Metrized algebras and their applications

Vladimir G. Tkachev

Linköping University, Sweden

Universidade da Beira Interior, Covilha, July 9th, 2024

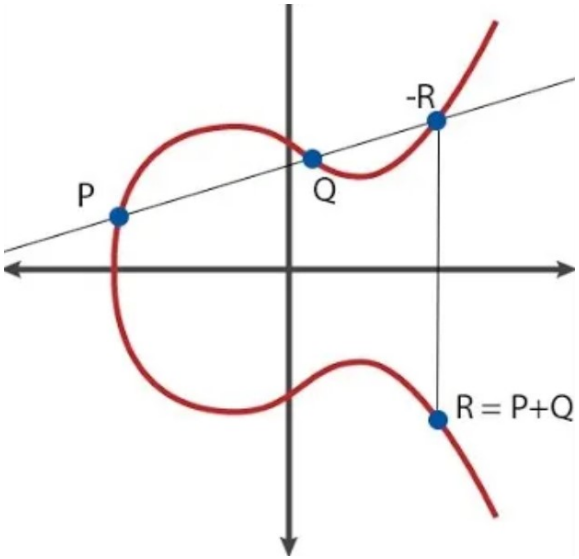
1. Motivations
2. Metrized algebras
3. Idempotents in metrized algebras
4. Hsiang algebras

Outline

1. Motivations
2. Metrized algebras
3. Idempotents in metrized algebras
4. Hsiang algebras

Why cubic forms?

- Classically known after Euler in two variables over complexes or arbitrary fields: the points obey a commutative associative operation with a unit, an **abelian group**



- Some further developments in three and more variables are known like loops etc (for example, Yu. Manin, Cubic forms)

But in fact, we shall discuss a different matter. The subject of metrized algebras is essentially a special approach to study cubic forms in presence of a certain bilinear form.

Main ideas

- We are interested in cubic forms via various problems from PDEs and Diff.Geometry
- Remarkably, these forms are connected to distinguished algebraic structures.
- To recover the algebraic structure, one replace a cubic form by a **metrized algebra**.
- The correspondence between cubic forms and metrized algebra is categorical.
- The metrized algebras in this context are **commutative** but nonassociative.
- Many problems lead to distinguished algebraic structures (eg. Clifford and Jordan). But the Jordan identity is difficult to recover, instead one search for an **admissible** cubic form
- The Freudenthal -Springer construction and the Peirce decomposition are principal tools in this analysis
- The classification of Jordan-v. Neumann-Wigner is crucial for further identifications

Three equivalent problems

(DG) (Hsiang, 1967) Find all algebraic homogeneous degree three **minimal cones** $u(x) = 0$ in $\mathbb{R}^n$.

(PDE) It is equivalent to finding all homogeneous degree 3 polynomial solutions to the nonlinear PDE

$$|\nabla u|^2 \Delta u - \frac{1}{2} \nabla u(x) \cdot \nabla(|\nabla u(x)|^2) = k\langle x; x\rangle u(x)$$

(NA) (V.Tk., 2013) This PDE induces a metrized commutative nonassociative algebra on $\mathbb{R}^n$, satisfying

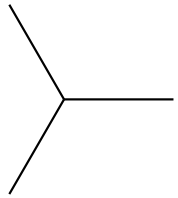
$$\begin{aligned} &\operatorname{tr} L_x = 0 \\ &x^3 x + \tfrac{1}{4} x^2 x^2 - |x|^2 x^2 - \tfrac{2}{3}\langle x^2; x\rangle x = 0. \end{aligned}$$

The correspondence between (PDE) and (NA):

$$u(x) = \tfrac{1}{6}\langle x; x^2\rangle \qquad \leftrightarrows \qquad \text{a 'composition law'} \quad x \bullet y = \operatorname{Hess} u(x) y$$

Three equivalent problems. An example in $\mathbb{R}^2$

(DG) A minimal cone (Steiner tree) of degree three in $\mathbb{R}^2$, $u(x_1, x_2) := x_1^3 - 3x_2^2 x_1 = 0$:



(PDE) $u(x_1, x_2) = \mathrm{Re}(x_1 + ix_2)^3$ satisfies (PDE) with $k = -54$:

$$|\nabla u|^2 \Delta u - \tfrac{1}{2}\nabla u(x) \cdot \nabla(|\nabla u(x)|^2) = -3 \cdot 18(x_1^2 + x_2^2)u(x)$$

(NA) A commutative algebra V generated by two idempotents c_1, c_2 satisfying $c_1 \bullet c_2 = -c_1 - c_2$. It is a metrized algebra of **Jordan type**, the invariant form $\langle ; \rangle$ is uniquely defined by $\langle c_i; c_i \rangle = 1$, $\langle c_1; c_2 \rangle = -\frac{1}{2}$.

To connect (NA) with (DG) and (NA), we define an ON basis in V: $e_1 = c_1$ and $e_2 = \frac{1}{\sqrt{3}}(c_1 + 2c_2)$. Then $e_1 \bullet e_2 = -e_1$ and $e_2 \bullet e_2 = -e_1$, so that if $x = x_1 e_1 + x_2 e_2$ then

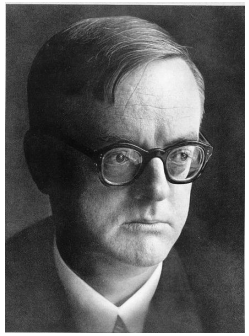
$$x^2 = (x_1^2 - x_2^2)e_1 - 2x_1 x_2 e_2$$

and $u(x) := \langle x; x^2 \rangle = u(x_1, x_2) = x_1^3 - 3x_2^2 x_1$.

Jordan algebras

In 1932 Pascual Jordan proposed a program to discover a new algebraic setting for quantum mechanics such that it is independent on an invisible but all-determining matrix structure. How to multiply two Hermitian matrices?... Unfortunately

$$\text{Matrix} \cdot \text{Matrix} = \text{Matrix},$$
$$\text{Hermitian} \cdot \text{Hermitian} \neq \text{Hermitian}$$



Jordan replaces the matrix multiplication by the anticommutator

$$x \bullet y = \frac{1}{2}(xy + yx)$$

A Jordan algebra is vector space defined equipped with a bilinear product $\bullet$ satisfying

$$x \bullet y = y \bullet x \qquad \text{Commutativity}$$
$$x^2 \bullet (x \bullet x) = x \bullet (x^2 \bullet y) \qquad \text{the Jordan identity}$$

For still unclear mystical reasons, all simple (irreducible) formally real Jordan algebras appears in many problems of Geometry, Algebra, Analysis, PDEs, Combinatorics and Theory Group.

