Some questions of nonassociative algebra from the idempotent point of $$\operatorname{view}$$

Vladimir G. Tkachev

Linköping University

Vladimir G. Tkachev

European Non-Associative Algebra Seminar, January 15th, 2024 (1 of 36)



Dedicated to Yakov Krasnov (1952-2023)

Dreams about 'spectral synthesis'...

Outline

Motivations

Generic algebras

- 3 Syzygies in generic algebras
- 4 Two-dimensional subalgebras
- 5 Inner isotopes and medial algebras

Properties "in large"

Given a two-dimensional surface in \mathbb{R}^3 , knowledge of its Gaussian curvature K provides a local information about S: K(x) > 0, K(x) = 0 or K(x) < 0 is exactly when S is convex, flat or saddle (locally) at x.

On the other hand, "geometry in large" studies the properties which hold under certain "completeness" assumptions (such as closed, complete or without boundary). In that case, one recover connections between geometry, analysis and topology/combinatorical structure.

Euler polyhedron formula for a convex polyhedron

 $\chi(P) = V - E + F = 2.$

Theorem (Gauss-Bonnet) Let S be a compact and orientable surface in \mathbb{R}^3 without boundary. If K is the Gaussian curvature of the surface and χ is its Euler characteristic then

$$\iint_S K \, dA = 2\pi \chi(S)$$

We also mention an example from group theory

Class equation If G is a finite group then

$$|G| = |Z(G)| + \sum_{i=1}^{r} |Cl(x_i)|$$

In order to find a natural meaning of properties "in large" in nonassociative algebra, we need to study the smallest subalgebras, idempotents and 2-nilpotents.

The word "idempotent" stems from the Latin words "idem" and "potent," which when put together, means "the same power." The term was originally introduced in mathematics to refer to mathematical operations that can occur multiple times while only altering the end result once.

We consider only commutative algebras over a field h of characteristic $\neq 2$.

An **idempotent** of an algebra (\mathbb{A}, \bullet) is an element x such that

 $x \bullet x = x,$

and a 2-nilpotent is a nonzero element satisfying

 $x \bullet x = 0.$

The set of idempotents is denoted by $Idm(\mathbb{A})$. In what follows we write $x \bullet x = x^2$. From the analytical point of view, an idempotent solves equation

$$x^2 - x = 0,$$

and its properties must depend on the "differential" at x, i.e. 2L(x) - I, this will be in focus of our discussion.

An immediate corollary of the definitions is the following

Observation. Any one-dimensional subalgebra is spanned either by an idempotent or a 2-nilpotent.

In some ways, idempotents are also analogous to \$12-subalgebras of Lie algebras. (Felix Rehren, 2016, Ind. Univ. Math. J.); similar ideas come back to Nielsen (1963) and Seligman (2003).

A map of similarities between Lie and Jordan algebras

algebra	$\mathfrak g$ is a semisimple Lie algebra over $\mathbb C$	${\mathbb A}$ is a (formally real) Jordan algebra		
class identity	[x,y] = -[y,x]	xy = yx		
defining identity	[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0	$[L(x^2), L(x)] = 0$		
adjoint operator	$\operatorname{ad}(x)y = [x, y]$	L(x)y = xy		
self-normalizer	h a Cartan subalgebra of g	$h^2 = h$, an idempotent		
associative form	$\operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y))$	$\operatorname{tr}(L(x)L(y))$		
eigensubspace	$\mathfrak{g}_{\alpha} = \{ x \in \mathfrak{g} : \mathrm{ad}(h)x = \alpha(h)x, h \in \mathfrak{h} \}$	$\mathbb{A}_h(\lambda) = \{ x \in \mathbb{A} : L(h)x = \lambda_\alpha x \}$		
roots/spectrum	$0 \neq \alpha \in \mathfrak{h}^*$ is a root if $\mathfrak{g}_{\alpha} \neq 0$ is nonzero	$\lambda \in \hbar$ is a Peirce eigenvalue if $\mathbb{A}_{\lambda} \neq 0$		
	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\sigma(\mathbb{A}) = \{0, \frac{1}{2}, 1\}$		
root/Peirce decomposition	$\mathfrak{g}=\mathfrak{h}\oplusigoplus_lpha\mathfrak{g}_lpha$	$\mathbb{A}=\mathbb{H}\oplusigoplus_{\lambda\in\sigma(\mathbb{A})}\mathbb{A}_h(\lambda)$		
fusion laws	$[\mathfrak{g}_{lpha},\mathfrak{g}_{eta}]\subset\mathfrak{g}_{lpha+eta}$	$\mathbb{A}_h(\lambda)\mathbb{A}_h(\mu) \subset \bigoplus_{\nu \in \lambda \star \mu} \mathbb{A}_h(\nu)$		
classification	regular families and exceptional	special algebras and the Albert algebra		

Below are some examples of appearance of **idempotents** in commutative nonassociative algebras outside a purely algebraic context:

- A projection operator (i.e. $P^2 = P$) in an associative algebra setting. The trace tr P is the dimension of the target space.
- An essential idempotent of the ∨-algebra defined in the famous 1962-paper on the "Eightfold way" by Gell-Mann on symmetries of baryons and mesons, corresponding to the equation

$$q \vee q + \eta(q)q = 0.$$

These essential idempotents are the directions of **symmetry breaking** for the adjoint representation: that is, they are the critical points of the invariant functional $\{A, A, A\}$ on the unit sphere $\{A, A\} = 1$.



- The bifurcation equations of the form B(x, x) + x = 0 considered by (D.H. Sattinger, 1977), where B is a bilinear mapping from $V \times V$ to V which is invariant under a certain group.
- In general, a stationary point of a distinguished cubic form u(x) on an inner product space V: the rule x y = Hess u(x)y induces a commutative algebra structure on V. Euler's homogeneous function theorem implies that x x = ½∇u(x), hence x is an idempotent if and only if x is a (non-degenerated) stationary point of a Lagrangian u(x) λ⟨x; x⟩.
- Idempotents in axial algebras: mimic a distinguished finite subset of involutions in a certain group generating a nice commutative algebra structure. Axes are specific idempotents with common fusion rules generating a whole algebra.

