Everything you always wanted to know about

... Hidden Jordan algebra structures

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A brief overview

1. Some motivations
2. Jordan algebras
3. Algebras of cubic forms
4. Example: Hidden Jordan algebras inside algebras of minimal cones

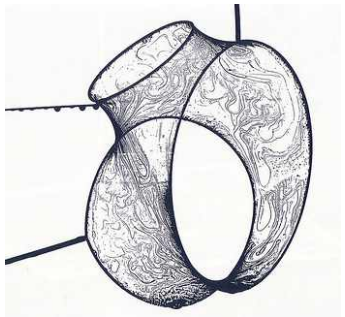
Outline

1. Some motivations
2. Jordan algebras
3. Algebras of cubic forms
4. Example: Hidden Jordan algebras inside algebras of minimal cones

Episode 1: Bernstein's theorem

The following definitions of a **minimal hypersurface** are equivalent:

- A critical point of the area functional
- The mean curvature $= 0$
- If $x_{n+1} = u(x)$ is a graph over $\mathbb{R}^n$ then $\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 0$



Theorem (S. Bernstein, 1915)

Any **entire** *analytic* solution u on $\mathbb{R}^2$ is a plane: $u = ax + by + c$.

The Bernstein result holds true for all $2 \leq n \leq 7$: Fleming (1962), De Giorgi (1965), Simons (1968) BUT: for $n \geq 8$ there are **non-linear** solutions. The construction (Bombieri-De Giorgi-Giusti, 1969) is based on geometric measure theory of **minimal cones** given by the norm of split octonions:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2 - x_7^2 - x_8^2$$

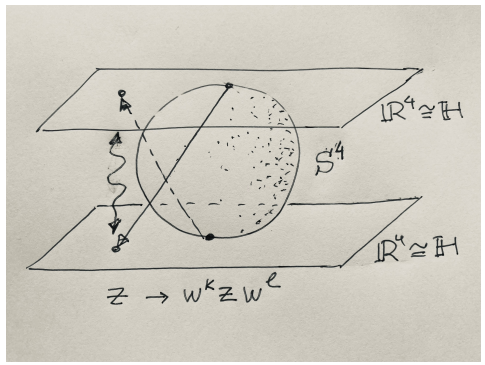
Why the octonions? No good acceptable explanation is found so far.

Episode 2: Milnor's exotic 7-spheres

The famous seven-page Milnor's proof (1956) was a virtuous combination of the Morse theory, Thom's cobordisms, and the **quaternionions**. He constructs the exotic 7-dimensional spheres M^7_k as S^3-bundles over S^4 in a very explicit way using **quaternionic multiplication**. A key ingredient is to

twist two copies of $\mathbb{R}^4 \cong \mathbb{H}$ by the **noncommutative** quaternion multiplication. Note that S^4 can be covered by an atlas with only two charts and the gluing maps

$$\boxed{z \to z' := w^k \cdot z \cdot w^l}$$



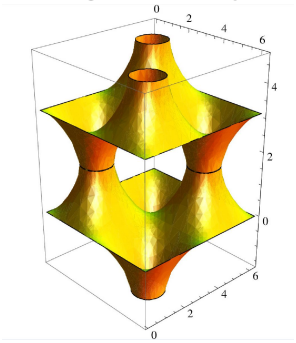
Thus obtained bundles are diffeomorphic to the standard S^7 iff

$$(k-l)^2 \equiv 1 \mod 7.$$

Why the quaternions? Do the exists any connection between these two results? My supervisor V. Miklyukov (with origins from global analysis) has believed that there should be a conceptual explanation for the 8-phenomenon.

Episode 3: Quaternions revisited...

