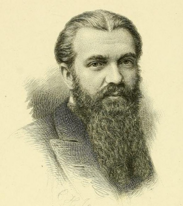
Geometry of Hsiang algebras, or Where Hsiang meets Clifford and Jordan?



Vladimir G. Tkachev (Linköping University)

(based on a joint work with D.J. Fox, Universidad Politécnica de Madrid)

During the writing my talk I realized that a more relevant title should be

Where Hsiang meets Elduque and Okubo?

Spoiler

Many distinguished algebras appear as the most natural places where certain interesting structures live in.

I will talk about Hsiang algebras. They have the following remarkable properties:

- They appear in various areas of mathematics, including Differential Geometry, Nonlinear PDEs, Algebra and Combinatorial Design.
- They are intimately related to the classical structures like Jordan, Clifford, Lie and axial algebras.
- Moreover, they are also very related and contain pseudo-composition and Elduque-Okubo (aka admissible cubic) algebras.
- Similarly to the latter, they have infinitely many regular families vs finitely many exceptional families.
- They are essentially the only known examples of commutative nonassociative algebras with infinitely many idempotents where all idempotents have the same length and the same algebraic spectrum.
- Furthermore, the set of idempotents has a nice structure of a smooth (Riemannian) homogeneous submanifold.

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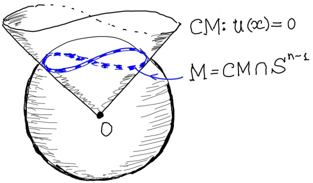
Initial geometrical context

- A geodesic line is a shortest curve between two points.
- A minimal surface is a surface that *locally minimizes its area*.

Malmö Live Concert, Sweden, https://www.fotosidan.se/blogs/wolfgang/offentlig-konst-i-malmo.htm

- Minimal submanifolds in spheres: $M \subset \mathbb{S}^{n-1}$ is (locally) minimal iff the cone $CM \subset \mathbb{R}^n$ is so. The majority of known minimal cones are algebraic



- W.-Y. Hsiang (1967): the cone $u^{-1}(0)$ is a minimal hypersurface in $\mathbb{R}^n$ iff

$$\sum_{j=1}^{n}\sum_{i=1}^{n} u''_{x_i x_j} u'_{x_i} u'_{x_j} - (\sum_{i=1}^{n} {u'_{x_i}}^2)(\sum_{j=1}^{n} u'_{x_j x_j}) = Q(x)u(x).$$

where $\deg Q(x) = 2\deg u - 4$

- The first **non-trivial case**: $\deg u = 3$ and

$$\tfrac{1}{2}\langle \nabla u; \nabla |\nabla u|^2\rangle - |\nabla u|^2 \Delta u = q(x)\cdot u(x) \qquad (*)$$

Hisang's Problem

Classify all cubic polynomials satisfying $(*)$.

In fact, all known **irreducible** solutions satisfy $q(x) = \theta \cdot \langle x; x \rangle$ for some real θ:

$$\frac{1}{2}\langle \nabla u; \nabla |\nabla u|^2 \rangle - |\nabla u|^2 \Delta u = \theta \cdot \langle x; x \rangle u(x) \qquad (\tilde{*})$$

Such a solution is called a *radial* Hsiang eigencubic.

Theorem 1.1

Any radial Hsiang eigencubic is harmonic, i.e. $\Delta u(x) = 0$, unless $u(x) = \langle x; a \rangle^3$. In other words, any nontrivial radial Hsiang eigencubic satisfies

$$\frac{1}{2}\langle \nabla u; \nabla |\nabla u|^2 \rangle = \theta \cdot \langle x; x \rangle u(x) \qquad (**)$$

So one can reformulate the original Hsiang problem as

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Problem

Classify all harmonic cubic polynomials satisfying $(**)$.

- A bilinear form h is **invariant** in a commutative algebra if $h(xy, z) = h(x, zy)$.
- An algebra with an invariant bilinear form is called **metrized**.
- An algebra is called **exact** if $\operatorname{tr} L(x) = 0$.

