$ the function

$$
\left(P_{12}(x) /|x|\right)_{\mid H}
$$

is a viscosity solution to a uniformly elliptic Hessian equation (1.2) in the unit ball $B \subset \mathbf{R}^{11}$.
4). For any $\delta, 1 \leq \delta<2$ the function

$$
w_{12, \delta}=P_{12}(x) /|x|^{\delta}
$$

is a viscosity solution to Isaacs equation (1.8) in the unit ball $B \subset \mathbf{R}^{12}$

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## Algebras with involution and associative bilinear form

Let $(\mathbb{A}, \circ, \sigma)$ be an algebra of dimension $n<\infty$ over a field $\mathbf{K}$, and an involution $\sigma$, i.e. $\sigma$ is an involutive anti-automorphism:

$$
\left(x^{\sigma}\right)^{\sigma}=x, \quad(x \circ y)^{\sigma}=y^{\sigma} \circ x^{\sigma} \quad \Leftrightarrow \quad \sigma L_{\circ}(x)=R_{\circ}\left(x^{\sigma}\right) \sigma
$$

(1) Let $(\mathbb{A}, \sigma)$ be an algebra with involution. If $\sigma=\mathbb{1}$ then $\mathbb{A}$ is commutative, and if $\sigma=-\mathbb{1}$ then $\mathbb{A}$ is anticommutative. (The converse is not true! examples on the next slide)
(2) If $\mathbb{A}$ is commutative (resp. anti-commutative) then $\sigma=\mathbb{1}_{\mathbb{A}}\left(\right.$ resp. $\left.\sigma=-\mathbb{1}_{\mathbb{A}}\right)$ is the standard involutions.
(3) A symmetric bilinear form $h: \mathbb{A} \times \mathbb{A} \rightarrow \mathbf{K}$ is called involutively invariant if

$$
h\left(x^{\sigma}, y^{\sigma}\right)=h(x, y), \quad h(x y, z)=h\left(y, x^{\sigma} z\right)
$$

(4) An algebra $(\mathbb{A}, \sigma, h)$ with nondegenerate involutively invariant $h$ is called metrized. If $h$ additionally is positive definite then $\mathbb{A}$ is Euclidean.
(5) An algebra $\mathbb{A}$ is called Killing metrized if its Killing form

$$
\tau(x, y):=\operatorname{tr} L_{\circ}(x) L_{\circ}(y)=\operatorname{tr} R_{\circ}(x) R_{\circ}(y)
$$

is well-defined and it is a nondegenerate involutively invatiant bilinear form.

## Lemma 2.1

If $h$ is involutively invariant then $L_{\circ}(x)^{*}=L_{\circ}\left(x^{\sigma}\right)$ and $R_{\circ}(y)^{*}=R_{\circ}\left(y^{\sigma}\right)$, in other words

$$
h(x y, z)=h\left(y, x^{\sigma} z\right)=h\left(y^{\sigma}, z^{\sigma} x\right)=h\left(z y^{\sigma}, x\right)=h\left(x, z y^{\sigma}\right) .
$$

Furthermore, in any metrized algebra, its Killing form is well-defined

$$
\operatorname{tr} L_{\circ}(x) L_{\circ}(y)=\operatorname{tr} L_{\circ}(x) \sigma R_{\circ}\left(x^{\sigma}\right) \sigma=\operatorname{tr} \sigma L_{\circ}(x) \sigma R_{\circ}\left(x^{\sigma}\right)=\operatorname{tr} R_{\circ}(x) R_{\circ}(y)
$$

## Algebras with involution and associative bilinear form

Below I give examples of standard and non-standard involutions for (anti-)commutative involutively metrized algebras:

## Example 2.2

(1) Consider a commutative algebra $\overline{\mathbb{C}}$ of para-complex numbers: $z \bullet w=\bar{z} \bar{w}$ with the standard involution $\sigma(z)=\bar{z}$. Then

$$
(z \bullet w)^{\sigma}=\overline{\bar{z} \bar{w}}=z w=w^{\sigma} \bullet z^{\sigma}
$$

Notice that $\sigma \neq \mathbb{1}$. The bilinear form $h(x, y)=\operatorname{Re}\left(x \cdot y^{\sigma}\right)$ is involutively invariant.
(2) The cross-product algebra $\mathfrak{s o}(3, \mathbf{K})$ with the standard involution $\sigma(x)=x$ and the Killing form $h(x, y)$.
(3) Define $\mathfrak{G}_{0}$ as the algebra with basis $\left(e_{1}, e_{2}, e_{3}\right)$ and the anticommutative $\bullet$-multiplication


Let $\alpha\left(e_{i}\right)=e_{i}, i=1,2$ and $\alpha\left(e_{3}\right)=-e_{3}$, then $(\alpha, \alpha, 1): \mathfrak{g}_{0} \rightarrow \mathfrak{s o}(3, \mathbf{K})$ is a principal isotopy, i.e.

$$
x \bullet y=\alpha(x) \times \alpha(y)
$$

Both $\mathfrak{s o}(3, \mathbf{K})$ and $\mathfrak{G}_{0}$ are Killing metrized algebras.

## Composition algebras

(1) An algebra $(\mathbb{A}, \circ)$ is a division algebra if $L_{\circ}(x)$ and $R_{\circ}(x)$ are invertible operators for all $0 \neq x \in \mathbb{A}$.
(2) An algebra $(\mathbb{A}, \circ)$ over $\mathbf{K}$ is a composition algebra if there exists a nondegenerate quadratic form called the norm $n: \mathbb{A} \rightarrow \mathbf{K}$ such that

$$
n(x \circ y)=n(x) n(y) .
$$

$n(x, y)=n(x+y)-n(x)-n(y)$ is the corresponding symmetric bilinear form.
(3) Unital composition algebras are called Hurwitz algebras.
(4) In any Hurwitz algebra with unit $e$, the endomorphism $x^{\sigma}=n(x, e) x-x$ is an involution.
(5) A unital algebra is quadratic if $e, x, x \circ x$ are linearly dependent. Any Hurwitz algebra is quadratic.
(6) Any Hurwitz algebra is isomorphic to one of the following (by the Cayley-Dickson doubling process):
(1D) The base field $\mathbf{K}$;
(2D) Generalized complex numbers $C(\alpha):=\operatorname{Dickson}(\mathbf{K}, \alpha)$
(4D) Generalized quaternions $H(\alpha, \beta):=\operatorname{Dickson}(C(\alpha), \beta)$
(8D) Generalized octonions Dickson $(H(\alpha, \beta), \gamma)$.
(7) Given $(\mathbb{A}, \circ, n, \sigma)$, its para-Hurwitz algebra $\overline{\mathbb{A}}$ w.r.t. $x \bullet y=x^{\sigma} \circ y^{\sigma}$ (obs. 1 acts as a para-unit, Elduque, 1996).
(8) A symmetric composition algebra if the norm is "associative" (Petersson, Okubo, Elduque, Myung, Osborn, Faulkner):

$$
n(x \circ y, z)=n(x, y \circ z),
$$

(9) Any Hurwitz or symmetric composition algebra is metized w.r.t. the standard involution and $h=n$.
(10) In any Hurwitz algebra, $x^{\sigma} \circ(x \circ y)=n(x, x) y$.

## Composition formulas

A composition formula of size $[r ; s ; n]$ is a formula (over a field) of the type

$$
\left(x_{1}^{2}+x_{2}^{2}+\ldots+x_{r}^{2}\right)\left(y_{1}^{2}+y_{2}^{2}+\ldots+y_{s}^{2}\right)=\left(z_{1}(x, y)^{2}+\ldots+z_{n}(x, y)^{2}\right)
$$

A composition formula of size $[r ; s ; n]$ exists iff there are $n \times s$-matrices $A_{1}, \ldots A_{r}$ over $\mathbf{K}$ satisfying

$$
A_{i}^{t} A_{j}+A_{j}^{t} A_{i}=2 \delta_{i j} \cdot \mathbb{1}_{s}, \quad 0 \leq i, j \leq r
$$

(1) (Hurwitz, 1898) A composition of size $[n ; n ; n]$ implies $n=1,2,4$, or 8 .
(2) (Radon 1922, Hurwitz 1923) A composition of size $[r ; n ; n]$ exists iff $r \leq \rho(n)$, where

$$
\rho\left(2^{s} \cdot \text { odd }\right)=8 a+2^{b}, \quad \text { where } s=4 a+b, \quad 0 \leq b \leq 3
$$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\ldots$ | 16 | $\ldots$ | 32 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\rho(n)$ | 1 | 2 | 1 | 4 | 1 | 2 | 1 | 8 | $\ldots$ | 9 | $\ldots$ | 10 |

(3) $\rho(n)=$ the maximum number of linearly independent vector fields of any homotopy $n$-sphere.
(4) the set of matrices as above generates a symmetric Clifford system, i.e. $A_{x}^{t} A_{x}=\langle x ; x\rangle I, A_{x}:=\sum x_{i} A_{i}$.