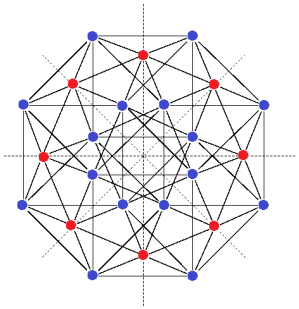
In 2008, I discovered a remarkable embedded **fourfold periodic** minimal hypersurface in $\mathbb{R}^4$ given by the lemniscatic sinus:



$$s(x_1)s(x_2) - s(x_3)s(x_4) = 0, \qquad \text{where } s(x) = \mathrm{sn}(x, \sqrt{-1})$$

which has isolated cone type singularities

$$x_1x_2 - x_3x_4 = 0 \qquad \text{or} \qquad x_1^2 + x_2^2 = x_3^2 + x_4^2$$



forming a lattice is isometric to the periodic F_4-lattice of the *Hurwitz quaternions* (the densest possible lattice packing of balls in $\mathbb{R}^4$). Notice that the above cones are given by the **split quaternion norms!**

This naturally led to the question: **How to construct embedded minimal hypersurfaces with given cone type singularities?**

When I started looking for available algebraic minimal cones, it turned out that there were only few known. The first series wss the so-called **isoparametric foliations** and the second was the **Hsiang minimal cones**.

Episode 4: Isoparametric foliations

A hypersurface M is called **isoparametric** if its principal curvatures are constant; by m one denotes the number of **distinct principal curvatures**. Münzner (1980) used algebraic topology to prove that $m \in \{1, 2, 3, 4, 6\}$, and M is a level set of a **homogeneous degree** m **polynomial** $u(x)$ in $\mathbb{R}^n$ satisfying

$$\begin{aligned} |\nabla u(x)|^2 &= m^2 \langle x, x \rangle^{m-1}, \\ \Delta u(x) &= C \langle x, x \rangle^{(m-2)/2}. \end{aligned} \tag{1}$$

The cases $m = 1, 2$ are trivial. E. Cartan (1938) completely characterized the case $m = 3$ and proved that there are exactly four isoparametric foliations in $\mathbb{R}^{3d+2}$

$$M^{3d} \subset S^{3d+1} \subset \mathbb{R}^{3d+2}, \qquad d = 1, 2, 4, 8,$$

For example, for $d = 1$:

$$u(x) := x_5^3 + \tfrac{3}{2} x_5 (x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \tfrac{3\sqrt{3}}{2} x_4 (x_2^2 - x_1^2) \quad \underbrace{+3\sqrt{3} x_1 x_2 x_3}_{\text{triality polynomial for } d = 1}$$

The values of d are the dimensions of Hurwitz composition algebras.
Why do the Hurwitz algebras and why the trialities appear here?

Episode 5: Hsiang's eigencubics

W.-Y. Hsiang (1967): Given a homogeneous polynomial $u(x)$, the cone $\{x \in \mathbb{R}^n : u(x) = 0\}$ is a **minimal hypersurface** in $\mathbb{R}^n$ iff

$$\Delta_1 u := |\nabla u|^2 \Delta u - \tfrac{1}{2}\langle \nabla u, \nabla |\nabla u|^2 \rangle = S(x)u(x).$$

where S is a **undetermined** homogeneous degree $(2 \deg u - 4)$ polynomial.

- The only minimal cones of $\deg u = 2$ are the Clifford cones
- The first **non-trivial case**: $\deg u = 3$ when

$$\Delta_1 u = S(x)u(x) = \text{a quadratic form} \cdot u(x)$$

  All known irreducible examples satisfy a simpler equation with $S(x) = \alpha\langle x, x\rangle$ ($\approx$ the Euclidean norm).

- Remarkable, the (four) available Hsiang examples were given by the generic determinants in appropriate Jordan algebras in dimensions $5, 9, 15, 18$. Why?

Hisang's Problem (ii): Classify all cubic minimal cones.

Episode 6: Truly viscosity solutions

Let $F(D^2u) = 0$ be uniformly elliptic. In general, there are no tools to guarantee the existence of a classical solution. Crandall and P.-L. Lions (Fields medal, 1994) introduced the notion of **viscosity solution**, which is a kind of generalized solution. Their definition is significant since they were able to establish

The existence and uniqueness (Evans, Crandall, Lions, Jensen, Ishii)

The Dirichlet problem for $F(D^2u) = 0$ has a unique **viscosity solution** $u \in C^0$.