Theorem 2.1 (V.T., 2010)

Let $\mathbb{R}^n$ be a Euclidean space with an inner product $\langle ; \rangle$. Any cubic polynomial solution $u(x)$ of $(**)$ can be written as

$$u(x) = \frac{1}{6}\langle x; x^2\rangle$$

in an exact commutative nonassociative metrized algebra on $\mathbb{R}^n$ satisfying

$$\langle x^3; x^2\rangle = \theta\langle x^2; x\rangle\langle x; x\rangle, \qquad (1)$$

or equivalently, satisfying the following degree 4 identity

$$4xx^3 + x^2x^2 - 3\theta\langle x; x\rangle x^2 - 2\theta\langle x^2; x\rangle x = 0. \qquad (2)$$

The multiplication in $\mathbb{A}$ is defined by $xy = x \cdot y = \mathrm{Hess}\, u(x) y$.

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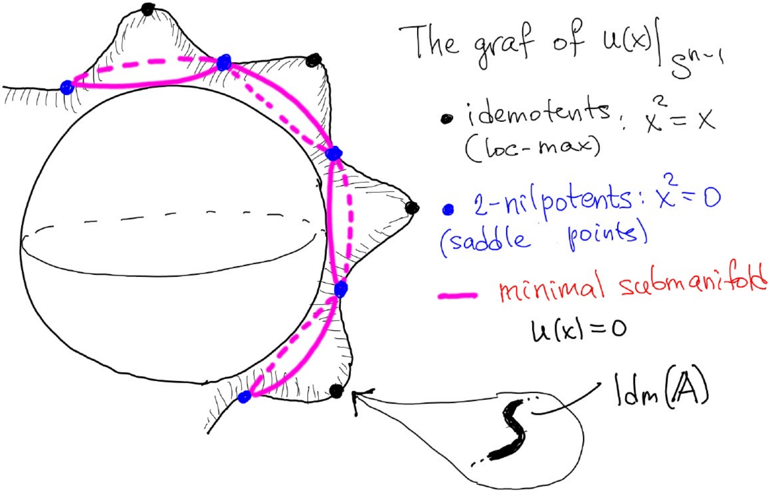
The graf of u(x)|_{S^{n-1}}
• idemotents: x^2 = x
(loc-max)
• 2-nilpotents: x^2 = 0
(saddle points)
— minimal submanifold
u(x) = 0
Idm(A)

Definition

An exact commutative h-metrized algebra $(\mathbb{A}, h)$ is called **Hsiang** if there exists $\theta \in \mathbb{K}$ such that the following identity holds:

$$h(x^3, x^2) = \theta h(x, x) h(x, x^2) \qquad (3)$$

An algebra satisfying the above identity only is called **almost-Hsiang**.

1. Any Hsiang algebra is a *almost* Hsiang algebra.
2. A subalgebra of a Hsiang algebra is an almost-Hsiang algebra.
3. A subalgebra of an almost-Hsiang algebra is also almost-Hsiang.
4. The equation $\langle x; x^2 \rangle = 0$ describes a minimal cone iff $\mathbb{A}$ a Hsiang algebra.
5. The exactness condition affects mostly'finer properties'.
6. $\theta = 0$ implies that $\mathbb{A}\mathbb{A} = 0$ (see [V.T., 2015]).
7. An immediate corollary of (3) is that any idempotent in $\mathbb{A}$ has length $1/\theta$.
8. If one scales $\langle x; y \rangle := \theta h(x, y)$ then (3) and (1) become respectively

$$\begin{aligned} \langle x^3; x^2 \rangle &= \langle x^2; x \rangle \langle x; x \rangle, \\ 4xx^3 + x^2x^2 - 3\langle x; x \rangle x^2 - 2\langle x^2; x \rangle x &= 0. \end{aligned} \qquad (4)$$

In the latter normalized setting, any idempotent in $\mathbb{A}$ has length 1.