## Lemma 2.3

Let $\mathbb{A}$ be an Euclidean involutive metrizied algebra, $f(x): B \rightarrow B^{\prime}$ be a linear homomorphism, $B, B^{\prime} \subset \mathbb{A}, f(x)$ depends linearly on $x \in C \subset \mathbb{A}$. Suppose that for all $x \in C$

$$
\begin{align*}
& B \xrightarrow{f(x)} B^{\prime} \xrightarrow{f^{*}(x)} B,  \tag{2}\\
& f^{*}(x) f(x)=h(x, x) \mathbb{1}_{B}, \tag{3}
\end{align*}
$$

then there exists a composition formula of size $\left[\operatorname{dim} C, \operatorname{dim} B, \operatorname{dim} B^{\prime}\right]$. If $\operatorname{dim} B=\operatorname{dim} B^{\prime}$ then $\operatorname{dim} C \leq \rho(\operatorname{dim} B)$.

## Constant rank algebras

An algebra $\mathbb{A}$ is said to satisfy the constant rank condition if the dimension of degeneracy of multiplication

$$
d(\mathbb{A}):=\operatorname{dim} \operatorname{ker} L(x)=\operatorname{dim} \operatorname{ker} R(x) \text { independently of a nonzero } x \in \mathbb{A}
$$

The classically known example is the class of division algebras: $d(\mathbb{A})=0$.
(1) (K.O. May, 1966) Originally, the question about the existence of division algebras behind dimension 2 was posed by Gauss in 1831: " The writer has reserved for himself . . . the question why the relations between things that make up a manifold of more than two dimensions cannot provide quantities admissible in universal arithmetic."
(2) (Hamilton, Cayley, Frobenious, Radon, Hurwitz) Several classical results under additional assumptions like the existence of associativity, composition law etc.
(3) Cartan's and Study's remark of 1908 ". . . a definitive answer, if one exists, can only be given by the whole ulterior development of algebra and analysis."
(4) (Bott, Milnor 1958) Any division algebra over the real numbers has dimension $\operatorname{dim} \mathbb{A}=1,2,4,8$.
(5) (Gabriel 1994) A division algebra over an algebraically closed field must be one dimensional.

Proof. Let $x, y \neq 0$ be non-proportional vectors in $\mathbb{A}$. Then the linear operator $L_{\circ}(x)^{-1} L_{\circ}(y)$ has an eigenvalue, say $\lambda \in \mathbf{K}$, therefore $L_{\circ}(x)^{-1} L_{\circ}(y) z=\lambda z, z \neq 0$, implying $\lambda x \circ z=y \circ z$, therefore $(\lambda x-y) \circ z=0$, a contradiction.

Unfortunately, the case $d(\mathbb{A}) \geq 1$ is almost unexplored even over $\mathbb{R}$. For example, what can be said in the case $d(\mathbb{A})=1$ ? Do there exist some distinguished properties in this case? Which possible values of $d(\mathbb{A})$ are possible in general?

Below we provide an extra motivation why the constant rank condition can be interesting.

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## Quasicomposition algebras

Definition. A metrized algebra $(\mathbb{A}, \sigma, h, \circ)$ is called a (QC) quasicomposition algebra if

$$
x \circ\left(x^{\sigma} \circ(x \circ y)\right)=h(x, x)(x \circ y), \quad \forall x, y \in \mathbb{A},
$$

holds for any $x \in \mathbb{A}$, or equivalently

$$
L_{\circ}(x) L_{\circ}\left(x^{\sigma}\right) L_{\circ}(x)=h(x, x) L_{\circ}(x)
$$

Under the metrized algebra assumption, the latter is equivalent to $R_{\circ}(x) R_{\circ}\left(x^{\sigma}\right) R_{\circ}(x)=h(x, x) R_{\circ}(x)$.
(1) The name is motivated by the observation that the quadratic form $n(x)=h(x, x)$ satisfies the quasicomposition property

$$
\begin{aligned}
n\left(x^{\sigma} \circ(x \circ y)\right) & =h\left(x^{\sigma} \circ(x \circ y), x^{\sigma} \circ(x \circ y)\right)=h\left(x \circ y, x \circ\left(x^{\sigma} \circ(x \circ y)\right)\right) \\
& =h(x, x) h(x \circ y, x \circ y)=n\left(x^{\sigma}\right) n(x \circ y)=n(x) n(x \circ y)
\end{aligned}
$$

(2) A Hurwitz algebra with its standard involution is a quasicomposition algebra: indeed, by (10)

$$
x^{\sigma} \circ(x \circ y)=n(x, x) y \quad \Rightarrow \quad x \circ\left(x^{\sigma} \circ(x \circ y)\right)=n(x, x)(x \circ y)
$$

(3) A semigroup context. Recall that if $S$ is a semigroup, then $a \in S$ is (Von Neumann) regular if there exists a generalized inverse of $a$, i.e. $\exists b \in S: a b a=a$ and $b a b=b$.
Thus, the definition of a quasicomposition algebra can be restated as: for every $h$-anisotropic $x \in \mathbb{A}, h(x, x)^{-1} L_{\circ}\left(x^{\sigma}\right)$ is the generalized inverse of $L_{\circ}(x)$ in the semigroup generated by $L_{\circ}(\mathbb{A}) \subset \operatorname{End}(\mathbb{A})$
4) Metrized anti-commutative algebras $\mathbb{A}$ were classified by Elduque in 1988 for the standard involution $\sigma=-\mathbb{1}$ : any such (non-zero) algebra over an algebraically closed field is either of the following: 1) $\mathfrak{s l}(2, \mathbf{K}), 2) \mathfrak{p s l}(3, \mathbf{K})$, char $\mathbf{K}=3,3)$ a simple non-Lie Maltsev algebra or 4) the anricommutative algebra of the vectors in the color algebra
(5) But originally QC-algebras appear in the context of exceptional Hsiang algebras of cubic minimal cones, see the next slide.

## Theorem (V.T.,2012-2016)

- There is a natural one-to-one correspondence between solution to Hsiang's equation $\Delta_{1} u=\langle x ; x\rangle u(x)$ in $\mathbb{R}^{n}$ and commutative metrized algebras $\mathcal{H}=\left(\mathbb{R}^{n}, \cdot, 1,\langle;\rangle\right)$, where $u(x)=\frac{1}{6}\left\langle x ; x^{2}\right\rangle, \operatorname{tr} L(x)=0$ and

$$
\langle x x ; x(x x)\rangle=k\langle x ; x\rangle\left\langle x^{2} ; x\right\rangle \quad \Leftrightarrow \quad x x^{3}+\frac{1}{4}(x x)(x x)-\langle x ; x\rangle(x x)-\frac{2}{3}\langle x x ; x\rangle x=0
$$

- Any Hsiang algebra $\mathcal{H}$ is either isomorphic to a polar algebra (i.e. a commutative metrised $\mathbb{Z}_{2}$-graded algebra $\mathbb{A}=\mathbb{A}_{0} \oplus \mathbb{A}_{1}$ such that $\mathbb{A}_{0} \mathbb{A}_{0}=\{0\}$ and $x_{0}\left(x_{0} x_{1}\right)=h\left(x_{0}, x_{0}\right) x_{1}$ for all $\left.x_{i} \in \mathbb{A}_{i}, i=0,1\right)$ or exceptional
- The class of polar algebras is in a natural 1-to-1 correspondence with Clifford summetric systems, well-understood.
- The set of idempotents in $\mathcal{H}$ is nonempty. For any idempotent $c$, the associated Peirce decomposition is

$$
\mathbb{A}=\mathbb{A}_{c}(1) \oplus \mathbb{A}_{c}(-1) \oplus \mathbb{A}_{c}\left(-\frac{1}{2}\right) \oplus \mathbb{A}_{c}\left(\frac{1}{2}\right) \quad \text { and } \quad \operatorname{dim} \mathbb{A}_{c}(1)=1
$$