Example. Jordan algebra of symmetric matrices

Let $\mathbb{A}$ be the 3D Jordan algebra of **real symmetric** 2×2-matrices with the Jordan multiplication $x \bullet y := \frac{1}{2}(xy + yx)$. Since $x \bullet x = xx = x^2$ and $x^2 \bullet y = \frac{1}{2}(x^2y + yx^2)$,

$$x \bullet (x^2 \bullet y) = \tfrac{1}{4}(x(x^2y + yx^2) + (x^2y + yx^2)x) = \tfrac{1}{4}(x^3y + xyx^2 + x^2yx + yx^3)$$
$$= x^2 \bullet (x \bullet y)$$

Let

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then $e_1 \bullet e_1 = e_1$, i.e. e_1 is an idempotent and:

$\bullet$	e_1	e_2	e_3
e_1	e_1	0	$\frac{1}{2}e_3$
e_2	0	e_2	$\frac{1}{2}e_3$
e_3	$\frac{1}{2}e_3$	$\frac{1}{2}e_3$	$e_1 + e_2$

Hence $\{e_1, e_2, e_3\}$ is an eigenbasis of $L(e_1)$, $\mathbb{A} = \mathbb{A}_1 \oplus \mathbb{A}_0 \oplus \mathbb{A}_{\frac{1}{2}}$ with $\mathbb{A}_1 = \langle e_1 \rangle$, $\mathbb{A}_0 = \langle e_2 \rangle$ and $\mathbb{A}_{\frac{1}{2}} = \langle e_3 \rangle$.

$\star$	1	0	$\frac{1}{2}$
1	1	$\emptyset$	$\frac{1}{2}$
0	$\emptyset$	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1, 0$

This example illustrates an important general fact:
$\mathbb{A}$ satisfies the **Jordan-type fusion law**:

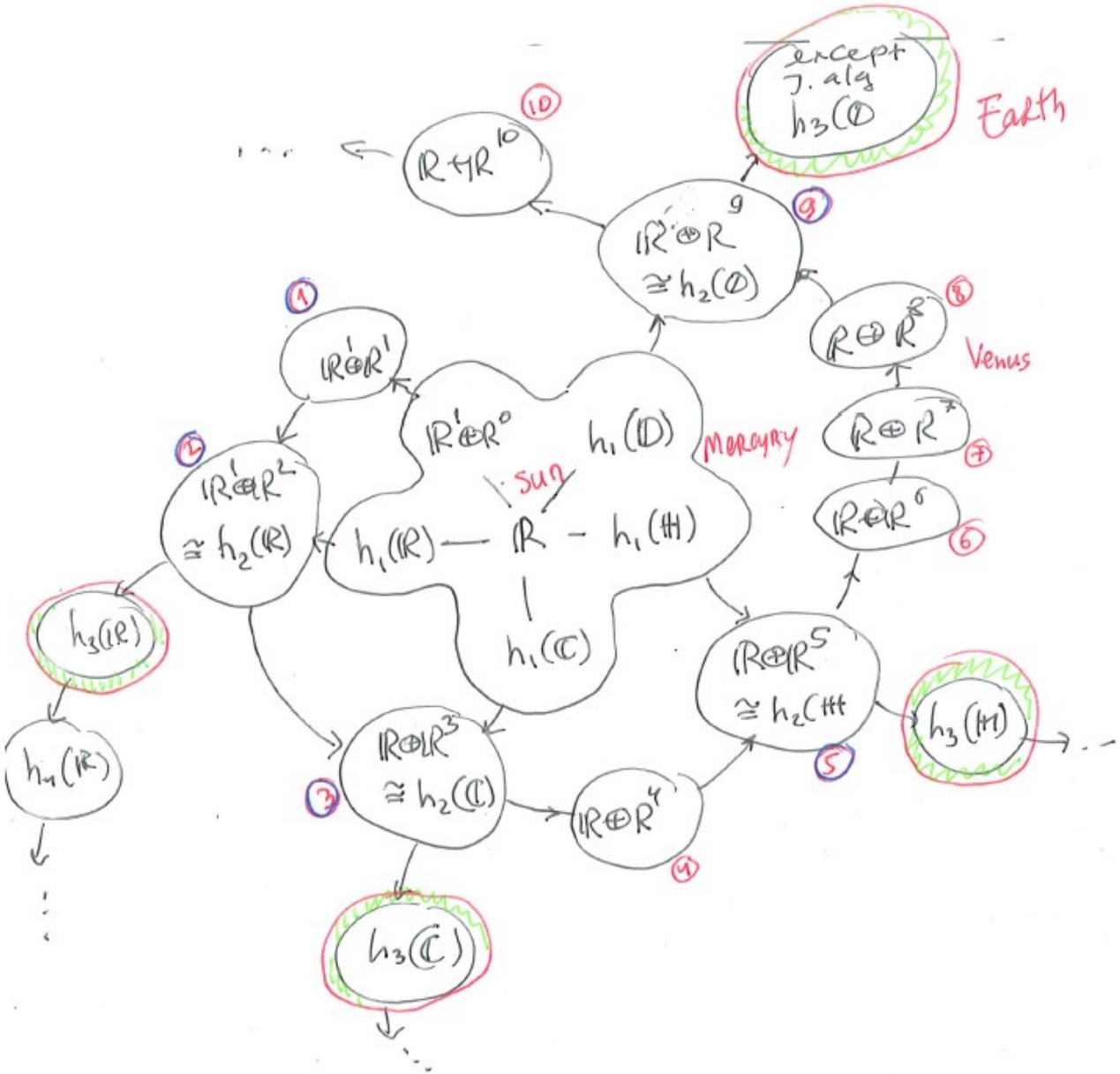
Classification of formally real Jordan algebras

P. Jordan, J. von Neumann, E. Wigner, *Annals of Math.*, **1934**

Any (finite-dimensional) formally real Jordan algebra is a direct sum of the following simple ones:

- Three special algebras coming from Hermitian matrices:
  - $\mathfrak{h}_n(\mathbb{R})$;
  - $\mathfrak{h}_n(\mathbb{C})$;
  - $\mathfrak{h}_n(\mathbb{H})$;
- Two new structures which met the Jordan axioms but were not themselves Hermitian matrices:
  - spin_n, the spin factors (not to be confused with *spinors*);
  - $\mathfrak{h}_3(\mathbb{O})$, nonassociative hermitian matrices of size 3×3 over the octonions $\mathbb{O}$, (also known as the Albert algebra)

The Jordan Universum



The Peirce decomposition

Sometimes there is a decomposition of an algebra into bigger 'blocks'

$$\mathbb{A} = \bigoplus_{\alpha \in I} \mathbb{A}_\alpha$$

such that the multiplication satisfies relatively simple **fusion rules** between the blocks:

$$\mathbb{A}_\alpha \mathbb{A}_\beta \subset \bigoplus_{\gamma \in \theta(\alpha,\beta)} \mathbb{A}_\gamma, \qquad \theta(\alpha,\beta) \text{ is a fusion rule}$$

The principal example is the Peirce decomposition relative to an idempotent c.