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Two important classes of almost-Hsiang algebras

A. A commutative algebra with a symmetric bilinear form h is called **pseudo-composition** (Meyberg, Osborn, Walcher, Röhrl, Gradl) if

$$x^3 = h(x,x)x \tag{5}$$

Elduque and Okubo (2000) proved that in fact h above must be an *invariant form*. This implies that *any pseudo-composition algebra is almost-Hsiang*:

$$(5) \quad \Rightarrow \quad h(x^3, x^2) = h(x,x)h(x,x^2), \quad \theta = 1.$$

The linearization of (5) gives $2L(x)^2 + L(x^2) = x \otimes_h x + h(x,x)\mathbf{1}$ implying for any idempotent c that

$$2L(c)^2 + L(c) - \mathbf{1} = 0$$

$\Rightarrow$ the spectrum of $L(c)$ on $c^\perp$ is $\{-1, \frac{1}{2}\}$ with the Peirce decomposition

$$\mathbb{A} = \mathbb{K}c \oplus \mathbb{A}_c(-1) \oplus \mathbb{A}_c(\tfrac{1}{2})$$

Definition 2.2

If additionally to (5), $\operatorname{tr} L(x) = 0$, such a **Hsiang** algebra is called **eikonal**.

Two important classes of almost-Hsiang algebras

B. Elduque and Okubo studied commutative *admissible cubic algebras*, i.e.

$$x^2x^2 = N(x)x = h(x^2, x)x \quad (6)$$

and proved that there exists an invariant bilinear form h:

$$N(x) = h(x^2, x).$$

Such an algebra is also almost-Hsiang:

$$(6) \quad \Rightarrow \quad h(x^2, x^3) = h(x^2x^2, x) = h(x^2, x)h(x, x), \quad \theta = 1.$$

The classification of Elduque and Okubo contains two relevant for us classes:

1. The simple Jordan algebras of **degree 3** with the new multiplication $x^2 := x^\sharp$
2. The contraction of **trace zero** elements of a simple **degree 4** Jordan algebra.

The Elduque-Okubo algebras have the following Peirce decomposition:

$$\mathbb{A} = \mathbb{K}c \oplus \mathbb{A}_c(-\tfrac{1}{2}) \oplus \mathbb{A}_c(\tfrac{1}{2})$$

The case 2 appears in the Hsiang paper **Hs'67**.

Assume for simplicity that $\mathbb{A}$ is **normalized** by (4).

Theorem (Basic facts on Hsiang algebras, I)

1. The set of nonzero idempotents in any Hsiang algebra $\mathbb{A}$ is nonempty
2. All idempotents have the *unit length* and share the *same fusion laws*.
3. For any idempotent c, the associated *Peirce decomposition* is

$$\mathbb{A} = \underbrace{\mathbb{A}_c(1)}_{\dim=1} \oplus \underbrace{\mathbb{A}_c(-1)}_{\dim=n_1} \oplus \underbrace{\mathbb{A}_c(-\tfrac{1}{2})}_{\dim=n_2} \oplus \underbrace{\mathbb{A}_c(\tfrac{1}{2})}_{\dim=n_3}$$

4. $\mathbb{A}_c(1) \oplus \mathbb{A}_c(-1)$ is a subalgebra of $\mathbb{A}$ and is a **pseudo-composition algebra**.
5. $\Lambda_c := \mathbb{A}_c(1) \oplus \mathbb{A}_c(-\frac{1}{2})$ is a subalgebra and is an **Elduque-Okubo algebra**
6. Λ_c carries a **rank 3 Jordan algebra structure** (an isotopy of multiplication)

Remark 1

The structure of the Peirce subspace $\mathbb{A}(\frac{1}{2})$ is much more involved and it is related to the structure of **idempotents** of $\mathbb{A}$.

A naïve philosophy

$$\text{Hsiang} \approx \underbrace{\text{Pseudo-composition}}_{\mathbb{A}_c(1)\oplus\mathbb{A}_c(-1)} \ltimes \underbrace{\text{Elduque-Okubo}}_{\mathbb{A}_c(1)\oplus\mathbb{A}_c(-\frac{1}{2})} \ltimes \mathbb{A}_c(\frac{1}{2})$$

These two cases are quite independent as the following lemma shows:

Observation

In an almost-Hsiang algebra, the elements satisfying both the Elduque-Okubo and the pseudo-composition conditions are exactly the cone over idempotents.