So, the principal question is the regularity of the viscosity solution:

$$\boxed{u(x) \text{ is a viscosity solution}} \overset{?}{\Longrightarrow} \boxed{u(x) \text{ is a classical } C^2 \text{ solution}}$$

- If $n = 2$ then u is classical ($C^{2,\alpha}$) solution (Nirenberg 1953)
- Hölder estimates imply that Krylov-Safonov (1980), Trudinger-Caffarelli (1989)

$$u \in C^{1,\alpha}, \qquad \alpha = \alpha(\text{ellipticity const}, n)$$

Episode 6: Truly viscosity solutions

For a long time it was unknown if the Krylov-Safonov-Cafarelli-Trudinger regularity is optimal until the main breakthrough has been done by Nadirashvili and Vlăduț in 2007:

(Nadirashvili, Vlăduț, 2007-2011)

Let $\mathbb{K}_d$ be a Hurwitz algebra of dimension $d \in \{4, 8\}$. Then for any $\delta \in [1, 2)$

$$u_\delta(\mathbf{x}) := \frac{\mathrm{Re}(xy)z}{\|\mathbf{x}\|^\delta}, \qquad \mathbf{x} = (x, y, z) \in \mathbb{K}_d^3,$$

is a viscosity solution to a certain uniformally elliptic equation $F(D^2u(x)) = 0$.

Note that the numerator is the **triality polynomial** over $\mathbb{K}_d$. The assumption that $d \neq 1, 2$ is essential and requires the "**non-commutativity**" of $\mathbb{K}_d$.

Theorem (Nadirashvili, Tkachev, Vlăduț, 2012). Let $C_5(\mathbf{x})$ be the Cartan isoparametric cubic in $\mathbb{R}^5$ and $u(\mathbf{x}) = \frac{C_5(\mathbf{x})}{|\mathbf{x}|}$ is a viscosity solution in $\mathbb{R}^5$. Furthermore,

$$F(D^2u) = (\Delta u)^5 + 2^8 3^2 (\Delta u)^3 + 2^{12} 3^5 \Delta u + 2^{15} \det D^2(u).$$

The existence of truly viscosity solutions is $\mathbb{R}^3$ and $\mathbb{R}^4$ is an **open problem**.

Let me summarize:

- The most difficult part is to find (if any) a conceptional explanation of the "8"-phenomenon in Episods 1 and 2, we don't have any good answer here.
- On the other hand, there are certain 'hidden' Jordan algebra structures behind the constructions described in Episodes 3,4, 5 and 6; we explain this below.

How it works

- An important observation here is that a homogeneous polynomial $u(x)$ on a vector space $\mathbb{A}$ (normally of degree 3) satisfies a certain **invariant Hessian type** (i.e. with D^2u) **PDE**.

- One can transform the cubic polynomial $u(x)$ into an appropriate commutative nonassociative **metrized** algebra with an **algebra identity**.

- If the PDE is 'good enough' (for example, arises from a geometrical problem) then the obtained algebra contains a **hidden Jordan algebra structure.**

- Algebraically this means that either $\mathbb{A}$ or its invariant subspace $\mathbb{B} \subset \mathbb{A}$, carries an admissible Jordan cubic form (normally the restriction of u on $\mathbb{B}$)

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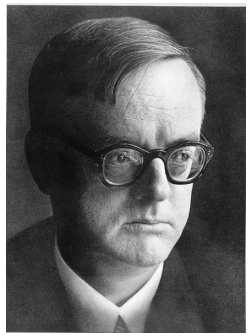
Jordan algebras

In 1932 Pascual Jordan proposed a program to discover a new algebraic setting for quantum mechanics such that it is independent on an invisible but all-determining matrix structure. How to multiply two Hermitian matrices?... Unfortunately

$$\text{Matrix} \cdot \text{Matrix} = \text{Matrix},$$
$$\text{Hermitian} \cdot \text{Hermitian} \neq \text{Hermitian}$$



The idea is to find a commutative algebra structure which is compatible with the 'squares' $x^2 = x \cdot x$ in the underlying associative structure and is power associative. Such a product can be found by

$$(x+y)^2 - x^2 - y^2 = x \cdot y + y \cdot x := 2x \bullet y.$$

This why Jordan replaces the matrix multiplication by the anticommutator

$$x \bullet y = \frac{1}{2}(x \cdot y + y \cdot x).$$

Absorbing the most important properties in the context of quantum mechanics, Jordan postulates the following definition:

Jordan algebras

A Jordan algebra $\mathbb{A}$ (over $\mathbb{F}$) is vector space defined equipped with a bilinear product $\bullet : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ satisfying the

$$x \bullet y = y \bullet x \qquad \text{Commutativity}$$

$$x^2 \bullet (x \bullet x) = x \bullet (x^2 \bullet y) \qquad \text{the Jordan identity}$$

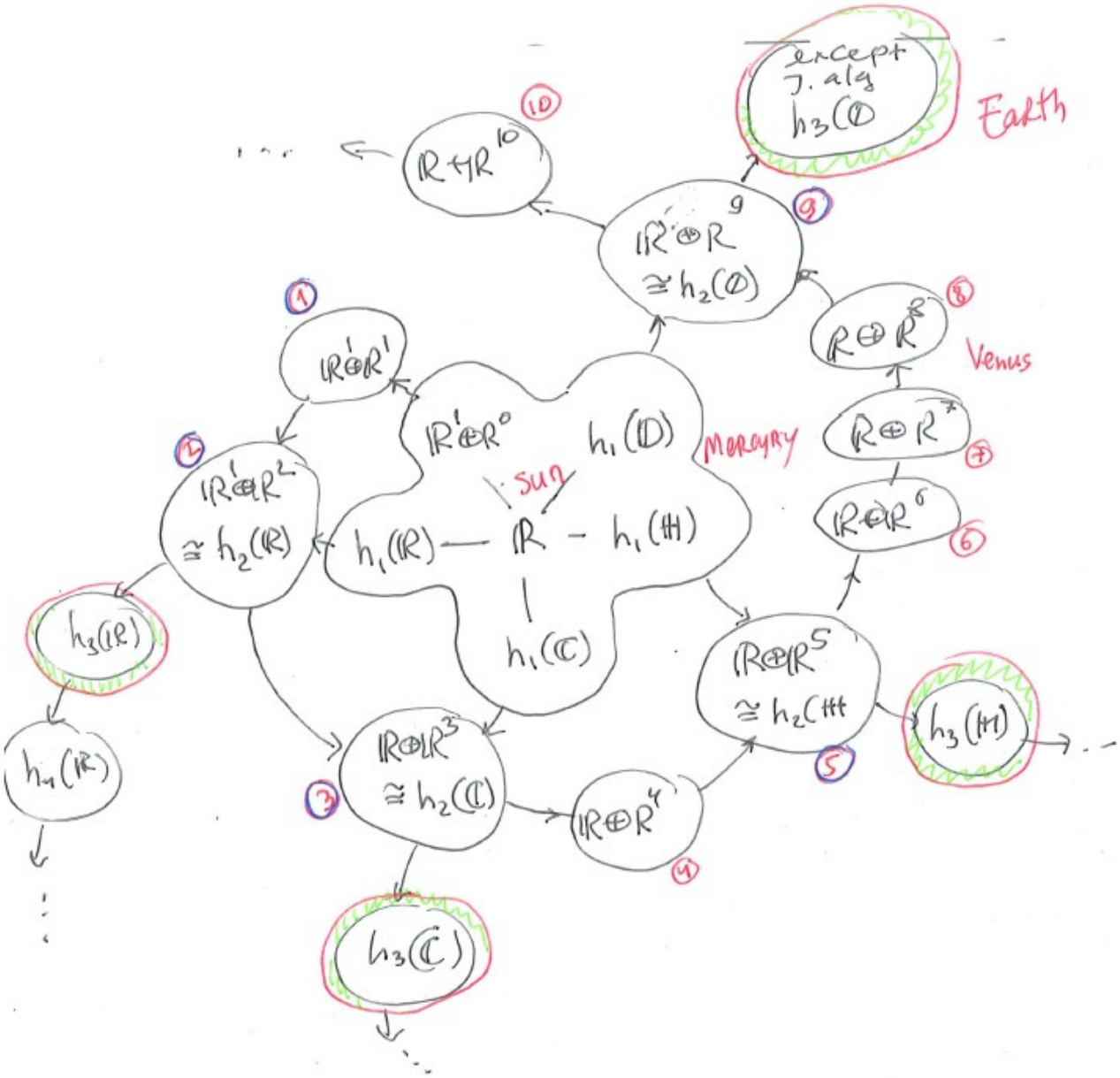
P. Jordan, J. von Neumann, E. Wigner, *Annals of Math.*, 1934