- $\mathbb{A}_{c}(1) \oplus \mathbb{A}_{c}(-1)$ is a subalgebra and its is isotopic to a Clifford type Jorand algebra;
- $\mathbb{A}_{c}(1) \oplus \mathbb{A}_{c}\left(-\frac{1}{2}\right)$ is a subalgebra and it is naturally isotopic to a rank 3 Jordan algebra structure, $n_{2}=\operatorname{dim} \mathbb{A}_{c}\left(-\frac{1}{2}\right)$.
- $\mathbb{A}$ is exceptional if and only if $\mathbb{A}_{c}(1) \oplus \mathbb{A}_{c}\left(-\frac{1}{2}\right)$ is (isotopy of) a simple Jordan algebra. In this case, either $n_{2}=0$ or $n_{2}=3 \mathbf{d}+2$ and the hidden simple Jordan algebra is $\operatorname{Herm}_{3}\left(\mathbf{F}_{\mathbf{d}}\right), \mathbf{d} \in\{1,2,4,8\}$.
- $\mathbb{A}$ is mutant iff $n_{2}=2$, this corresponds to $\mathbf{d}=0$. $\mathbb{A}$ is exceptional or mutant iff $\operatorname{tr} L(x)^{2}=m\langle x ; x\rangle$ for some real $m$. In this case, $m=2\left(n_{1}+\mathbf{d}+1\right)$.
- There are finitely many dimensions $n$ of $\mathbb{A}$ where exceptional Hsiang algebras can exist. Except the case $n_{2}=0$, in all other cases, $\operatorname{dim} \mathbb{A}=3\left(n_{1}+2 \mathbf{d}+1\right)$, where $\operatorname{dim} \mathbb{A}_{c}\left(-\frac{1}{2}\right)=3 \mathbf{d}+2, \mathbf{d} \in\{0,1,2,4,8\}$.

| $n$ | 2 | 5 | 8 | 14 | 26 | 9 | 12 | 15 | 21 | 15 | 18 | 21 | 24 | 30 | 42 | 27 | 30 | 33 | 36 | 51 | 54 | 57 | 60 | 72 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n_{1}$ | 1 | 2 | 3 | 5 | 9 | 0 | 1 | 2 | 4 | 0 | 1 | 2 | 3 | 5 | 9 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 7 |
| $n_{2}$ | 0 | 0 | 0 | 0 | 0 | 5 | 5 | 5 | 5 | 8 | 8 | 8 | 8 | 8 | 8 | 14 | 14 | 14 | 14 | 26 | 26 | 26 | 26 | 26 |



$$
V=\underbrace{\mathbb{R}\left(x_{\alpha}, w_{\alpha}\right)}_{-1,1} \oplus \underbrace{\mathbb{R}\left(w_{\beta}, w_{\gamma}\right)}_{0^{2}} \oplus(\underbrace{S_{\beta} \oplus S_{\gamma}}_{-\frac{1}{\sqrt{2}}^{d}, \frac{1}{\sqrt{2}}^{d}}) \oplus(\underbrace{S_{-\beta} \oplus S_{-\gamma}}_{-\frac{1}{\sqrt{2}}^{d}, \frac{1}{\sqrt{2}^{d}}}) \oplus(\underbrace{\left(S_{\alpha} \cap x_{\alpha}^{\perp}\right) \oplus D_{-\alpha}}_{0^{d+n_{1}-1}}) \oplus(\underbrace{M_{\alpha} \oplus D_{\alpha} \oplus S_{-\alpha}}_{-1^{n_{1}, 1^{n_{1}}, 0^{d}}}),
$$

The QC algebras birthday: March 23, 2022


[^0]
## The triple of an algebra

Let $(\mathbb{A}, \sigma, h, \circ)$ be a metrized algebra. Define a commutative algebra structure on $\mathbb{A} \times \mathbb{A} \times \mathbb{A}$ (the triple) by virtue of

$$
\mathbf{x} \circledast \mathbf{y}=\left(x_{1}, x_{2}, x_{3}\right) \circledast\left(y_{1}, y_{2}, y_{3}\right):=\left(x_{3}^{\sigma} \circ y_{2}^{\sigma}+y_{2}^{\sigma} \circ x_{2}^{\sigma}, x_{1}^{\sigma} \circ y_{3}^{\sigma}+y_{1}^{\sigma} \circ x_{3}^{\sigma}, y_{2}^{\sigma} \circ x_{1}^{\sigma}+x_{2}^{\sigma} \circ y_{1}^{\sigma}\right)
$$

The triple construction generalizes the Nahm algebra construction of Kinyuon-Sagle (2002), which is the special case where the algebra $\mathbb{A}$ is a Lie algebra $\mathfrak{g}$ and the involution $\sigma=-\mathbb{1}$. More precisely, this is motivated by the Nahm ODE system

$$
\left\{\begin{array}{l}
\dot{x}=[y, z], \\
\dot{y}=[z, x], \quad x, y, z \in \mathfrak{g} \\
\dot{z}=[x, y] .
\end{array}\right.
$$

Then the Nahm algebra is the commutative algebra obtain by tripling of $(\mathfrak{k},[], \mathbb{1},\langle;\rangle)$ with

$$
\left(x_{1}, x_{2}, x_{3}\right) \circledast\left(y_{1}, y_{2}, y_{3}\right)=\frac{1}{2}\left(\left[x_{2}, y_{3}\right]+\left[y_{2}, x_{3}\right],\left[x_{3}, y_{1}\right]+\left[y_{3}, x_{1}\right],\left[x_{1}, y_{2}\right]+\left[y_{1}, x_{2}\right]\right) \quad \text { on } \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}
$$

In the general case: think of the Jordan multiplication $\left(X \bullet Y=\frac{1}{2}(X Y+Y X)\right)$ for formal 'Hermitian matrices' over $\mathbb{A}$ :

$$
\left(\begin{array}{ccc}
0 & x_{3} & x_{2}^{\sigma} \\
x_{3}^{\sigma} & 0 & x_{1} \\
x_{2} & x_{1}^{\sigma} & 0
\end{array}\right) \bullet\left(\begin{array}{ccc}
0 & y_{3} & y_{2}^{\sigma} \\
y_{3}^{\sigma} & 0 & y_{1} \\
y_{2} & y_{1}^{\sigma} & 0
\end{array}\right)=\left(\begin{array}{ccc}
* & z_{3} & z_{2}^{\sigma} \\
z_{3}^{\sigma} & * & z_{1} \\
z_{3} & z_{1}^{\sigma} & *
\end{array}\right)
$$

then

$$
\mathbf{z}=\mathbf{x} \circledast \mathbf{y}
$$

## Proposition 3.1

The algebra $T(\mathbb{A})=(\mathbb{A} \times \mathbb{A} \times \mathbb{A}, \cdot, \mathbb{1}, H)$ is metrized, $H(x, y):=\sum_{i=1}^{3} h\left(x_{i}, y_{i}\right)$.

$$
\begin{aligned}
& H(x \cdot y, z)=h\left(x_{3}^{\sigma} \circ y_{2}^{\sigma}+y_{2}^{\sigma} \circ x_{2}^{\sigma}, z_{1}\right)+h\left(x_{1}^{\sigma} \circ y_{3}^{\sigma}+y_{1}^{\sigma} \circ x_{3}^{\sigma}, z_{2}\right)+h\left(y_{2}^{\sigma} \circ x_{1}^{\sigma}+x_{2}^{\sigma} \circ y_{1}^{\sigma}, z_{3}\right) \\
& =h\left(x_{1}, y_{3}^{\sigma} \circ z_{2}^{\sigma}+z_{3}^{\sigma} \circ y_{2}^{\sigma}\right)+h\left(x_{2}, z_{1}^{\sigma} \circ y_{3}^{\sigma}+y_{1}^{\sigma} \circ z_{3}^{\sigma}\right)+h\left(x_{3}, z_{2}^{\sigma} \circ y_{1}^{\sigma}+y_{2}^{\sigma} \circ z_{1}^{\sigma}\right)=H(x, y \cdot z)
\end{aligned}
$$

## Theorem 3.1 (Hsiang algebras vs QC-algebras)

(A) $T((\mathbb{A}, \sigma, h, \circ))$ is a Hsiang algebra if and only if $(\mathbb{A}, \sigma, h, \circ)$ is a quasi-composition algebra.
(B) In that case, $T((\mathbb{A}, \sigma, h, \circ))$ is a exceptional or mutant Hsiang algebra with $d(\mathbb{A}) \in\{0,1,2,4,8\}$.

