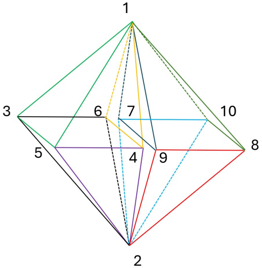
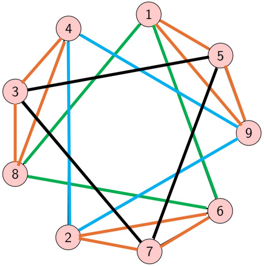
Exceptional Hsiang algebras and Steiner triple systems





Finite groups, Geometries, Algebras and Related Objects

Birmingham, 18/06/2025

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(based on a joint work with Daniel Fox, Universidad Politécnica de Madrid)

Erdös(S.S., V.T.)=4 (noncommutative!):

MR Collaboration Distance = 4

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Many groups are best described as the group of automorphlsms of some natural object. M. Aschbacher, [Asc88]

I will talk about Hsiang algebras. They have the following remarkable properties:

- They appear in various areas of mathematics, including Differential Geometry, Nonlinear PDEs, Algebra and Combinatorial Design.
- Intimately related to the classical structures like Jordan, Clifford, Lie algebras
- Hsiang algebras are axial algebras [HRS15].
- Moreover, they are also very related and contain pseudo-composition and Elduque-Okubo algebras.
- There are infinitely many regular families vs finitely many exceptional families.
- There are infinitely many idempotents with the same length, the same algebraic spectrum and the same fusion laws.
- Furthermore, the set of idempotents has a nice structure of a smooth (Riemannian) homogeneous submanifold.
- Finally, Hsiang algebras have infinite (continuous) automorphism groups.

Cherchez la algebra

Good news:

Popov, Gordeev, Ann. Math., 2003

If a field $\Bbbk$ contains sufficiently many elements and $\mathbf{K}$ is an algebraically closed field containing $\Bbbk$, then *every linear algebraic $\Bbbk$-group over $\mathbf{K}$ is $\Bbbk$-isomorphic to* $\mathrm{Aut}(\mathbb{A} \otimes_{\Bbbk} \mathbf{K})$, where $\mathbb{A}$ is a finite dimensional simple algebra over $\Bbbk$.

Bad news:

Popov, 2015

Let $\mathbf{K}$ be an algebraically closed field and let $\mathcal{M} = V^* \otimes V^* \otimes V$ denote the set of all (nonassociative) $\mathbf{K}$-algebra structures. Then for points m in general position in $\mathcal{M}$, the algebras (V, m) are simple and have trivial automorphism group.

Moral:

A nonassociative algebra taken at random is simple and has trivial automorphism group. So you have to be lucky/happy if you find an algebra with a nontrivial Aut!

Notations

- $\Bbbk$ is a field of characteristic not equal to 2 and 3
- $\mathbb{A}$ is a commutative but maybe nonassociative algebra over $\Bbbk$
- $L(x)y := xy = yx$
- $\operatorname{Idm}(\mathbb{A})$ is the set of nonzero algebra idempotents
- $\langle\langle E \rangle\rangle$ denotes the **subalgebra** generated by $E \subset \mathbb{A}$
- for a semi-simple idempotent c, $\mathbb{A}$ decomposes into a direct sum of eigenspaces of $L(c)$ (the Peirce subspaces):

$$\mathbb{A} = \bigoplus_i \mathbb{A}_c(\lambda_i), \quad L(c) = \lambda_i \operatorname{id}_{\mathbb{A}_c(\lambda_i)}, \quad 1 \le i \le s.$$

- A bilinear form $\langle ; \rangle$ is **invariant** in an algebra if $\langle xy; z \rangle = \langle x; zy \rangle$.
- An algebra with an invariant bilinear form is called **metrized**.
- An algebra is **exact** if $\operatorname{tr} L(x) = 0$.

An elementary observation

cubic forms on inner product vector spaces = metrized algebras

Spoiler: a 27-dimensional Hsiang algebra

Dickson (1901) studied the following E_6-invariant cubic form Φ on $\mathbb{k}^{27}$:

$$\Phi(t) = \sum_{1\le i,j\le 6} x_i y_j z_{ij} + \sum z_{ab}z_{cd}z_{ef} = \underbrace{\mathrm{Pf}(Z)}_{15 \text{ terms}} + \underbrace{x^t Z y}_{30 \text{ terms}} = \sum_{\alpha\in B} \pm t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} \qquad (1)$$

where $t = (x, y, Z) \in \mathbb{k}^6 \times \mathbb{k}^6 \times \mathrm{Skew}(6, \mathbb{k}) = \mathbb{k}^{6+6+\frac{6\cdot 5}{2}} = \mathbb{k}^{27}$,

From [Dickson1901], [Aschbacher1987]

12 34 56.	14 23 56.	16 25 34.	13 24 65.	15 26 43.
12 35 64.	14 25 63.	16 23 45.	13 26 54.	15 24 36.
12 36 45.	14 26 35.	16 24 53.	13 25 46.	15 23 64.