Proof. Substitution $x^3 = \langle x; x\rangle x$ and $x^2x^2 = \langle x^2; x\rangle x$ into (4) yields $\langle x; x\rangle x^2 = \langle x^2; x\rangle x$. We have $x^2 \neq 0$ because otherwise $0 = x^3 = \langle x; x\rangle x$, a contradiction. This yields $x^2 = \frac{\langle x^2;x\rangle}{\langle x;x\rangle}x$, and thus the desired conclusion. □

A naïve philosophy

$$\text{Hsiang} \approx \underbrace{\text{Pseudo-composition}}_{\mathbb{A}_c(1)\oplus\mathbb{A}_c(-1)} \ltimes \underbrace{\text{Elduque-Okubo}}_{\mathbb{A}_c(1)\oplus\mathbb{A}_c(-\frac{1}{2})} \ltimes \mathbb{A}_c(\frac{1}{2})$$

These two cases are quite independent as the following lemma shows:

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A toy example I: the simplicial algebra $\mathbb{E}_2$

1. $u(x) = \frac{1}{6}(x_1^3 - 3x_2^2 x_1)$, $x = (x_1, x_2) \in \mathbb{K}^2$, $\mathrm{Hess}\, u(x) = \begin{pmatrix} x_1 & -x_2 \\ -x_2 & -x_1 \end{pmatrix}$
2. $x \diamond y := \mathrm{Hess}\, u(x) y = (x_1 y_1 - x_2 y_2, -x_2 y_1 - x_1 y_2) = \overline{x \cdot y}$, where $\cdot$ is the complex number multiplication on $\mathbb{R}^2 \cong \mathbb{C}$.
3. $(\mathbb{A}, \diamond) \cong (\widehat{\mathbb{C}}, \cdot) \cong \mathbb{E}_2$ (**para-complex** numbers, resp. the $2D$ **simplicial** algebra) with an invariant bilinear form $\langle x; y \rangle := \mathrm{Re}(\bar{x} \cdot y)$.
4. $x \diamond (x \diamond x) = \langle x; x \rangle x$ (a **pseudo-composition** algebra!)
5. $\mathbb{A}$ is a **Hsiang algebra** for $\theta = \frac{3}{2}$.
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The simplicial algebra $\mathbb{E}_3$ plays a prominent role in the Hsiang algebras.

1. $\mathbb{E}_3$ is metrized w.r. to the natural Killing form $h(x, y) := \frac{2}{3} \operatorname{tr} L(x)L(y)$
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1. $\mathbb{E}_3$ is metrized w.r. to the natural Killing form $h(x, y) := \frac{2}{3} \operatorname{tr} L(x)L(y)$
2. $\mathbb{E}_3$ is **exact**
3. $\mathbb{E}_3$ is an Elduque-Okubo algebra: $x^2x^2 = \langle x; x^2\rangle x$
4. Totally there are 4 nonzero idempotents in $\mathbb{E}_3$ (with $e_0 := -(e_1 + e_2 + e_3)$)
5. $(\dim \mathbb{A}_c(-1), \dim \mathbb{A}_c(-\frac{1}{2}), \dim \mathbb{A}_c(\frac{1}{2})) = (0, 2, 0)$
6. $\mathbb{E}_3$ is a Hsiang algebra (a mutant) with the Peirce decomposition
$$\mathbb{E}_3 = \mathbb{A}_{c_i}(1) \oplus \mathbb{A}_{c_i}(-\tfrac{1}{2}), \quad \dim \mathbb{A}_{c_i}(-\tfrac{1}{2}) = 2$$
7. $\mathbb{E}_3 \cong \mathrm{Tri}(\mathbb{K})$.
8. $\mathrm{Aut}(\mathbb{E}_3) = S_4$ (permuting the four idempotents).

A toy example II: the simplicial algebra $\mathbb{E}_3$

Definition 3.1

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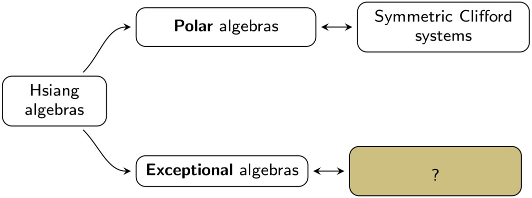
Further examples of Hsiang algebras

Definition 3.2

A commutative h-metrized algebra $\mathbb{A}$ is **polar** if there is a nontrivial $\mathbb{Z}_2$-grading $\mathbb{A} = \mathbb{A}_0 \oplus \mathbb{A}_1$ such that $\mathbb{A}_0\mathbb{A}_0 = 0$ and $x(xy) = h(x,x)y$, $x \in \mathbb{A}_0$, $y \in \mathbb{A}_1$.