Example

Any Jordan algebra satisfy

$\star$	1	0	$\frac{1}{2}$
1	1	$\emptyset$	$\frac{1}{2}$
0	$\emptyset$	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1,0$

Here 0, $\frac{1}{2}$ and 1 are the Peirce numbers, i.e. the eigenvalues of $L(c)$.

Outline

1. Motivations
2. Metrized algebras
3. Idempotents in metrized algebras
4. Hsiang algebras

Metrized algebras: motivations

Let $u(x)$ be a function (or a polynomial) on a vector space $\mathbb{A}$. Note that the **differential** of $u(x)$ has an invariant meaning, while the **gradient** (covariant derivative) appears in the presence of some **metric**. More precisely,

$$u(x+h) = u(x) + \omega_1(h) + \text{terms of higher order},$$

where $\omega_1(h) \in \mathbb{A}^*$ is a linear form, called the **differential** of u at x, denoted by $Du(x)(h)$

If $b(x,y)$ is a bilinear symmetric **non-degenerate** form (sometimes is called **metric**) then $\mathbb{A}^* \cong \mathbb{A}$, where the isomorphism is given for any $\omega \in \mathbb{A}^*$ by

$$\omega(h) = b(z_\omega, h), \quad \forall h \in \mathbb{A}.$$

This isomorphism is called the **duality**. Thus, in the presence of a metric b, the differential (as an element in $\mathbb{A}^*$) has its analogue, called the **gradient** $\nabla u(x)$:

$$Du(x)(h) = b(\nabla u(x),\, h)$$

For polynomials,these concepts have a clear algebraic meaning. The differential is recovered by **linearization**. We explain this using the example of a cubic form.

Metrized algebras

Consider a vector space $\mathbb{A}$ with a distinguished metric $b(x,y) = \langle x; y\rangle$. Recall that a homogeneous degree 3 function u on $\mathbb{A}$ is called a **cubic form** if

$$U(x,y,z) = u(x+y+z) - u(x+y) - u(x+z) - u(y+z) + u(x) + u(y) + u(z)$$

is a **symmetric trilinear form** on $\mathbb{A}$. Note that

$$U(x,x,x) = u(3x) - 3u(2x) + 3u(x) = (3^3 - 3\cdot 2^3 + 3)\cdot u(x) = 6u(x)$$

hence

$$u(x) = \tfrac{1}{6}U(x,x,x).$$

To determine the differential, we apply the above to $x = y$:

$$\begin{aligned} U(x,x,z) &= u(2x+z) - u(2x) - 2u(x+z) + 2u(x) + u(z) \\ &= u(2x) + Du(2x)(z) - u(2x) - 2u(x) - 2Du(x)(z) + 2u(x) + z^2 \ldots \\ &= 2Du(x)(z) + 0 \quad \text{(notice that the LHS is linear in } z\text{!)} \end{aligned}$$

Therefore

$$Du(x)(z) = \tfrac{1}{2}U(x,x,z) = b(\nabla u(x),\, z)$$

Similarly, one obtains for the second differential

$$D^2u(x)(y,z) = U(x,y,z) = b(x, .?.)$$

Metrized algebras

In the above notation with $b = \langle ; \rangle$, let us define a **multiplication** on $\mathbb{A}$ by virtue of

$$D^2u(x)(y,z) = U(x,y,z) = \langle x; y \bullet z \rangle. \tag{1}$$

Similarly to the duality between the 1st differential and gradient, there is the duality between the 2nd differential and the Hessian:

$$D^2u(x)(y,z) = \langle x; \operatorname{Hess} u(y)z \rangle.$$

Using the duality, we find an explicit expression for the new multiplication

$$y \bullet z := \operatorname{Hess} u(y)z.$$

An important corollary

Thus defined algebra on $\mathbb{A}$ is denoted by $\mathrm{CMA}(u)$. Since U is symmetric trilinear form, this implies the following important properties:

(i) $x \bullet y = y \bullet x$ (the algebra $\mathbb{A}(u)$ is **commutative**)

(ii) $\langle x \bullet y; z \rangle = \langle y; x \bullet z \rangle$ ($\mathbb{A}(u)$ is a **Euclidean metrized algebra**)

(iii) $\langle x \bullet x; x \rangle = 6u(x)$ $\left(\frac{1}{6}b(x \bullet x,\, x) = u(x)\right)$

(iv) $x \bullet x = 2\nabla u(x)$ $\left(\frac{1}{2}b(x \bullet x,\, z) = \frac{1}{2}U(x,x,z) = b(\nabla u(x),\, z)\right)$

Examples

- **A trivial example.** $V = \mathbb{R}^1$ with $\langle x; y\rangle = xy$ and $u(x) = \frac{1}{6}x^3$. One has

$$U(x; y, z) = xyz, \qquad \Rightarrow \qquad x \bullet y = xy$$

- **A less trivial example (a Jordan spin-factor)** Let $V = \mathbb{R}^2$ with the standard Euclidean inner product $\langle x; y\rangle$. Let $u(x) = \frac{1}{2}x_1^2 x_2$. Then

$$U(x; y; z) = x_2 y_1 z_1 + x_1 y_2 z_1 + x_1 y_1 z_2 \quad \Rightarrow \quad x \bullet y = (x_1 y_2 + x_2 y_1,\, x_1 y_1)$$

- **A non-trivial example (H. Freudenthal 1954, T. Springer 1961)** Let $V = \mathscr{H}_3(\mathbb{F}_d)$ be the vector space of *self-adjoint* 3×3-matrices with coefficients in a normed division algebra $\mathbb{F}_d$, obtained by the **Freudenthal-Springer construction** from the cubic form

$$N(x) := \tfrac{1}{6}((\operatorname{tr} x)^3 - 3 \operatorname{tr} x \operatorname{tr} x^2 + 2 \operatorname{tr} x^3).$$

$V(u)$ is a Jordan algebra w.r.t. the multiplication $x \bullet y = \frac{1}{2}(xy + yx)$, where

$$u(x) := \tfrac{1}{6} \operatorname{tr} x^3$$

Examples

- **A trivial example.** $V = \mathbb{R}^1$ with $\langle x; y\rangle = xy$ and $u(x) = \frac{1}{6}x^3$. One has

$$U(x; y, z) = xyz, \qquad \Rightarrow \qquad x \bullet y = xy$$

- **A less trivial example (a Jordan spin-factor)** Let $V = \mathbb{R}^2$ with the standard Euclidean inner product $\langle x; y\rangle$. Let $u(x) = \frac{1}{2}x_1^2 x_2$. Then

$$U(x; y; z) = x_2 y_1 z_1 + x_1 y_2 z_1 + x_1 y_1 z_2 \quad \Rightarrow \quad x \bullet y = (x_1 y_2 + x_2 y_1,\, x_1 y_1)$$