There are several partial Steiner systems here:

- (1) is a Steiner triple decomposition, where B is a *decorated* partial Steiner triple system on $|B| = 27$ elements with replication number $r = 5$.
- the set of the Pfaffian variables Z itself is a partial triple Steiner on $\frac{6\cdot 5}{2} = 15$ elements $z_{ij} = -z_{ji}$ with replication number $r = 3$

Spoiler: from a cubic form to an algebra

$$\Phi(t) = \sum_{1\le i,j\le 6} x_i y_j z_{ij} + \sum z_{ab}z_{cd}z_{ef} = \underbrace{\mathrm{Pf}(Z)}_{15 \text{ terms}} + \underbrace{x^t Z y}_{30 \text{ terms}} = \sum_{\alpha\in B} \pm t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} \qquad (1)$$

- i. Φ is exactly the defining form of a 27-dim exceptional Hsiang algebra $\mathbb{A}$
- ii. $\mathbb{A}$ is a pseudo-composition algebra (i.e. $x^3 = \langle x; x\rangle x$).
- iii. (1) is equivalent to the following $\mathbb{Z}_2$-graded Peirce decomposition

$$\mathbb{A} = (\mathbb{A}_c(1) \oplus \mathbb{A}_c(-\tfrac{1}{2})) \oplus \mathbb{A}_c(\tfrac{1}{2}) = \Lambda_c \oplus \Lambda_c^{\perp}$$

- iv. The zero locus $\Phi(t) = 0$ defines a zero mean curvature cone in $\Bbbk^{27}$.
- v. $\mathrm{Der}(\mathbb{A}) = \mathfrak{e}_6$
- vi. $\mathrm{Idm}(\mathbb{A})$ is the homogeneous space $Sp(4)/(Sp(1)\times Sp(3))$.
- vii. Λ_c is isotopic to a 15-dim Jordan algebra.
- viii. $\mathbb{A}_c(1) \oplus \mathbb{A}_c(-\frac{1}{2})$ is a 15-dimensional exceptional Hsiang (sub)algebra (also a pseudo-composition algebra).

Initial geometrical context

- A geodesic line is a shortest curve between two points.
- A minimal surface is a surface that *locally minimizes its area*.

Malmö Live Concert, Sweden, https://www.fotosidan.se/blogs/wolfgang/offentlig-konst-i-malmo.htm

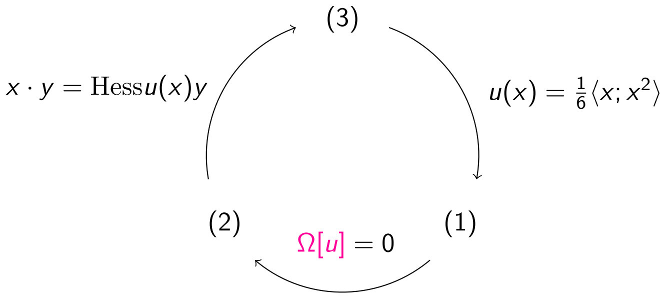
The context: three equivalent problems

1. [Hsi67] **Zero mean curvature** cubic cones in $\mathbb{R}^n$
2. [Hsi67] Homogeneous degree 3 polynomial solutions of

$$\Omega[u] := \frac{1}{2}\nabla u(x) \cdot \nabla(|\nabla u(x)|^2) - |\nabla u|^2 \Delta u - \theta\langle x; x\rangle u(x) = 0$$

3. [Tka10b] Commutative nonassociative exact metrized algebras on $\mathbb{R}^n$ with

$$\langle x^2; x^3\rangle = \theta\langle x^2; x\rangle\langle x; x\rangle \tag{2}$$



Definition

An exact commutative h-metrized algebra $(\mathbb{A}, h)$ is called **Hsiang** if there exists $\theta \in \Bbbk$ such that the following identity holds:

$$\langle x^2; x^3\rangle = \theta\langle x^2; x\rangle\langle x; x\rangle \tag{3}$$

An algebra satisfying (3) is called **almost-Hsiang**. Notice that (3) is equivalent to

$$4xx^3 + x^2x^2 - 3\theta\langle x; x\rangle x^2 - 2\theta\langle x^2; x\rangle x = 0. \tag{4}$$

Remarks

- A subalgebra of a Hsiang algebra is almost-Hsiang.
- (3) $\Rightarrow$ all idempotents have the same length, i.e. lie on a sphere.
- There are many distinguished algebras where a relevant subset of idempotents (not all!) has this property.

Three important classes of almost-Hsiang algebras: I

A commutative algebra with a bilinear form $\langle ; \rangle$ is called **pseudo-composition** (Meyberg, Osborn, Walcher, Röhrl, Gradl) if

$$x^3 = \langle x; x\rangle x \tag{5}$$

Elduque and Okubo [EO00] proved that in fact h above must be an invariant form. This implies that any pseudo-composition algebra is almost-Hsiang:

$$(5) \quad \Rightarrow \quad \langle x^3; x^2\rangle = \langle x; x\rangle\langle x; x^2\rangle, \quad \theta = 1.$$

The linearization of (5) implies for any idempotent c that the spectrum of $L(c)$ on $c^{\perp}$ is $\{-1, \frac{1}{2}\}$ with the Peirce decomposition

$$\mathbb{A} = \Bbbk c \oplus \mathbb{A}_c(-1) \oplus \mathbb{A}_c(\tfrac{1}{2})$$

Three important classes of almost-Hsiang algebras: II

In [EO00], Elduque and Okubo studied commutative admissible cubic algebras, i.e.

$$x^2x^2 = N(x)x \tag{6}$$

[EO00] states that in general there exists an invariant bilinear form $\langle ; \rangle$ such that $N(x) = \langle x^2; x\rangle$. This implies that any Elduque-Okubo algebra is almost-Hsiang:

$$(6) \quad \Rightarrow \quad \langle x^2; x^3\rangle = \langle x^2x^2; x\rangle = \langle x^2; x\rangle\langle x; x\rangle, \quad \theta = 1.$$

The Elduque-Okubo algebras have the following Peirce decomposition:

$$\mathbb{A} = \Bbbk c \oplus \mathbb{A}_c(-\tfrac{1}{2}) \oplus \mathbb{A}_c(\tfrac{1}{2})$$

A typical example is the algebra of 3×3 matrices with the adjugate multiplication: $x\#y := \operatorname{adj}(x+y) - \operatorname{adj}(x) - \operatorname{adj}(y)$, then $x\#x = 2\operatorname{adj}(x)$, hence

$$(x\#x)\#(x\#x) = 2\operatorname{adj}(x\#x) = 2^3\operatorname{adj}(\operatorname{adj}(x)) = 8\det(x)x$$