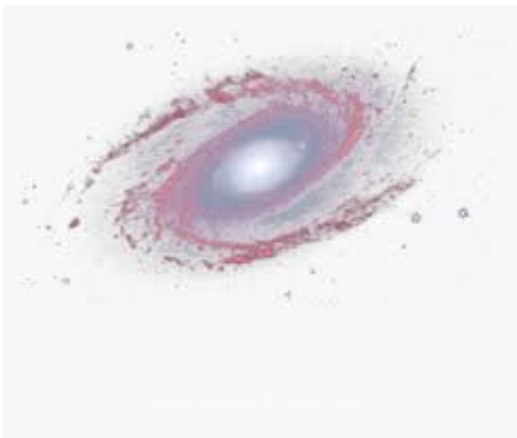
Any finite-dimensional formally real **simple** Jordan algebra is isomorphic to one of

- three special algebras coming from Hermitian matrices:
  - $\mathfrak{h}_n(\mathbb{R})$;
  - $\mathfrak{h}_n(\mathbb{C})$;
  - $\mathfrak{h}_n(\mathbb{H})$;
- or two new structures:
  - $\mathrm{spin}_n \cong \mathbb{R}^1 \oplus \mathbb{R}^n$, the spin factors (not to be confused with *spinors*);
  - $\mathfrak{h}_3(\mathbb{O})$ over the octonions $\mathbb{O}$ (the Albert algebra)

For some (still unclear) mystical reasons, all simple formally real Jordan algebras appear in many problems of geometry, algebra, analysis, pdes, combinatorics and group theory.

The Jordan Galaxy





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Nonassociative algebra of a cubic form

Let $u(x)$ be an **arbitrary cubic form** on a vector space $\mathbb{A}$ with a distinguished symmetric bilinear form $\langle , \rangle$. Recall that any cubic form u comes from a **symmetric** trilinear form U (the converse is also true): if

$$U(x,y,z) = u(x+y+z) - u(x+y) - u(x+z) - u(y+z) + u(x) + u(y) + u(z)$$

then $u(x) = \frac{1}{6}U(x,x,x)$. By the definition of the Hessian:

$$\langle (D^2 u(x))y, z\rangle = U(x,y,z) = \langle x \bullet y, z\rangle,$$

where the multiplication $\bullet$ uniquely determined and can be given explicitly by

$$x \bullet y := (D^2 u(x))y.$$

This implies have the following important properties:

(i) $x \bullet y = y \bullet x$ (the algebra $\mathbb{A}(u)$ is **commutative**)

(ii) $\langle x \bullet y, z\rangle = \langle y, x \bullet z\rangle$ ($\mathbb{A}(u)$ is a **Euclidean metrized algebra**)

(iii) $x \bullet x = 2\nabla u(x)$ (by Euler's homogeneous function theorem)

(iv) $\langle x \bullet x, x\rangle = 6u(x)$ (by Euler's homogeneous function theorem)

Example. Consider $u(x) = \frac{1}{6}x_1(x_1^2 - 3x_2^2 - 3x_3^2)$ on $\mathbb{A} = \mathbb{K}^3$. Then

$$H(x) := D^2u(x) = \begin{pmatrix} x_1 & -x_2 & -x_3 \\ -x_2 & -x_1 & 0 \\ -x_3 & 0 & -x_1 \end{pmatrix}$$

The algebra $\mathbb{A}(u)$ on $\mathbb{K}^3$ with commutative multiplication

$$x \bullet y := H(x)y = (x_1y_1 - x_2y_2 - x_3y_3,\ -x_2y_1 - x_1y_2,\ x_3y_1 - x_1y_3)$$

A simple examination reveals the identities in the spirit of "3D-complex numbers":

$$x \bullet x = (x_1^2 - x_2^2 - x_3^2,\ -2x_1x_2,\ -2x_1x_3)$$
$$(x \bullet x) \bullet x = (x_1^2 + x_2^2 + x_3^2)x = \langle x, x\rangle x$$

Remarks

- The latter identity implies that $\mathbb{A}(u)$ is a pseudocomposition algebra (in the sense of Meyberg-Osborn, Elduque). See also very recent results by Gorshkov, Mamontov and Staroletov arXiv:2309.05237.
- In fact, $u(x)$ solves also the Hsiang equation.