Proof. (A): the if-part. If $\mathbf{v}_{1}=\left(x_{1}, 0,0\right) \in \mathbb{T}_{1}, \mathbf{v}_{2}=\left(0, x_{2}, 0\right) \in \mathbb{T}_{2}, \mathbf{v}_{3}=\left(0,0, x_{3}\right) \in \mathbb{T}_{3}$, then

$$
\mathbf{v}_{1} \mathbf{v}_{1}=0, \quad \mathbf{v}_{1} \mathbf{v}_{2}=\mathbf{v}_{2} \mathbf{v}_{1}=\left(0,0, x_{2}^{\sigma} x_{1}^{\sigma}\right), \quad \mathbf{v}_{1} \mathbf{v}_{3}=\mathbf{v}_{3} \mathbf{v}_{1}=\left(0, x_{1}^{\sigma} x_{3}^{\sigma}, 0\right) \quad \text { etc. }
$$

This implies that $T((\mathbb{A}, \sigma, h, \circ))$ is naturally decomposed into an $H$-orthogonal sum

$$
\begin{aligned}
T(\mathbb{A}) & =\mathbb{T}_{1} \oplus \mathbb{T}_{2} \oplus \mathbb{T}_{3}, \quad \text { where } \mathbb{T}_{i} \mathbb{T}_{i}=0, \quad \mathbb{T}_{i} \mathbb{T}_{j}=\mathbb{T}_{k}, \quad i, j, k \sim 1,2,3 . \\
\mathbf{v}_{1}\left(\mathbf{v}_{1} \mathbf{v}_{2}\right) & =\left(0, x_{1}^{\sigma}\left(x_{1} x_{2}\right), 0\right) \\
\mathbf{v}_{1}\left(\mathbf{v}_{1}\left(\mathbf{v}_{1} \mathbf{v}_{2}\right)\right. & \left.=\left(0,0,\left(x_{2}^{\sigma} x_{1}^{\sigma}\right) x_{1}\right) x_{1}^{\sigma}\right)=\left(0,0, R\left(x_{1}^{\sigma}\right) R\left(x_{1}\right) R\left(x_{1}^{\sigma}\right) x_{2}^{\sigma}\right)
\end{aligned}
$$

For any $x=\mathbf{v}_{1} \oplus \mathbf{v}_{2} \oplus \mathbf{v}_{3}$ we have $x^{2}=2\left(\mathbf{v}_{2} \mathbf{v}_{3} \oplus \mathbf{v}_{3} \mathbf{v}_{1} \oplus \mathbf{v}_{1} \mathbf{v}_{2}\right)$, hence

$$
\frac{1}{2} x^{3}=\left(\mathbf{v}_{2}\left(\mathbf{v}_{2} \mathbf{v}_{1}\right)+\mathbf{v}_{3}\left(\mathbf{v}_{3} \mathbf{v}_{1}\right)\right) \oplus\left(\mathbf{v}_{1}\left(\mathbf{v}_{1} \mathbf{v}_{2}\right)+\mathbf{v}_{3}\left(\mathbf{v}_{3} \mathbf{v}_{2}\right)\right) \oplus\left(\mathbf{v}_{1}\left(\mathbf{v}_{1} \mathbf{v}_{3}\right)+\mathbf{v}_{2}\left(\mathbf{v}_{2} \mathbf{v}_{3}\right)\right)
$$

A nontrivial point: using the invariance of $H$ and the fact that $(\mathbb{A}, \sigma, h, \circ)$ is a QC-algebra implies

$$
H\left(x^{3}, x^{2}\right)=4 \sum H\left(\mathbf{v}_{i}\left(\mathbf{v}_{i}\left(\mathbf{v}_{i} \mathbf{v}_{k}\right), \mathbf{v}_{j}\right)=4 \sum_{\text {cyclic permutations }} h\left(\left(x_{k}^{\sigma} x_{i}^{\sigma}\right) x_{i}\right) x_{i}^{\sigma}, x_{j}\right)=\ldots=\frac{4}{3} H\left(x^{2}, x\right) H(x, x)
$$

The proof of $(B)$ is indirect and makes an essential use of the Hsiang algebra theory.

## Some important questions arise:

- Which of Hsiang algebras are QC-generated, i.e. obtained from QC-algebras?
- How to classify Euclidean QC algebras over $\mathbb{R}$ ?
- How to classify general QC-algebras?


## Remarks

- Even for the particular case of composition algebras, the classification in the general case is more involved than for $\mathbb{R}$. That is why we discuss below only Euclidean QC algebras over the reals. However, some important properties are still valid in the general case (over infinite field of charK $\neq 2$ ).
- We also assume that $\mathbb{A} \mathbb{A} \neq 0$ and will omit $\circ$ (use the juxtaposition)


## Examples of QC algebras

## Example 1

The complex numbers $(\mathbb{C}, \sigma, h)$, with $x^{\sigma}=\bar{x}$ and $h(x, y)=\operatorname{Re}(x \bar{y})$ satisfies $x\left(x^{\sigma}(x y)\right)=h(x, x) x y$. In general, any Hurwitz algebra $\mathbb{A}$ is a quasicomposition algebra.

## Example 2

A cross product algebra is a metrized anticommutative algebra $(\mathbb{A}, \times, \sigma, h)$, such that $x \times(x \times y)=-h(x, x) y+h(x, y) x$ and $x^{\sigma}=-x$. Then the latter implies

$$
x^{\sigma} \times(x \times y)=h(x, x) y-h(x, y) x \quad \Rightarrow \quad x \times\left(x^{\sigma} \times(x \times y)\right)=h(x, x) x \times y
$$

which together with $h(x \times y, z)=-h(y, x \times z)=h\left(y, x^{\sigma} \times z\right)$ shows that $(\mathbb{A}, \times, \sigma, h)$ is a quasicomposition algebra.

## Example 3

Domokos and Kövesi-Domokos (1978) introduced the color algebra Col as a unique 7D algebra (over $\mathbb{C}$ ) with unity $e$ and a basis $u_{i}, i= \pm 1, \pm 2, \pm 3$ with multiplication

$$
u_{ \pm i} \circ u_{ \pm j}=\epsilon_{i j k} u_{\mp k}, \quad u_{ \pm i} \circ u_{\mp j}=\delta_{i j} e,
$$

where $\epsilon_{i j k}$ is the totally skew-symmetric tensor with $\epsilon_{123}=1$. The algebra Col is metrized with respect to a natural $h$. Elduque (1988) studied in particular the 'imaginary' subspace $\operatorname{Col}_{0}=e^{\perp}$ and proved that it satisfies the QC identity which implies an example of a 6D quasicomposition algebra. Explicitly,

$$
\mathrm{Col}_{0} \cong \mathbf{K}^{3} \oplus \mathbf{K}^{3} \quad \text { with } \quad\left(x^{\prime}, x^{\prime \prime}\right) \times\left(y^{\prime}, y^{\prime \prime}\right)=\left(-x^{\prime} \times y^{\prime}+x^{\prime \prime} \times y^{\prime \prime}, x^{\prime} \times y^{\prime \prime}+x^{\prime \prime} \times y^{\prime}\right)
$$

## Outline

(1) Introduction and motivations
2) Algebras with involution and associative bilinear form
(3) Quasicomposition algebras and triple algebras

4 The degeneracy index and the triad principle
(5) A hidden Hurwitz algebra structure
(5) The Petterson autotopy and the classification of 3D QC-algebras


## Theorem 4.1

Let $(\mathbb{A}, \sigma, h)$ be a $Q C$ Euclidean algebra, $\operatorname{dim} \mathbb{A}=n, x \neq 0$. Then there exists $0 \leq d(\mathbb{A}) \leq n-1$ :
(1) $\frac{1}{h(x, x)} M(x) M\left(x^{\sigma}\right)$ is the orthogonal projection onto $\operatorname{Im} M(x)$, here and below $M \in\{L, R\}$;
(2) $\operatorname{dim} \operatorname{ker} M(x)=d(\mathbb{A})$
(3) $\operatorname{rank} M(x)=n-d(\mathbb{A})$
(The constant rank condition)
(4) $\operatorname{ker} L(x)=\left(\operatorname{ker} R\left(x^{\sigma}\right)\right)^{\sigma}=\left(\operatorname{Im} L\left(x^{\sigma}\right)\right)^{\perp}$
(5) $\operatorname{Im} M(x)=\operatorname{Im} M(x) M\left(x^{\sigma}\right)$ and $\quad \operatorname{ker} M\left(x^{\sigma}\right)=\operatorname{ker} M(x) M\left(x^{\sigma}\right)$, in particular:

$$
\operatorname{Im} M(x) \xrightarrow{M\left(x^{\sigma}\right)} \operatorname{Im} M\left(x^{\sigma}\right) \xrightarrow{M(x)} \operatorname{Im} M(x), \quad \text { the composition }=h(x, x) \cdot \mathbb{1}_{\operatorname{Im} M(x)}
$$

Proof. The principal observation is that $P(x):=M(x) M\left(x^{\sigma}\right)$ satisfies

$$
P(x)^{2}=M(x) M\left(x^{\sigma}\right) M(x) M\left(x^{\sigma}\right)=h(x, x) M(x) M\left(x^{\sigma}\right)=h(x, x) P(x)
$$

therefore $P / h(x, x)$ is a projection. In particular, $x \rightarrow \operatorname{tr} P(x) / h(x, x) \in \mathbb{Z}$ must be constant. This proves (1)-(3). Next

$$
h(x y, \mathbb{A})=h\left(y, x^{\sigma} \mathbb{A}\right) \quad \Rightarrow \quad \operatorname{ker} L(x)=\left(\operatorname{Im} L\left(x^{\sigma}\right)\right)^{\perp}
$$

implying (4). Therefore

$$
L(x) L\left(x^{\sigma}\right) y=0 \Rightarrow x^{\sigma} y \in \operatorname{ker} L(x) \cap \operatorname{Im} L\left(x^{\sigma}\right)=\operatorname{ker} L(x) \cap(\operatorname{ker} L(x))^{\perp}=(\text { by }(4))=0
$$

i.e. $y \in \operatorname{ker} L\left(x^{\sigma}\right)$, therefore proving essentially (5).