Three important classes of almost-Hsiang algebras: III

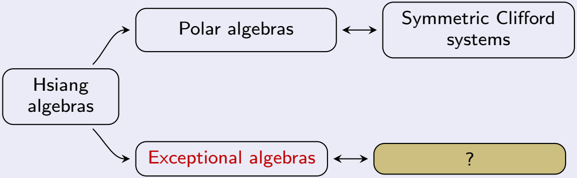
Definition

A metrized $\mathbb{Z}_2$-graded algebra $\mathbb{A} = \mathbb{A}_0 \oplus \mathbb{A}_1$ is said to be **polar** if $\mathbb{A}_0\mathbb{A}_0 = \{0\}$ and

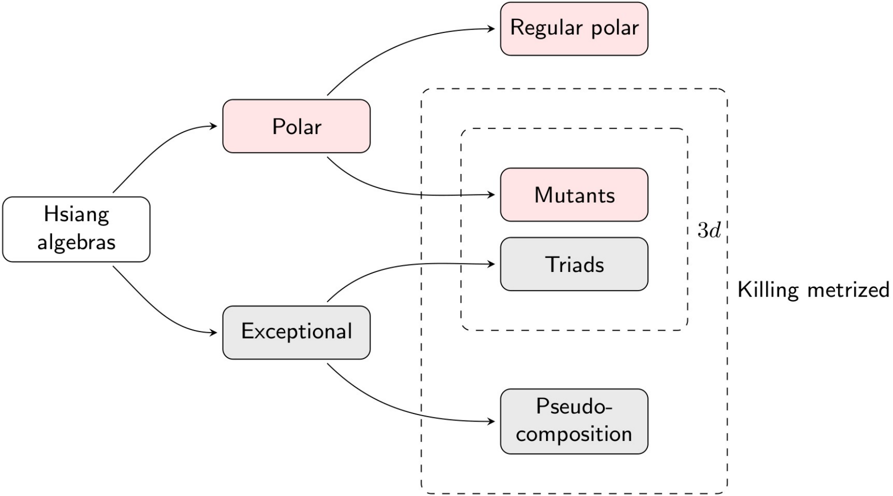
$$x_0(x_0x_1) = \langle x_0; x_0\rangle x_1, \qquad x_0 \in \mathbb{A}_0,\ x_1 \in \mathbb{A}_1.$$

Theorem V.T. 2010 [Tka10a]

1. Any polar algebra is Hsiang.
2. There is a natural isomorphism between the category of polar algebras and the category of symmetric Clifford system. This yields an effective classification.
3. Polar algebras exist **in almost all dimensions**.



On the Origin of Species



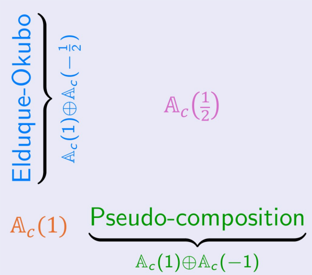
Theorem 1.1 (Basic facts on Hsiang algebras [NTV14], [FT25])

1. $\emptyset \neq \mathrm{Idm}(\mathbb{A}) \subset S^{n-1}$ *and all idempotents share the same fusion laws.*
2. *For any idempotent c, the associated Peirce decomposition is*

$$\mathbb{A} = \underbrace{\Bbbk c}_{\dim=1} \oplus \underbrace{\mathbb{A}_c(-1)}_{\dim=n_1} \oplus \underbrace{\mathbb{A}_c(-\tfrac{1}{2})}_{\dim=n_2} \oplus \underbrace{\mathbb{A}_c(\tfrac{1}{2})}_{\dim=n_3}$$

3. *the Elduque-Okubo subalgebra $\Lambda_c = \Bbbk c \oplus \mathbb{A}_c(-\frac{1}{2})$ carries an isotopical degree three Jordan algebra structure w r t the isotopy*

$$x \bullet y = xy + \tfrac{1}{2}(\langle x; c\rangle y + \langle y; c\rangle x) - \langle xy; c\rangle c$$



The Finiteness of Exceptional Hsiang Algebras, [NTV14]

There are **finitely many dimensions** n of $\mathbb{A}$ where exceptional Hsiang algebras can exist. Here $n_1 = \dim \mathbb{A}_c(-1)$ and $n_2 = \dim \mathbb{A}_c(-1/2) = 3d + 2$, $d \in \{-\frac{2}{3}, 0, 1, 2, 4, 8\}$

n	2	5	8	14	26	3	6	12	24	9	12	21	15	18	24	30	42	27	30	54
n_1	1	2	3	5	9	0	1	3	7	0	1	4	0	1	3	5	9	0	1	1
n_2	0	0	0	0	0	2	2	2	2	5	5	5	8	8	8	8	8	14	14	26
d	–	–	–	–	–	0	0	0	0	1	1	1	2	2	2	2	2	4	4	8

The colours correspond to

- pseudo-composition algebras,
- mutants
- Elduque-Okubo algebras
- para-complexifications
- unsettled cases.

Questions and motivations

An important tool to study a bilinear form is to diagonalize it. For cubic forms the situation is more subtle. In the context of metrized algebras, there are at least two distinguished ways to write $u(x)$ in some orthonormal coordinates in $\mathbb{R}^n$:

1. **Normal decomposition (corresponds to idempotents)**:

$$u(x) = x_1^3 + 3\underbrace{\left(-1 \cdot |\xi|^2 - \tfrac{1}{2} \cdot |\eta|^2 + \tfrac{1}{2} \cdot |\zeta|^2\right)x_1}_{\text{the Peirce decomposition}} + \underbrace{\psi(\xi,\eta,\zeta)}_{\text{fusion laws}}.$$

   Here $x = (0, 1)$ corresponds to an idempotent in $\mathbb{A}(u)$, see for example (**??**).