Some immediate naive (=natural) questions arise:

- As we have seen, idempotents appear in several contexts outside the conventional area of nonassociative algebra. One could expect that some corresponding analogues holds true for higherdimensional subalgebras. What are analogues for two-(or three-) dimensional subalgebras?
- The number of idempotents of a commutative algebra in a generic situation is $2^{\dim \mathbb{A}}$. What is an expectable number of two-dimensional subalgebras of a commutative algebra in a generic situation?
- The spectrum of idempotents in an associative algebra is quite simple: 0 and 1 are the only spectral values. In general, the (Peirce) spectrum can be very variegated.
- Do there exist any obstructions for the spectrum of idempotents? any exceptional eigenvalues? How much freedom one can use to construct idempotents a *priori*?

So, what is a "generic algebra"?

If you start with a basis in a vector space then you have a full freedom to define an algebra: for example, in the multiplication table

$$e_i \bullet e_j = \sum_k \gamma_{ij,k} e_k$$

the structure constants $\gamma_{ij,k}$ maybe arbitrary. For a commutative algebra there should be $\gamma_{ij,k} = \gamma_{ji,k}$. This, in particular, yields that the dimension of the variety of all commutative *n*-dimensional algebras is

dim ComAlg
$$(k, n) = \frac{n^2(n+1)}{2}$$
.

For example, dim $\operatorname{ComAlg}(k, 2) = 6$.

But one has to be careful with **idempotents**. In general, a commutative algebra may have **any number** of idempotents, even infinitely many. The problem comes back to B. Segre (1938), who interpreted idempotents in commutative algebras over **complex numbers** as solutions of a quadratic system

$$\sum_{i,j=1}^{n} \gamma_{ij,k} x_i x_j = x_k, \qquad 1 \le k \le n.$$

where $x = \sum_{i} x_i e_i$ is a basis decomposition. It turns out that idempotents, if they form a "complete" set, they must satisfy some **explicit constraints**. A nontrivial issue here is how to interpret the completeness, as we shall see the regular idempotents plays a special role here.

Definition

An algebra idempotent c is called singular if det(2L(c) - I) = 0, otherwise c is regular, i.e.

$$\Phi(c) := 2L(c) - I$$

is invertible.

- The definition of regular idempotents in the context of algebras of rank three were introduced by S. Walcher (1999). In his terminology, a singular idempotent is "of multiplicity 2".
- The trivial zero idempotent is regular.
- Note that for a singular idempotent, the number ¹/₂ is always an eigenvalue, while for a non-singular idempotent the set of nontrivial (≠ 1) eigenvalues maybe empty.
- The operator $\Phi(c)$ acts as the squaring on idempotent differences:

$$\Phi(c)(c-c_1) = 2c \bullet (c-c_1) - (c-c_1) = (c-c_1)^2.$$

- Idempotents in associative algebras are always regular.
- If a Jordan algebra contains a regular idempotent $\neq 0$ and the unit then the algebra is **not simple**.
- If the field k is the complex or real numbers then non-singular idempotents are always isolated points.
- The value ¹/₂ is distinguished for commutative algebras with identities (V.T., J. Algebra, 2021), it appears
 important for automorphism of axial algebras, construction exotic solutions of PDE etc.

Lemma 1

If c is an idempotent in a two-dimensional algebra, u is a nonzero non-collinear with c vector then

 $c \bullet c = c$ $c \bullet u = \alpha c + \lambda u.$

where λ is independent of u.

Proof. Indeed, $L(c) = \begin{pmatrix} 1 & \alpha \\ 0 & \lambda \end{pmatrix}$ in the basis (c, u), that implies tr $L(c) - 1 = \lambda$ has an invariant meaning. In fact, λ is the second (except for 1) eigenvalue of L(c).

Corollary 1

Let \mathbb{A} be a two-dimensional algebra and c_1, c_2, c_3 are distinct nonzero idempotents. Then

$$c_1 \bullet c_2 = \lambda_2 c_1 + \lambda_1 c_2,$$

$$c_2 \bullet c_3 = \lambda_3 c_2 + \lambda_2 c_3,$$

$$c_3 \bullet c_1 = \lambda_1 c_3 + \lambda_3 c_1,$$
(1)

where (a multiset) $\{1, \lambda_i\}$ is the spectrum of $L(c_i)$. Moreover $\lambda_j c_k - \lambda_k c_j$ is an eigenvector of $L(c_i)$ with eigenvalue λ_i .

Indeed, $L(c_i)(\lambda_j c_k - \lambda_k c_j) = \lambda_i (\lambda_j c_k - \lambda_k c_j).$

A presence of $3 = 2^2 - 1$ distinct nonzero idempotents in a two-dimensional **subspace** of a commutative algebra considerably affects the algebra structure.

Lemma 2

If c_1, c_2, c_3 are three distinct nonzero idempotents in a two-dimensional subspace V of a commutative algebra (\mathbb{A}, \bullet) then V is a subalgebra of \mathbb{A} , and either of the following holds:

- V contains exactly three non-collinear idempotents c_1, c_2, c_3 , or
- (i) c_1, c_2, c_3 are singular idempotents, (ii) the line $\{c_1t + (1 t)c_2 : t \in \hbar\}$ consists of idempotents, (iii) $c_1 c_2$ is a 2-nilpotent, and (iv) all idempotents of V belong to the line.

Proof. Any pair of the idempotents is a basis of V, hence $c_3 = \alpha c_1 + \beta c_2$, $\alpha \beta \neq 0$, and therefore $\alpha c_1 + \beta c_2 = (\alpha c_1 + \beta c_2)^2 = \alpha^2 c_1 + 2\alpha\beta c_1 \bullet c_2 + \beta^2 c_2$, implying

$$c_1 \bullet c_2 = \frac{1-\alpha}{2\beta}c_1 + \frac{1-\beta}{2\alpha}c_2.$$
 (2)

Hence V is a subalgebra and $\frac{1-\beta}{2\alpha}$ is an eigenvalue of $L(c_1)$. Note that $\frac{1-\beta}{2\alpha} = \frac{1}{2}$ iff $\alpha + \beta = 1$, which is equivalent to that c_1, c_2, c_3 lie on the same line, in which case (2) becomes $2c_1 \bullet c_2 = c_1 + c_2$, therefore $(c_1 - c_2)^2 = 0$, i.e. $c_1 - c_2$ is a 2-nilpotent and also the whole line consists of idempotents:

$$(c_1t + (1-t)c_2)^2 = c_1t^2 + 2t(1-t)c_1 \bullet c_2 + (1-t)^2c_2 = c_1t + (1-t)c_2,$$

and conversely, any nonzero idempotent of V belong to the same line. However, if $\alpha + \beta \neq 1$ then a similar argument implies that $xc_1 + yc_2$, $xy \neq 0$, is an idempotent iff a linear system with determinant $\frac{\alpha+\beta-1}{\alpha\beta}$ is (uniquely) solvable, implying that $xc_1 + yc_2 = c_3$.