Definition. To any $Q C$ algebra one can associate its degeneracy index $d(\mathbb{A}) \in\{0,1, \ldots, \operatorname{dim} \mathbb{A}-1\}$.

- $d(\mathbb{A})=0:$ for the classical Hurwitz algebras in dimensions $n=1,2,4,8$;
- $d(\mathbb{A})=1:$ for $n=3$ the cross product algebra $\mathfrak{s o}(3)$ (with Petersson's isotopes and $\mathfrak{G}_{0}$ )
- $d(\mathbb{A})=1$ : for $n=7$, one has a cross product anti-commutative algebra of imaginary octonions;
- $d(\mathbb{A})=2$ : an 'imaginary' subspace of the color anti-commutative algebra (with isotopes)


## Corollary 4.1

(i) Every Euclidean QC-algebra satisfies the constant rank condition.
(ii) A unital QC (non-necessarily Euclidean) algebra over infinite field of char $\mathbf{K} \neq 2$ is a Hurwitz algebra.
(iii) If $\mathbb{A}$ is a Euclidean $Q C$-algebra and $\operatorname{dim} \mathbb{A} \leq 2$ then $\mathbb{A}$ is a division algebra.

Proof. (iii) follows from $2 \geq \operatorname{dim} \mathbb{A}>2 d(\mathbb{A})$ implying that $d(\mathbb{A})=0$.

## Theorem 4.2 (The triad principle)

Let $\mathbb{A}$ be a Euclidean QC-algebra and $x_{1}, x_{2} \neq 0$. Then
(1) If $x_{1} x_{2}=0$ then $\operatorname{ker} R\left(x_{1}\right)=\operatorname{ker} L\left(x_{2}\right)$ and $\operatorname{Im} L\left(x_{1}\right)=\operatorname{Im} R\left(x_{2}\right)$.
(2) For any $x \neq 0: \operatorname{ker} L(x)$ ker $R(x)=0$.
(3) $x_{1} x_{2}=x_{2} x_{3}=0$ implies $x_{3} x_{1}=0$.
(4) Conversely, if $x_{1} x_{2}=0$ then there exists $x_{3} \neq 0: x_{1} x_{2}=x_{2} x_{3}=x_{3} x_{1}=0$.
(5) For any triple $x_{1}, x_{2}, x_{3}$ satisfying (4) there holds for any $i \in \mathbb{Z} / 3 \mathbb{Z}$

$$
\operatorname{ker} R\left(x_{i}\right)=\operatorname{ker} L\left(x_{i+1}\right), \quad \operatorname{Im} L\left(x_{i}\right)=\operatorname{Im} R\left(x_{i+1}\right)
$$

We denote this by the infinite cyclic diagram: . . . $\rightsquigarrow x_{1} \rightsquigarrow x_{2} \rightsquigarrow x_{3} \rightsquigarrow x_{1} \rightsquigarrow \ldots$

Proof. The linearization of the QC-identity yields

$$
\left(L(x) L\left(z^{\sigma}\right)+L\left(z^{\sigma}\right) L(x)\right)(x y)+L(x) L\left(x^{\sigma}\right)(z y)=2 h(x, z) x y+h(x, x) z y
$$

If $x y=0, x, y \neq 0$ then $\left[L(x) L\left(x^{\sigma}\right)-h(x, x)\right] z y$, i.e. $z y \in \operatorname{Im} L(x)$ for any $z \in \mathbb{A}$, i.e. $\operatorname{Im} R(y) \subset \operatorname{Im} L(x)$, implying for the dimensional reasons $\operatorname{Im} R(y)=\operatorname{Im} L(x)$. Since $y^{\sigma} x^{\sigma}=(x y)^{\sigma}=0$ we have $\operatorname{Im} R\left(x^{\sigma}\right)=\operatorname{Im} L\left(y^{\sigma}\right)$, therefore

$$
\begin{equation*}
\text { ker } R(x)=\left(\operatorname{Im} R\left(x^{\sigma}\right)\right)^{\perp}=\left(\operatorname{Im} L\left(y^{\sigma}\right)\right)^{\perp}=\operatorname{ker} L(x) \Leftarrow \tag{1}
\end{equation*}
$$

If $\operatorname{ker} L(x) \neq 0$ then for any $0 \neq y \in \operatorname{ker} L(x): x y=0$, hence by (1) $\operatorname{ker} L(y)=\operatorname{ker} R(x)$, thus $y \operatorname{ker} R(x)=0$ implying (2). Now, if $x_{1} x_{2}=x_{2} x_{3}=0$ then $x_{3} x_{1} \in \operatorname{ker} L\left(x_{2}\right)$ ker $R\left(x_{2}\right)=0$, implying (3) and similarly (4)-(5).

## Corollary 4.2

If $\mathbb{A}$ is a Euclidean $Q C$ algebra with $\sigma=\mathbb{1}$ then $\mathbb{A}$ is a commutative division algebra, in particular, $\operatorname{dim} \mathbb{A} \leq 2$.

Proof. The assumption $\sigma=\mathbb{1}$ implies that $\mathbb{A}$ is commutative, hence $L(x)=L\left(x^{\sigma}\right)=R(x)=R\left(x^{\sigma}\right)$ and therefore ker $L(x)$ ker $L(x)=0$ for any $x \in \mathbb{A}$. If $d(\mathbb{A}) \geq 1$ then given an arbitrary nonzero $y \in \mathbb{A}, y \in \operatorname{ker} R(x)=\operatorname{ker} L(x)$ for $0 \neq x \in \operatorname{ker} L(y)$, hence $y y=0$. Since $\mathbb{A}$ is commutative then polarization of $y y=0$ implies $\mathbb{A} \mathbb{A}=0$, a contradiction.

## Corollary 4.3

The following statements are equivalent:
(1) $\operatorname{ker} L(x) \cap \operatorname{ker} L(y) \neq 0$;
(2) $\operatorname{ker} R(x) \cap \operatorname{ker} R(y) \neq 0$.
(3) $\operatorname{ker} R(x)=\operatorname{ker} R(y)$;
(4) $\operatorname{ker} L(x)=\operatorname{ker} L(y)$;
(5) $\operatorname{Im} L(x)=\operatorname{Im} L(y)$;
(6) $\operatorname{Im} R(x)=\operatorname{Im} R(y)$;

In particular, the left (or right) kernels is a projective partition of $\mathbb{A} \backslash\{0\}$.

Observe that for example $\operatorname{Im} L(x) \cap \operatorname{Im} L(y) \neq \emptyset$ does not imply that $\operatorname{Im} L(x)=\operatorname{Im} L(y)$.

## Corollary 4.4

Let $\mathbb{A}_{1}$, resp. $\mathbb{A}_{-1}$, denote the subspace of symmetric, resp. skew-symmetric elements w.r.t. action of the involution $\sigma$. If $\operatorname{dim} \mathbb{A} \geq 3$ then $\mathbb{A}_{-1} \neq 0$. In particular, if $\operatorname{dim} \mathbb{A} \geq 2$ and $\operatorname{dim} \mathbb{A}_{-1} \geq 1$. Then

$$
n-d(\mathbb{A}) \equiv 0 \quad \bmod 2
$$

Proof. The first part: if $\mathbb{A}_{-1}=0$ then $\sigma=\mathbb{1}$, hence $\mathbb{A}$ is commutative. By Corollary 4.2: $\operatorname{dim} \mathbb{A} \leq 2$, a contradiction. Next, fix a nonzero $w \in \mathbb{A}_{-1}$. Then $L\left(w^{\sigma}\right)=-L(w)$ and

$$
\begin{aligned}
& \operatorname{Im} L(w) \stackrel{-L(w)}{\longrightarrow} \operatorname{Im} L(w) \stackrel{L(w)}{\longrightarrow} \operatorname{Im} L(w) \quad(\operatorname{Im} L(w) \text { is an invariant subspace }), \\
& L(w) L\left(w^{\sigma}\right)=-L(w)^{2}=\mathbb{1}_{\operatorname{Im} L(w)} \\
\Rightarrow & {\left[\left.\operatorname{det} L(w)\right|_{\operatorname{Im} L(w)]^{2}=(-1)^{\operatorname{dim} \operatorname{Im} L(w)}}\right.}
\end{aligned}
$$

hence $\operatorname{dim} \operatorname{Im} L(w)$ is an even number.