2. **Steiner decomposition (corresponds to 2-nilpotents)**:

$$u(x) = \sum_{\alpha \in B} \epsilon_\alpha \quad \underbrace{(x_{\alpha_1} x_{\alpha_2} x_{\alpha_3})}_{\text{the skeleton idempotents}} \qquad \Rightarrow \quad \mathbb{A}(u) \approx \sum_\alpha \mathbb{E}_3(\alpha),$$

   where $\epsilon_\alpha \in \Bbbk$ and B a **partial Steiner triple system** ($PSTS$) on $\mathbb{N}_n$. Not every cubic form admits a Steiner decomposition. But, for a Hsiang eigencubic, if a Steiner form exists then the coefficients ϵ_α must be ± 1.

Questions and motivations

1. There are various (equivalent or not) relevant contexts, e.g. '*trieders*' of E. Dickson (1901), combinatorial designs, axial algebras etc.
2. For many Hsiang algebras, their defining form $\langle x^2; x\rangle$ admits a Steiner decomposition in some ON.. to basis. What is the general statement here?
3. How the combinatorial invariants of the associated PST systems are related to the algebraic invariants of the corresponding Hsiang algebra? its geometrical properties?
4. Any Steiner decomposition of $\langle x; x^2\rangle$ (if exists) distinguishes a finite family of idempotents and the corresponding $\mathbb{E}_3$-cells. It would be interesting to clarify the algebraic side of this simplicial 'complex'.
5. The above questions are also relevant for arbitrary metrized algebras admiting a PSTS decomposition (see also a recent paper [Fox22])

Definition

A collection B of triples from $\mathbb{N}_n := \{1, \dots, n\}$, so that each element $i \in \mathbb{N}_n$ occurs in at least one triple and each unordered pair $i \neq j$ occurs in at most one triple of B, is called a partial Steiner triple system, or PSTS. Given a PSTS, there is smallest subset $R \subset \mathbb{N}$ (the set of replication numbers) such that each $i \in \mathbb{N}_n$ is contained in exactly $r \in R$ blocks. If $R = \{r\}$ then the *PSTS* is called regular.

Example

A 2-regular PSTS on $\mathbb{N}_6$:

$$\{(1,3,6),(1,4,5),(2,3,5),(2,4,6)\},$$

A $\{2,4\}$-regular PSTS on $\mathbb{N}_{10}$:

$$\{(\mathbf{1},3,5),(\mathbf{1},4,6)(\mathbf{1},7,9),(\mathbf{1},8,10),(\mathbf{2},3,6),(\mathbf{2},4,5),(\mathbf{2},7,10),(\mathbf{2},9,8)\}, \quad (7)$$

A 3-regular STS the Fano plane on $\mathbb{N}_7$ (all pairs occur!):

$$\{(1,2,3),(1,4,5),(1,6,7),(2,4,6),(2,5,7),(3,4,7),(3,5,6)\},$$

In fact regular STS exist iff $n = 6k + 1$ or $6k + 3$.

Definition

A cubic form $u(x)$ on an inner product vector space $(V, \langle ; \rangle)$ is said to admit a (partial) Steiner decomposition if there exist a *PSTS* B on $\mathbb{N}_{\dim V}$ and a basis e_i such that

$$u(x) = \sum_{\alpha \in B} a_\alpha x_{\alpha_1} x_{\alpha_2} x_{\alpha_3}, \qquad x_i = \langle x; e_i \rangle, \quad a_\alpha \in \Bbbk.$$

An orthonormal basis $\{e_i\}$ of a metrized algebra $(\mathbb{A}, \langle ; \rangle)$ is a Steiner basis if

- $e_i e_i = 0$ for all i,
- $e_i e_j$ is proportional to some e_k for all $i \neq j$.

Proposition 2.1

Let $(\mathbb{A}, \langle ; \rangle)$ be a metrized algebra, $(\mathbb{A}\mathbb{A})^\perp = 0$. The defining form $u_{\mathbb{A}}(x) = \frac{1}{6}\langle x^2; x \rangle$ admits a Steiner form if and only $\mathbb{A}$ admits a Steiner basis.

Three different ways to write 27

1. $\frac{6\cdot 5}{2}+6+6$: Recall the 27-dimensional Dickson cubic form

$$\Phi(t) = \mathrm{Pf}(Z) + x^t Z y, \qquad t = (x, y, Z) \in \Bbbk^6 \times \Bbbk^6 \times \mathrm{Skew}(6, \Bbbk)$$

2. $9+9+9$: equivalently (Tits, McCrimmon)

$$\Psi(x) = \det x' + \det x'' + \det x''' - \mathrm{tr}(x' x'' x'''), \quad x = (x', x'', x''') \in M(3, \Bbbk)^{\times 3}$$

3. $3 + \frac{4\cdot(4-1)}{2} \cdot \dim \mathbf{H}$: Hermitean trace-less matrices $\mathrm{Herm}_0(4, \mathbf{H})$ over quaternions $\mathbf{H}$

$$\Omega(x) = \mathrm{tr}\, x^3, \quad x \in \mathrm{Herm}_0(4, \mathbf{H})$$

The Steiner decomposition can be written explicitly by (**45 terms**)