Theorem (V.T. 2010)

There exists a bijection between polar algebras symmetric Clifford systems.



For example, any exact pseudo-composition algebra $\mathbb{A}$ is exceptional. Indeed, we have $\langle x^2; x^2\rangle = \langle x^3; x\rangle = \langle x; x\rangle \neq 0$ for $x \neq 0$. This contradicts to $\mathbb{A}_0\mathbb{A}_0 = 0$.

Basic facts on Hsiang algebras, II

1. $\mathbb{A}$ is **exceptional if and only if** $\mathbb{A}_c(1) \oplus \mathbb{A}_c(-\frac{1}{2})$ is (isotopy of) a simple Jordan algebra. In this case, either $n_2 = 0$ or $n_2 = 3\mathbf{d} + 2$ and the hidden simple Jordan algebra is $\mathrm{Herm}_3(\mathbf{H_d})$, $\mathbf{d} \in \{1, 2, 4, 8\}$.
2. $\mathbb{A}$ is **mutant iff** $n_2 = 2$, this corresponds to $\mathbf{d} = 0$.
3. $\mathbb{A}$ is **exceptional or mutant iff** $\operatorname{tr} L(x)^2 = m\langle x; x\rangle$ for some real m. In this case, $m = 2(n_1 + \mathbf{d} + 1)$.
4. There are **finitely many dimensions** n of $\mathbb{A}$ where exceptional Hsiang algebras can exist. Except the case $n_2 = 0$, in all other cases, $\dim \mathbb{A} = \mathbf{3}(n_1 + 2\mathbf{d} + 1)$, where $\dim \mathbb{A}_c(-\frac{1}{2}) = 3\mathbf{d} + 2$, $\mathbf{d} \in \{0, 1, 2, 4, 8\}$.

n	2	5	8	14	26	3	6	12	24	9	12	21	15	18	30	42	27	30	54
n_1	1	2	3	5	9	0	1	3	7	0	1	4	0	1	5	9	0	1	1
n_2	0	0	0	0	0	2	2	2	2	5	5	5	8	8	8	8	14	14	26
$\mathbf{d}$	–	–	–	–	–	0	0	0	0	1	1	1	2	2	2	2	4	4	8

The colours correspond to pseudo-composition, mutants and two unsettled cases.

Tripling construction

Let $(\mathbb{A}, h, \sigma)$ be a metrized algebra with involution σ. Define a **commutative** algebra structure on $\mathbb{A} \times \mathbb{A} \times \mathbb{A}$ (the **tripling**) with a $\circ$-invariant form T by

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) := (x_3^\sigma y_2^\sigma + y_2^\sigma x_2^\sigma, x_1^\sigma y_3^\sigma + y_1^\sigma x_3^\sigma, y_2^\sigma x_1^\sigma + x_2^\sigma y_1^\sigma),$$

$$H((x_1, x_2, x_3), (y_1, y_2, y_3)) := \sum_{i=1}^{3} h(x_i, y_j).$$

When $\mathbb{A} = \mathfrak{g}$ is a **Lie algebra** endowed with the **standard** involution $\sigma = -\mathbb{1}$, its triple is the Nahm algebra construction of Kinyuon-Sagle (2002).

Definition

A metrized algebra $(\mathbb{A}, \sigma, h, \circ)$ is called a **quasicomposition algebra** if

$$x \circ (x^\sigma \circ (x \circ y)) = h(x, x)(x \circ y), \quad x, y \in \mathbb{A}.$$

Theorem 4.1 (D.J. Fox, V.T., 2024)

$T(\mathbb{A})$ is a **Hsiang algebra** *iff $\mathbb{A}$ is a* **quasicomposition algebra**. *In that case, $T(\mathbb{A})$ is a* **exceptional** *or* **mutant** *Hsiang algebra with $d(\mathbb{A}) \in \{0, 1, 2, 4, 8\}$.*

Contractions of Jordan algebras

Theorem (V.T., 2015)