- **A non-trivial example (H. Freudenthal 1954, T. Springer 1961)** Let $V = \mathscr{H}_3(\mathbb{F}_d)$ be the vector space of *self-adjoint* 3×3-matrices with coefficients in a normed division algebra $\mathbb{F}_d$, obtained by the **Freudenthal-Springer construction** from the cubic form

$$N(x) := \tfrac{1}{6}((\operatorname{tr} x)^3 - 3 \operatorname{tr} x \operatorname{tr} x^2 + 2 \operatorname{tr} x^3).$$

$V(u)$ is a Jordan algebra w.r.t. the multiplication $x \bullet y = \frac{1}{2}(xy + yx)$, where

$$u(x) := \tfrac{1}{6} \operatorname{tr} x^3$$

Examples

- **A trivial example.** $V = \mathbb{R}^1$ with $\langle x; y\rangle = xy$ and $u(x) = \frac{1}{6}x^3$. One has

$$U(x; y, z) = xyz, \qquad \Rightarrow \qquad x \bullet y = xy$$

- **A less trivial example (a Jordan spin-factor)** Let $V = \mathbb{R}^2$ with the standard Euclidean inner product $\langle x; y\rangle$. Let $u(x) = \frac{1}{2}x_1^2 x_2$. Then

$$U(x; y; z) = x_2 y_1 z_1 + x_1 y_2 z_1 + x_1 y_1 z_2 \quad \Rightarrow \quad x \bullet y = (x_1 y_2 + x_2 y_1,\, x_1 y_1)$$

- **A non-trivial example (H. Freudenthal 1954, T. Springer 1961)** Let $V = \mathscr{H}_3(\mathbb{F}_d)$ be the vector space of *self-adjoint* 3×3-matrices with coefficients in a normed division algebra $\mathbb{F}_d$, obtained by the **Freudenthal-Springer construction** from the cubic form

$$N(x) := \tfrac{1}{6}((\operatorname{tr} x)^3 - 3 \operatorname{tr} x \operatorname{tr} x^2 + 2 \operatorname{tr} x^3).$$

  $V(u)$ is a Jordan algebra w.r.t. the multiplication $x \bullet y = \frac{1}{2}(xy + yx)$, where

$$u(x) := \tfrac{1}{6} \operatorname{tr} x^3$$

Example. Consider $u(x) = \frac{1}{6}x_1(x_1^2 - 3x_2^2 - 3x_3^2)$ on $\mathbb{A} = \mathbf{K}^3$. Then

$$\operatorname{Hess} u(x) := D^2u(x) = \begin{pmatrix} x_1 & -x_2 & -x_3 \\ -x_2 & -x_1 & 0 \\ -x_3 & 0 & -x_1 \end{pmatrix}$$

The algebra $\mathbb{A}(u)$ on $\mathbf{K}^3$ with commutative multiplication

$$x \bullet y := \operatorname{Hess} u(x)y = (x_1y_1 - x_2y_2 - x_3y_3,\ -x_2y_1 - x_1y_2,\ x_3y_1 - x_1y_3)$$

has the following **Peirce decomposition** and **fusion rules**:

$\bullet$	e_1	e_2	e_3
e_1	e_1	$-e_2$	$-e_3$
e_2	$-e_2$	$-e_1$	0
e_3	$-e_3$	0	e_1

$\bullet$	$\mathbb{A}_1$	$\mathbb{A}_{-1}$
$\mathbb{A}_1$	$\mathbb{A}_1$	$\mathbb{A}_{-1}$
$\mathbb{A}_{-1}$	$\mathbb{A}_{-1}$	$\mathbb{A}_1$

$$\mathbb{A} = \mathbb{A}_1 \oplus \mathbb{A}_{-1}, \qquad \mathbb{A}_\lambda = \{x : e_1 \bullet x = \lambda x\}$$

A simple examination reveals the identities in the spirit of "3D-complex numbers":

$$x \bullet x = (x_1^2 - x_2^2 - x_3^2,\ -2x_1x_2,\ -2x_1x_3)$$
$$(x \bullet x) \bullet x = (x_1^2 + x_2^2 + x_3^2)x = \langle x; x\rangle x$$

The dictionary

In summary: the algebra $\mathrm{CMA}(u)$ is commutative, nonassociative, metrized and

- $u(x) = \frac{1}{6}\langle x; x \bullet x\rangle$
- $\nabla u(x) = \frac{1}{2} x^{\bullet 2}$
- $\operatorname{Hess} u(x)\, y = x \bullet y$
- $\operatorname{Hess} u(x) = L_\bullet(x)$ the multiplication operator by x
- $L_\bullet(x)$ is self-adjoint: $\langle L_\bullet(x) y; z\rangle = \langle y; L_\bullet(x) z\rangle$

The latter property implies that if $\mathbf{K} = \mathbb{R}$ then $L_\bullet(x)$ is diagonalizable.

Outline

1. Motivations
2. Metrized algebras
3. Idempotents in metrized algebras
4. Hsiang algebras

Idempotents

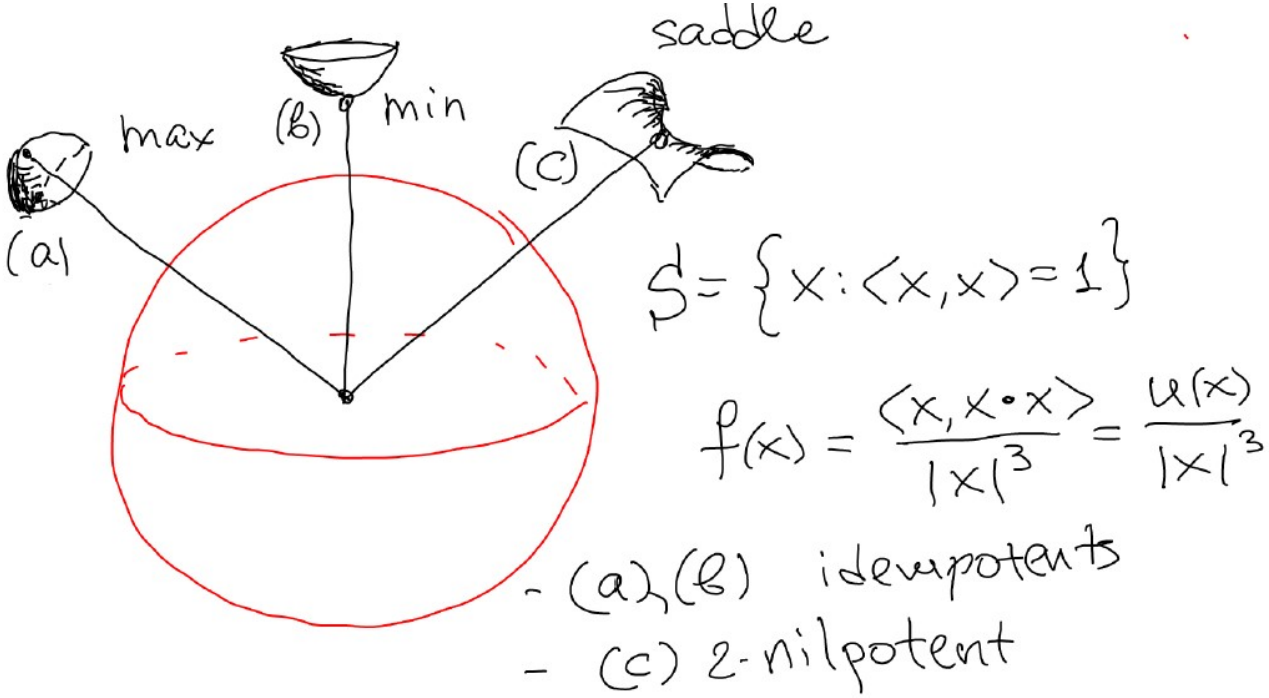
In order to study finer properties of an algebra it is natural to look at its idempotents. Below are some examples of **idempotents** outside the standard algebraic context:

- A **projection operator** (i.e. $P^2 = P$) in an associative algebra setting. The trace $\operatorname{tr} P$ is the dimension of the target space.
- An **essential idempotent** of the $\vee$-algebra defined in the famous 1962-paper on the "Eightfold way" by Gell-Mann on symmetries of baryons and mesons, corresponding to the equation
$$q \vee q + \eta(q)q = 0.$$
These essential idempotents are the directions of **symmetry breaking** for the adjoint representation: that is, they are the critical points of the invariant functional $\{A, A, A\}$ on the unit sphere $\{A, A\} = 1$.
- The **bifurcation equations** $B(x, x) + x = 0$ considered by Sattinger (1977), where B is a bilinear mapping $V \times V \to V$ which is invariant under a certain group.
- In general, a **stationary point** of a distinguished cubic form $u(x)$ on an inner product space V (see below).
- Idempotents in **axial algebras**: mimic a distinguished finite subset of **involutions** in a certain group generating a nice commutative algebra structure. Axes are specific idempotents with common fusion rules generating a whole algebra.

Idempotents

It turns out that the idempotent structure in a metrized algebra is very distinguished, especially if the metric is positive definite. The latter situation is very common in the problems coming from differential geometry and group theory (for example, the Conway–Griess–Norton algebra of the Monster finite simple sporadic group)

So, what does it mean to be an idempotent in $\mathrm{CMA}(u)$*?*



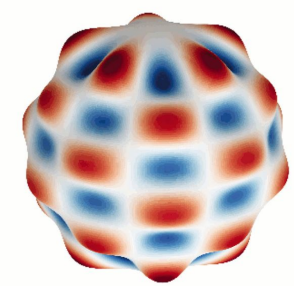
Idempotents

Let $(\mathbb{A}, \langle ; \rangle)$ be a **Euclidean** space, $u \not\equiv 0$ and let S be the sphere $\langle x; x\rangle = 1$. Then

$$x \text{ is a } \textbf{stationary point} \text{ of } u(x) \text{ on } S \Leftrightarrow \quad \nabla u(x) = k\nabla\langle x; x\rangle = 2kx,\ k \in \mathbb{R}$$
$$\Leftrightarrow \quad x \bullet x = 4kx$$

By Euler's homogeneous function theorem, $3u(x) = \langle x; \nabla u(x)\rangle = 2k\langle x; x\rangle = 2k$, hence

$$x \bullet x = 6u(x)x$$



Thus, if $u(x) \neq 0$, this implies that $c_0 = \frac{1}{6u(x)}x$ is a **nonzero idempotent** (or a 2-nilpotent if $u(x) = 0$).

Next, if c is an nonzero idempotent then for its projection $x := c/|c|$

$$u(x) = \frac{1}{6}\langle x^2; x\rangle = \frac{1}{6|c|}$$

Definition

An idempotent c in a $\mathrm{CMA}(u)$ is called **extremal** if u attains a local maximum at $x := c/|c|$, and it is called **minimal** if the maximum is global (obviously exists). In the latter case, c has **the shortest possible length among other idempotents.**

Minimal idempotents: Example

Let $\mathbb{A}$ be the algebra of $n \times n$ symmetric matrices over reals (or in general, Hermitian matrices over a Hurwitz algebra) with Jordan multiplication

$$x \bullet y = \underbrace{\tfrac{1}{2}(x \cdot y + y \cdot x)}_{\cdot \text{ is the matrix product}} .$$

This Jordan algebra is a metrized algebra of $u(x) = \frac{1}{6} \operatorname{tr} x^3$ with respect to the canonical metric

$$\langle x; y \rangle := \operatorname{tr}(x \cdot y) = \operatorname{tr}(x \bullet y).$$

If c is a nonzero idempotent then $c \bullet c = c \cdot c = c$. Since the matrix c is symmetric, it is diagonalizable, which implies that its eigenvalues are either 1 or 0. Therefore, for any idempotent c:

$$\langle c; c \rangle = \operatorname{tr} c = \operatorname{rank} c \in \{1, 2, \ldots n\}.$$

Therefore, the **minimal** idempotents are those with the norm $= 1$, i.e. the **primitive** idempotents. It is well-known that any idempotent in a Euclidean Jordan algebra can be written as a sum of primitive ($=$ minimal) idempotents.

Extremal idempotents

Suppose $\mathbb{A}$ be a finite-dimensional **unital** metrized commutative nonassociative algebra with positive definite bilinear form $\langle ; \rangle$, e being the algebra unit. Then for any idempotent c, $c' = e - c$ is also an idempotent:

$$c' \bullet c' = e \bullet e - 2e \bullet c + c \bullet c = e - 2c + c = c',$$

and since

$$\langle c; c' \rangle = \langle c; e - c \rangle = \langle c \bullet c; e - c \rangle = \langle c; (e - c) \bullet c \rangle = \langle c; c - c \rangle = 0,$$

we have $|c|^2 + |c'|^2 = |e|^2$ (where the norm $|c|^2 := \langle c; c \rangle$). Now suppose that c a minimal idempotent (i.e. it has the shortest possible length). Then

$$2|c|^2 \leq |e|^2.$$

Theorem (V.Tk., Comm. Alg., 2019)

If the equality $2|c|^2 = |e|^2$ holds for some minimal idempotent c in $\mathbb{A}$ then $\mathbb{A}$ is a Jordan algebra of Clifford type (a spin-factor).

Theorem A

Any minimal idempotent c in a Euclidean metrized algebra is semisimple, primitive and except for the single eigenvalue $\lambda = 1$, all other eigenvalues are $\leq \frac{1}{2}$.