One of the crucial tools

Recall that $N(x;y) := \frac{1}{2}N(x;x;y) \approx \frac{\partial}{\partial y}N(x)$.

The Springer-Freudenthal Construction (McCrimmon, 1969)

Let $N : \mathbb{A} \to \mathbb{F}$ be a cubic form on a vector space $\mathbb{A}$ with a distinguished basepoint e, $N(e) = 1$. The cubic form N is called a admissible if the bilinear form

$$T(x;y) = N(e;x)N(e;y) - N(e;x;y)$$

is a nondegenerate and the map $\# : \mathbb{A} \to \mathbb{A}$ (uniquely) determined by $T(x^\#;y) = N(x;y)$ satisfies the adjoint identity

$$(x^\#)^\# = N(x)x.$$

If N is admissible and $x\#y = (x+y)^\# - x^\# - y^\#$ then

$$x \bullet y = \tfrac{1}{2}(x\#y + N(e;x)y + N(e;y)x - N(e;x;y)e)$$

defines a Jordan algebra structure on $\mathbb{A}$.

An important ingredient of the Springer-Freudenthal construction is the 2nd derivative

$$T(x;y) := \frac{\partial^2}{\partial x \partial y} \log N \Big|_{\text{at } e} = \frac{N(e;x)N(e;y) - N(e;x;y)}{N(e)^2}$$

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Algebras of minimal cones

u on $\mathbb{A}$ is a solution of a PDE $\Rightarrow$ $\mathbb{A}(u)$ satisfies an identity.

Let $u(x)$ be a Hsiang cubic, i.e. for some real constant θ:

$$|\nabla u|^2 \Delta u - \nabla u(x)^\top \operatorname{Hess}(u(x)) \nabla u(x) = \theta |x|^2 u(x).$$

One can show (nontrivial!) that any non-trivial Hsiang algebra is trace-free, i.e. satisfies $\operatorname{tr} L(x) = 0$, so that the metrized algebra of the cubic form $u(x)$ satisfies the following nice **identity**:

$$\langle x^2, x^3 \rangle = -\frac{2}{3} \theta \langle x, x \rangle \langle x^2, x \rangle$$

with $u(x) = \frac{1}{6}\langle x, x^2 \rangle$. Linearization yields an algebra identity in one variable

$$\boxed{x^3 x + \tfrac{1}{4} x^2 x^2 - \langle x, x \rangle x^2 - \tfrac{2}{3} \langle x^2, x \rangle x = 0} \tag{2}$$

Definition

Any commutative $\langle , \rangle$-metrized algebra satisfying (2) is called is a **Hsiang algebra**.