Given a simple unital rank 3 Jordan algebra $(\mathbb{A}, \bullet)$ *and its arbitrary unital subalgebra* $\mathbb{B} \trianglelefteq \mathbb{A}$*, the contraction of* $\mathbb{A}$ *onto* $\mathbb{B}^\perp$ *is a Hsiang algebra.*
More precisely, let e be the unit of $\mathbb{A}$, $\mathrm{Tr}(x)$ be the trace form such that $\mathrm{Tr}(e) = 3$, $h(x, y) := \mathrm{Tr}(x \bullet y)$, and

$$\mathbb{A} = \mathbb{B} \oplus_h \mathbb{B}^\perp, \quad \pi : \mathbb{A} \to \mathbb{B}^\perp \text{ be the orthogonal projection.}$$

If $x \diamond y := \pi(x \bullet y)$ then $(\mathbb{B}^\perp, \diamond, h)$ is a Hsiang algebra with $\theta = 1/6$.

- If $\mathbb{B} = \langle e \rangle$ then $(e^\perp, \bullet)$ is the *deunitalization* of $\mathbb{A}$ and it is known as an **eikonal algebra**. The classes of eikonal algebras and **trace-less pseudo-composition algebras** coincide (and appear only in dimensions $n = 5, 8, 14, 26$). These algebras occur in many contexts including the Cartan **isoparametric** hypersurfaces with 3 distinct principal curvatures.
- Another important case is a "diagonal frame" $\mathbb{B} = \mathbb{D} = \langle e_1, e_2, e_2 \rangle$, $\sum e_i = e$; then $\mathbb{D}^\perp$ is known as a **mutant Hsiang algebra**, i.e. the **tripling** of a Hurwitz algebra.

The contractions of cubic Jordan algebras

Let $\mathcal{H}_3(\mathrm{Hurv}_d)$ denote the Jordan algebra of Hermitean 3×3-matrices over the Hurwitz algebra Hurv_d, $d \in \{1, 2, 4, 8\}$. Then

	$\mathcal{H}_3(\Omega_1)$	$\mathcal{H}_3(\Omega_2)$	$\mathcal{H}_3(\Omega_4)$	$\mathcal{H}_3(\Omega_8)$	
$\langle e \rangle$	$\mathbf{K}^5$	$\mathbf{K}^8$	$\mathbf{K}^{14}$	$\mathbf{K}^{26}$	Eikonals
$\langle e, e_1 \rangle$	$\mathbf{K}^4$	$\mathbf{K}^7$	$\mathbf{K}^{13}$	$\mathbf{K}^{25}$	
$\langle e, e_1, e_2 \rangle$	$\mathbf{K}^3$	$\mathbf{K}^6$	$\mathbf{K}^{12}$	$\mathbf{K}^{24}$	Mutants
$\mathcal{H}_3(\Omega_1)$	$-$	$-$	$\mathbf{K}^9$	$\mathbf{K}^{21}$	Triplings
$\mathcal{H}_3(\Omega_2)$	$-$	$-$	$-$	$\mathbf{K}^{18}$	Triplings
$\mathcal{H}_3(\Omega_4)$	$-$	$-$	$-$	$-$	

In the remained $(-)$ cases, the contraction is a zero algebra.

Mutants

A borderline case of polar algebras, called mutants, share important properties of exceptional algebras.
Given a Hurwitz algebra Hurv_d with unity e, conjugation $\bar{x}$, norm $n(x)$, $\dim \mathbf{H}_d = d \in \{1, 2, 4, 8\}$, the **tripling**

$$\mathrm{Tri}(\mathbf{H}_d) := \mathbf{H}_d \times \mathbf{H}_d \times \mathbf{H}_d = V_1 \oplus V_2 \oplus V_3$$

with **commutative** multiplication

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = (\bar{x}_3\bar{y}_2 + \bar{y}_3\bar{x}_2, \bar{x}_1\bar{y}_3 + \bar{y}_1\bar{x}_3, \bar{x}_2\bar{y}_1 + \bar{y}_2\bar{x}_1)$$

and an invariant bilinear form $H((x_1, x_2, x_3), (y_1, y_2, y_3)) = \sum_{i=1}^{3} t(\bar{x}_i y_j)$, where $t(x) = n(x + e) - n(x) - n(e)$ is the trace form ('the real part'). Note that