Proof. Since L_c is self-adjoint, $\mathbb{A}$ is decomposable (over $\mathbb{R}$) into a direct sum of eigesubspaces. Let

$$f(x) = \frac{\langle x^2; x\rangle}{|x|^3} : \mathbb{A}^\times \to \mathbb{R}.$$

It is a homogeneous of degree zero function, regular outside the origin, and

$$\tfrac{1}{3}\operatorname{Hess}(f(x)) = \frac{2|x|^4 L_x - \langle x^2; x\rangle |x|^2 - 3|x|^2(x\otimes x^2 + x^2 \otimes x) + 5\langle x^2; x\rangle x\otimes x}{|x|^5},$$

where $(a\otimes b)(x) = a\langle b; x\rangle$. *Since c_0 is a minimal idempotent, it is a local maximum of* $f(x)$, hence

$$0 \geq \tfrac{1}{3}\operatorname{Hess}(f(c)) = \frac{(2L_c - 1)|c|^2 - c\otimes c}{|c|^3}.$$

Since $c\otimes c = 0$ on $c^\perp$ we conclude that $2L_c \leq \frac{1}{2}$ on $c^\perp$, as desired.

Minimal idempotents

The Peirce eigenvalue $\frac{1}{2}$ is very special. It appears in any commutative nonassociative algebra (over an arbitrary field char$\neq 2,3$) possessing a polynomial identity. Let a commutative algebra $\mathbb{A}$ satisfy an identity of the kind

$$P := \sum_{\alpha} \phi_\alpha(x) x^\alpha = 0, \qquad x^\alpha \in N(x), \tag{2}$$

where $N(x) = \{x^\alpha : x,\ x^2,\ x^3,\ x^2x^2,\ xx^3,\ x^2x^3,\ x(x^2x^2), \ldots\}$ is the **nonassociative commutative groupoid** generated by x and $\phi_\alpha(x)$ are polynomial functions.

Theorem (V.T., *The universality of one half ...*, J. Algebra, 2021)

Let $\mathbb{A}$ be a finite dimensional commutative nonassociative algebra over a field of characteristic $\neq 2,3$ and let $\mathbb{A}$ satisfy a **nontrivial** identity $P = 0$ in one nonassociative indeterminate x. Then the following holds:

(A) $\frac{1}{2}$ is the Peirce number for *any* idempotent $c \neq 0$;

(B) if additionally c is semi-simple and λ is a single root of the Peirce polynomial then

$$\mathbb{A}_c(\lambda)\mathbb{A}_c(\tfrac{1}{2}) \subset \bigoplus_{\nu \in \sigma(c), \nu \neq \lambda} \mathbb{A}_c(\nu). \tag{3}$$

In fact, $\frac{1}{2}$ is also very distinguished in **metrized algebras**.

Theorem B

If c is a minimal idempotent then $\mathbb{A}_c(\frac{1}{2})\mathbb{A}_c(\frac{1}{2}) \subset \mathbb{A}_c(\frac{1}{2})^\perp$ ('Jordan type fusion law').

Proof. We claim that $x \in \mathbb{A}_c(\frac{1}{2})$ implies $\langle x^2; x\rangle = 0$. Since $c \bullet x = \frac{1}{2}x$, we have

$$(c+x)^2 = c + 2c \bullet x + x \bullet x = c + x + x^2.$$

Also,

$$\langle c; x\rangle = \langle c \bullet c; x\rangle = \langle c; x \bullet c\rangle = \tfrac{1}{2}\langle c; x\rangle,$$

hence$\langle c; x\rangle = 0$, and similarly $\langle c; x^2\rangle = \langle c \bullet x; x\rangle = \frac{1}{2}\langle x; x\rangle = \frac{1}{2}|x|^2$, thus

$$\langle (c+tx); (c+tx)^2\rangle = |c|^2 + \tfrac{3}{2}|x|^2t^2 + \langle x; x^2\rangle t^3, \qquad \forall t \in \mathbb{R}.$$

Since c is minimal, $f(c+tx) \le f(c)$ for small t, where $f(z) := \frac{\langle z^2; z\rangle}{|z|^3}$. This yields

$$\frac{1}{|c|} = f(c) \ge f(c+tx) = \frac{|c|^2 + \frac{3}{2}|x|^2t^2 + \langle x; x^2\rangle t^3}{(|c|^2 + |x|^2t^2)^{3/2}} = \frac{1}{|c|} + \frac{\langle x; x^2\rangle}{|c|^3}t^3 + O(t^4),$$

hence $\langle x; x^2\rangle = 0$ for $x \in \mathbb{A}_c(\frac{1}{2})$. Then polarization yields $\langle xy; z\rangle = 0$ for all $x, y, z \in \mathbb{A}_c(\frac{1}{2})$, the conclusion follows.

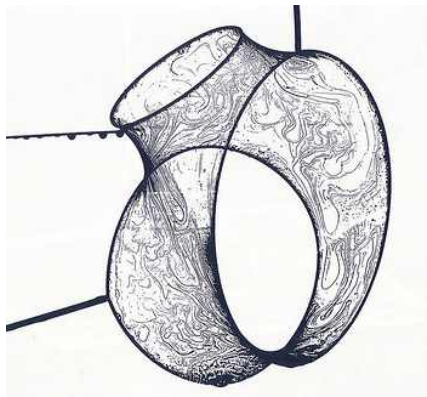
Outline

1. Motivations
2. Metrized algebras
3. Idempotents in metrized algebras
4. Hsiang algebras

Minimal surfaces

A **minimal surface** (a soap film) is a critical point of the area functional. Geometrically, this means that the **mean curvature vanishes totally**. If $x_{n+1} = u(x)$ is a minimal graph over $\mathbb{R}^n$ then

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{u'_{x_i}}{\sqrt{1+|\nabla u|^2}} = 0.$$



Bernstein's theorem (1915): u is an affine function $n = 2$. The result holds still true for $n \leq 7$ (Almgren, De Giorgi, Simons) but it fails for $n \geq 8$ (Bombieri-de Giorgi-Giusti).

A **cone** is a typical singularity of a minimal surface. All known minimal cones are algebraic, i.e. zero level sets of a homogeneous polynomial $u \in \mathbb{R}[x_1, \dots, x_n]$:

- $u = (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$ (**the norm of split octonions**).
- $u = \mathrm{Re}((z_1 z_2) z_3), z_i \in \mathbf{K}_d, d = 1, 2, 4, 8$ in $\mathbb{R}^{3d}$ (the **triality polynomials**).
- $u = \det X$ (X in the trace free subspace of the **Jordan algebra** $\mathscr{H}_3'(\mathbf{K}_d)$)

Cubic minimal cones

W.-Y. Hsiang (1967): Let u be a homogeneous polynomial in $\mathbb{R}^n$. Then $u^{-1}(0)$ is a minimal cone iff

$$\Delta_1 u := |\nabla u|^2 \Delta u - \nabla u(x)^\top \operatorname{Hess}(u(x)) \nabla u(x) \equiv 0 \mod u.$$

- The first non-trivial case: $\deg u = 3$ and then

$$\Delta_1 u = \text{a quadratic form} \cdot u(x) \tag{4}$$

For example, in dimension $n = 2$ one has a quasilinear PDE

$$u_y'^2 u_{xx}'' - 2u_x' u_y' u_{xy}'' + u_x'^2 u_{yy}'' = (ax^2 + 2bxy + cy^2)u$$

where a, b, c are unknown constants.