$$\begin{aligned}\Psi = {} & x_1x_5x_9 - x_1x_6x_8 - x_2x_4x_9 + x_2x_6x_7 + x_3x_4x_8 - x_3x_5x_7 + x_{10}x_{14}x_{18} - x_{10}x_{15}x_{17} - x_{11}x_{13}x_{18} \\ & + x_{11}x_{15}x_{16} + x_{12}x_{13}x_{17} - x_{12}x_{14}x_{16} + x_{19}x_{23}x_{27} - x_{19}x_{24}x_{26} - x_{20}x_{22}x_{27} + x_{20}x_{24}x_{25} + x_{21}x_{22}x_{26} \\ & - x_{21}x_{23}x_{25} - x_1x_{16}x_{21} - x_1x_{10}x_{19} - x_1x_{13}x_{20} - x_2x_{10}x_{22} - x_2x_{13}x_{23} - x_2x_{16}x_{24} - x_3x_{10}x_{25} \\ & - x_3x_{13}x_{26} - x_3x_{16}x_{27} - x_4x_{11}x_{19} - x_4x_{14}x_{20} - x_4x_{17}x_{21} - x_5x_{11}x_{22} - x_5x_{14}x_{23} - x_5x_{17}x_{24} \\ & - x_6x_{11}x_{25} - x_6x_{14}x_{26} - x_6x_{17}x_{27} - x_7x_{12}x_{19} - x_7x_{15}x_{20} - x_7x_{18}x_{21} - x_8x_{12}x_{22} - x_8x_{15}x_{23} \\ & - x_8x_{18}x_{24} - x_9x_{12}x_{25} - x_9x_{15}x_{26} - x_9x_{18}x_{27}\end{aligned}$$

Remark

$$\Psi = x_1x_5x_9 - x_1x_6x_8 - x_2x_4x_9 + x_2x_6x_7 + x_3x_4x_8 - x_3x_5x_7 + x_{10}x_{14}x_{18} - x_{10}x_{15}x_{17} - x_{11}x_{13}x_{18}$$
$$+ x_{11}x_{15}x_{16} + x_{12}x_{13}x_{17} - x_{12}x_{14}x_{16} + x_{19}x_{23}x_{27} - x_{19}x_{24}x_{26} - x_{20}x_{22}x_{27} + x_{20}x_{24}x_{25} + x_{21}x_{22}x_{26}$$
$$- x_{21}x_{23}x_{25} - x_1x_{16}x_{21} - x_1x_{10}x_{19} - x_1x_{13}x_{20} - x_2x_{10}x_{22} - x_2x_{13}x_{23} - x_2x_{16}x_{24} - x_3x_{10}x_{25}$$
$$- x_3x_{13}x_{26} - x_3x_{16}x_{27} - x_4x_{11}x_{19} - x_4x_{14}x_{20} - x_4x_{17}x_{21} - x_5x_{11}x_{22} - x_5x_{14}x_{23} - x_5x_{17}x_{24}$$
$$- x_6x_{11}x_{25} - x_6x_{14}x_{26} - x_6x_{17}x_{27} - x_7x_{12}x_{19} - x_7x_{15}x_{20} - x_7x_{18}x_{21} - x_8x_{12}x_{22} - x_8x_{15}x_{23}$$
$$- x_8x_{18}x_{24} - x_9x_{12}x_{25} - x_9x_{15}x_{26} - x_9x_{18}x_{27}$$

Dickson writes (p. 64): *Any two triangles ABC and A'B'C' having no side in common determine uniquely a third triangle A"B"C", such that the corresponding sides of the three triangles intersect and form three new triangles AA'A", BB'B", CC'C". The former set of three triangles is said to constitute a* ***trieder****. Each triangle lies in exactly 16 trieders.*

For example,

$$ABC = 159 \qquad AA'A'' = 168$$
$$A'B'C' = 672 \qquad BB'B'' = 573$$
$$A''B''C'' = 834 \qquad CC'C'' = 924$$

Any trieder gives rise to a 9-dim Hsiang exceptional algebra.

Mutants

Given a Hurwitz algebra $\mathbf{H}_d$, $d \in \{1,2,4,8\}$ with conjugation $\bar{x}$, norm $n(x)$, a mutants is the **tripling** (see [FT25] for the general case)

$$\mathrm{Tri}(\mathbf{H}_d) := \mathbf{H}_d \times \mathbf{H}_d \times \mathbf{H}_d = V_1 \oplus V_2 \oplus V_3$$

with commutative multiplication

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = (\bar{x}_3\bar{y}_2 + \bar{y}_3\bar{x}_2, \bar{x}_1\bar{y}_3 + \bar{y}_1\bar{x}_3, \bar{x}_2\bar{y}_1 + \bar{y}_2\bar{x}_1)$$

and an invariant bilinear form $\langle (x_1, x_2, x_3); (y_1, y_2, y_3)\rangle = \sum_{i=1}^{3} t(\bar{x}_i y_j)$, where $t(x) = n(x+e) - n(x) - n(e)$ is the trace form ('the real part'). Note that

$$V_i V_i = 0, \qquad V_i V_j = V_k$$

Proposition 2.2 ([FT25])

$\mathrm{Tri}(\mathbf{H}_d)$ *is a Killing metrized polar algebra with respect to any of the three decompositions* $V_k \oplus V_k^{\perp}$, $k = 1,2,3$. *The corresponding defining form* $u(x) = u(x_1, x_2, x_3) = t(x_1(x_2x_3))$ *admits a Steiner decomposition in the canonical coordinates.*