Polar vs exceptional

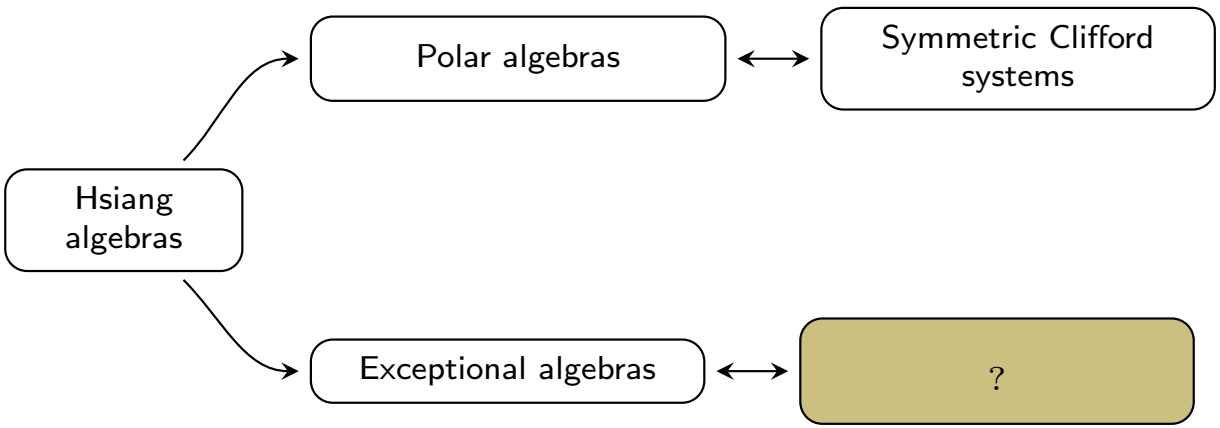
Definition

A commutative b-metrised $\mathbb{Z}_2$-graded algebra $V = V_0 \oplus V_1$ is called polar if

$$V_0 \bullet V_0 = 0 \qquad \text{and} \qquad L(x)^2 = b(x,x)\mathbf{1} \text{ on } V_1, \quad \forall x \in V_0.$$

Definition

A Hsiang algebra $\mathbb{A}$ isomorphic to a polar algebra is of **Clifford type**; otherwise it is **exceptional**.



The hidden Jordan algebra inside Hsiang algebras

Theorem (V. Tkachev, 2013-2019)

Let $\mathbb{A}$ be a Hsiang algebra. Then the set of nonzero idempotents is nonempty and

- For each idempotent c, the spectrum of $L(c)$ is a subset of $\{\pm 1, \pm\frac{1}{2}\}$, the multiplicity of an eigenvalue is independent of a choice of an idempotent.
- The decomposition into $L(c)$-invariant subspaces holds:

$$\mathbb{A} = \mathbb{R}c \oplus \mathbb{A}_c(-1) \oplus \mathbb{A}_c(-\tfrac{1}{2}) \oplus \mathbb{A}_c(\tfrac{1}{2})$$

- The invariant subspace $\Lambda_c := \mathbb{R}c \oplus \mathbb{A}_c(-1)$ caries a hidden Clifford algebra structure.
- The invariant subspace $\Lambda_c := \mathbb{R}c \oplus \mathbb{A}_c(-\frac{1}{2})$ is a subalgebra. The new multiplication

$$x \bullet y = \tfrac{1}{2}xy + \langle x, c\rangle y + \langle y, c\rangle x - 2\langle xy, c\rangle c, \qquad x, y \in \mathbb{R}c \oplus \mathbb{A}_c(-\tfrac{1}{2}),$$

  turns Λ_c into a rank 3 **Euclidean Jordan algebra** with the unit $2c$
- Any element $x \in \Lambda_c$ satisfies the cubic identity

$$x^{\bullet 3} - 2\langle x, c\rangle x^{\bullet 2} + (2\langle x, c\rangle^2 - \tfrac{1}{2}|x|^2)x - \tfrac{1}{3}\langle x, x^2\rangle c = 0,$$

- $\mathbb{A}$ is exceptional if and only if Λ_c is a simple Jordan algebra.
- There are finitely many exceptional Jordan algebras.

Idea of the proof

We know that Λ_c is a subalgebra and since $e = 2c$ and $c^2 = c$, $e^2 = 2e$, $\langle e, e\rangle = 3$,

$$x = \lambda e \oplus z \in \Lambda_c \quad \Rightarrow \quad \text{manipulations!} \quad \Rightarrow \quad L(e) = e \otimes e - 1$$

in particular, $xe \in \operatorname{span}(e, x)$. Using the lineartization of the Hsiang identity (2), yields

$$x^2x^2 = \tfrac{4}{3}\langle x, x^2\rangle x \quad \text{on } \Lambda_c. \tag{$*$}$$

Let $N(x) := \frac{1}{6}\langle x, x^2\rangle$. Then $N(e) = 1$ and $N(x, y, z) = \langle xy, z\rangle$,

$$\begin{aligned}
&N(x; y) = \tfrac{1}{2}\langle x^2, y\rangle \\
&N(e; x) = \langle e, x\rangle \\
&T(x; y) = N(e; x)N(e; y) - N(e; x; y) = \langle e, x\rangle\langle e, y\rangle - \langle e, xy\rangle = \langle \langle e, x\rangle e - xe, y\rangle,
\end{aligned}$$

Therefore $T(x; y) = \langle x, y\rangle$ and $T(x^\#; y) = N(x; y)$ is equivalent to $\langle x^\#, y\rangle = \frac{1}{2}\langle x^2, y\rangle$ implying $x^\# = \frac{1}{2}x^2$. Then $(*)$ yields

$$x^{\#\#} = \tfrac{1}{8}x^2x^2 = \tfrac{1}{6}\langle x^2, x\rangle = N(x)x,$$

therefore the claim follows.