We write $(\lambda_1,\lambda_2,\lambda_3)\in\Lambda$ iff

$$4\lambda_1\lambda_2\lambda_3 - \lambda_1 - \lambda_2 - \lambda_3 + 1 = 0.$$

Proposition 1

$$\begin{split} & \text{Suppose } (\lambda_1, \lambda_2, \lambda_3) \in \Lambda. \text{ Then} \\ & (a) \quad (1 - 4\lambda_i\lambda_j)(2\lambda_k - 1) = (2\lambda_i - 1)(2\lambda_j - 1) \text{ for any permutation } (i, j, k) \text{ of } (1, 2, 3); \\ & (b) \text{ either all } \lambda_i \neq \frac{1}{2} \text{ or at least two of them are } \frac{1}{2}; \\ & (c) \quad (\frac{1}{2}, \frac{1}{2}, \lambda) \in \Lambda \text{ for any } \lambda; \\ & (d) \text{ if all } \lambda_i \neq \frac{1}{2} \text{ then} \\ & \quad \frac{1}{1 - 2\lambda_1} + \frac{1}{1 - 2\lambda_2} + \frac{1}{1 - 2\lambda_3} = 1. \end{split}$$

Theorem 1 (Krasnov, V.T., 2018)

Let A be a commutative algebra over \hbar , dim_{\hbar} A = 2. Suppose c_1, c_2, c_3 are distinct nonzero idempotents, spect $(c_i) = \{1, \lambda_i\}$. Then $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda$, i.e.

$$4\lambda_1\lambda_2\lambda_3 - \lambda_1 - \lambda_2 - \lambda_3 + 1 = 0.$$
(3)

Proof. Write $c_k = x_j c_i + x_i c_j$, where $x_j, x_i \in K$, $x_j x_i \neq 0$. Then (1) gives

$$(x_jc_i + x_ic_j) \bullet c_i = \lambda_i(x_jc_i + x_ic_j) + \lambda_kc_i,$$

and then $(x_j(1-\lambda_i) + x_i\lambda_j - \lambda_k)c_i = 0$, therefore $\lambda_k = (1-\lambda_i)x_j + \lambda_j x_i$. Arguing similarly, we get $\lambda_k = \lambda_i x_j + (1-\lambda_j)x_i$, hence

$$x_i + x_j = 2\lambda_k$$

$$x_j(1 - 2\lambda_i) = x_i(1 - 2\lambda_j).$$

If $\lambda_i = \frac{1}{2}$ then $\lambda_j = \frac{1}{2}$, which implies (3) by Proposition 1(c). If $\lambda_i \neq \frac{1}{2}$ then $\lambda_j \neq \frac{1}{2}$ and we have

$$\frac{x_i}{x_j} = \frac{1 - 2\lambda_i}{1 - 2\lambda_j}.$$

Since $c_i = -\frac{1}{x_j}c_k + \frac{x_i}{x_j}c_j$ we find $x_i = \frac{2\lambda_k - 1}{1 - 2\lambda_j}$ and $x_j = \frac{2\lambda_k - 1}{1 - 2\lambda_i}$. Summing up we obtain

$$2\lambda_k = x_i + x_j = -(1 - 2\lambda_k) \left(\frac{1}{1 - 2\lambda_i} + \frac{1}{1 - 2\lambda_j}\right)$$

which readily yields (3).

Theorem 2

Let $\dim_K A = 2$ and c_1, c_2, c_3 be distinct regular nonzero idempotents. Then

(i)
$$\frac{1}{1-2\lambda_1} + \frac{1}{1-2\lambda_2} + \frac{1}{1-2\lambda_3} = 1$$

(ii)
$$\frac{c_1}{1-2\lambda_1} + \frac{c_2}{1-2\lambda_2} + \frac{c_3}{1-2\lambda_3} = 0$$

(iii)
$$c_k = \frac{1}{1 - 4\lambda_i \lambda_j} (c_i - c_j)^2;$$

- (iv) there are exactly three nonzero idempotents in \mathbb{A} ;
- (v) there are no 2-nil elements;
- (vi) c_1, c_2, c_3 are non-collinear.

• (iii) follows from (i) and Proposition 1(a):

$$c_k = x_j c_i + x_i c_j = \frac{2\lambda_k - 1}{(1 - 2\lambda_i)(1 - 2\lambda_j)} \left(c_i + c_j - \underbrace{(2\lambda_j c_i + 2\lambda_i c_j)}_{=2c_i \bullet c_j} \right) = \frac{1}{1 - 4\lambda_i \lambda_j} (c_i - c_j)^2$$

• (iv)-(vi) follow from Lemma 2.

Generic algebras

Since a generic (in the Zariski sense) polynomial system has always *Bézout's number* of solutions, a *generic* algebra must have exactly $2^{\dim A}$ **distinct** idempotents (including 0). The subset of the corresponding nonassociative algebra structures on a vector space V is an open Zariski subset in $V^* \otimes V^* \otimes V$.

Definition 1

An *n*-dimensional algebra $(\mathbb{A}, \bullet, \hbar)$ is said to be **generic** if it contains 2^n distinct regular idempotents.

It immediately follows from Theorem 2 that

Corollary 2

A two-dimensional algebra is generic if and only if it contains three distinct regular nonzero idempotents.

- When K is the complex numbers or an algebraically closed field, one can prove that a generic algebra does not contain 2-nilpotents (a result "in large"). A very closed set of algebras in dimension 2 was discussed by S. Walcher (1999).
- Many axial algebras (excluding Jordan type with $\eta = \frac{1}{2}$) are generic. We believe that the invariant algebras of sporadic finite simple groups (as the Griess-Conway-Norton algebra) are generic.
- If A is a generic axial algebra then Aut(A) is finite (Gorshkov/McInroy/Shumba/Mudziiri/Shpectorov arXiv:2311.18538)
- An example of a non-generic algebra is any commutative algebra satisfying a nontrivial identity, for instance, Jordan algebras or Hsiang algebras (algebras of cubic minimal cones).