- In fact, all known irreducible cubic minimal cones satisfy very special equation:

$$\Delta_1 u = \theta |x|^2 \cdot u(x) \tag{5}$$

Hsiang problem: Classify all cubic polynomial solutions of (5).

An explicit form of the Hsiang solutions

1) In $\mathbb{R}^5$ one obtains a Cartan isoparametric cubic (1939)

$$u(x) = x_5^3 + \frac{3}{2}x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \frac{3\sqrt{3}}{2}x_4(x_2^2 - x_1^2) + 3\sqrt{3}x_1x_2x_3,$$

(1939) Cartan's "Sur des familles remarquables..."

2) In $\mathbb{R}^9$:

$$u(x) = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$$

Question

There are several other examples, all having strange connections with Jordan algebras. Why??

Algebras of minimal cones

u on $\mathbb{A}$ is a solution of a PDE $\Rightarrow$ $\mathbb{A}(u)$ satisfies an identity.

Let $u(x)$ be a Hsiang cubic, i.e.

$$|\nabla u|^2 \Delta u - \nabla u(x)^\top \operatorname{Hess}(u(x)) \nabla u(x) = \theta |x|^2 u(x).$$

This is equivalent to that a metrized algebra $\mathbb{A}(u)$ satisfies the identity

$$\langle x^2; x^2 \rangle \operatorname{tr} L(x) - \langle x^2; x^3 \rangle = \frac{2}{3} \theta \langle x; x \rangle \langle x^2; x \rangle$$

with $u(x) = \frac{1}{6} \langle x; x^2 \rangle$.

Definition

Any commutative metrized algebra satisfying the latter identity is called is a **Hsiang algebra**.

One can show (nontrivial!) that in fact any non-trivial Hsiang algebra is trace-free, i.e. satisfies $\operatorname{tr} L(x) = 0$

The defining equation

We can assume that u is a **normalized** Hsiang eigencubic, i.e.

$$\Delta_1 u = -2|x|^2 u(x), \qquad x \in V = \mathbb{R}^n$$

Theorem C

Given a **nontrivial** normalized u, there exists a commutative metrized algebra such that

1. $u(x) = \frac{1}{6}\langle x; x^2\rangle$ (the recovering property)
2. $x^3 x + \frac{1}{4}x^2 x^2 - |x|^2 x^2 - \frac{2}{3}\langle x^2; x\rangle x = 0$ (the defining identity)
3. $\operatorname{tr} L_x = 0$ (the harmonicity property)

In the converse direction, given a commutative metrized algebra with properties (2) and (3) above, the cubic form u defined by (1) is a (normalized) Hsiang eigencubic.

Theorem D

(i) For any nonzero idempotent c, the associated Peirce decomposition is

$$\mathbb{A} = \mathbb{A}_c(1) \oplus \mathbb{A}_c(-1) \oplus \mathbb{A}_c(-\tfrac{1}{2}) \oplus \mathbb{A}_c(\tfrac{1}{2}) \quad \text{and} \quad \dim \mathbb{A}_c(1) = 1;$$

and the following fusion laws hold:

	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
1	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
-1	-1	1	$\frac{1}{2}$	$-\frac{1}{2} \oplus \frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$1 \oplus -\frac{1}{2}$	$-1 \oplus -\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2} \oplus \frac{1}{2}$	$-1 \oplus -\frac{1}{2}$	$1 \oplus -1 \oplus -\frac{1}{2}$

(ii) The Peirce dimensions $n_1 = \dim \mathbb{A}_c(-1)$, $n_2 = \dim \mathbb{A}_c(-\frac{1}{2})$ and $n_3 = \dim \mathbb{A}_c(\frac{1}{2})$ do not depend on a particular choice of c and $n_3 = 2n_1 + n_2 - 2$;

(iii) All idempotents have **the same length and satisfy the same fusion law**

The Peirce decomposition

	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
1	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
-1	-1	1	$\frac{1}{2}$	$-\frac{1}{2} \oplus \frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$1 \oplus -\frac{1}{2}$	$-1 \oplus -\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2} \oplus \frac{1}{2}$	$-1 \oplus -\frac{1}{2}$	$1 \oplus -1 \oplus -\frac{1}{2}$

- The fusion law is not graded in general, but $\mathbb{Z}/2$-graded a posteriori for many particular cases
- $V_c(1) \oplus V_c(-1)$ is a subalgebra. It carries a **hidden Clifford algebra structure**.
- $V_c(1) \oplus V_c(-\frac{1}{2})$ is a subalgebra. It carries a **hidden rank 3 Jordan algebra structure**.

An example: polar algebras

Definition. A commutative metrised $\mathbb{Z}_2$-graded algebra $V = V_0 \oplus V_1$ is called polar if

$$V_0 \bullet V_0 = 0 \qquad \text{and} \qquad L(x)^2 = |x|^2 \text{ on } V_1, \quad \forall x \in V_0.$$

Definition. $A : X \to \mathrm{End}^S(Y)$ is a *symmetric Clifford system* ($A \in \mathrm{Cliff}(X, Y)$) if

$$A(x)^2 = |x|^2 \mathbf{1}_Y \quad \forall x \in X.$$

Proposition

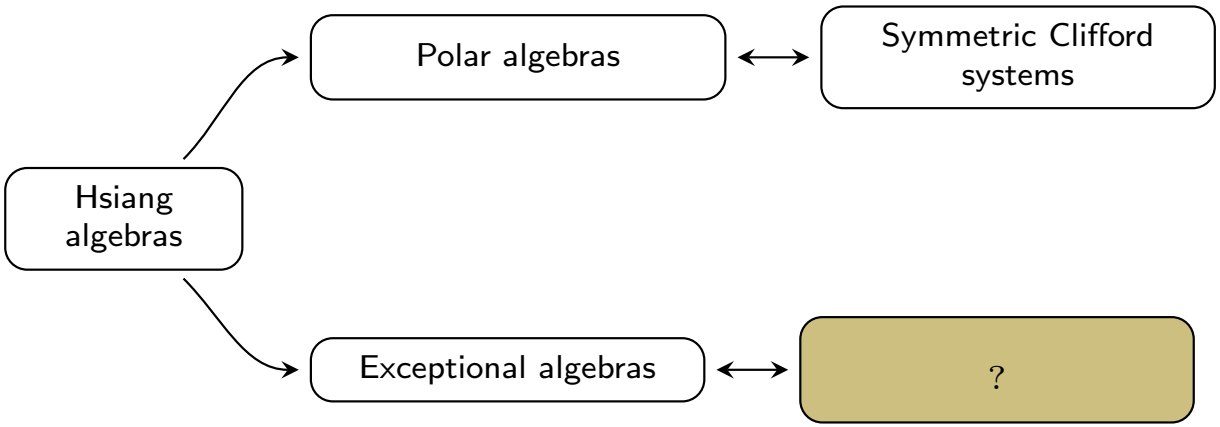
- If $A \in \mathrm{Cliff}(X, Y)$ then the metrized algebra $(X \times Y)(u)$, where $u(x, y) = \frac{1}{2}\langle y; A(x)y\rangle$, is a polar algebra with $V_0 = X \times \{0\}$, $V_1 = \{0\} \times Y$.
- Conversely, if $V = V_0 \oplus V_1$ is a polar algebra then $L_x \in \mathrm{Cliff}(V_0, V_1)$.
- Any polar algebra is a Hsiang algebra.