## Proposition 4.1 (Classification of Euclidean QC algebras in 2D)

If $\operatorname{dim} \mathbb{A}=2$ then the only three following possibilities hold:
(A) $\mathbb{A}$ is a symmetric composition algebra, and in this case
(a) $\mathbb{A}$ contains a nonzero idempotent and is isomorphic to para-complex numbers,
(b) $\mathbb{A}$ does not contain nonzero idempotents: there is a basis $(e, f)$ of $\mathbb{A}$ with $e^{2}=f, e f=f e=e$ and $f^{2}=\lambda e-f, f \in \mathbf{K}$
(B) $\mathbb{A}$ is the unital algebra of complex numbers.

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The main tool is the so-called pre-idempotent triples which represent idempotents 'upstairs' ( $=$ in the triple of $\mathbb{A}$ ).

$$
\begin{aligned}
\mathbb{A} & \longrightarrow T(\mathbb{A}) \\
\left(x_{1}, x_{2}, x_{3}\right) & \longrightarrow \operatorname{Idm}(T(\mathbb{A}))
\end{aligned}
$$

Definition. Let $\mathbb{A}$ be a quasi-composition algebra. A triple $\left(x_{1}, x_{2}, x_{3}\right), h\left(x_{i}, x_{i}\right)=1$, is called an pre-idempotent triple or $\left(x_{1}, x_{2}, x_{3}\right) \in J(\mathbb{A})$, if

$$
x_{i}^{\sigma}=x_{i+1} x_{i+2}, \quad i \in \mathbb{Z} / 3 \mathbb{Z}, \quad \Leftrightarrow \quad \begin{cases}x_{1}^{\sigma} & =x_{2} x_{3}  \tag{4}\\ x_{2}^{\sigma} & =x_{3} x_{1} \\ x_{3}^{\sigma} & =x_{1} x_{2}\end{cases}
$$

## How it works:

Theorem 5.4 below on the commutator relations $\left[L_{i}^{\sigma} L_{i}, R_{i+2}^{\sigma} R_{i+2}\right]=\left[R_{i} R_{i}^{\sigma}, L_{i+2} L_{i+2}^{\sigma}\right]=0$ implies the following "three-kernels" decomposition:
$\mathbb{A}=\operatorname{ker} L\left(x_{i+2}\right) \oplus \operatorname{ker} R\left(x_{i+1}\right) \oplus \mathscr{M}\left(x_{i}\right) \oplus \mathscr{N}\left(x_{i}\right), \quad$ where $\quad \mathscr{M}\left(x_{i}\right) \oplus \mathscr{N}\left(x_{i}\right)=\operatorname{Im} L\left(x_{i+2}^{\sigma}\right) \cap \operatorname{Im} R\left(x_{i+1}^{\sigma}\right)$
Then a deeper result is that the principal kernel $\mathscr{M}\left(x_{i}\right)$ can be isotopically made into a Hurwitz algebra w.r.t.

$$
x \bullet y:=\left(x u_{2}\right)\left(u_{3} y\right), \quad \forall x, y \in \mathscr{M}\left(u_{1}\right) \subset \operatorname{Im} L\left(u_{3}^{\sigma}\right) \cap \operatorname{Im} R\left(u_{2}^{\sigma}\right)
$$

For instance, it is easy to see that $u_{1}$ is the $\bullet$-unity:

$$
\begin{aligned}
& x \bullet u_{1}=\left(x u_{2}\right)\left(u_{3} u_{1}\right)=(\text { by }(4))=\left(x u_{2}\right) u_{2}^{\sigma}=R\left(u_{2}^{\sigma}\right) R\left(u_{2}\right) x=x \\
& u_{1} \bullet y=\left(u_{1} u_{2}\right)\left(u_{3} y\right)=(\text { by }(4))=u_{3}^{\sigma}\left(u_{3} y\right)=L\left(u_{3}^{\sigma}\right) L\left(u_{3}\right) y=y .
\end{aligned}
$$

But the proof of the composition property and the closeness by the $\bullet$-multiplication is nontrivial.

For any nonzero $u \in \mathbb{A}$ there are nonzero solutions $\xi, \eta$ of

$$
u \xi=\eta u=0 \quad \Rightarrow \quad(\text { by the triad principle }) \quad \xi \eta=0
$$

hence the following is well-defined:

$$
\begin{aligned}
\mathscr{M}(u) & :=\operatorname{ker} R(\xi)=\operatorname{ker} L(\eta) \\
\mathscr{E}(u) & :=\operatorname{Im} R\left(\xi^{\sigma}\right)=\operatorname{Im} L\left(\eta^{\sigma}\right) \\
\mathbb{A} & =\mathscr{M}(u) \oplus \mathscr{E}(u)
\end{aligned}
$$

## Theorem 5.1

Let $\left(u_{1}, u_{2}, u_{3}\right) \in J(\mathbb{A})$ and $d(\mathbb{A}) \geq 1$. Then

$$
\begin{equation*}
u_{i} \in \mathscr{M}\left(u_{i}\right) \subset \operatorname{Im} L\left(u_{i+2}^{\sigma}\right) \cap \operatorname{Im} R\left(u_{i+1}^{\sigma}\right) \tag{5}
\end{equation*}
$$

Let us define for any $\forall x, y \in \mathscr{M}\left(u_{1}\right)$

$$
x \bullet y:=\left(x u_{2}\right)\left(u_{3} y\right) \in \mathscr{M}\left(u_{1}\right)
$$

Then

$$
\begin{aligned}
x \bullet y & \in \mathscr{M}\left(u_{1}\right) \\
u_{1} \bullet x & =x \bullet u_{1}=x \\
h(x \bullet y, x \bullet y) & =h(x, x) h(y, y),
\end{aligned}
$$

In particular, $\left(\mathscr{M}\left(u_{1}\right), \bullet\right)$ is a Hurwitz algebra and $d(\mathbb{A}) \in\{1,2,4,8\}$.

Let us consider the realization of the symmetric group $S_{3}$ as the general affine group $\mathrm{Aff}(\mathbb{Z} / 3 \mathbb{Z})$ :

$$
S_{3} \cong \mathrm{Aff}(\mathbb{Z} / 3 \mathbb{Z}) \cong\left\{g=\left(\begin{array}{cc}
m & i \\
0 & 1
\end{array}\right): m \in(\mathbb{Z} / 3 \mathbb{Z})^{\times}, i \in \mathbb{Z} / 3 \mathbb{Z}\right\}
$$

where the determinant is a multiplicative homomorphism det : $S_{3} \cong \operatorname{Aff}(\mathbb{Z} / 3 \mathbb{Z}) \rightarrow(\mathbb{Z} / 3 \mathbb{Z})^{\times} \cong \mathbb{Z} / 2 \mathbb{Z}$ is the sign of $g \in S_{3}$ :

$$
\operatorname{det} g=m=\operatorname{sign} g
$$

Given $g \in \operatorname{Aff}(\mathbb{Z} / 3 \mathbb{Z})$, define the corresponding nondegenerate linear endomorphism

$$
[g](x)=m x+i: \mathbb{Z} / 3 \mathbb{Z} \rightarrow \mathbb{Z} / 3 \mathbb{Z}
$$

$S_{3}$ acts on permutation triples $\tau=(i, j, k)$ of $\{1,2,3\}$ coordinate-wisely: $\tau \rightarrow g(\tau)=(g(i), g(j), g(k))$. Any involutive operator $\sigma$ generates a cyclic group $\left\{\sigma, \sigma^{2}=e\right\} \cong \mathbb{Z} / 2 \mathbb{Z}$, this implies the multiplicative group homomorphism

$$
\chi: \operatorname{Aff}(\mathbb{Z} / 3 \mathbb{Z}) \xrightarrow{\text { det }}(\mathbb{Z} / 3 \mathbb{Z})^{\times} \xrightarrow{\tau} \mathbb{Z} / 2 \mathbb{Z} \cong\{e, \sigma\},
$$

Notice that $\{e, \sigma\}$ is abelian, hence

$$
\chi(g h)=\chi(g) \chi(h)=\chi(h) \chi(g)=\chi(h g) .
$$

We illustrate the above explicitly below

| $(m, k)$ | $g$ | $[g](t)$ | $g((1,2,3))$ | $\chi(g)$ |
| :--- | :--- | :--- | :--- | :--- |
| $(1,0)$ | $e$ | $t$ | $(1,2,3)$ | $e$ |
| $(1,1)$ | $\epsilon$ | $t+1$ | $(2,3,1)$ | $e$ |
| $(1,2)$ | $\epsilon^{2}$ | $t+2$ | $(3,1,2)$ | $e$ |
| $(2,2)$ | $\alpha_{1}$ | $2 t+2$ | $(1,3,2)$ | $\sigma$ |
| $(2,1)$ | $\alpha_{2}$ | $2 t+1$ | $(3,2,1)$ | $\sigma$ |
| $(2,0)$ | $\alpha_{3}$ | $2 t$ | $(2,1,3)$ | $\sigma$ |

Here $\alpha_{i}$ acts on $\tau$ by interchanging $i+1$ and $i+2$ (as elements of $\mathbb{Z} / 3 \mathbb{Z}$ ) followed by involution $\sigma$ on all coordinates.