Example 3 (Associative)

Let $V = \hbar^2$ with componentwise multiplication. Then the only nonzero idempotents are $c_1 = (1, 0)$, $c_2 = (0, 1)$, $c_3 = c_1 + c_2 = (1, 1)$, where $c_1c_2 = 0$, and $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$, so that (i)-(ii) become

$$\frac{1}{1-2\cdot 0} + \frac{1}{1-2\cdot 0} + \frac{1}{1-2\cdot 1} = 1$$
$$\frac{c_1}{1} + \frac{c_2}{1} + \frac{c_3}{-1} = 0.$$

Example 4 (Harada-Hsiang algebra)

Let V be a two-dimensional algebra generated by three idempotents subject to the condition $c_1+c_2+c_3=0.$ Then

$$c_k = c_k^2 = (-c_i - c_j)^2 = c_i + c_j + 2c_i c_j = -c_k + 2c_i c_j, \qquad \{i, j, k\} = \{1, 2, 3\},\$$

hence $c_i c_j = c_k = -c_i - c_j$. This implies $\lambda_1 = \lambda_2 = \lambda_3 = -1$, hence (i)-(ii) turn into

$$\frac{1}{1-2\cdot(-1)} + \frac{1}{1-2\cdot(-1)} + \frac{1}{1-2\cdot(-1)} = 1$$
$$\frac{c_1}{3} + \frac{c_2}{3} + \frac{c_3}{3} = 0.$$

Example 5 (Non-generic, a Hsiang algebra)

Let $\mathbb A$ be the three-dimensional algebra generated by four idempotents $c_i,\,i=0,1,2,3$ subject to the condition:

• $c_i + c_j$ is a 2-nilpotent $\Leftrightarrow (c_i + c_j)^2 = 0$ for any $i \neq j$

Then the following Peirce decomposition holds:

 $\mathbb{A} = \mathbb{A}_{c_i}(1) \oplus \mathbb{A}_{c_i}(-\frac{1}{2}),$

where $\dim \mathbb{A}_{c_i}(1)=1$ and $\dim \mathbb{A}_{c_i}(-\frac{1}{2})=2.$ The corresponding fusion rules are

*	1	$-\frac{1}{2}$
1	1	$-\frac{1}{2}$
$-\frac{1}{2}$		$1, -\frac{1}{2}$

A simple analysis reveals that besides the initial four idempotents are the only idempotents in \mathbb{A} , in particular, \mathbb{A} is not generic.

In other words, there exist non-generic algebras with only non-singular idempotents, but they are **for few** and they are compensated by the presence of 2-nilpotents.

 c_0

Example 6 (A mix of generic and nongeneric)

The three-dimensional Matsuo algebra $\mathbb{A} = 3C_{\alpha}$ is spanned by three idempotents c_1, c_2, c_3 subject to the following conditions:

$$c_i \bullet c_j = \frac{\alpha}{2}(c_i + c_j - c_k), \quad \{i, j, k\} = \{1, 2, 3\}.$$

• If $\alpha \neq -1, \frac{1}{2}$ then the Matsuo algebra $3C_{\alpha}, \alpha \in \hbar$, is a 3-dimensional generic unital algebra. More precisely, there exists exactly $7 = 2^3 - 1$ distinct nonzero idempotents: $c_7 = \frac{1}{\alpha+1}(c_1 + c_2 + c_3)$ is the algebra unit and $c_{3+i} = c_7 - c_i$, i = 1, 2, 3 ares the conjugate idempotents. The spectrum is given as follows:

$$\sigma(c_7) = \{1, 1, 1\}, \quad \sigma(c_i) = \{0, \alpha, 1\}, \qquad \sigma(\bar{c}_i) = \{0, 1 - \alpha, 1\}.$$

• In the exceptional case $\alpha = \frac{1}{2}$, there exists an infinite family of idempotents $c_x := x_1c_1 + x_2c_2 + x_3c_3$ on the circle

$$(x_1 - \frac{1}{3})^2 + (x_2 - \frac{1}{3})^2 + (x_3 - \frac{1}{3})^2 = \frac{2}{3}, \quad x_1 + x_2 + x_3 = 1$$

and have the same Peirce spectrum $\sigma(c) = \{1/2, 1, 0\}$. It is known that the Matsuo algebra $3C_{\frac{1}{2}}$ is power associative.

 In the case α = -1, there are exactly three nonzero idempotents c_i, and also a one-dimensional zero subalgebra.



Example 7 (An isospectral algebra in 3D, Krasnov, V.T., 2019)

Let \hbar be a field containing primitive roots of unity of degrees 3 and 7, denoted by ϵ and ζ , respectively. Then $\gamma := \zeta + \zeta^2 + \zeta^4$ is a Klein integer unit, i.e. it satisfies $2\gamma^2 - \gamma + 1 = 0$. Consider the algebra \mathbb{A}_3 over \hbar spanned by three **idempotents** c_1, c_2, c_3 satisfying

$$c_1c_2 = (\gamma - 1)c_1 - \gamma c_2 + \gamma c_3 =: c_5,$$

$$c_2c_3 = \gamma c_1 + (\gamma - 1)c_2 - \gamma c_3 =: c_6,$$

$$c_3c_1 = -\gamma c_1 + \gamma c_2 + (\gamma - 1)c_3 =: c_7,$$

One can show that

• The only nonzero idempotents in \mathbb{A}_3 are $\{c_1,c_2,\ldots,c_7\}$, where $c_4:=-\gamma(c_1+c_2+c_3),$ and their spectrum is

$$\sigma(c_i) = \{1, \epsilon, \epsilon^2\}, \qquad \epsilon^2 + \epsilon + 1 = 0,$$

i.e. A₃ is isospectral and generic.

• The nonzero idempotents in A₃ are closed under multiplication:

$$\forall c_i, c_j \in \mathrm{Idm}(\mathbb{A}_3) \Rightarrow c_i c_j \in \mathrm{Idm}(\mathbb{A}_3).$$

Since the eigenvalue 1 is single, $c_i c_j \neq c_j, c_i$. Thus, \mathbb{A}_3 is a **quasigroup**.