It is well-known that $\mathrm{Cliff}(X, Y) \neq \emptyset \Leftrightarrow \dim X \leq 1 + \rho(\frac{1}{2}\dim Y)$, where $\rho(m) = 8a + 2^b$, if $m = 2^{4a+b} \cdot \mathrm{odd}$, $0 \leq b \leq 3$ is the Hurwitz-Radon function.

Polar vs exceptional

Definition

A Hsiang algebra $\mathbb{A}$ isomorphic to a polar algebra is of **Clifford type**; otherwise it is **exceptional**.



Example

Any commutative **pseudocomposition algebra** $\mathbb{A}$, i.e. an algebra with $x^3 = |x|^2 x$ and $\operatorname{tr} L_x = 0$ is an *exceptional* Hsiang algebra. The trace free subspace of the Jordan algebra $(M_3, \circ)$ of all 3×3 real symmetric matrices with the Jordan multiplication $x \circ y = \frac{1}{2}(xy + yx)$ is a pseudocomposition algebra.

Theorem E

Let $\mathbb{A}$ be a Hsiang algebra and $c \in \mathrm{Idm}(V)$. Let us introduce on the invariant subspace $\Lambda_c := \mathbb{R}c \oplus \mathbb{A}_c(-\frac{1}{2})$ the new multiplication

$$x \bullet y = \tfrac{1}{2}xy + \langle x; c\rangle y + \langle y; c\rangle x - 2\langle xy; c\rangle c, \qquad x, y \in \mathbb{R}c \oplus \mathbb{A}_c(-\tfrac{1}{2}), \tag{6}$$

Then Λ_c is a rank 3 Euclidean Jordan algebra with unit $2c$ and any element $x \in \Lambda_c$ satisfies the cubic identity

$$x^{\bullet 3} - 2\langle x; c\rangle x^{\bullet 2} + (2\langle x; c\rangle^2 - \tfrac{1}{2}|x|^2)x - \tfrac{1}{3}\langle x; x^2\rangle c = 0, \tag{7}$$

Furthermore, $\mathbb{A}$ is exceptional if and only if Λ_c is a simple Jordan algebra!

Theorem F

Let V be a Hisang algebra. The Peirce dimensions $n_1 = \dim \mathbb{A}_c(-1)$, $n_2 = \dim \mathbb{A}_c(-\frac{1}{2})$ and $n_3 = \dim \mathbb{A}_c(\frac{1}{2})$ do not depend on a particular choice of c and

$$n_3 = 2n_1 + n_2 - 2;$$

Furthermore,

$$n_1 - 1 \leq \rho(n_1 + n_2 - 1),$$

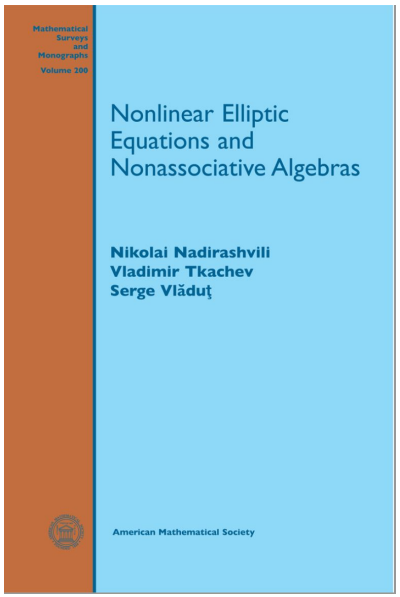
where ρ is the Hurwitz-Radon function. In particular, for each n_2 there exist only finitely many possible values of n_1.

Theorem H

If V is an exceptional Hsiang algebra then $\dim \Lambda_c \in \{0, 5, 8, 14, 26\}$ for any $c \in \mathrm{Idm}(V)$ and the possible Peirce dimensions $n_1 = \dim V_c(-1)$, $n_2 = \dim V_c(-\frac{1}{2})$ are displayed below.

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

(Trondheim, Norway, October 2016)



1. N. Nadirashvili,v, and S. Vlăduţ. Nonlinear elliptic equations and nonassociative algebras, vol. 200, Math. Surveys and Monographs. AMS, Providence, RI, 2014.
2. V. Tkachev, The universality of one half in commutative nonassociative algebras with identities. *J. Algebra*, 569(2021), 466–510
3. V. Tkachev, Spectral properties of nonassociative algebras and breaking regularity for nonlinear elliptic type PDEs *Algebra i Analiz* 31 (2019), no. 2, 51–74
4. V. Tkachev, Miklyukov: From dimension 8 to nonassociative algebras, *Math. Phys. Comp. Sim.*, 22(2019).
5. Y. Krasnov, V. Tkachev, Idempotent geometry in generic algebras, Adv. Appl. Clifford Alg., 28(2018):84.
6. Y. Krasnov, V. Tkachev, Variety of idempotents in nonassociative algebras, Topics in Clifford Analysis in Trends Math., Birkhäuser/Springer, Cham, 2019, 405–436,
7. V. Tkachev, On the non-vanishing property for real analytic solutions of the p-Laplace equation, *Proc. Amer. Math. Soc.*, 144(2016), 2375-2382
8. N. Nadirashvili, V. Tkachev, and S. Vlăduţ. A non-classical solution to a Hessian equation from Cartan isoparametric cubic, Adv. Math., 231(2012), 4:1589–1597.
9. V. Tkachev, On an extremal property of Jordan algebras of Clifford type, Comm. Algebra, 47(2018), 1057–1066.
10. V. Tkachev, A correction of the decomposability result in a paper by Meyer-Neutsch. J. Algebra, 504(2018), 432-439.
11. V. Tkachev, A Jordan algebra approach to the cubic eiconal equation, J. Algebra, 419(2014), 34-51.
12. V. Tkachev, Minimal cubic cones via Clifford algebras, Complex Anal. Oper. Theory, 4(2010), 3:685–700.