## Lemma 5.2

$S_{3} \cong \operatorname{Aff}(\mathbb{Z} / 3 \mathbb{Z})$ acts faithfully on $J(\mathbb{A})$ : if $\tau=\left(x_{1}, x_{2}, x_{3}\right) \in J(\mathbb{A})$ then $g(\tau) \in J(\mathbb{A})$. More explicitly, the $S_{3}$-action consists of
(a) three right-shifts $x_{i} \rightarrow x_{i+1}$ and
(b) three conj-flips: interchanging any pair $x_{i}$ and $x_{k}$ followed by $\sigma$-action coordinate-wise.

Furthermore, the $S_{3}$-action can be naturally extended to an $S_{4}$-action on $J(\mathbb{A})$ by adding sign involutions on $\mathbb{A}$.

For example, if $\left(x_{1}, x_{2}, x_{3}\right) \in J(\mathbb{A})$ then so do also $\left(x_{2}, x_{3}, x_{1}\right)$ etc. and $\left(x_{1}^{\sigma}, x_{3}^{\sigma}, x_{2}^{\sigma}\right)$ etc.

Example. Let $\mathbb{A}=\mathfrak{s o}(3)$. Then $\left(x_{1}, x_{2}, x_{3}\right) \in J(\mathbb{A})$ if and only if $\left(x_{1}, x_{3}, x_{2}\right)$ is the right-handed orthonormal basis:

$$
\operatorname{Im} L\left(x_{1}\right)=\operatorname{span}\left(x_{2}, x_{3}\right), \quad \operatorname{ker} L\left(x_{1}\right)=\operatorname{span}\left(x_{1}\right)
$$



## Theorem 5.3 (The existence)

Let $\mathbb{A}$ be a Euclidean quasi-composition algebra. Let
(1) $x_{1} \in \mathbb{A}$ and $h\left(x_{1}, x_{1}\right)=1$
(2) $x_{2} \in \operatorname{Im} L\left(x_{1}^{\sigma}\right)$ and $h\left(x_{2}, x_{2}\right)=1$.

Then $\left(x_{1}, x_{2}, x_{2}^{\sigma} x_{1}^{\sigma}\right) \in J(\mathbb{A})\left(\right.$ recall that the latter means that $x_{i}^{\sigma}=x_{i+1} x_{i+2}, i \in \mathbb{Z} / 3 \mathbb{Z}$.)

Proof. Let $x_{3}:=x_{2}^{\sigma} x_{1}^{\sigma}$, implying the 3rd identity in (4). Next, by Theorem 4.1, since $x_{2} \in \operatorname{Im} L\left(x_{1}^{\sigma}\right)$ :

$$
x_{1}^{\sigma} x_{3}^{\sigma}=x_{1}^{\sigma}\left(x_{2}^{\sigma} x_{1}^{\sigma}\right)^{\sigma}=x_{1}^{\sigma}\left(x_{1} x_{2}\right)=L\left(x_{1}^{\sigma}\right) L\left(x_{1}\right) x_{2}=L\left(x_{1}^{\sigma}\right) L\left(x_{1}\right)=\text { projection on } \operatorname{Im} L\left(x_{1}^{\sigma}\right)=x_{2},
$$

implying the 2 nd identity in (4). It remains to show that $h\left(x_{3}, x_{3}\right)=1$ and $y:=x_{3}^{\sigma} x_{2}^{\sigma}$ is equal to $x_{1}$. To this end, note that

$$
h\left(x_{3}, x_{3}\right)=h\left(x_{2}^{\sigma} x_{1}^{\sigma}, x_{2}^{\sigma} x_{1}^{\sigma}\right)=h\left(x_{1} x_{2}, x_{1} x_{2}\right)=h\left(x_{2}, x_{1}^{\sigma}\left(x_{1} x_{2}\right)\right)=h\left(x_{2}, x_{2}\right)=1 .
$$

Furthermore, by the above $x_{2} \in \operatorname{Im} R\left(x_{3}^{\sigma}\right)$ and $x_{3} \in \operatorname{Im} L\left(x_{2}^{\sigma}\right)$ hence

$$
\begin{aligned}
y^{\sigma} x_{3}^{\sigma} & =\left(x_{2} x_{3}\right) x_{3}^{\sigma}=R\left(x_{3}^{\sigma}\right) R\left(x_{3}\right) x_{2}=x_{2}\left(\text { but also }=x_{1}^{\sigma} x_{3}^{\sigma}\right) \\
x_{2}^{\sigma} y^{\sigma} & =x_{2}^{\sigma}\left(x_{2} x_{3}\right)=L\left(x_{2}^{\sigma}\right) L\left(x_{2}\right) x_{3}=x_{3}\left(\text { but also }=x_{2}^{\sigma} x_{1}^{\sigma}\right) .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left(y^{\sigma}-x_{1}^{\sigma}\right) x_{3}^{\sigma} & =0 \\
x_{2}^{\sigma}\left(y^{\sigma}-x_{1}^{\sigma}\right) & =0,
\end{aligned}
$$

therefore by the triad principle either $y^{\sigma}-x_{1}^{\sigma}=0$ or $0=x_{3}^{\sigma} x_{2}^{\sigma}=\left(x_{2} x_{3}\right)^{\sigma}=y$. The latter is impossible because

$$
h(y, y)=h\left(x_{3}^{\sigma} x_{2}^{\sigma}, x_{3}^{\sigma} x_{2}^{\sigma}\right)=h\left(x_{2} x_{3}, x_{2} x_{3}\right)=h\left(x_{3}, x_{2}^{\sigma}\left(x_{2} x_{3}\right)\right)=h\left(x_{3}, x_{3}\right)=1 .
$$

Therefore we have $y^{\sigma}-x_{1}^{\sigma}=0$, i.e. $x_{1}^{\sigma}=x_{2} x_{3}$ which implies the 1 st identity in (4). This proves the theorem.

## Theorem 5.4 (The commutator relations)

Let $\left(x_{1}, x_{2}, x_{3}\right) \in J(\mathbb{A})$. Then for any $i \in \mathbb{Z} / 3 \mathbb{Z}$,
(1) $R\left(x_{i+2}\right) L\left(x_{i}^{\sigma}\right) L\left(x_{i}\right)=L\left(x_{i+1}\right) L\left(x_{i+1}^{\sigma}\right) R\left(x_{i+2}\right)$,
(2) $\operatorname{ker} L\left(x_{i}\right) \xrightarrow{R\left(x_{i+2}\right)} \operatorname{ker} L\left(x_{i+1}^{\sigma}\right) \xrightarrow{R\left(x_{i+2}^{\sigma}\right)} \operatorname{ker} L\left(x_{i}\right)$ are bijections and $R\left(x_{i+2}^{\sigma}\right) R\left(x_{i+2}\right)=\mathbb{1}_{\text {ker } L\left(x_{i}\right)}$.
(3) $\operatorname{ker} L\left(x_{i}\right) \subset \operatorname{Im} R\left(x_{i+2}^{\sigma}\right)$ and $\operatorname{ker} L\left(x_{i}^{\sigma}\right) \subset \operatorname{Im} R\left(x_{i+1}\right)$

Example. Illustrate this by $\mathbb{A}=\mathfrak{s o}(3)$ and $\left(x_{1}, x_{2}, x_{3}\right) \in J(\mathbb{A})$ :

- $\operatorname{Im} L\left(x_{1}\right)=\operatorname{span}\left(x_{2}, x_{3}\right), \quad$ ker $L\left(x_{1}\right)=\operatorname{span}\left(x_{1}\right)$
- $\operatorname{ker} L\left(x_{1}\right) \xrightarrow{R\left(x_{3}\right)} \operatorname{ker} L\left(x_{2}^{\sigma}\right) \xrightarrow{R\left(x_{3}^{\sigma}\right)} \operatorname{ker} L\left(x_{1}\right)$
- $\operatorname{ker} L\left(x_{1}\right) \subset \operatorname{Im} R\left(x_{3}^{\sigma}\right), \quad \operatorname{ker} L\left(x_{1}^{\sigma}\right) \subset \operatorname{Im} R\left(x_{2}\right)$



## Corollary 5.5

If $\mathbb{A}$ is a nonzero Euclidean $Q C$-algebra then $\operatorname{dim} \mathbb{A} \geq 2 d(\mathbb{A})+1$. In particular, for any two nonzero $x, y$ : $\operatorname{dim}(\operatorname{Im} L(x) \cap \operatorname{Im} L(y)) \geq n-2 d(\mathbb{A}) \geq 1$. In particular, any Euclidean $Q C$-algebra is simple.