• The idempotents, and therefore all elements of A_3 satisfy the medial algebra identity

 $(x \bullet y) \bullet (z \bullet w) = (x \bullet z) \bullet (y \bullet w), \quad \forall x, y, z, w \in \mathbb{A}.$

There are no two-dimensional subalgebras.

Proposition 2 (An explanation)

 \mathbb{A}_3 is an isotope of the associative algebra $\mathscr{A}_3:= \hbar[z]/(z^3-1)$ with the new multiplication

$$[p(z)] \bullet [q(z)] = [p(\epsilon z)q(\epsilon z)],$$

where $[\cdot]: \hbar[z] \to \hbar[z]/(z^3-1)$ is the standard projection.

Proof. Indeed, let ζ be a primitive root of unity of degree 7 and $\rho(z)$ be a polynomial of degree two with the Lagrange data $\rho(\epsilon^k) = \zeta^{2^{4-k}}$, k = 0, 1, 2. Then $[\rho(z)^m] \bullet [\rho(z)^m] = [\rho^{2m}(\epsilon z)]$ and

$$\rho^{2m}(\epsilon \epsilon^k) = (\zeta^{2^{4-k-1}})^{2m} = (\zeta^{2^{4-k}})^m = \rho^m(\epsilon^k),$$

thus $[\rho^m] \bullet [\rho^m] = [\rho^m]$, i.e. $c_m := [\rho^m(z)]$ is an idempotent in \mathscr{A}_3 , $1 \le m \le 7$. This gives the only possible 7 distinct nonzero idempotents in \mathscr{A}_3 . Note that $c_7 = [\rho^7 7] = [1]$ and $[\rho(\epsilon z)] = [\rho(z)^4]$, hence

$$c_m \bullet c_n = [\rho^m] \bullet [\rho^n] = [\rho^{m+n}(\epsilon z)] = [\rho(z)^{4(n+m)}] = c_{4(m+n)}$$

We have for the values at the node points:

$$\begin{array}{c|ccccc} & 1 & \epsilon & \epsilon^2 \\ \hline \rho & \zeta & \zeta^4 & \zeta^2 \\ \rho^2 & \zeta^2 & \zeta & \zeta^4 \\ \rho^4 & \zeta^4 & \zeta^2 & \zeta \end{array}$$

therefore $c_1 + c_2 + c_4 = [\rho + \rho^2 + \rho^4] = (\zeta + \zeta^2 + \zeta^4)[1] = \gamma c_7$ where $2\gamma^2 - \gamma + 1 = 0$. Also $c_1 \bullet c_2 = c_5$, $c_2 \bullet c_4 = c_3$ and $c_4 \bullet c_1 = c_6$, readily implying the claim.

Given a generic algebra $(\mathbb{A}, \bullet, \hbar)$ over a general field we address several principle questions:

- Do there exist any nonzero idempotents in A except for the 2ⁿ 1 postulated? (an expected answer: NO)
- Do there exist any 2-nilpotents in A? (an expected answer: NO)
- Is any proper subalgebra of A generic? (an expected answer: unclear)
- Given a proper subalgebra of A, how many idempotents it contain? (an expected answer: unclear)
- How many proper subalgebras of A in each dimension exist? For example, how many 2D subalgebras do exist?
- Do there exist some natural restrictions ("syzygies") for idempotents? (an expected answer: YES)
- Does there exist a natural stratification of the set of idempotents in generic algebras?

Syzygy: from Greek $\sigma u \zeta v \gamma u \alpha$ "conjunction, yoked together"

The classical syzygies:

Let us consider the matrix

$$X = \left(\begin{array}{ccc} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{array}\right)$$

where the x_{ij} are indeterminates over a field h. With $[ij] = x_{1i}x_{2j} - x_{1j}x_{2i}$, one has the Plücker relation

$$[12][34] - [13][24] + [14][23] = 0.$$

Let $P \in \mathbb{C}[x]$ be a polynomial of degree n with only simple roots a_1, \ldots, a_n and deg $H(x) \leq n - 1$. Then

$$\sum_{i=1}^{n} \frac{H(a_i)}{P'(a_i)} = 0$$

This is a partial case of what is called Euler-Jacobi formula, see below.

Generic algebras over complex numbers

Following to Segre (1938), given a commutative algebra \mathbb{A} over \mathbb{C} , let $x = \sum_i x_i e_i$ be an element written in a basis. The idempotent equation $\Psi(x) - x = 0$, where $\Psi := x^2$, becomes in the homogeneous coordinates

$$\{\mathbf{F}_{k}(\mathbf{x}) := \sum_{i,j=1}^{n} \gamma_{ijk} x_{i} x_{j} - x_{k} x_{0} = 0, \ 1 \le k \le n\}, \qquad \mathbf{x} = (x_{0} : x_{1} : \ldots : x_{n}) \in \mathbb{CP}^{n}.$$
(4)

Then $x_0 = 1$ (resp. $x_0 = 0$) corresponds to idempotents (resp. to 2-nilpotents). By the Bézout theorem, the quadratic system has exactly 2^n solutions in \mathbb{CP}^n , counted with multiplicities. Recall that a (finite) root is simple if and only if the Jacobi matrix of (F_1, \ldots, F_n) , where $F_k(x) = \mathbf{F}_k(1:x)$, is nondegerate at x.

Proposition 3

A commutative algebra \mathbb{A} over \mathbb{C} is generic iff (4) has exactly 2^n simple rootd.

Proof. Since A is commutative, the multiplication is uniquely determined from Ψ by polarization

$$2L(x)y = 2x \bullet y = \Psi(x+y) - \Psi(x) - \Psi(y) = (DF(x) + I)y,$$

hence

$$DF(x) = 2L(x) - I \tag{5}$$

hence det DF(x) = 0 whenever det(2L(x) - I) = 0.

We recall a classical result (a generalization of the residue theorem on multidimensional case).