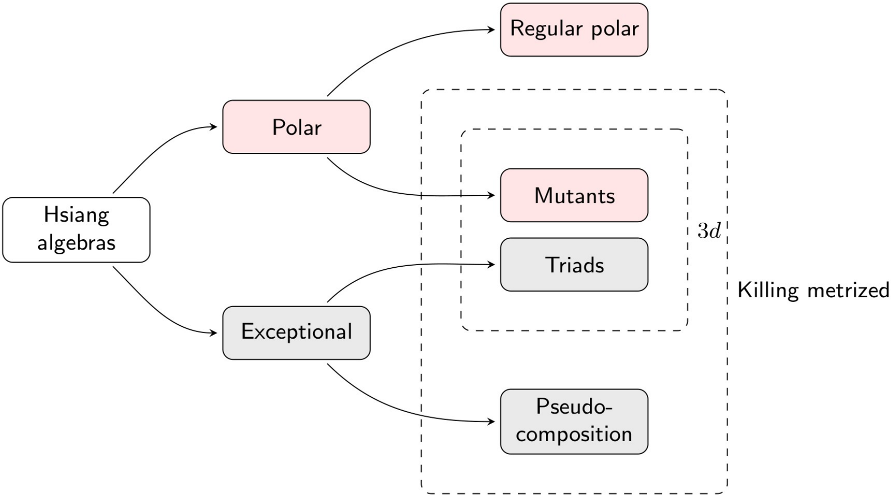
$$V_i V_i = 0, \qquad V_i V_j = V_k$$

(cf. the concept of Cartan's triality)

Proposition (D. Fox, V.T., 2024)

$\mathrm{Tri}(\mathbf{H}_d)$ is a **polar algebra** w.r.t. to any of the three decompositions $V_k \oplus V_k^{\perp}$.
The corresponding defining form $u(x) = u(x_1, x_2, x_3) = t(x_1(x_2x_3))$.

On the Origin of Species



All examples of Hsiang algebras (including the infinite family of polar algebras) reveal a remarkable structure of idempotents:

Not a theorem yet

1. Except the dimensions 2 and 3, the set of idempotents $\mathrm{Idm}(\mathbb{A})$ of a Hsiang algebra is a **homogeneous subspace** of the Euclidean sphere.
2. The dimension of the submanifold $\mathrm{Idm}(\mathbb{A})$ is $\dim A_c(\frac{1}{2})$.
3. For finer properties and $\mathrm{Aut}(\mathbb{A})$ one need to work with pre-idempotents (i.e. the elements x satisfying $x^2x^2 = -x^2$)

A heuristic argument explaining item (2) (Krasnov-V.T., 2018): Let $c \in \mathrm{Idm}(\mathbb{A})$ be a fixed idempotent, then for $c + x \in \mathrm{Idm}(\mathbb{A})$ close to c enough:

$$c + x = (c + x)^2 = c + 2cx + x^2 \approx c + 2cx \quad \Rightarrow \quad 2cx = c$$

$$\Rightarrow \quad x \in \mathbb{A}_c(\tfrac{1}{2}) \quad \Rightarrow \quad \dim \mathrm{Idm}(\mathbb{A}) = \dim \mathbb{A}_c(\tfrac{1}{2})$$

Idempotents in Triple($\mathbb{B}$)

Proposition 5.1

Let $\mathbb{B}$ *be a quasicomposition algebra. Then* $c \in \mathsf{Idm}(\mathsf{Triple}(\mathbb{B}))$ *iff* $c = (x, y, 2(xy)^\sigma)$, *where* $|x| = |y| = \frac{1}{2}$ *and* $y \in \mathsf{im}(L(x^\sigma))$

Example 5.1

1. If $\mathbb{B} = \mathrm{Hurv}_d$ is a Hurwitz algebra then $\mathsf{im}(L(x^\sigma)) = \mathrm{Hurv}_d$, therefore

$$\mathsf{Idm}(\mathsf{Triple}(\mathrm{Hurv}_d)) \cong S^{d-1} \times S^{d-1}.$$

2. If $(\mathfrak{so}(3, \mathbf{K}), \times)$ then $\mathsf{im}(L(x^\sigma)) = x^\perp$

$$\mathsf{Idm}(\mathsf{Triple}(\mathfrak{so}(3, \mathbf{K})) \cong \mathrm{St}(2, \mathbf{K}^3) = SO(3, \mathbf{K}).$$