Proof. For any ideal $I \neq \mathbb{A}, 0: \operatorname{dim} I \geq \operatorname{dim} I \mathbb{A} \geq n-d$ and similarly $\operatorname{dim} I^{\perp} \geq n-d$, hence $2 d \geq n$, a contradiction with Corollary 5.5.

From the left and right kernels to the "principal kernel"

For any nonzero $u \in \mathbb{A}$ there are nonzero solutions $\xi, \eta$ of

$$
u \xi=\eta u=0 \quad \Rightarrow \quad \xi \eta=0
$$

hence the triad principle, the following is well-defined:

$$
\begin{aligned}
\mathscr{M}(u) & :=\operatorname{ker} R(\xi)=\operatorname{ker} L(\eta) \\
\mathscr{E}(u) & :=\operatorname{Im} R\left(\xi^{\sigma}\right)=\operatorname{Im} L\left(\eta^{\sigma}\right) \\
\mathbb{A} & =\mathscr{M}(u) \oplus \mathscr{E}(u)
\end{aligned}
$$

## Definition 5.6

If $\mathbb{A}$ is a Euclidean QC algebra, then $\mathscr{M}(x)$ is the principal kernel and $\mathscr{E}(x)$ is the principal image of $x \in \mathbb{A}$.

## Proposition 5.1

For any nonzero $x \in \mathbb{A}, \operatorname{dim} \mathscr{M}(x)=d(\mathbb{A})$ and
(1) if $0 \neq t \in \mathscr{M}(x)$ then $\mathscr{M}(t)=\mathscr{M}(x)$;
(2) if $\mathscr{M}(t) \cap \mathscr{M}(x) \neq \emptyset$ for a nonzero $t$, then $\mathscr{M}(t)=\mathscr{M}(x)$;
(3) $t \in \mathscr{M}(x) \Leftrightarrow x \in \mathscr{M}(t)$.


Figure 1. Vertical lines between orthogonal elements, wave arrows shows kernels. I.e. one makes a general triple $\mathbf{u}=\left(u_{1}, u_{2}, u_{3}\right)$ into an orthogonal idempotent triple $\left(\xi_{i+1}, u_{i}, \eta_{i-1}\right)$

## Some useful numerology

| Hsi.dim $=3 n$ | 3 | 6 | 12 | 24 | 9 | 12 | 15 | 21 | 15 | 18 | 21 | 24 | 30 | 42 | 27 | 30 | 33 | 36 | 51 | 54 | 57 | 60 | 72 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| QC-dim $=n$ | 1 | 2 | 4 | 8 | 3 | 4 | 5 | 7 | 5 | 6 | 7 | 8 | 10 | 14 | 9 | 10 | 11 | 12 | 17 | 18 | 19 | 20 | 24 |
| $n_{1}=n-2 d-1$ | 0 | 1 | 3 | 7 | 0 | 1 | 2 | 4 | 0 | 1 | 2 | 3 | 5 | 9 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 7 |
| $d=d(\mathbb{A})$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 4 | 4 | 4 | 4 | 8 | 8 | 8 | 8 | 8 |
| $n+d$ | 1 | 2 | 4 | 8 | 4 | 5 | 6 | 8 | 7 | 8 | 9 | 10 | 12 | 16 | 13 | 14 | 15 | 16 | 25 | 26 | 27 | 28 | 32 |
| $n-3 d$ | 1 | 2 | 4 | 8 | 0 | 1 | 2 | 4 | - | 0 | 1 | 2 | 4 | 8 | - | - | - | 0 | - | - | - | - | 0 |
| QC-algebra? | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | - | $?$ | $\checkmark$ | - | $\checkmark$ | - | - | $?$ | $?$ | - | - | - | $?$ | - | - | - | - | $?$ |

Table 3. Permitted Peirce dimensions of exceptional Hsiang algebras. The four first columns correspond to mutants. The Peirce dimension $\operatorname{dim} \mathbb{A}_{c}\left(-\frac{1}{2}\right)=3 d+2$

- In dimensions $n=1,2,4,8$ (corresponding the first 4 columns) one has classical Hurwitz algebras, $d=0$
- For $n=3$ one has a cross product algebra $\mathfrak{s o}(3), d=1$ (and its isotopes)
- For $n=6$ one has a color anti-commutative algebra $C_{6}, d=2$ (and its isotopes)
- For $n=7$ one has a cross product anti-commutative algebra of imaginary octonions, $d=1$


## Outline

(1) Introduction and motivations
2) Algebras with involution and associative bilinear form
(3) Quasicomposition algebras and triple algebras
(4) The degeneracy index and the triad principle
(5) A hidden Hurwitz algebra structure

6 The Petterson autotopy and the classification of 3D QC-algebras
(1) $(\mathbb{A}, \circ)$ and $(\mathbb{B}, \diamond)$ are isotopic, if there exist invertible linear maps $(\alpha, \beta, \gamma)$ from $\mathbb{A}$ to $\mathbb{B}$, such that

$$
\alpha(x) \diamond \beta(y)=\gamma(x \circ y) \quad \text { holds for any } x, y \in \mathbb{A} \text {. }
$$

If $\gamma=1$ then an isotopy is called principal. If $(\alpha, \beta, \gamma)$ is an isotopy $(\mathbb{A}, \circ) \rightarrow(\mathbb{B}, \diamond)$ then $\left(\alpha \gamma^{-1}, \beta \gamma^{-1}, \mathbb{1}\right)$ is a principal isotopy $\left(\mathbb{B}, \diamond^{\prime}\right) \rightarrow(\mathbb{B}, \diamond)$.
(2) An isotopy of $(\mathbb{A}, \circ)$ to itself is called an autotopy. The set $\operatorname{Atp}(\mathbb{A}, \circ)$ of all isotopies $(a, b, c) \in G L(\mathbb{A}) \times 3$ of $\mathbb{A}$ is the autotopy group of the algebra $\mathbb{A}$. An automorphism $\phi$ is an autotopy where $a=b=c$.

Let $(\mathbb{A}, \circ, \sigma, h)$ be a metrized algebra. Given $a \in G L(\mathbb{A})$, we denote by $a^{-1}$ its inverse, $a^{b}=\sigma a \sigma$ its $\sigma$-conjugate and by $a^{*}$ its $h$-adjoint, i.e. $h(a(x), y)=h\left(x, a^{*}(y)\right), \forall x, y \in \mathbb{A}$. Then

- $a \rightarrow a^{*}, a \rightarrow a^{-1}$ and $a \rightarrow a^{b}$ are involutions which commute pairwisely (the first two are anti-automorphisms)


## Lemma 6.1 (A $D_{6}$-action on the autotopy group $\left.\operatorname{Atp}(\mathbb{A})\right)$

Let $\theta:=(a, b, c) \in \operatorname{Atp}(\mathbb{A}, \circ, \sigma, h)$ then $T \theta, S \theta, Z \theta \in \operatorname{Atp}(\mathbb{A}, \circ, \sigma, h)$ and $\langle S, Z, T\rangle \cong D_{6}$, where

$$
\begin{aligned}
& S(a, b, c)=\left(b^{b}, a^{b}, c^{b}\right) \\
& Z(a, b, c)=\left(a^{-1}, b^{-1}, c^{-1}\right) \\
& T(a, b, c)=\left(c^{*},\left(b^{b}\right)^{-1}, a^{*}\right)
\end{aligned}
$$

## Proof.

- $S^{2}=Z^{2}=T^{2}=\mathbb{1}$
- $S Z=Z S, \quad T Z=Z T$
- $S T Z=T S T Z,(S T)^{3}=Z$


## Proposition 6.1 (Petersson's isotopes)

Let $(\mathbb{A}, \sigma, h, 0)$ be a quasicomposition algebra and $\tau \in G L(\mathbb{A})$ satisfy
(1) $\tau^{3}=\mathbb{1}$
(2) $\tau^{*}=\tau^{2}=\tau^{b} \quad$ (notice that $\tau^{*} \tau=\mathbb{1} \Leftrightarrow \tau$ is orthogonal)
(3) $\left(\tau, \tau, \tau^{2}\right) \in \operatorname{Atp}(\mathbb{A}, \sigma, h, \circ)$.

Then $(\mathbb{A}, \sigma, h, \circ)$ is also a quasicomposition algebra, where $x \circ_{\tau} k y=\tau^{k}(x) \circ \tau^{2 k}(y)$.

## Theorem 6.2 (Classification of 3D (Euclidean) QC algebras)

Any 3D Euclidean quasi-composition algebra is isomorphic to one of the following:
(1) one of the three Petersson's isotopes of the anticommutative cross-product algebra on $\mathfrak{s o}(3, \mathbf{K})$
(2) the algebra $\mathfrak{G}_{0}$ in Example 2.2

Thank you for your attention!


[^0]:    