Euler-Jacobi Formula

Let $F(x) = (F_1(x), \ldots, F_n(x))$ be a polynomial map and let \widetilde{F} be the polynomial map, whose components are the *highest homogeneous terms of the components* of F. Denote by $S_{\mathbb{C}}(F)$ the locus of all complex roots of $F_1(x) = F_2(x) = \ldots = F_n(x) = 0$ and suppose that any root $a \in S_{\mathbb{C}}(F)$ is simple and, furthermore, that $S_{\mathbb{C}}(\widetilde{F}) = \{0\}$. Then, for any polynomial h of degree less than the degree of the Jacobian: deg $h < N := -n + \sum_{i=1}^n \deg F_i$, one has

$$\sum_{a \in S(F)} \frac{h(a)}{\det[DF(a)]} = 0 \tag{6}$$

where $D(\cdot)$ denotes the Jacobi matrix.

In our notation:

$$\begin{split} F_k(\mathbf{x}) &= \sum_{i,j=1}^n \gamma_{ijk} x_i x_j - x_k, \qquad S_{\mathbb{C}}(F) = \{x : F_k(x) = 0\} \text{ is the set of all idempotents of } \mathbb{A}, \\ \widetilde{F}_k(\mathbf{x}) &= \sum_{i,j=1}^n \gamma_{ijk} x_i x_j, \qquad S_{\mathbb{C}}(\widetilde{F}) = \{x : \widetilde{F}_k(x) = 0\} \text{ is the set of 2-nilpotents of } \mathbb{A} \end{split}$$

By Bézout theorem, the total cardinality of $S_{\mathbb{C}}(F) \cup S_{\mathbb{C}}(\widetilde{F})$ is 2^n , hence for a generic algebra $S_{\mathbb{C}}(\widetilde{F}) = \emptyset$. Furthermore, in our case $N = -n + \sum_{i=1}^n \deg F_i = 2n - n = n$, so that we arrive at

The Syzygy Theorem (Krasnov Y., V.T., 2018)

Let \mathbb{A} be a generic commutative algebra over \mathbb{C} and $\mathrm{Idm}_0(A)$ be the set of its idempotents counting zero. If $H(x) : \mathbb{A} \to \hbar^s$ be a vector-valued polynomial map, $\deg H \leq n-1$, then

$$\sum_{e \in \mathrm{Idm}_0(A)} \frac{H(c)}{\chi_c(\frac{1}{2})} = 0,$$
(7)

In particular,

$$\sum_{\in \mathrm{Idm}(A)} \frac{c}{\chi_c(\frac{1}{2})} = 0.$$

Furthermore, if $\chi_c(t) = \det(tI - L(c))$ and $\operatorname{Idm}(\mathbb{A})$ the set of all nonzero idempotents of \mathbb{A} then

$$\sum_{c \in \mathrm{Idm}_0(\mathbb{A})} \frac{\chi_c(t)}{\chi_c(\frac{1}{2})} = 2^n, \quad \forall t \in \mathbb{C}.$$

Remark. In other words, if an algebra A is generic then its spectrum is overdetermined, i.e. satisfies syzygies. For n = 2 we have $\chi_{c_i}(t) = (t - 1)(t - \lambda_i)$, i = 1, 2, 3 (nonzero idempotents) and $\chi_0(t) = t^2$, hence $\chi_{c_i}(\frac{1}{2}) = (2\lambda_i - 1)/4$, implying the claims (i)-(ii) of Theorem 2 above

$$\frac{1}{1-2\lambda_1} + \frac{1}{1-2\lambda_2} + \frac{1}{1-2\lambda_3} = 1$$
$$\frac{c_1}{1-2\lambda_1} + \frac{c_2}{1-2\lambda_2} + \frac{c_3}{1-2\lambda_3} = 0.$$

Proof. Idempotents $c \in \mathbb{A}$ are exactly the zeroes of $F(x) = \Psi(x) - x$, hence by the Euler-Jacobi formula

$$0 = \sum_{c:F(c)=0} \frac{H(c)}{\det[DF(c)]} = \sum_{c\in\mathrm{Idm}(\mathbb{A})\cup\{0\}} \frac{H(c)}{(-2)^n \det[\frac{1}{2}I - L_c]} = (-2)^{-n} \sum_{c\in\mathrm{Idm}(\mathbb{A})\cup\{0\}} \frac{H(c)}{\chi_c(\frac{1}{2})}$$

which proves the first claim.

$$X_c(t) := \chi_c(\frac{1}{2} + t) = t^n - a_1 t^{n-1} + \ldots + (-1)^n a_n = \prod_{i=1}^n (t - t_i),$$

where a_k is an elementary symmetric function of the roots t_1, \ldots, t_n of $X_c(t)$. Then $a_1 = p_1$, $a_2 = \frac{1}{2}(p_1^2 - p_2)$ etc, by the Newton's identities, $a_k = T_k(p_1, \ldots, p_k)$, where the power sums

$$p_{i} = p_{i}(c) = t_{1}^{i} + \ldots + t_{n}^{i} = \left/ \text{by (5):} \quad \chi_{c}(\frac{1}{2} + t) = \det(tI - \frac{1}{2}DF(c))\right/ = 2^{-i}\operatorname{tr}(DF(c))^{i}.$$
$$h_{k}(c) = T_{k}(\operatorname{tr} DF(c), \ldots, \operatorname{tr}(DF(c))^{k}) = 2^{k}a_{k}(c).$$

Therefore the Euler-Jacobi formula yields for a fixed $k \leq n-1$ (note that $k \neq n$):

$$\sum_{c \in \mathrm{Idm}_0(A)} \frac{h_k(c)}{\chi_c(\frac{1}{2})} = 2^k \sum_{c \in \mathrm{Idm}_0(A)} \frac{a_k(c)}{\chi_c(\frac{1}{2})} = 0 \quad \Rightarrow \quad \sum_{c \in \mathrm{Idm}_0(A)} \frac{X_c(t) - \chi_c(\frac{1}{2})}{\chi_c(\frac{1}{2})} = 0$$

Definition 8

An algebra is called isospectral if all its idempotents have the same spectrum.

The algebras in examples 4, 5 and 7 are isospectral. But their structure is quite different. Note that the latter example is a generic algebra. In such a case, the syzygy theorem helps to understand which spectrum is possible for generic algebras.