3. If $\mathbb{M}$ the Malcev algebra of imaginary octonions then $\mathsf{im}(L(x^\sigma)) = x^\perp$ and

$$\mathsf{Idm}(\mathsf{Triple}(\mathbb{M}) \cong \mathrm{St}(2, \mathbf{K}^7) = G_2/SU(2)$$

dim $\mathbb{A}$	dim Idm($\mathbb{A}$)	Idm($\mathbb{A}$)	type	tripling
5	2	$\mathbb{RP}^2$	E.-O. (eikonal)	
6	2	$S^1 \times S^1$	mutant	yes
8	4	$\mathbb{CP}^2$	E.-O. (eikonal)	
9	3	$SO(3)$	pseudo-comp.	yes
12	5	$SU(3)/SO(3)$	para-complex	
12	6	$S^3 \times S^3$	mutant	yes
14	8	$\mathbb{HP}^2$	E.-O. (eikonal)	
15	6	$SU(4)/S(U(1) \times U(3))$	pseudo-comp.	
18	8	$SU(4)$	para-complex	yes
21	11	$G_2/SU(2)$	contraction	yes
24	14	$S^7 \times S^7$	mutant	yes
26	16	$\mathbb{OP}^2$	E.-O. (eikonal)	
27	12	$Sp(4)/Sp(1) \times Sp(3)$	pseudo-comp.	
30	14		para-complex	
54	26	E_6/F_4	para-complex	

項武義

Hsiang Wu-Yi Nov

十一月四日

2016

Thank you!

n	n1(-1)	n2(-1/2)	n3(1/2)	d	replica	terms	m QuasiCLiff	n/3	chi	rho		Tripl	Idm	[illegible]	Aut	Comments	
2	1	0	0													Eikonal	
5	2	0	2										$\mathbb{R}P^2$	3	3 $SO(3)$	Eikonal	
8	3	0	4										$\mathbb{C}P^2$	6	8 $SU(3)$	Eikonal	
14	5	0	8										$\mathbb{H}P^2$	12	21 $Sp(3)$	Eikonal	
26	9	0	16										$\mathbb{O}P^2$	24	52 E_6	Eikonal	
3	0	2	0	0	1	1	1	1	1	1	1	y	$S^0 \times S^0$	—		Mutant	
6	1	2	2	0	2	4	2	2	2	2	2	y	$S^1 \times S^1$	2		Mutant	
12	3	2	6	0	4	16	4	4	4	4	4	y	$S^3 \times S^3$	8		Mutant	
24	7	2	14	0	8	64	8	8	8	8	8	y	$S^7 \times S^7$	20		Mutant	
9	0	5	3	1	2	6	2	3	4	4	4	y	$SO(3)=SO(4)/SO(1)\times SO(3)$	—		A	
12	1	5	5	1		12	3	4	5	1	5		$SU(3)/SO(3)$	5		Lagrange	grassma,
15	2	5	7	1		20	4	5	6	2	6						
15	0	8	6	2	3	15	3	5	7	1	7		$SU(4)/S(U(1)\times U(3))$	—		A	
18	1	8	8	2	4	24	4	6	8	8	8	y	$SU(4)$	8			
21	4	5	11	1	6	42	6	7	8	8	8	y	$V_2(\mathbb{R}^7)=G_2/SU(2)$	14			—
21	2	8	10	2		35	5	7	9	1	9						
24	3	8	12	2		48	6	8	10	2	10			—			
27	0	14	12	4	5	45	5	9	13	1	13		$Sp(4)/Sp(1)\times Sp(3)$	—		A	
30	1	14	14	4	6	60	6	10	14	2	14			14			
30	5	8	16	2		80	8	10	12	4	12						78-26
33	2	14	16	4		77	7	11	15	1	15						
36	3	14	18	4		96	8	12	16	9	16						
42	9	8	24	2		168	12	14	16	9	16						
51	0	8	6	8		153	9	17	25	10	7		$\nexists$				
54	1	26	26	8	10	180	10	18	26	2	26		E_6/F_4 ?	26			
57	2	26	28	8		209	11	19	27	1	27						
60	3	26	30	8		240	12	20	28	4	28						
72	7	26	38	8		384	16	24	32	10	32						