The defining forms for mutants

The cubic forms u_{3d} of Steiner type $PSTS(3d, d)$, $d \in \{1, 2, 4, 8\}$:

$$u_3 = x_1 x_2 x_3$$

a Steiner type $n = 3 \cdot 2^0$, $|B| = 4^0$, replication number $r = 2^0$

$$u_6 = x_1x_3x_6 - x_1x_4x_5 + x_2x_3x_5 + x_2x_4x_6$$

a Steiner type $n = 3 \cdot 2^1$, $|B| = 4^1$, replication number $r = 2^1$

$$\begin{aligned} u_{12} = {} & x_1x_5x_9 - x_1x_6x_{10} - x_1x_7x_{11} - x_1x_8x_{12} - x_2x_5x_{10} - x_2x_6x_9 - x_2x_7x_{12} + x_2x_8x_{11} \\ & - x_3x_5x_{11} + x_3x_6x_{12} - x_3x_7x_9 - x_3x_8x_{10} - x_4x_5x_{12} - x_4x_6x_{11} + x_4x_7x_{10} - x_4x_8x_9 \end{aligned}$$

a Steiner type $n = 3 \cdot 2^2$, $|B| = 4^2$, replication number $r = 2^2$

$$\begin{aligned} u_{24} = {} & x_1x_9x_{17} - x_{18}x_1x_{10} - x_1x_{11}x_{19} - x_{20}x_1x_{12} - x_1x_{13}x_{21} - x_{22}x_1x_{14} - x_{23}x_1x_{15} \\ & - x_{24}x_1x_{16} - x_{18}x_2x_9 - x_{17}x_2x_{10} - x_2x_{11}x_{20} + x_{19}x_2x_{12} - x_{22}x_2x_{13} + x_{21}x_2x_{14} \\ & + x_{24}x_2x_{15} - x_{23}x_2x_{16} - x_{19}x_3x_9 + x_{20}x_3x_{10} - x_{17}x_3x_{11} - x_{18}x_3x_{12} - x_{23}x_3x_{13} \\ & - x_{24}x_3x_{14} + x_{21}x_3x_{15} + x_{22}x_3x_{16} - x_4x_9x_{20} - x_{19}x_4x_{10} + x_{18}x_4x_{11} - x_{17}x_4x_{12} \\ & - x_{24}x_4x_{13} + x_{23}x_4x_{14} - x_{22}x_4x_{15} + x_{21}x_4x_{16} - x_5x_9x_{21} + x_5x_{10}x_{22} + x_{23}x_5x_{11} \\ & + x_{24}x_5x_{12} - x_5x_{13}x_{17} - x_{18}x_5x_{14} - x_{19}x_5x_{15} - x_{20}x_5x_{16} - x_{22}x_6x_9 - x_6x_{10}x_{21} \\ & + x_6x_{11}x_{24} - x_{23}x_6x_{12} + x_{18}x_6x_{13} - x_{17}x_6x_{14} + x_{20}x_6x_{15} - x_{19}x_6x_{16} - x_{23}x_7x_9 \\ & - x_{24}x_7x_{10} - x_{21}x_7x_{11} + x_{22}x_7x_{12} + x_{19}x_7x_{13} - x_7x_{14}x_{20} - x_{17}x_7x_{15} + x_{18}x_7x_{16} \\ & - x_8x_9x_{24} + x_8x_{10}x_{23} - x_8x_{11}x_{22} - x_8x_{12}x_{21} + x_8x_{13}x_{20} + x_8x_{14}x_{19} - x_8x_{15}x_{18} - x_8x_{16}x_{17} \end{aligned}$$

a Steiner type $n = 3 \cdot 2^3$, $|B| = 4^3$, replication number $r = 2^3$

The defining forms in 15 and 18 dimensions

A Steiner type form: $n = 15$, $|B| = 15$, replication number $r = 3$ (this form is exactly the *contraction* of the 27-dimensional Dickson form above)

$$x_1x_6x_{15} - x_1x_9x_{14} + x_1x_{10}x_{13} - x_2x_5x_{15} + x_2x_8x_{14} - x_2x_{10}x_{12} + x_3x_4x_{15} - x_3x_7x_{14} + x_3x_{10}x_{11} - x_4x_8x_{13} + x_4x_9x_{12} + x_5x_7x_{13} - x_5x_9x_{11} - x_6x_7x_{12} + x_6x_8x_{11}$$

A Steiner type form: for $n = 18$, $|B| = 24$, replication number $r = 4$

$$x_1x_5x_9 - x_1x_6x_8 - x_1x_{14}x_{18} + x_1x_{15}x_{17} - x_4x_2x_9 + x_7x_2x_6 + x_{13}x_2x_{18} - x_{16}x_2x_{15} + x_4x_3x_8 - x_7x_3x_5 - x_{13}x_3x_{17} + x_{16}x_3x_{14} + x_4x_{11}x_{18} - x_4x_{12}x_{17} - x_{10}x_5x_{18} + x_{16}x_{12}x_5 + x_{10}x_6x_{17} - x_{16}x_{11}x_6 - x_7x_{11}x_{15} + x_7x_{12}x_{14} + x_{10}x_{15}x_8 - x_{13}x_{12}x_8 - x_{10}x_{14}x_9 + x_{13}x_{11}x_9$$

These examples suggest that for exceptional and mutant Hsiang cubics holds $|B| = \frac{n \cdot r}{3}$ We shall see below that this holds true.

Example: the determinant eigencubic in $9D$

The following are Hsiang eigencubics in $\Bbbk^{6d+3}$ [Hsi67], [HT19]:

$$u_d(X) := \operatorname{tr} X^3, \qquad X \in \operatorname{Herm}(\mathbf{H}_d, 4), \quad \operatorname{tr} X = 0, \quad d \in \{1, 2, 4\}$$

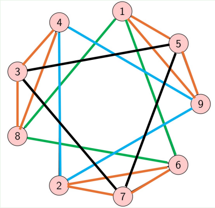
For $d = 1$, this provides an **exceptional** eigencubic $u_1(x)$ in $\Bbbk^9$ which in some orthonormal coordinates can be written as the determinant:

$$u_1(x) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{vmatrix} = \underbrace{\begin{matrix} x_1x_5x_9 + x_2x_6x_7 + x_3x_4x_8 \\ -x_1x_6x_8 - x_2x_4x_9 - x_3x_5x_7 \end{matrix}}_{\text{a Steiner form}},$$

where the set of **unordered** triples

$$B = \{(1,5,9), (2,6,7), (3,4,8), (1,6,8), (2,4,9), (3,5,7)\}$$

is a **regular** partial Steiner triple system on $\mathbb{N}_9$ with **replication number** $r = 2$. One can naturally assign to B a 4-regular graph (two vertices i, j are incident iff $\{i, j, \star\} \in B$):