To this end suppose we have a generic algebra \mathbb{A} of dimension $n \geq 2$. Then

$$\begin{split} 2^n &= \sum_{c \in \mathrm{Idm}(\mathbb{A})} \frac{\chi_c(t)}{\chi_c(\frac{1}{2})} = /\mathrm{totally} \; 2^n - 1 \; \mathrm{equal \; terms} / \\ &= (2^n - 1) \frac{\chi(t)}{\chi(\frac{1}{2})} + \frac{t^n}{(1/2)^n}, \end{split}$$

therefore

$$\frac{\chi(t)}{\chi(\frac{1}{2})} = \frac{2^n}{2^n - 1}(1 - t^n) = \frac{t^n - 1}{(1/2)^n - 1}$$

implying that the common characteristic polynomial is $\chi(t) = t^n - 1$.

Theorem 3 (Krasnov Y., V.T., 2018)

If \mathbb{A} is an isospectral generic algebra then its spectrum is generated by a primitive root of unity of degree n.

Remark. It is however completely unclear how to characterize the spectrum of **nongeneric** isospectral algebras. The only known examples come from Hsiang algebras (algebras of minimal cones), with the Peirce spectrum $\{1, -1, -\frac{1}{2}, \frac{1}{2}\}$.

The proof of the syzygy theorem crucially depends on the Euler-Jacobi formula. Clearly, its statement is a typical result "**in large**", i.e. it depends on a certain completeness, like Cauchy residue theorem, where the sum of residues is taken over "all" zeroes, counting multiplicities. Of course, the statement holds true for algebras over subfields of complex numbers.

On the other hand, the **statement** of the syzygy theorem does not depend on a field. Therefore we believe that the general fact holds still true for generic algebras (maybe some natural assumption on the characteristic of k should be added):

Conjecture

Let \mathbb{A} be a generic commutative algebra over an arbitrary field \hbar . If $H(x) : \mathbb{A} \to \hbar^s$ be a vector-valued polynomial map, $\deg H \leq n-1$, then

$$\sum_{e \in \mathrm{Idm}_0(A)} \frac{H(c)}{\chi_c(\frac{1}{2})} = 0.$$
 (8)

Any generic algebra over $\mathbb C$ contains exactly 2^n regular idempotents and does not contain 2-nilpotents.

Lemma 3

Let $(\mathbb{A}, \bullet, \hbar)$ be a generic algebra over $\hbar = \mathbb{C}$, dim $\mathbb{A} \ge 3$. If two distinct nonzero idempotents generate a two-dimensional subalgebra then this subalgebra contains exactly three nonzero idempotents and it is generic.

Proof. Let $c_1 \neq c_2$ be the two nonzero idempotents generating the two dimensional subalgebra $\operatorname{span}(c_1, c_2)$. Then $c_1 \bullet c_2 = \mu_2 c_1 + \mu_1 c_2$, where μ_i are eigenvalues of $L(c_i)$, hence $\mu_i \neq \frac{1}{2}$. Since

$$(c_1 - 2\mu_1 c_2)^2 = c_1 - 4\mu_1(\mu_2 c_1 + \mu_1 c_2) + 4\mu_1^2 c_2 = (1 - 4\mu_1 \mu_2)c_1,$$

therefore by the assumptions, $1 - 4\mu_1\mu_2 \neq 0$, hence there exist uniquely determined $x_1, x_2 \neq 0$ such that

$$(x_1c_1 + x_2c_2)^2 = x_1\underbrace{(x_1 + 2x_2\mu_2)}_{=1}c_1 + x_2\underbrace{(2x_1\mu_1 + x_2)}_{=1}c_2 = x_1c_1 + x_2c_2$$

Proposition 2. Let $(\mathbb{A}, \bullet, \hbar)$ be an algebra, \mathscr{A} be its two-dimensional generic subalgebra, and \mathscr{B} is an arbitrary subalgebra of \mathbb{A} . Then either of the following holds:

- $\mathscr{A} \cap \mathscr{B} = 0;$
- $\mathscr{A} \subset \mathscr{B};$

• $\mathscr{A} \cap \mathscr{B}$ is a one-dimensional subspace spanned by a nonzero idempotent.

Proof. Since $\mathscr{A} \cap \mathscr{B}$ is a subalgebra of \mathscr{A} , Theorem 2 implies the conclusion.

Corollary 2. Let $(\mathbb{A}, \bullet, \hbar)$ be an algebra, $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be its two-dimensional generic subalgebras. Then

Example 9

Let \hbar^3 be the standard direct product algebra with (orthogonal) basis e_1, e_2, e_3 . Then it contains exactly $8 = 2^3$ distinct idempotents

$$c_{\alpha} = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \qquad \alpha \in \mathbb{F}_2^3$$

There are exactly 6 two-dimensional subalgebras, all generic: $\mathbb{E}_{ij} = \operatorname{span}(e_i, e_j)$, $\mathbb{D}_{ij} = \operatorname{span}(e_i + e_j, c_{111})$. Any subalgebra contains exactly the following idempotents:

(100)	(010)	(110)
(100)	(001)	(101)
(100)	(011)	(111)
(010)	(001)	(011)
(010)	(101)	(111)
(001)	(110)	(111)
(011)	(101)	(110)



 ${\bf Q} {\bf uestion:}$ do there exist a generic algebra such that all its two-dimensional subalgebras form the coincidence relations isomorphic to the Fano plane?



Note that S_3 acts naturally on (k^3, \bullet) . Given $\sigma \in S_3$, let (k^3, \bullet_{σ}) be the corresponding inner isotope. If $\epsilon^3 = e$ then $(k^3, \bullet_{\epsilon})$ is isomorphic to isospectral algebra in Example 7.

Inner isotopes and medial algebras

We come back to Example 7 and its explanation in Proposition 2. Note that in the latter case, the substitution $P(z) \rightarrow P(\epsilon z)$ is an **automorphism** of the commutative associative algebra $\mathscr{A}_3 := \hbar[z]/(z^3 - 1)$. This is a part of a general construction in arXiv: 2308.16284.

Definition. Given (\mathscr{A}, \bullet) and $h \in Aut(\mathscr{A})$ its inner isotope is the algebra with the new multiplication

 $x \bullet_h y := h(x \bullet y) = h(x) \bullet h(y).$

Proposition 4

Let (\mathbb{A}, \bullet) satisfy $\mathbb{A} \bullet \mathbb{A} = \mathbb{A}$ and $h, f \in Aut(\mathbb{A}, \bullet)$. Then (\mathbb{A}, \bullet_h) is isomorphic to (\mathbb{A}, \bullet_f) if and only if h and f conjugate in $Aut(\mathbb{A}, \bullet)$.