Theorem (V. Tkachev, 2014)

There is a natural correspondence between

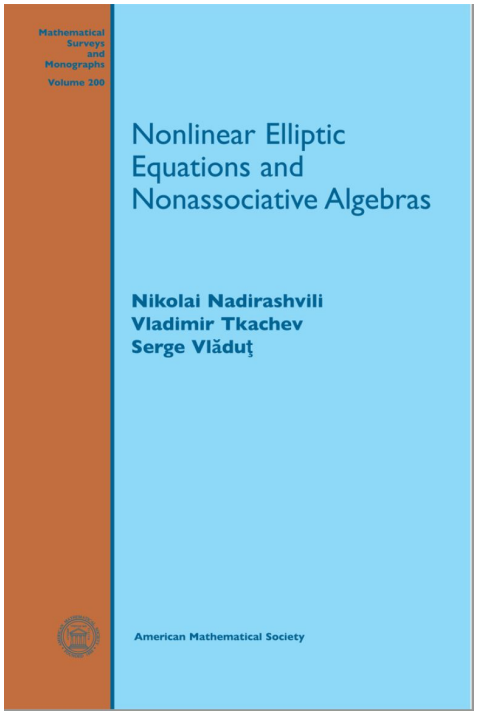
- cubic solutions of $|\nabla u(x)|^2 = 9|x|^4$ in $\mathbb{R}^n$, and
- rank 3 formally real semisimple Jordan algebras

Proof. Let $\mathbb{A} = V(u)$, then $u(x) = \frac{1}{6}\langle x^2, x\rangle$, $\nabla u(x) = \frac{1}{2}x^2$, hence $\langle x^2, x^2\rangle = 36|x|^4$. Let $W = \mathbb{R} \oplus \mathbb{A}$ and define a cubic form

$$N(\boldsymbol{x}) = x_0^3 - \frac{3}{2}x_0|x|^2 + \tfrac{1}{6\sqrt{2}}\langle x^2, x\rangle, \quad \boldsymbol{x} = (x_0, x) \in W.$$

Then $\boldsymbol{e} = (1, 0)$ is a base point: $N(\boldsymbol{e}) = 1$. The polarization yields:

$$\begin{aligned}
& N(\boldsymbol{x};\boldsymbol{y}) && = 3x_0^2y_0 - 3x_0\langle x, y\rangle - \tfrac{3}{2}|x|^2y_0 + \tfrac{1}{2\sqrt{2}}\langle x^2, y\rangle \\
\Rightarrow\quad & N(\boldsymbol{x};\boldsymbol{e}) && = 3x_0^2 - \tfrac{3}{2}|x|^2 \qquad \text{and} \qquad N(\boldsymbol{e};\boldsymbol{x}) = -3x_0 \\
\Rightarrow\quad & T(\boldsymbol{x},\boldsymbol{y}) && = N(\boldsymbol{e};\boldsymbol{x})N(\boldsymbol{e};\boldsymbol{y}) - N(\boldsymbol{x};\boldsymbol{y};\boldsymbol{e}) = 3(x_0y_0 + \langle x, y\rangle) = 3\langle \boldsymbol{x}, \boldsymbol{y}\rangle \\
& N(\boldsymbol{x};\boldsymbol{y}) && = T(\boldsymbol{x}^{\#}, y) \quad \text{implies} \quad \boldsymbol{x}^{\#} = (x_0^2 - \tfrac{1}{2}|x|^2, \tfrac{1}{6\sqrt{2}}x^2 - x_0x\,) \\
\Rightarrow\quad & (\boldsymbol{x}^{\#})^{\#} && = N(\boldsymbol{x})\boldsymbol{x} \quad \Rightarrow \quad N(\boldsymbol{x}) \text{ is admissible} \quad \square
\end{aligned}$$



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