A handicap graph (Kovar, Kravčenko) et al, 2017

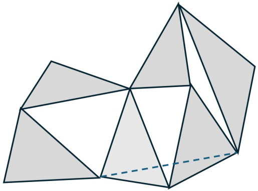
Simplicial algebras

Definition

A commutative algebra generated by $n \geq 2$ idempotents $\{e_i\}$ such that $e_i e_j = -\frac{1}{n-1}(e_i + e_j)$ is called the Griess-Harada (simplicial) algebra $\mathbb{E}_n$.

1. $e_{n+1} := -\sum_{i=1}^{n} e_i$ is an idempotent.
2. $\mathbb{E}_n$ is Killing metrized with an invariant form $\langle x; y \rangle := \operatorname{tr} L(x)L(y)$
3. $\{e_i\}$ have the same norm and share the same fusion laws [Fox21]
4. $\operatorname{Aut}(\mathbb{E}_n) = S_{n+1}$ [Har81].

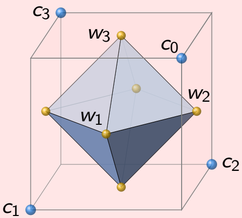
The simplicial algebra $\mathbb{E}_3$ plays a prominent role for the structure of Hsiang algebras (cf. Dickson's 'trieders').



The simplicial algebra $\mathbb{E}_3$ (a cell)

1. $\mathbb{E}_3$ is a Hsiang algebra (a mutant, also an Elduque-Okubo algebra)
2. $\mathrm{Idm}(\mathbb{E}_3) = \{c_0 := -(c_1 + c_2 + c_3), c_1, c_2, c_3\}$
3. There are totally three 2-nilpotent 1D-subalgebras spanned resp. by
$$w_i := \tfrac{1}{2}(c_0 + c_i), \qquad 1 \le i \le 3$$
$$w_i w_j = w_k, \qquad \{i, j, k\} = \{1, 2, 3\}$$
4. $\mathrm{Aut}(\mathbb{E}_3) = S_4$ (permuting the four idempotents).
5. $\mathbb{E}_3$-cells play the role analogues to $\mathfrak{sl}_2$ triples in semi-simple Lie algebras.

Cube (4 Idempotents + 4 anti-idempotents) and Octahedron (2-Nilpotents)



Definition

Let $n_2 = \dim A_c(-\frac{1}{2})$. Hsiang algebras with $n_2 = 0$, resp. $n_2 = 1$ are called *eikonal*, resp. *sub-eikonal* ($\dim \mathbb{A} = n = 3d + 1 + n_2$, where $d \in \{0, 1, 2, 4, 8\}$).

Theorem [NTV14], [FT25]

Let $\mathbb{A}$ be neither eikonal nor eikonal Hsiang algebra. Then

1. The Jordan algebra Λ_c is a *simple* if and only if $\mathbb{A}$ is *exceptional*.
2. The following Jacobson–Morozov type **Theorem** holds:
   - The set $N_2(\mathbb{A})$ of nonzero 2-nilpotents of $\mathbb{A}$ is nonempty.
   - Any $w \in N_2(\mathbb{A})$ can be expanded to a $\mathbb{E}_3$-cell inside of $\mathbb{A}$
   - Any nonzero 3D Hsiang subalgebra of $\mathbb{A}$ is a $\mathbb{E}_3$-cell.
3. Any $\mathbb{E}_3$-cell $\mathbb{B} \subset \mathbb{A}$ gives rise to a germ of Steiner type decomposition: if $\mathbb{B} = \langle\langle w_1, w_2, w_3 \rangle\rangle$ given by an orthonormal basis of 2-nilpotents then

$$\frac{1}{6}\langle x^2; x\rangle = x_1x_2x_3 + ax_1x_ix_j + \ldots + bx_2x_kx_l + \text{etc}$$

Theorem 3.1

Let a Hsiang algebra $\mathbb{A}$ *admit a Steiner form* B *on* $\mathbb{N}_n$ *and* $n_1 = \dim \mathbb{A}_c(-1)$. *Then*

$$n_2 = \dim \mathbb{A}_c(-1/2) \geq 2, \quad n_2 = 3d + 2,$$

and either of the following holds:

1. $\mathbb{A}$ *is* **exceptional or mutant**, B *is* r*-regular, where*

$$r = n_1 + d + 1, \qquad |B| = (n_1 + d + 1)(n_1 + 2d + 1).$$

2. $\mathbb{A}$ *is a* **polar algebra** $\mathbb{A}_0 \oplus \mathbb{A}_1$, B *is* R*-regular, where*

$$R = \{\dim \mathbb{A}_0, \tfrac{1}{2} \dim \mathbb{A}_1\}, \qquad |B| = \tfrac{1}{2} \dim \mathbb{A}_1 \dim \mathbb{A}_0.$$

Simplicial complex for a polar algebra in $n = 10$

Example

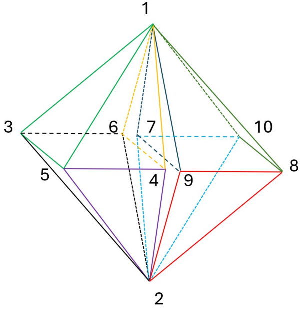
$\mathbb{E}_3$-complex: each triple

$$x_{\alpha_1} x_{\alpha_2} x_{\alpha_3}$$

corresponds to a simplicail cell

$$\mathbb{E}_3 = \mathrm{span}(e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3})$$

See (7) above for $n = 10 \Longrightarrow$



$n = 10$
$R = \{2, 4\}$
$\mathbb{A} = \mathbb{A}_0 \oplus \mathbb{A}_1$

The sign decoration does matter!