An important corollary of the definition of medial algebra is that

 $(x \bullet y) \bullet (x \bullet y) = (x \bullet x) \bullet (y \bullet y).$

which implies that the product of two idempotents in a medial algebra is an idempotent again. A medial algebra $(\mathbb{A}, *)$ is called **special** if the set idempotents c with L(c) invertible is non-empty.

Proposition 5

Any inner isotope (\mathbb{A}, \bullet_h) of a (unital) commutative associative algebra (\mathbb{A}, \bullet) is a (special) medial algebra.

Proof.

$$(x \bullet_h y) \bullet_h (z \bullet_h w) = h(h(x) \bullet h(y)) \bullet h(h(z) \bullet h(w)) = h^2(x) \bullet h^2(y) \bullet h^2(z) \bullet h^2(w).$$

Let us define $(\mathcal{C}_n, \bullet) := (\hbar[z]/(z^n - 1), \bullet)$. Given $\sigma \in S_n$ let $\sigma = \sigma_1 \dots \sigma_r$ be its disjoint cycle decomposition and $s_i = |\sigma_i|$. We assume that \hbar is a splitting field for all polynomials $z^t - 1$, where $t \in \{n, s_1, \dots, s_r, 2^{s_1} - 1, \dots, 2^{s_r} - 1\}$.

Theorem 4

Let a permutation $\sigma \in S_n$ have the disjoint cycle decomposition $\sigma = \sigma_1 \dots \sigma_r$. Then

- (a) an inner isotope $(\mathcal{C}_n, \bullet_{\tau})$ is isomorphic to $(\mathcal{C}_n, \bullet_{\sigma})$ if and only if τ has the same cyclic type as σ ;
- (b) there are exactly 2ⁿ distinct idempotents in (C_n, •_σ) and they are naturally divided in 2^r classes I_α, enumerated by binary codes α ∈ ℝ^{×r}₂.
- (c) For any idempotent $c \in I_{\alpha}$, the characteristic polynomial of $L_{\bullet_{\sigma}}(c)$ is given by

$$\chi_c(\lambda) = \prod_{i=1}^r (\lambda^{|\sigma_i|} - \alpha(i)).$$
(9)

- (d) If σ consists of one cycle (i.e. σ is a shift), the set of idempotents of (C_n, \bullet_{σ}) form a commutative medial idempotent quasigroup w.r.t. the original multiplication.
- (e) the algebra $(\mathcal{C}_n, \bullet_{\sigma})$ is generic.

Remark. Deunitalization of \hbar^n : the latter can be thought of as various 'bifurcations' of the unital algebra \hbar^n by inner isotopy.

Three possibilities for n = 3

By Prop. 4, inner isotopes are isomorphic iff the corresponding elements of S_3 conjugate, hence there are exactly three distinct inner isotopes coded by the conjugacy classes of S_3 .

- The unity in S₃ gives rise to a unital commutative associative algebra, see Example 9.
- The conjugacy class of τ = (2 3 1) ∈ S₃, i.e. (left or right) shifts, give rise to a commutative isospectral medial algebra, see Example 7. This algebra has many remarkable properties. For instance,

$$(x \bullet_{\tau} (x \bullet_{\tau} (x \bullet_{\tau} y))) = \Delta(x)y$$

where Δ is a multiplicative homogeneous degree 3:

$$\Delta(a_0e_0 + a_1e_1 + a_2e_2) = \begin{vmatrix} a_0 & a_2 & a_1 \\ a_1 & a_0 & a_2 \\ a_2 & a_1 & a_0 \end{vmatrix}$$

 $(\mathcal{C}_3, \bullet_\tau)$ is a (primitive) axial algebra and the set of nonzero idempotents $\mathrm{Idm}(\mathcal{C}_3, \bullet_\tau)$ is a medial idempotent quasigroup.

Theorem 10 (V.T., 2023)

In the above notation, the idempotent quasigroup and the algebra automorphism groups are:

$$\operatorname{Aut}(\operatorname{Idm}(\mathcal{C}_3, \bullet_{\tau})) \cong \mathbb{Z}_7 \rtimes_{\operatorname{id}} \mathbb{Z}_7^{\times}$$
$$\operatorname{Aut}(\mathcal{C}_3, \bullet_{\tau}) \cong \mathbb{Z}_7 \rtimes_{\delta} \mathbb{Z}_3^{\times},$$

where $\delta(i) = 2^i : \mathbb{Z}_3 \to \mathbb{Z}_7^{\times}$.

Three possibilities for n = 3

• When $\omega = (21)(3) \in S_3$, the corresponding inner isotope is a generic Harada-type algebra. Its idempotents relation can be described as follows:

$$\chi_0 = \lambda^3,$$

$$\chi_1 = \chi_2 = \chi_3 = (\lambda - 1)(\lambda + 1)\lambda,$$

$$\chi_4 = (\lambda - 1)\lambda^2,$$

$$\chi_5 = \chi_6 = \chi_7 = (\lambda - 1)^2(\lambda + 1)$$

such that the syzygy relations become

$$\sum_{c} \frac{\chi_c(\lambda)}{\chi_c(\frac{1}{2})} = \frac{\lambda^3}{\frac{1}{2^3}} + 3 \cdot \frac{(\lambda^2 - 1)\lambda}{-\frac{3}{2^3}} + \frac{(\lambda - 1)\lambda^2}{-\frac{1}{2^3}} + 3 \cdot \frac{(\lambda^2 - 1)(\lambda - 1)}{\frac{3}{2^3}} = 2^3.$$

The multiplication table of the idempotents is given explicitly by

\bullet_{ω}	1	2	3	4	5	6	7
1	1	3	2	0	1	3	2
2	3	2	1	0	3	2	1
3	2	1	3	0	2	1	3
4	0	0	0	4	4	4	4
5	1	3	2	4	5	7	6
6	3	2	1	4	7	6	5
7	2	1	3	4	6	5	7

where $X := \operatorname{span}(c_1, c_2) = \operatorname{span}(c_1, c_2, c_3)$ is the 2D Harada-Hsiang algebra from Example 4.

THANK YOU FOR YOUR ATTENTION!