$H_6 = \{(1,3,6),(1,4,5),(2,3,5),(2,4,6)\}$

$u = x_1x_3x_6 - x_1x_4x_5 + x_2x_3x_5 + x_2x_4x_6$

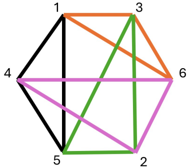
$v = x_1x_3x_6 + x_1x_4x_5 + x_2x_3x_5 + x_2x_4x_6$

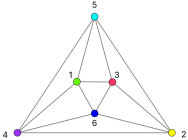
The corresponding algebras are

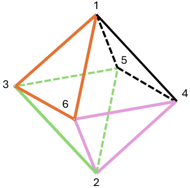
$\mathbb{A}(u) \cong \mathrm{Tri}(\widehat{\mathbb{C}}) \cong \mathbb{E}_2 \otimes \mathbb{E}_3$

$\mathbb{A}(v) \cong \mathrm{Tri}(\mathrm{Cl}(1,0)) \cong \mathrm{Cl}(1,0) \otimes \mathbb{E}_3$

($\mathrm{Cl}(1,0)$ is the algebra of split-complex numbers)







The sign decoration does matter!

The cubic forms

$$\begin{aligned} \text{perm}(A) &= x_1x_5x_9 + x_2x_6x_7 + x_3x_4x_8 + x_1x_6x_8 + x_2x_4x_9 + x_3x_5x_7, \\ \det(A) &= x_1x_5x_9 + x_2x_6x_7 + x_3x_4x_8 - x_1x_6x_8 - x_2x_4x_9 - x_3x_5x_7, \end{aligned}$$

have the same KIlling invariant form $\operatorname{tr} L(x)L(y)$ but they give rise to non-isomorphic algebras! More precisely,

Theorem 3.2 ([Fox22], [FT25])

The corresponding algebras

$$\begin{aligned} \mathrm{Det} &\cong \mathfrak{so}(3, \Bbbk) \otimes \mathfrak{so}(3, \Bbbk) \\ \mathrm{Perm} &\cong \mathbb{E}_3 \otimes \mathbb{E}_3, \end{aligned}$$

where $\mathbb{E}_3$ is the simplicial algebra. Moreover, both Det and $\mathbb{E}_3$ are Hsiang algebras, while Perm is **not***.*

Furthermore, the set of idempotents of Det is infinite homogeneous space $S^1 \times S^1$, while Perm contains only finitely many nonzero idempotents

$\dim \mathbb{A}$	$\dim \mathrm{Idm}(\mathbb{A})$	$\mathrm{Idm}(\mathbb{A})$	type	tripling
5	2	$\mathbb{RP}^2$	E.-O. (eikonal)	
6	2	$S^1 \times S^1$	mutant	yes
8	4	$\mathbb{CP}^2$	E.-O. (eikonal)	
9	3	$SO(3)$	pseudo-comp.	yes
12	5	$SU(3)/SO(3)$	para-complex	
12	6	$S^3 \times S^3$	mutant	yes
14	8	$\mathbb{HP}^2$	E.-O. (eikonal)	
15	6	$SU(4)/S(U(1) \times U(3))$	pseudo-comp.	
18	8	$SU(3)$	para-complex	yes
21	11	$G_2/SU(2)$	contraction	yes
24	14	$S^7 \times S^7$	mutant	yes
26	16	$\mathbb{OP}^2$	E.-O. (eikonal)	
27	12	$Sp(4)/Sp(1) \times Sp(3)$	pseudo-comp.	
30	14		para-complex	
54	26	E_6/F_4	para-complex	

References

M. Aschbacher, Some multilinear forms with large isometry groups, Geometries and Groups: Proc. Workshop Geom. Groups, Finite and Algebraic, Noordwijkerhout, Holland, March 1986, Springer, 1988, pp. 417–465.

A. Elduque and S. Okubo, On algebras satisfying $x^2x^2 = N(x)x$, Math. Z. **235** (2000), no. 2, 275–314.

D. J. F. Fox, The commutative nonassociative algebra of metric curvature tensors, Forum Math. Sigma **9** (2021), Paper No. e79, 48. MR 4350139

______, Killing metrized commutative nonassociative algebras associated with Steiner triple systems, J. Algebra **608** (2022), 186–213. MR 4436483

D. J. F. Fox and V.G. Tkachev, Algebraic constructions of cubic minimal cones, Pure App. Funct.Anal. **10** (2025), no. 2.

K. Harada, On a commutative nonassociative algebra associated with a multiply transitive group, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 843–849 (1982).

J. I. Hall, F. Rehren, and S. Shpectorov, Universal axial algebras and a theorem of Sakuma, J. Algebra **421** (2015), 394–424.

W.-Y. Hsiang, Remarks on closed minimal submanifolds in the standard Riemannian m-sphere, J. Differential Geometry **1** (1967), 257–267.

J. Hoppe and V. G. Tkachev, New construction techniques for minimal surfaces, Complex Variables and Elliptic Equations **64** (2019), 1–18,

N. Nadirashvili, V. G. Tkachev, and S. Vlăduț, Nonlinear elliptic equations and nonassociative algebras, Math. Surveys and Monographs, vol. 200, AMS, Providence, RI, 2014.

V. G. Tkachev, Minimal cubic cones via Clifford algebras, Complex Anal. Oper. Theory **4** (2010), no. 3, 685–700.

______, On a classification of minimal cubic cones in $\mathbb{R}^n$, Preprint, arxiv:1009.5409. 42. p. (2010).

Happy Birthday Sergey!